Copenhagen Conference: Strings, Gauge Theory and the LHC

Quantum strings on $AdS_4 \times \mathbb{C}P^3$

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Quantizing strings on $AdS_4 \times \mathbb{C}P^3$: a long story

- **G. G., T. Harmark and M. Orselli**,
  "The $SU(2) \times SU(2)$ sector in the string dual of $\mathcal{N} = 6$ superconformal Chern-Simons theory", arXiv:0806.4959 [hep-th], Nucl. Phys. B **810**, 115 (2009)
- **G. G., T. Harmark, M. Orselli and G. W. Semenoff**,
  "Finite size Giant Magnons in the string dual of $\mathcal{N} = 6$ superconformal Chern-Simons theory", arXiv:0807.0205 [hep-th], JHEP **0812**, 008 (2008)
- **D. Astolfi, V. G. M. Puletti, G. G., T. Harmark and M. Orselli**,
  "Finite-size corrections in the SU(2) x SU(2) sector of type IIA string theory on $AdS_4 \times \mathbb{C}P^3$", arXiv:0807.1527 [hep-th], Nucl. Phys. B **810**, 150 (2009)
- **D. Astolfi, V. G. M. Puletti, G. G., T. Harmark and M. Orselli**,
  "Full Lagrangian and Hamiltonian for quantum strings on $AdS_4 \times \mathbb{C}P^3$ in a near plane wave limit.", arXiv:0912.2257 [hep-th], JHEP 1004 (2010) 079
- **D. Astolfi, V. Giangreco M. Puletti, G.G., T. Harmark, M. Orselli**
  "Finite-size corrections for quantum strings on $AdS_4 \times \mathbb{C}P^3$", arXiv:1101.0004 [hep-th] , JHEP 1105 (2011) 128
- **D. Astolfi, G. G., A. Zayakin**
  "The complete two oscillator spectrum for quantum strings on $AdS_4 \times \mathbb{C}P^3$". In preparation

Outline

Motivations and Overview

- ABJM (Aharony, Bergman,Jafferis and Maldacena, 0806.1218) theory $\rightarrow$ AdS_4/CFT_3 $\rightarrow$ should, in a suitable limit, provide another example of integrability in gauge/string duality.
- Minahan and Zarembo showed that the SU(4) sector (which contains an $SU(2) \times SU(2) \subset SU(4)$ subsector) of ABJM theory is integrable at two loop order (4-loops: Minahan, Sax, Sieg 0908.2463, 6-loops: Bak, Min, Rey, 0911.0689).
- Giombi, Gaiotto and Yin and G.G., Harmark and Orselli proposed an expression for the all loop fundamental magnon dispersion relation.

$$E = \sqrt{\frac{1}{4} + 4h^2(\lambda)\sin^2\frac{p}{2}}$$

(Compare with AdS_5/CFT_4: $E \sim \sqrt{1 + \lambda \sin^2 \frac{p}{2}}$)

with a strong-weak coupling interpolating function $h(\lambda)$.
λ is the t'Hooft coupling.

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- Suppose $h(\lambda)$ at strong coupling has the form

$$h(\lambda) = \sqrt{\lambda} + a_1 + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)$$

then from the pp-wave dispersion relation for large J fixed $\lambda' = \frac{\lambda}{J^2}$ we would get a correction of the form

$$E = \sqrt{\frac{1}{4} + 2\pi^2 n^2 \lambda'} + 4\pi^2 n^2 a_1^2 \frac{\sqrt{\lambda'}}{J}$$

namely a $1/J$ correction.

- This can be obtained by a sigma model calculation of the finite size corrections to the energy of oscillator states.
- Such a correction has been obtained for the spinning string with a one loop calculation. Some calculations give $a_1 = 0$ others $a_1 = -\frac{log 2}{2\pi}$ (McLoughlin,Roiban;Alday,Arutyunov,Bykov;Krishnan;Gromov,Mikhaylov;McLoughlin,Roiban,Tseytlin, ...)
- Gromov and Vieira proposed an algebraic curve and an all loop Bethe ansatz for the whole OSp(6|4) group. Type IIA string theory on $AdS_4 \times \mathbb{C}P^3$ seems to be integrable at the quantum level.
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1. Quantize type IIA superstring on $AdS_4 \times \mathbb{C}P^3$ in suitable limits, pp-wave, Landau-Lifshits, giant magnon and derive the fundamental magnon dispersion relation.
2. Derive the full interacting Lagrangian and Hamiltonian for quantum strings in $AdS_4 \times \mathbb{C}P^3$ that fluctuate around a null curve on $\mathbb{C}P^3$. There are both cubic and quartic terms.
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ABJM theory (Aharony, Bergman, Jafferis, Maldacena, Jun 08)

It is a 2+1 dimensional $\mathcal{N}=6$ superconformal Chern-Simons theory with gauge group $\mathrm{U}(N)_k \times \mathrm{U}(N)_{-k}$, the component action reads

$$\mathcal{S} = \frac{k}{4\pi}\int d^3x \Big[\epsilon^{\mu\nu\lambda}\,\mathrm{Tr}\big(A_\mu\partial_\nu A_\lambda + \tfrac{2i}{3}A_\mu A_\nu A_\lambda - \hat{A}_\mu\partial_\nu\hat{A}_\lambda - \tfrac{2i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda\big) - \mathrm{Tr}(D_\mu Y)^\dagger D^\mu Y - i\,\mathrm{Tr}\,\psi^\dagger \not{D}\psi - V_{\rm ferm} - V_{\rm bos}\Big]\,,$$

where the sextic bosonic and quartic mixed potentials are

$$V^{\rm bos} = -\frac{1}{12}\,\mathrm{Tr}\Big[Y^A Y_A^\dagger Y^B Y_B^\dagger Y^C Y_C^\dagger + Y_A^\dagger Y^A Y_B^\dagger Y^B Y_C^\dagger Y^C + 4 Y^A Y_B^\dagger Y^C Y_A^\dagger Y^B Y_C^\dagger - 6 Y^A Y_B^\dagger Y^B Y_A^\dagger Y^C Y_C^\dagger\Big]$$

$$V^{\rm ferm} = \frac{i}{2}\,\mathrm{Tr}\Big[Y_A^\dagger Y^A \psi^{\dagger B}\psi_B - Y^A Y_A^\dagger \psi_B \psi^{\dagger B} + 2 Y^A Y_B^\dagger \psi_A \psi^{\dagger B} - 2 Y_A^\dagger Y^B \psi^{\dagger A}\psi_B - \epsilon^{ABCD} Y_A^\dagger \psi_B Y_C^\dagger \psi_D + \epsilon_{ABCD} Y^A \psi^{\dagger B} Y^C \psi^{\dagger D}\Big]\,.$$

The covariant derivative acts on bi-fundamental fields as

$$D_\mu Y = \partial_\mu Y + i A_\mu Y - i Y \hat{A}_\mu\,,$$

while on anti-bi-fundamental fields it acts with A_μ and $\hat{A}_\mu$ interchanged.

ABJM model: content and symmetries

- Fields

		$\mathrm{U}(N)_k$	$\mathrm{U}(N)_{-k}$	$\mathrm{SU}(2)$	$\mathrm{SU}(2)$
2 Gauge fields	A_μ	adj	1	1	1
	$\hat{A}_\mu$	1	adj	1	1
4+4 Scalars	Y^A	N	$\bar{N}$	2	1
	$Y_A^\dagger$	$\bar{N}$	N	1	2
Fermions	Ψ^A	N	$\bar{N}$	2	1
	$\Psi_A^\dagger$	$\bar{N}$	N	1	2

with $\mu = 0, 1, 2,\ A = 1, 2, 3, 4$

- 4 complex scalars in the bifundamental representations of $\mathrm{U}(N) \times \mathrm{U}(N)$

$$Y^A = (A_1, A_2, B_1^\dagger, B_2^\dagger)\ ,\quad Y_A^\dagger = (A_1^\dagger, A_2^\dagger, B_1, B_2)$$

grouped into multiplets (fundamental and anti-fundamental) of the R-symmetry group $\mathrm{SU}(4)$ and their superpartners

- Symmetries: it is superconformal the symmetry group is

$$\text{Full symmetry}\ \ \mathrm{OSp}(6|4) \begin{cases} \mathrm{SU}(4)_R \text{ R-symmetry} \\ \text{conformal in 3D: } \mathrm{SO}(3,2) \\ \mathcal{N} = 6 \text{ supersymmetries} \end{cases}$$

- Effective theory of a stack of M2-branes probing $\mathbb{C}^4/\mathbb{Z}_k$ singularity
- The theory contains two parameters: k, N
- Large-N limit: gravitational dual is M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$
- Large-k limit: IIA superstring theory on $AdS_4 \times \mathbb{C}P^3$

AdS_4/CFT_3 correspondence

- k,N parameters
- $1/k \sim$ coupling constant
- Existence of a weakly coupled string dual guaranteed by the field theory adjustable parameter that enables to go to weak coupling for fixed N

$\Rightarrow$ 't Hooft coupling in the ABJM model

$$\boxed{\lambda = \frac{N}{k}}$$ $\Rightarrow$ large N, k and λ= fixed :

$\mathcal{N}=6$ planar super CS gauge theory at level k and $-k$ gauge group $\mathrm{U}(N)\times\mathrm{U}(N)$ $\lambda \ll 1$	$\Leftrightarrow$	type IIA strings on $AdS_4 \times \mathbb{C}P^3$ N units of $F_{(4)}$ flux on AdS_4 k units of $F_{(2)}$ flux on a $\mathbb{C}P^1 \subset \mathbb{C}P^3$ $\lambda \gg 1$

AdS_4/CFT_3 correspondence

the 't Hooft coupling

$$\boxed{\lambda = \frac{N}{k}}$$

- if $\lambda \ll 1 \Longrightarrow$ gauge theory description
- When k increases the compactification circle of the M-theory 11th dimension becomes small and one reduces to weakly coupled type IIA string theory
- Type IIA is a good description $\frac{N}{k^5} \ll 1$ is valid when $1 \ll \lambda \ll k^4$
- the 11 dimensional supergravity approximation is valid when $k^4 \ll \lambda$ the realtion between the curvature radius and the 't Hooft coupling is

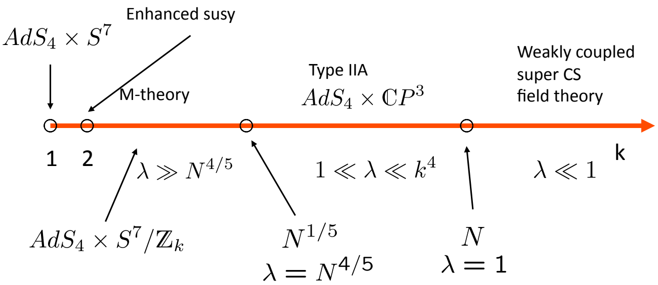
$$e^{2\phi} = \frac{R^2}{k^2} \propto \sqrt{\frac{N}{k^5}} \qquad R^2 = 2^{5/2}\pi\sqrt{\lambda}$$

Notice that in AdS_5/CFT_4 we have $R^2 = \sqrt{\lambda}$

- When $\lambda \sim 1$ the type IIA approximation by supergravity breaks down since the curvature radius becomes of the order of the string scale

details

ABJM Theory



$$\lambda = \frac{N}{k}$$

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Outline

Metric of $\mathrm{AdS}_4 \times \mathbb{C}P^3$

- The metric in AdS_4 is

$$ds^2_{\mathrm{AdS}} = \frac{R^2}{4}\left(-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\hat{\Omega}_2^2\right)$$

- The metric in $\mathbb{C}P^3$ is

$$ds^2_{\mathbb{C}P^3} = R^2\Big(d\theta^2 + \frac{1}{4}\cos^2\theta \overbrace{d{\Omega'}_2^2}^{S^2} + \frac{1}{4}\sin^2\theta \overbrace{d\Omega_2^2}^{S^2} + 4\cos^2\theta\sin^2\theta(d\delta+\omega)^2\Big)$$

with

$$\omega = \frac{1}{4}\sin\theta_1 d\varphi_1 + \frac{1}{4}\sin\theta_2 d\varphi_2$$

with $0 \le \theta \le \pi/2$, $0 \le \delta \le 2\pi$, $0 \le \varphi_i \le 2\pi$ and $0 \le \theta_i \le \pi$ for $i = 1, 2$.

- To zoom on a particular null curve on $\mathbb{C}P^3$ that selects the $SU(2) \times SU(2)$ sector, giving to the states of this sector the lightest energies, it is useful to make the coordinate transformation

$$t' = t \ , \quad \chi = \delta - \frac{1}{2} t$$

This gives the following metric for $AdS_4 \times \mathbb{C}P^3$

$$ds^2 = -\frac{R^2}{4} dt'^2 (1 - 4\cos^2\theta \sin^2\theta + \sinh^2\rho) + \frac{R^2}{4}(d\rho^2 + \sinh^2\rho d\hat{\Omega}_2^2)$$
$$+ R^2 \left[d\theta^2 + \frac{\cos^2\theta}{4} d\Omega_2^2 + \frac{\sin^2\theta}{4} d\Omega_2'^2 + 4\cos^2\theta \sin^2\theta (dt' + d\chi + \omega)(d\chi + \omega) \right]$$

We have that

$$E \equiv \Delta - J = i\partial_{t'} \ , \quad 2J = -i\partial_\chi$$

$\Delta - J$ is the energy we are interested in.

- In order to take the pp-wave limit zoom into the $SU(2) \times SU(2)$ sector we set

$$\rho = 0, \qquad \theta = \frac{\pi}{4}$$

This can be justified further since in the limit we will take one can check that the transverse excitations in the ρ and θ directions become infinitely heavy.

- To take the pp-wave limit it is convenient to define the rescaled coordinates

$$v = R^2\chi \ , \quad x_1 = R\varphi_1 \ , \quad y_1 = R\theta_1 \ , \quad x_2 = R\varphi_2 \ , \quad y_2 = R\theta_2 \ , \quad u_4 = R\left(\theta - \frac{\pi}{4}\right)$$

$$\frac{R}{2}\sinh\rho = \frac{u}{1-\frac{u^2}{R^2}} \ , \quad \frac{R^2}{4}(d\rho^2 + \sinh^2\rho d\hat{\Omega}_2^2) = \frac{\sum_{i=1}^3 du_i^2}{(1-\frac{u^2}{R^2})^2} \ , \quad u^2 = \sum_{i=1}^3 u_i^2$$

- The metric of $AdS_4 \times \mathbb{C}P^3$ becomes

$$\begin{aligned} ds^2 = & -dt'^2\left(\frac{R^2}{4}\sin^2\frac{2u_4}{R} + \frac{u^2}{(1-\frac{u^2}{R^2})^2}\right) + \frac{\sum_{i=1}^3 du_i^2}{(1-\frac{u^2}{R^2})^2} + du_4^2 \\ & + \frac{1}{8}\left(\cos\frac{u_4}{R} - \sin\frac{u_4}{R}\right)^2\left(dy_1^2 + \cos^2\frac{y_1}{R}dx_1^2\right) + \frac{1}{8}\left(\cos\frac{u_4}{R} + \sin\frac{u_4}{R}\right)^2\left(dy_2^2 + \cos^2\frac{y_2}{R}dx_2^2\right) \\ & + R^2\cos^2\frac{2u_4}{R}\left[dt' + \frac{dv}{R^2} + \frac{1}{4}\left(\sin\frac{y_1}{R}\frac{dx_1}{R} + \sin\frac{y_2}{R}\frac{dx_2}{R}\right)\right] \cdot \\ & \cdot \left[\frac{dv}{R^2} + \frac{1}{4}\left(\sin\frac{y_1}{R}\frac{dx_1}{R} + \sin\frac{y_2}{R}\frac{dx_2}{R}\right)\right] \end{aligned}$$

a very convenient form to expand around $R \to \infty$.

- The $SU(2) \times SU(2)$ Penrose limit $R \to \infty$ gives now the pp-wave metric

$$ds^2 = dvdt' + \sum_{i=1}^{4}(du_i^2 - u_i^2 dt'^2) + \frac{1}{8}\sum_{i=1}^{2}(dx_i^2 + dy_i^2 + 2dt' y_i dx_i)$$

- This metric has two flat directions x^1 and x^2, and $v = x^-$ and $t' = x^+$ are the light-cone coordinates.
- The background in the limit $R \to \infty$ selects automatically the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ subsector of $AdS_4 \times \mathbb{C}P^3$.
- Then the spin is rescaled by R^2

$$E \equiv \imath\partial_{t'} = \Delta - J \qquad \frac{2J}{R^2} = -\imath\partial_v$$

Integrability on the string side. Dispersion relation for the magnons in the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ sector of type II A string theory on $\mathrm{AdS}_4 \times \mathbb{C}P^3$

a) pp-wave

$\Longrightarrow$

$$H_{\mathrm{lc}} = \Delta - J = \sqrt{\frac{1}{4} + \frac{\lambda p^2}{2}} - \frac{1}{2} \qquad p = \frac{2\pi n}{J}$$

holds in the limit $p \to 0$, $\lambda p^2 = \lambda' 4\pi^2 n^2$, $\lambda' \equiv \frac{\lambda}{J^2}$, fixed.

b) Giant magnon

$\Longrightarrow$

$$\Delta - J = \sqrt{2\lambda}\left|\sin\frac{p}{2}\right|$$

holds in the limit $\lambda \to \infty$ but fixed p.

c) Landau-Lifshitz model

$\Longrightarrow$

$$\Delta - J = \frac{\lambda}{2} p^2 \qquad p = \frac{2\pi n}{J}$$

holds in the limit $p \to 0$ with large but fixed λ.

Dispersion relation

Putting together the information, we could derive the general form for the dispersion relation of the single (light) magnon

$$E = \sqrt{\frac{1}{4} + 4h^2(\lambda)\sin^2\frac{p}{2}}$$

$$h(\lambda) \sim \sqrt{\frac{\lambda}{2}} \text{ for } \lambda \gg 1$$

$$\Delta = \sqrt{\frac{1}{4} + 4h^2(\lambda)\sin^2\frac{p}{2}}$$

$$h(\lambda) \sim \lambda \text{ for } \lambda \ll 1$$

(Compare with AdS_5/CFT_4: $E \sim \sqrt{1 + \lambda \sin^2\frac{p}{2}}$)
(Gaiotto, Giombi, Yin, 08), (GG, Harmark, Orselli, 08)

$$h(\lambda) = \begin{cases} \lambda\left[1 + c_1\lambda^2 + c_2\lambda^4 + \ldots\right] & \text{for } \lambda \ll 1 \\ \sqrt{\frac{\lambda}{2}} + a_1 + \frac{a_2}{\sqrt{\lambda}} & \text{for } \lambda \gg 1 \end{cases}$$

Finite-size corrections: Procedure

- Goal: Finite-size corrections of the string energy for states with two oscillators
- How? Approach used by Callan et al. for the AdS_5/CFT_4 case (Callan, Swanson, McLoughlin, Schwarz, Wu, 03, 04)
- The light-cone gauge must be fixed in order to remove the unphysical bosonic degrees of freedom, namely

$$t' = c\tau \qquad p_v = \frac{\partial \mathcal{L}}{\partial \dot{v}} = constant, \quad \frac{\partial \mathcal{L}}{\partial v'} = 0\ , \qquad c = \frac{4J}{R^2} = \frac{J}{\pi\sqrt{2\lambda}} = \frac{1}{\pi\sqrt{2\lambda'}}$$

where v plays the role of the light-cone coordinate x^-
The constant is fixed using $\frac{2J}{R^2} = \int_0^{2\pi} \frac{d\sigma}{2\pi} p_v$ and $p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}$.

$$\mathcal{H}_{lc} = -p_{t'}$$

- Penrose limit $R \to \infty$: string with large angular momentum J in $\mathbb{C}P^3$

$$\lambda\,, J \to \infty \text{ and } \lambda' \equiv \frac{\lambda}{J^2} \equiv \text{ fixed and } \Delta - J = \text{fixed}$$

- Beyond the Penrose limit:

  λ' is fixed, i.e. J is large but finite: $R^2 = 4\pi J\sqrt{2\lambda'}$ $\Rightarrow$ Finite-size corrections as inverse curvature radius corrections.

- The fermions for the type IIA superstring are real Majorana-Weyl spinors with 32 components: $\theta = \theta^1 + \theta^2$ with $\Gamma_{11}\theta^1 = \theta^1$ and $\Gamma_{11}\theta^2 = -\theta^2$. However, since the $AdS_4 \times \mathbb{C}P^3$ background preserves only 24 supercharges out of the initial 32, in order to work with the fermionic d. o. f. corresponding to the unbroken supersymmetries, namely the 24 physical fermionic d.o.f., the appropriate κ-symmetry gauge must be fixed. ▸ details
- The world-sheet metric can be fixed to a Minkowski metric only to leading order, since in general the w.s. conformal gauge does no commute with the e.o.m. for v. The world-sheet metric should then be derived as a series expansion in powers of $1/R$.

- The Virasoro constraints can be used to solve for $\dot{v}$ and v' order by order in $1/R$. These should also be used to compute the corrections to the world-sheet metric.
- The gauge fixed Lagrangian, $\mathcal{L}_{\rm gf} = \mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{v}}\dot{v}$, is obtained using the solutions for $\dot{v}$ and v' and has the following expansion in powers of $1/R$

$$\mathcal{L}_{\rm gf} = \mathcal{L}_{2,B} + \mathcal{L}_{2,F} + \frac{1}{R}(\mathcal{L}_{3,B} + \mathcal{L}_{3,BF}) + \frac{1}{R^2}(\mathcal{L}_{4,B} + \mathcal{L}_{4,BF} + \mathcal{L}_{4,F}) + \mathcal{O}(R^{-3})$$

B = purely bosonic, F = purely fermionic and BF = mixed terms.
There are cubic terms that go like $\sim 1/R \sim 1/\sqrt{J}$!! $\Rightarrow \neq \mathrm{AdS}_5/CFT_4$.
$(R^2 = \pi J \sqrt{2\lambda'})$

After the light-cone gauge fixing we are left with the 8 transverse coordinates

$$\overbrace{u_1\,,\,u_2\,,\,u_3}^{\mathrm{AdS}_4}\,,\,\overbrace{u_4\,,x_1\,,y_1\,,x_2\,,y_2}^{\mathbb{C}P^3}$$

The light cone Hamiltonian is

$$H_{\rm gf} = H_{2,B} + H_{2,F} + \frac{1}{R}(H_{3,B} + H_{3,BF}) + \frac{1}{R^2}(H_{4,B} + H_{4,BF} + H_{4,F}) + \mathcal{O}(R^{-3})$$

Quantizing the pp-wave bosonic Hamiltonian

- The leading order is the pp-wave Hamiltonian in the light-cone gauge.

$$\mathcal{H}_{2,B} = \frac{1}{16c}\left[(x_a')^2 + (y_a')^2 + (\dot{x}_a)^2 + (\dot{y}_a)^2\right] + \frac{1}{2c}\sum_{i=1}^{4}\left[(\dot{u}_i)^2 + (u_i')^2 + c^2 u_i^2\right]$$

- We need to quantize it. The mode expansion for the coordinates are:

$$u_i(\tau,\sigma) = i\frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}}\frac{1}{\sqrt{\Omega_n}}\left[\hat{a}_n^i e^{-i(\Omega_n\tau - n\sigma)} - (\hat{a}_n^i)^\dagger e^{i(\Omega_n\tau - n\sigma)}\right]$$

$$z_a(\tau,\sigma) = 2\sqrt{2}\, e^{i\frac{c\tau}{2}}\sum_{n\in\mathbb{Z}}\frac{1}{\sqrt{\omega_n}}\left[a_n^a e^{-i(\omega_n\tau - n\sigma)} - (\tilde{a}^a)_n^\dagger e^{i(\omega_n\tau - n\sigma)}\right]$$

with the free dispersion relations $\boxed{\Omega_n = \sqrt{c^2+n^2},\ \omega_n = \sqrt{\frac{c^2}{4}+n^2}}$

and $z_a(\tau,\sigma) = x_a(\tau,\sigma) + iy_a(\tau,\sigma)$.

Results for the bosonic pp-wave Hamiltonian

- The quantized free bosonic Hamiltonian is, ($c = \frac{1}{\pi\sqrt{2\lambda'}}$)

$$
\begin{aligned}
cH_{2,B} &= \sum_{i=1}^{4}\sum_{n\in\mathbb{Z}} \sqrt{n^2+c^2}\,\hat{N}_n^i \\
&+ \sum_{a=1}^{2}\sum_{n\in\mathbb{Z}} \left(\sqrt{n^2+\frac{c^2}{4}}-\frac{c}{2}\right) M_n^a + \sum_{a=1}^{2}\sum_{n\in\mathbb{Z}} \left(\sqrt{n^2+\frac{c^2}{4}}+\frac{c}{2}\right) N_n^a
\end{aligned}
$$

  with the number operators $\hat{N}_n^i = (\hat{a}_n^i)^\dagger \hat{a}_n^i$, $M_n^a = (a^a)_n^\dagger a_n^a$ and $N_n^a = (\tilde{a}^a)_n^\dagger \tilde{a}_n^a$
- and with the level-matching condition

$$
\sum_{n\in\mathbb{Z}} n\left[\sum_{i=1}^{4}\hat{N}_n^i + \sum_{a=1}^{2}(M_n^a+N_n^a)\right] = 0
$$

- 4 "heavy bosons", $\Omega_n = \sqrt{n^2+c^2}$, and 4 "light bosons", $\sqrt{n^2+\frac{c^2}{4}}$. In the gauge theory there are only $4_B + 4_F$ scattering states.

Pp-wave fermion Hamiltonian

Using the pp-wave metric, the fermion Hamiltonian reduces to

$$\mathcal{H}_{2,F} = \frac{i}{4c^2}(c^2\psi_+\psi'_+ - 4\rho_+\rho'_+ + 2c^2\psi_-\psi'_- - 2\rho_-\rho'_-) - \frac{i}{2}\psi_+\rho_+ + i\psi_-\rho_- + \frac{1}{2}\psi_-\Gamma_{56}\rho_-$$

where the conjugate momenta are

$$\rho = -\frac{ic}{2}(2\mathcal{P}_- + \mathcal{P}_+)\psi^*$$

$\rho_\pm = \mathcal{P}_\pm\rho$ and it is given in terms of a 16 components complex spinor

$$\psi = \theta^1 + i\Gamma_{049}\theta^2 \qquad \psi^* = \theta^1 - i\Gamma_{049}\theta^2$$

In the following we split up the spinor as

$$\psi = \psi_+ + \psi_- \quad \text{with} \quad \psi_\pm = \mathcal{P}_\pm\psi$$

The mode expansions which follow from the plane-wave Lagrangian are

$$\psi_{+,\alpha} = \frac{\sqrt{2\alpha'}}{\sqrt{c}}\sum_n \left[f_n^+ d_{n,\alpha}e^{-i(\omega_n\tau - n\sigma)} - f_n^- d_{n,\alpha}^\dagger e^{i(\omega_n\tau - n\sigma)}\right]$$

$$\psi_{-,\alpha} = \frac{\sqrt{\alpha'}}{\sqrt{c}}\left(e^{-\frac{c}{2}\Gamma_{56}\tau}\right)_{\alpha\beta}\sum_n \left[-g_n^- b_{n,\beta}e^{-i(\Omega_n\tau - n\sigma)} + g_n^+ b_{n,\beta}^\dagger e^{i(\Omega_n\tau - n\sigma)}\right]$$

$$f_n^\pm = \frac{\sqrt{\omega_n + n} \pm \sqrt{\omega_n - n}}{2\sqrt{\omega_n}}\ , \quad g_n^\pm = \frac{\sqrt{\Omega_n + n} \pm \sqrt{\Omega_n - n}}{2\sqrt{\Omega_n}}$$

- The quantized free fermionic Hamiltonian is

$$cH_{2,F} = \sum_n \omega_n \sum_{f=1}^{4} F_n^{(f)} + \sum_n \left(\Omega_n + \frac{c}{2}\right) \sum_{f=1}^{2} \widetilde{F}_n^{(f)} + \sum_n \left(\Omega_n - \frac{c}{2}\right) \sum_{f=3}^{4} \widetilde{F}_n^{(f)}$$

where $F_n^{(f)} = d_{n,\alpha}^\dagger d_{n,\alpha}$ and $\widetilde{F}_n^{(f)} = b_{n,\alpha}^\dagger b_{n,\alpha}$, with dispersion relations

$$\boxed{\omega_n = \sqrt{\frac{c^2}{4} + n^2},\ \Omega_n = \sqrt{c^2 + n^2}}$$

- and level-matching condition

$$\sum_{n\in\mathbb{Z}} n \left[\sum_{f=1}^{4} F_n^{(f)} + \sum_{f=1}^{4} \widetilde{F}_n^{(f)}\right] = 0$$

- Like for the bosons, there are 4 fermions with frequency ω_n and 4 with frequency Ω_n.

4 "light" fermions and 4 "heavy" fermions.

Beyond the pp-wave Hamiltonian

The interaction Hamiltonian contains two terms $\boxed{\mathcal{H}_{\rm int} = \mathcal{H}_3 + \mathcal{H}_4}$:

- A cubic term that goes like $\frac{1}{R}$: $\mathcal{H}_3$, ($\sim 1/R \sim 1/\sqrt{J}$)
- A quartic term that goes like $\frac{1}{R^2}$: $\mathcal{H}_4$
- The bosonic part of $\mathcal{H}_3$ reads

$$\mathcal{H}_{3,B} = \frac{u_4}{8c}\left[p_{x_1}^2 + p_{y_1}^2 - p_{x_2}^2 - p_{y_2}^2 - x_1'^2 - y_1'^2 + x_2'^2 + y_2'^2\right]$$

It is cubic and it goes like $\sim 1/R$!! $\Rightarrow \neq \mathrm{AdS}_5/CFT_4$.

- The quartic bosonic Hamiltonian is

$$\mathcal{H}_{4,B} = \frac{2}{c^3}(\sum_{i=1}^{8} p_i X'^i)^2 - \frac{1}{2c^3}\left(\sum_{i=1}^{8}(p_i^2 + (X'^i)^2) - c^2\sum_{i=1}^{3} u_i^2 + c^2 u_4^2\right)^2$$
$$+c(\sum_{i=1}^{3} u_i^2)^2 + \frac{4}{3}c u_4^4 + \frac{1}{c}\sum_{i,j=1}^{3} u_i^2(u_j'^2 - p_j^2) + \frac{2}{c}u_4^2\sum_{i=5}^{8} p_i^2$$
$$+\frac{1}{12\sqrt{2}}(p_5 y_1^3 + p_7 y_2^3) + \frac{1}{2c}y_1^2(p_5^2 - X_5'^2) + \frac{1}{2c}y_2^2(p_7^2 - X_7'^2)$$
$$p_{i=1\ldots4} = (p_{u_1}, p_{u_2}, p_{u_3}, p_{u_4}), \quad p_{i=5\ldots8} = \frac{\sqrt{2}}{4}(p_{x_1}, p_{y_1}, p_{x_2}, p_{y_2})$$
$$X'^{i=1\ldots4} = (u_1', u_2', u_3', u_4'), \quad X'^{i=5\ldots8} = \frac{\sqrt{2}}{4}(x_1', y_1', x_2', y_2')$$

Beyond the Penrose limit: Fermionic sector

In order to deal with the fermions it is useful to introduce the following fermionic bilinears

$$A_{a,A} = \bar{\theta}\Gamma_a\partial_A\theta \ , \quad \tilde{A}_{a,A} = \bar{\theta}\Gamma_{11}\Gamma_a\partial_A\theta$$

$$B_{abc} = \bar{\theta}\Gamma_a\Gamma_{bc}\theta \ , \quad \tilde{B}_{abc} = \bar{\theta}\Gamma_{11}\Gamma_a\Gamma_{bc}\theta$$

$$C_{ab} = \bar{\theta}\Gamma_a P\Gamma_{0123}\Gamma_b\theta \ , \quad \tilde{C}_{ab} = \bar{\theta}\Gamma_{11}\Gamma_a P\Gamma_{0123}\Gamma_b\theta$$

$$\mathcal{B}_{abc;d} = \bar{\theta}\Gamma_{abc}(\mathcal{P}_+ + \frac{1}{2}\mathcal{P}_-)\Gamma^0\Gamma_d\theta \ , \quad \tilde{\mathcal{B}}_{abc;d} = \bar{\theta}\Gamma_{11}\Gamma_{abc}(\mathcal{P}_+ + \frac{1}{2}\mathcal{P}_-)\Gamma^0\Gamma_d\theta$$

$$\mathcal{C}_{ab;c} = \bar{\theta}\Gamma_a P\Gamma_{0123}\Gamma_b(\mathcal{P}_+ + \frac{1}{2}\mathcal{P}_-)\Gamma^0\Gamma_c\theta \ , \quad \tilde{\mathcal{C}}_{ab;c} = \bar{\theta}\Gamma_{11}\Gamma_a P\Gamma_{0123}\Gamma_b(\mathcal{P}_+ + \frac{1}{2}\mathcal{P}_-)\Gamma^0\Gamma_c\theta$$

$$E_{ab} = \bar{\theta}\Gamma_a(\mathcal{P}_+ + \frac{1}{2}\mathcal{P}_-)\Gamma^0\Gamma_b\theta \ , \quad \tilde{E}_{ab} = \bar{\theta}\Gamma_{11}\Gamma_a(\mathcal{P}_+ + \frac{1}{2}\mathcal{P}_-)\Gamma^0\Gamma_b\theta$$

Cubic Hamiltonian

$$\mathcal{H}_{3,BF} = \frac{i}{2}\sum_{i=1}^{8}(C_{+i}p_i + \tilde{C}_{+i}X'^i) - \frac{ic}{4}(B_{+56} - B_{+78})u_4 - \frac{ic}{4}B_{+-4}u_4$$
$$-\frac{i}{4}\sum_{i=5}^{8} s_i(B_{+4i}p_i + \tilde{B}_{+4i}X'^i) - \frac{i}{8}\sum_{i,j=5}^{8}\epsilon_{ij}(B_{+-i}p_j + \tilde{B}_{+-i}X'^j)$$

- The quartic purely fermionic Hamiltonian

$$\mathcal{H}_{4,F} = -\frac{i}{24}\left(\bar{\theta}\Gamma_{11}\Gamma_+\mathcal{M}^2\theta' + \bar{\theta}\Gamma_+\mathcal{M}^2\Gamma_{11}\theta'\right) - \frac{1}{2c}(A^2_{+,\sigma} - \tilde{A}^2_{+,\sigma})$$
$$-\frac{1}{4}A_{+,\sigma}(\tilde{C}_{+-} + \tilde{B}_{+56} + \tilde{B}_{+78}) + \frac{1}{4}\tilde{A}_{+,\sigma}(C_{+-} - C_{++} + B_{+56} + B_{+78})$$
$$-\frac{c}{8}\sum_{i=1}^{4} C^2_{+i} - \frac{c}{32}\sum_{i=5}^{8}\left[2C_{+i} - s_i B_{+4i} + \frac{1}{2}\sum_{j=5}^{8}\epsilon_{ij}B_{+-j}\right]^2$$

The mixed quartic Hamiltonian is

$$
\begin{aligned}
\mathcal{H}_{4,BF} =& \frac{i}{c^2}\sum_{i=1}^{8}\left(p_i^2+(X'^i)^2\right)\left[\tilde{A}_{+,\sigma}+\frac{c}{4}(B_{+56}+B_{+78}-C_{++}+C_{+-})\right]-i\tilde{A}_{+,\sigma}\left[\sum_{i=1}^{3}u_i^2-u_4^2\right]\\
&+\frac{2i}{c^2}\sum_{i=1}^{8}p_iX'^i\left[A_{+,\sigma}+\frac{c}{4}(\tilde{B}_{+56}+\tilde{B}_{+78})+\frac{c}{4}\tilde{C}_{+-}\right]+\frac{ic}{2}\sum_{i=1}^{3}u_i^2C_{++}-\frac{ic}{4}\sum_{i=1}^{4}u_i^2(B_{+56}+B_{+78})\\
&+\frac{i}{2}u_4\sum_{i=5}^{8}s_i\left[C_{+i}p_i-\tilde{C}_{+i}X'^i\right]-\frac{i}{c}\sum_{i,j=1}^{8}\left[C_{ij}(X'^iX'^j-p_ip_j)+2\tilde{C}_{ij}X'^ip_j\right]-i\sum_{i,j=1}^{3}u_i'u_j\tilde{B}_{+ij}\\
&-\frac{i}{8}u_4\sum_{i,j=5}^{8}s_i\epsilon_{ij}(3B_{+-i}p_j+\tilde{B}_{+-i}X'^j)+\frac{i}{4}(B_{+56}p_{x_1}y_1+\tilde{B}_{+56}x_1'y_1+B_{+78}p_{x_2}y_2+\tilde{B}_{+78}x_2'y_2)\\
&-\frac{i}{2}\sum_{i=1}^{4}\sum_{j=1}^{8}u_i\left[B_{-ij}p_j-\tilde{B}_{-ij}X'^j\right]+\frac{i}{2c}\sum_{i=1}^{8}\sum_{j=5}^{8}s_j\left[(p_ip_j-X'^iX'^j)B_{4ij}+(p_iX'^j-X'^ip_j)\tilde{B}_{4ij}\right]\\
&-\frac{i}{2}\sum_{i=1}^{3}\sum_{j=4}^{8}u_i\left[B_{+ij}p_j-\tilde{B}_{+ij}X'^j\right]-\frac{i}{4}u_4\sum_{i=5}^{8}(B_{+4i}p_i+3\tilde{B}_{+4i}X'^i)+\frac{i}{2}u_4\sum_{i=1}^{3}(B_{+4i}p_i-\tilde{B}_{+4i}u_i')\\
&-\frac{i}{4c}\sum_{i=1}^{8}\sum_{j,k=5}^{8}\epsilon_{jk}\left[(B_{+ij}-B_{-ij})(p_ip_k-X'^iX'^k)+(\tilde{B}_{+ij}-\tilde{B}_{-ij})(p_iX'^k-X'^ip_k)\right]\\
&+\frac{i}{2c^2}\sum_{i,j=1}^{8}(p_ip_j'+X'^iX''^j)\tilde{E}_{ij}-\frac{i}{2c^2}\sum_{i,j=1}^{8}(X'^ip_j'+p_iX''^j)E_{ij}-\frac{3i}{4c}\sum_{i,j=1}^{8}(p_ip_j-X'^iX'^j)\mathcal{C}_{i+;j}\\
&+\frac{3i}{4c}\sum_{i,j=1}^{8}(X'^ip_j-p_iX'^j)\tilde{\mathcal{C}}_{i+;j}-\frac{i}{4c}\sum_{i,j=1}^{8}(p_ip_j+X'^iX'^j)\mathcal{C}_{+i;j}-\frac{i}{4c}\sum_{i,j=1}^{8}(X'^ip_j+p_iX'^j)\tilde{\mathcal{C}}_{+i;j}\\
&+\frac{iu_4}{2}\sum_{i=1}^{8}(p_j\mathcal{B}_{+-4;i}-X'^j\tilde{\mathcal{B}}_{+-4;i})+\frac{i}{2c}\sum_{i=5}^{8}\sum_{j=1}^{8}s_i\left[(p_ip_j-X'^iX'^j)\mathcal{B}_{+4i;j}+(X'^ip_j-p_iX'^j)\tilde{\mathcal{B}}_{+4i;j}\right]\\
&+\frac{i}{4c}\sum_{i,j=5}^{8}\sum_{k=1}^{8}\epsilon_{ij}\left[(p_ip_k+X'^iX'^k)(\mathcal{B}_{+-i;k}+E_{jk})+(X'^ip_k+p_iX'^k)(\tilde{\mathcal{B}}_{+-i;k}-\tilde{E}_{jk})\right]
\end{aligned}
$$

One oscillator spectrum

- We want to compute the energy for a one oscillator state with the level matching condition relaxed up to $\frac{1}{R^2}$. This will give the finite size corrections to the dispersion relation of the single magnon. Start with the example of a state $\in \mathrm{SU}(2) \sim S^2$

$$|s\rangle = (a_n)^\dagger |0\rangle \in S^2, \quad 1 \text{ oscillator in one } \mathrm{SU}(2)$$

- Using perturbation theory

$$E_s^{(1)} = \langle s|H_3|s\rangle = 0$$

$$E_s^{(2)} = \underbrace{\langle s|H_4|s\rangle}_{A} + \underbrace{\sum_{|i\rangle} \frac{|\langle i|H_3|s\rangle|^2}{E^{(0)}_{|s\rangle} - E^{(0)}_{|i\rangle}}}_{B}$$

where $|i\rangle$ is an intermediate state with zeroth order energy $E^{(0)}_{|i\rangle}$.

- Notice that $|i\rangle$ has to contain 2 or 4 oscillators.

Divergences, H_3

The cubic Hamiltonian written in terms of oscillators reads

$$H_{3,B} = \frac{i}{c\sqrt{2}} \sum_{m,\,l,\,r} \Bigg\{ \frac{\delta(m+l+r)(\hat{a}^4_{-r})^\dagger}{\sqrt{\omega_m \omega_l \Omega_r}} \left[\left(\omega_m - \frac{c}{2}\right)\left(\omega_l - \frac{c}{2}\right) + ml\right] \cdot \left[(a^2_{-m})^\dagger (a^2_l) - (a^1_{-m})^\dagger (a^1_l)\right] + \left[\left(\omega_m + \frac{c}{2}\right)\left(\omega_l - \frac{c}{2}\right) - ml\right] \cdot \left[(\tilde{a}^1_m)(a^1_l) + (\tilde{a}^1_{-m})^\dagger (a^1_{-l})^\dagger - (\tilde{a}^2_m)(a^2_l) + (\tilde{a}^2_{-m})^\dagger (a^2_{-l})^\dagger\right] \Bigg\}$$

- The cubic Hamiltonian produces divergent results $\Longrightarrow$ it is natural to handle the sums over mode numbers by introducing cutoffs and removing the cutoffs only at the end of the calculations.
- Under the quite natural assumption that all the light modes have the same cutoff N, and that all the heavy modes have the same cutoff M the form of the interaction naturally suggests $M = 2N$.
- $\hat{a}^4$ is the oscillator of a heavy mode whereas a and $\tilde{a}$ are oscillators of light modes. $\Longrightarrow$ the Hamiltonian contains one heavy oscillator and two light oscillators, if the sums over m and l have cutoff N, the sum over r has cutoff $2N$.

The motivation behind this regularization prescription is that heavy excitations are not fundamental but rather bound states of two light fundamental modes.

Divergences, H_4

Consider $\langle s|H_4|s\rangle$, where $H_4 = H_{4,B} + H_{4,F} + H_{4,BF}$.

- We got a divergent result from the cubic terms $\Longrightarrow$ to obtain a finite result for the energy, this divergence must be canceled by the H_4 term .
- This cancelation can happen only if **H_4 is not normal ordered** (for type IIB superstring on $AdS_5 \times S^5$ instead it is normal ordered) otherwise its mean value would just be finite $\Longrightarrow$ we shall thus introduce for it an appropriate and consistent normal ordering prescription.
- Assume that the vacuum is a protected state $\Longrightarrow$ $H_{4,F}$ does not contribute to the energy of the state $|s\rangle$, since if it would, it would also change the vacuum energy

$$\Longrightarrow \qquad \langle s|H_4|s\rangle = \langle s|H_{4,B}|s\rangle + \langle s|H_{4,BF}|s\rangle$$

- Requiring that the pp-wave algebra is not affected by normal ordering constants and that the spectrum of string states is finite, will fix uniquely the normal ordering prescription $\Longrightarrow$ **the symmetric prescription!**

All the divergences cancel leaving a finite result for the energy!

Light Magnon Dispersion relation

For the state $|s\rangle = (a_n)^\dagger |0\rangle$

$$E_s^{(2)} = \mathcal{S}(n)$$

$$\begin{aligned}\mathcal{S}(n) &= \frac{1}{4\,c\,R^2\omega_n}\left[8n^2\left(\sum_{q=-2N}^{2N}\frac{1}{\Omega_q} - \sum_{q=-N}^{N}\frac{1}{\omega_q}\right)\right.\\ &+ \left.\left(\frac{9}{2}c^2 - 9c\,\omega_n + 8n^2\right)\left(\sum_{q=-2N}^{2N}\frac{1}{\Omega_q} - \sum_{q=-N}^{N}\frac{1}{\Omega_{q+n}}\right)\right]\end{aligned}$$

Two different prescriptions for performing the sums:

1. First manipulate the sum so that all the sums have the same cutoff, then send $N \to \infty$.
   All the sums can be computed by standard ζ-function techniques

$$\mathcal{S}(n) = \frac{4n^2}{c\,R^2\omega_n}\sum_{p=1}^{\infty}\left[(-1)^p - 1\right]K_0(\pi c p)$$

In the limit for large c it is exponentially suppressed

This corresponds to $\boxed{a_1 = 0 \text{ in } \; h(\lambda)}$

$$E_l = \sqrt{\frac{1}{4} + 4\left(\sqrt{\frac{\lambda}{2}} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)\right)^2 \frac{n^2\pi^2}{J^2}} - \frac{\pi n^2\sqrt{2\lambda'}}{J\sqrt{\frac{1}{4} + 2\lambda' n^2\pi^2}}\sum_{q=1}^{\infty}\left[1 - (-1)^q\right]K_0\left(\frac{q}{\sqrt{2\lambda'}}\right)$$

2. There is however another proposal for how to regularize the sum which was used in the context of the semiclassical world-sheet computation of the folded and spinning string in $AdS_4 \times \mathbb{C}P^3$. This prescription was also adopted in the AdS_5/CFT_4 case and it does not distinguish between the world-sheet heavy and light excitations. This implies that we simply remove the cutoff by sending $N \to \infty$ to get

$$\mathcal{S}(n) = \frac{2\,n^2}{R^2 c\,\omega_n} \sum_{q=-\infty}^{\infty} \left(\frac{1}{\Omega_q} - \frac{1}{\omega_q} \right) \tag{1}$$

This gives

$$\mathcal{S}(n) = -\frac{4n^2}{c\,R^2 \omega_n} \left(\log 2 - \sum_{p=1}^{\infty} \left[(-1)^p - 1 \right] K_0(\pi c p) \right) \tag{2}$$

This corresponds to $\boxed{a_1 = -\frac{\log 2}{2\pi} \text{ in } h(\lambda)}$

but since the cutoffs are different for light and heavy modes, it is more natural to first write all the sums in terms of a single cutoff and then remove it. Note that the second term in (2) is independent on the regularization, in (1) it appears the same term.

Heavy Magnon Dispersion relation

Also for heavy magnons we proved cancelation of divergences!

- With the procedure (1.) for the single oscillator states, non level matched
$$|s\rangle = (a_n^{u_1,u_2,u_3,u_4})^\dagger |0\rangle$$
we got the finite size dispersion relation

$$E_h = \sqrt{1 + 4\left(\sqrt{\frac{\lambda}{2}} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)\right)^2 \frac{n^2\pi^2}{J^2}} - \frac{\pi n^2 \sqrt{2\lambda'}}{J\sqrt{1+2\lambda' n^2\pi^2}} \sum_{q=1}^{\infty} [1-(-1)^q]\, K_0\left(\frac{q}{\sqrt{2\lambda'}}\right)$$

- $u_i, i = 1,2,3 \in \mathrm{AdS}_4$ whereas $u_4 \in \mathbb{C}P^3$ but they have the same dispersion relation
- The finite size correction of the dispersion relation for the heavy modes can be obtained from that of the light modes by

$n \longrightarrow \frac{n}{2}$ and multiplying by 2

Heavy magnons look like bound states of light magnons

Light and heavy fermions have the same dispersion relations of the bosons!

2 oscillator spectrum: no mixing

The energy of two oscillator states tests the scattering matrix of the Bethe equations

- Start with the example of states $\in \mathrm{SU}(2) \times \mathrm{SU}(2) \sim S^2 \times S^2$

$$|s\rangle = (a^1_n)^\dagger (a^1_{-n})^\dagger |0\rangle \in S^2, \quad \text{2 oscillators in one SU(2)}$$
$$|t\rangle = (a^1_n)^\dagger (a^2_{-n})^\dagger |0\rangle \in S^2 \times S^2, \quad \text{1 oscillator for each } S^2$$

- These states are degenerate but we can use non degenerate perturbation theory

$$E^{(2)}_s = \underbrace{\langle s,t|H_4|s,t\rangle}_{A} + \underbrace{\sum_{|i\rangle} \frac{|\langle i|H_3|s,t\rangle|^2}{E^{(0)}_{|s\rangle,|t\rangle} - E^{(0)}_{|i\rangle}}}_{B}$$

because the interaction hamiltonian do not mixes them

- Notice that $|i\rangle$ has to contain 3 or 5 oscillators.
- **Divergences cancel!! what's left?**

The finite size corrections to the energies of the states $|s\rangle$ and $t\rangle$ are

$$E_s^{(2)} = -\frac{8\,n^2\left[\left(\omega_n - \frac{c}{2}\right)^2 - \frac{c^2}{2}\right]}{R^2c^3\omega_n^2} - \frac{8n^2}{R^2c\,\omega_n}\sum_{q=1}^{\infty}[1-(-1)^q]\,K_0(\pi c q)$$

$$\simeq \frac{8n^2\pi^2\lambda'}{J} - \frac{64n^4\pi^4\lambda'^2}{J} + \frac{448n^6\pi^6\lambda'^3}{J} + \mathcal{O}\left(\lambda'^4\right)$$

and

$$E_t^{(2)} = -\frac{8\,n^2\left(\omega_n - \frac{c}{2}\right)^2}{R^2c^3\omega_n^2} - \frac{8n^2}{R^2c\,\omega_n}\sum_{q=1}^{\infty}[1-(-1)^q]\,K_0(\pi c q) \simeq -\frac{64n^6\pi^6\lambda'^3}{J} + \dots$$

where $\omega_n = \sqrt{n^2 + \frac{c^2}{4}}$ is the pp-wave energy of a light mode

- The Bessel function term appear here to be twice that on a single oscillator state and give exponentially suppressed terms in the small λ' limit.
- For the state $|s\rangle$ the finite-size corrections for $1/J$ and λ', λ'^2 are analogous to those of the $\mathrm{SU}(2)$ sector in $\mathrm{AdS}_5 \times S^5$

- For the state $|t\rangle$ at order λ'^3 the two spheres S^2 start to interact (case $|t\rangle$)
- The energy of $|t\rangle$ starts from $\lambda'^3! \neq$ AdS_5/CFT_4

2 oscillator spectrum: mixing

The zeroth order energy of many two oscillator states is degenerate $\Longrightarrow$ mixing matrix

Using the basis of four single-oscillator light bosonic state

$$\{a_n^{1+}|0\rangle, \tilde{a}_n^{1+}|0\rangle, a_n^{2+}|0\rangle, \tilde{a}_n^{2+}|0\rangle\},$$

and the light fermionic states $d_\alpha^+|0\rangle$.
From the 4 single-oscillator bosonic states we can construct 32 two-oscillator states of the type $\mathfrak{a}_{i,-n}^+\mathfrak{a}_{i,n}^+|0\rangle$, where $\mathfrak{a}_i = a_i, \tilde{a}_i$. The tree-level degenerate states are

$$\begin{aligned} v_n^1 &= a_n^{1+}\tilde{a}_{-n}^{1+}|0\rangle, \\ v_n^2 &= a_n^{1+}\tilde{a}_{-n}^{2+}|0\rangle, \\ v_n^3 &= a_n^{2+}\tilde{a}_{-n}^{1+}|0\rangle, \\ v_n^4 &= a_n^{2+}\tilde{a}_{-n}^{2+}|0\rangle. \end{aligned}$$

where $n > 0$, and equally states with $n \to -n$.
Requiring definite parities with respect to $\mathbb{Z}_2$ symmetries:

1. the momentum reflection symmetry $P_n : n \to -n$,
2. the symmetry between the two $SU(2)$'s $P_a : a_1 \to a_2$,
3. the symmetry $\tilde{P} : a_i \to \tilde{a}_i$.

Such states are constructed as follows: first symmetrize and antisymmetrize in P_a

$$\begin{aligned} s_n &= \frac{1}{\sqrt{2}}(v_n^1 + v_n^4) \\ p_n &= \frac{1}{\sqrt{2}}(-v_n^1 + v_n^4) \\ q_n &= \frac{1}{\sqrt{2}}(v_n^2 + v_n^3) \\ r_n &= \frac{1}{\sqrt{2}}(-v_n^2 + v_n^3). \end{aligned}$$

The full basis of $s = 0$ scalar light two-oscillator boson-boson tree-level degenerate states has then dimension 8 and is, after due decomposition into P_n even and odd states

state	definition	P_n	$\tilde{P}$	P_a
u_1	$\frac{1}{\sqrt{2}}(s_n + s_{-n})$	1	1	1
u_2	$\frac{1}{\sqrt{2}}(-s_n + s_{-n})$	-1	-1	1
u_3	$\frac{1}{\sqrt{2}}(p_n + p_{-n})$	1	1	-1
u_4	$\frac{1}{\sqrt{2}}(-p_n + p_{-n})$	-1	-1	-1
u_5	$\frac{1}{\sqrt{2}}(q_n + q_{-n})$	1	1	1
u_6	$\frac{1}{\sqrt{2}}(-q_n + q_{-n})$	-1	-1	1
u_7	$\frac{1}{\sqrt{2}}(r_n + r_{-n})$	1	-1	-1
u_8	$\frac{1}{\sqrt{2}}(-r_n + r_{-n})$	-1	1	-1

The two-fermion-oscillator states

States of the type $d^+_\alpha A_{\alpha\beta} d^+_\beta |0\rangle$.
The states with zero AdS spin s can potentially mix with the light bosonic states as well, where $A_{\alpha\beta}$ is an arbitrary matrix with fermionic indices.
Choosing the linearly independent states by the following projection criteria

$$\Gamma^T_{11} A \Gamma_{11} \neq 0,$$

$$\Gamma^T_{+} A \Gamma_{+} \neq 0,$$

$$\mathcal{P}^T A \mathcal{P} \neq .0$$

we find the fermionic-fermionic basis has dimension 16.
Therefore, the total basis in the mixing sector has 24 dimensions; the remaining terms $u_{9..24} = \{d^+ A_i d^+ |0\rangle\}$ can be chosen as

$$A_{i=9\ldots 24} = \{\tfrac{1}{2}, \tfrac{1}{2}\Gamma_{65}, \tfrac{1}{2}\Gamma_{21}, \tfrac{1}{2}\Gamma_{31}, \tfrac{1}{2}\Gamma_{32}, \tfrac{1}{2}\Gamma_{6521}, \tfrac{1}{2}\Gamma_{6531}, \tfrac{1}{2}\Gamma_{6532}, \tfrac{1}{2}\Gamma_{7541}, \tfrac{1}{2}\Gamma_{7542}, \tfrac{1}{2}\Gamma_{7543}, \tfrac{1}{2}\Gamma_{7641}, \tfrac{1}{2}\Gamma_{7642}, \tfrac{1}{2}\Gamma_{7643}, \tfrac{1}{2}\Gamma_{9750}, \tfrac{1}{2}\Gamma_{9760}\}.$$

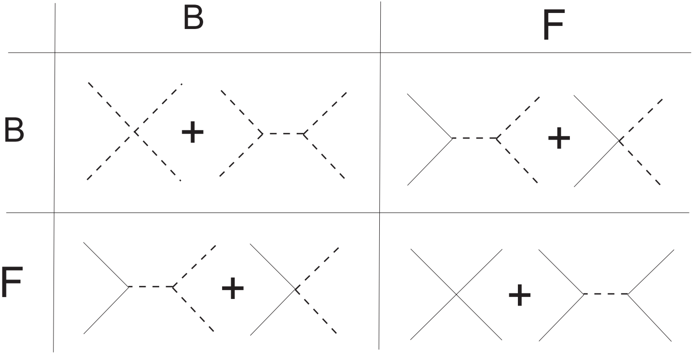
where the index i numbering basis states runs from 9 to 24. It can be explicitly seen that this basis is orthonormal.

details

Schematically, we may picture the mixing matrix as follows

	B	F
B	$H_{4B} + H_{3B}^2$	$H_{2B2F} + H_{FFB} H_{BBF}$
F	$H_{2B2F} + H_{FFB} H_{BBF}$	$H_{4F} + H_{FFB}^2$

where F and B represent the two-fermion and two-boson oscillator states. This Hamiltonian is symbolically depicted in.



- Amazingly in the basis $u_1 \dots u_{24}$ the 24×24 mixing matrix in the bosonic and fermionic sector is diagonal, we get the set of eigenvalues as power series in λ'
- The string spectrum becomes as shown in the Table below:

$$E^{(2)} = \frac{8}{J} \sum_i a_i \left(\lambda' \pi^2 n^2\right)^i$$

a_1	a_2	a_3	a_4	a_5
1	−12	96	−768	6144
1	−12	96	−768	6144
0	−4	32	−256	2048
0	−4	32	−256	2048
0	−4	32	−256	2048
0	−4	32	−256	2048
0	−4	32	−256	2048
0	−4	32	−256	2048
0	−4	32	−256	2048
0	−4	32	−256	2048
0	−4	32	−256	2048
0	−4	32	−256	2048
−1	4	−32	256	−2048
−1	4	−32	256	−2048
−1	4	−32	256	−2048
−1	4	−32	256	−2048
−1	4	−32	256	−2048
−1	4	−32	256	−2048
−1	4	−32	256	−2048
−1	4	−32	256	−2048
−1	4	−32	256	−2048
−1	4	−32	256	−2048
−2	12	−96	768	−6144
−2	12	−96	768	−6144

This spectrum must be compared to the solutions of the Bethe Equations.

Outline

The Bethe roots are quantized through the algebraic Bethe equations **(Gromov, Vieira, 2008)**

$$
\begin{aligned}
1 &= \prod_{j=1}^{K_2} \frac{u_{1,k}-u_{2,j}+\frac{i}{2}}{u_{1,k}-u_{2,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{1-1/x_{1,k}x^+_{4,j}}{1-1/x_{1,k}x^-_{4,j}} \prod_{j=1}^{K_{\bar 4}} \frac{1-1/x_{1,k}x^+_{\bar 4,j}}{1-1/x_{1,k}x^-_{\bar 4,j}}\,, \\
1 &= \prod_{j\neq k}^{K_2} \frac{u_{2,k}-u_{2,j}-i}{u_{2,k}-u_{2,j}+i} \prod_{j=1}^{K_1} \frac{u_{2,k}-u_{1,j}+\frac{i}{2}}{u_{2,k}-u_{1,j}-\frac{i}{2}} \prod_{j=1}^{K_3} \frac{u_{2,k}-u_{3,j}+\frac{i}{2}}{u_{2,k}-u_{3,j}-\frac{i}{2}}\,, \\
1 &= \prod_{j=1}^{K_2} \frac{u_{3,k}-u_{2,j}+\frac{i}{2}}{u_{3,k}-u_{2,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3,k}-x^+_{4,j}}{x_{3,k}-x^-_{4,j}} \prod_{j=1}^{K_{\bar 4}} \frac{x_{3,k}-x^+_{\bar 4,j}}{x_{3,k}-x^-_{\bar 4,j}} \\
\left(\frac{x^+_{4,k}}{x^-_{4,k}}\right)^L &= \prod_{j\neq k}^{K_4} \frac{u_{4,k}-u_{4,j}+i}{u_{4,k}-u_{4,j}-i} \prod_{j=1}^{K_1} \frac{1-1/x^-_{4,k}x_{1,j}}{1-1/x^+_{4,k}x_{1,j}} \prod_{j=1}^{K_3} \frac{x^-_{4,k}-x_{3,j}}{x^+_{4,k}-x_{3,j}} \times \\
&\times \prod_{j=1}^{K_4} \sigma_{\mathrm{BES}}(u_{4,k},u_{4,j}) \prod_{j=1}^{K_{\bar 4}} \sigma_{\mathrm{BES}}(u_{4,k},u_{\bar 4,j})\,, \\
\left(\frac{x^+_{\bar 4,k}}{x^-_{\bar 4,k}}\right)^L &= \prod_{j=1}^{K_{\bar 4}} \frac{u_{\bar 4,k}-u_{\bar 4,j}+i}{u_{\bar 4,k}-u_{\bar 4,j}-i} \prod_{j=1}^{K_1} \frac{1-1/x^-_{\bar 4,k}x_{1,j}}{1-1/x^+_{\bar 4,k}x_{1,j}} \prod_{j=1}^{K_3} \frac{x^-_{\bar 4,k}-x_{3,j}}{x^+_{\bar 4,k}-x_{3,j}} \times \\
&\times \prod_{j\neq k}^{K_{\bar 4}} \sigma_{\mathrm{BES}}(u_{\bar 4,k},u_{\bar 4,j}) \prod_{j=1}^{K_4} \sigma_{\mathrm{BES}}(u_{\bar 4,k},u_{4,j})\,,
\end{aligned}
$$

The spectrum of string energies is expressed in terms of the roots u_4 and $u_{\bar{4}}$, which carry momentum, as follows:

$$E = h(\lambda)\mathcal{Q}_2 \,,$$

being the conserved charges $\mathcal{Q}_n$ expressed in terms of the roots as

$$\mathcal{Q}_n = \sum_{j=1}^{K_4} \mathbf{q}_n(u_{4,j}) + \sum_{j=1}^{K_4} \mathbf{q}_n(u_{\bar{4},j}) \;\;,\;\; \mathbf{q}_n = \frac{i}{n-1}\left(\frac{1}{(x^+)^{n-1}} - \frac{1}{(x^-)^{n-1}}\right) .$$

The Zhukovsky variables x, $x^{\pm}$ are defined in terms of the roots as

$$x + \frac{1}{x} = \frac{u}{h(\lambda)} \;\;,\;\; x^{\pm} + \frac{1}{x^{\pm}} = \frac{1}{h(\lambda)}\left(u \pm \frac{i}{2}\right) .$$

Recalling that $p_j = \frac{1}{i}\log\frac{x^+_{4,j}}{x^-_{4,j}}$ and $\bar{p}_j = \frac{1}{i}\log\frac{x^+_{\bar{4},j}}{x^-_{\bar{4},j}}$, we have

$$E = \textstyle\sum_{j=1}^{K_4} \frac{1}{2}\left(\sqrt{1 + 16h(\lambda)^2\sin^2\frac{p_j}{2}} - 1\right) + \sum_{j=1}^{K_{\bar{4}}} \frac{1}{2}\left(\sqrt{1 + 16h(\lambda)^2\sin^2\frac{\bar{p}_j}{2}} - 1\right)$$

The rapidity variable expressed in terms of the momentum of the roots is given by

$$u_{4,j} = \frac{1}{2} \cot\left(\frac{p_j}{2}\right) \sqrt{1 + 16h(\lambda)^2 \sin\left(\frac{p_j}{2}\right)^2}$$

In the near plane wave limit, the BES kernel reduces to the AFS phase factor:

$$\sigma_{\mathrm{AFS}}(u_j, u_k) = e^{i\theta_{jk}} ,$$

where

$$\theta_{jk} = \sum_{r=2} h(\lambda) \left[\mathbf{q}_r(x_j)\mathbf{q}_{r+1}(x_k) - \mathbf{q}_r(x_k)\mathbf{q}_{r+1}(x_j)\right] .$$

Recall that at large 't Hooft coupling we have

$$h(\lambda) \simeq \sqrt{\lambda/2} .$$

Procedure

1. Express the Bethe equations in terms of the momenta p_j
2. Assume a perturbative ansatz for the momentum of the form
$$p_j = \frac{2\pi n_j}{J} + \frac{A}{J^2} + \frac{B\lambda'}{J^2} + \frac{C\lambda'^2}{J^2} + \frac{D\lambda'^3}{J^2} + \frac{E\lambda'^4}{J^2} + \dots ,$$
where $\lambda' = \frac{\lambda}{J^2}$
3. Derive the solutions for the coefficients A, B, ... and solve the equations for the momenta.
4. Plug the solution for the momenta in the dispersion relation to get the spectrum.

One and two magnon states

- The 8 one-magnon states are $(K_{u_4}, K_{u_{\bar{4}}}, K_{u_1}, K_{u_2}, K_{u_3}) = (1, 0, 0, 0, 0)$, $(0, 1, 0, 0, 0)$, $(1, 0, 1, 1, 1)$, $(0, 1, 1, 1, 1)$ in the boson sector, and $(1, 0, 1, 0, 0)$, $(0, 1, 1, 0, 0)$, $(1, 0, 1, 1, 0)$, $(0, 1, 1, 1, 0)$.
- Out of these, 32 two-magnon states may be formed. Of special interest are those which have degenerate energies at tree level.
  - In the boson-boson sector these are (11111), (20111), (02111)
  - In the fermion-fermion sector these are (11200), (11210), (11220), (20200), (20210), (20220), (02200), (02210), (02220).

Warm up: recap of the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ subsector: $(2,0,0,0,0)$ and $(0,2,0,0,0)$

- Magnons of the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ group are described by $(K_{u_4},K_{u_{\bar 4}},K_{u_1},K_{u_2},K_{u_3})=(2,0,0,0,0)$ and $(K_{u_4},K_{u_{\bar 4}},K_{u_1},K_{u_2},K_{u_3})=(0,2,0,0,0)$, which clearly are identical .
- Due to the level matching condition, we have only one independent momentum, p. Plugging the expansion (3) in the Bethe equations, one gets, up to order λ'^2 and $\frac{1}{J}$:

$$\frac{1}{J}\left[A-2\pi n+\lambda'\left(B+8n^3\pi^3\right)+\lambda'^2\left(C-32n^5\pi^5\right)\right]=0\,,$$

which completely determines the momentum up to the desired perturbative order. We have

$$A=2n\pi,\quad B=-8n^3\pi^3,\quad C=32n^5\pi^5\,,$$

This plugged in the dispersion relation gives the spectrum

$$E_{20000}=4n^2\pi^2\lambda'-8n^4\pi^4\lambda'^2+32n^6\pi^6\lambda'^3+\frac{1}{J}\left(8n^2\pi^2\lambda'-64n^4\pi^4\lambda'^2+448n^6\pi^6\lambda'^3\right)+\ldots$$

which is the spectrum of the string states $|s\rangle=\left(a_n^{1,2}\right)^\dagger\left(a_{-n}^{1,2}\right)^\dagger|0\rangle$.

$(1, 1, 0, 0, 0)$

Consider the case $(K_{u_4}, K_{u_{\bar{4}}}, K_{u_1}, K_{u_2}, K_{u_3}) = (1, 1, 0, 0, 0)$.

- Level matching condition $\Longrightarrow$ there is only one independent momentum, p.
- Yet we can build two different configurations, which shall be degenerate: the u_4 root carrying momentum p and the $u_{\bar{4}}$ carrying $-p$, or viceversa.

The perturbative expansion of the Bethe equations read:

$$\frac{1}{J}\left[A + \lambda' B + \lambda'^2 \left(C + 16 n^5 \pi^5\right)\right] = 0\,,$$

which gives

$$A = 0, \quad B = 0, \quad C = -16 n^5 \pi^5\,,$$

and therefore

$$E_{11000} = 4 n^2 \pi^2 \lambda' - 8 n^4 \pi^4 \lambda'^2 + 32 n^6 \pi^6 \lambda'^3 - \frac{1}{J}\left(64 n^6 \pi^6 \lambda'^3\right) + \dots\,,$$

which is the spectrum of the string states $|t_{1,2}\rangle = \left(a_n^{1,2}\right)^\dagger \left(a_{-n}^{2,1}\right)^\dagger |0\rangle$.

$(1,1,1,1,1)$: how to deal with auxiliary roots

Consider the case $(K_{u_4}, K_{u_{\bar{4}}}, K_{u_1}, K_{u_2}, K_{u_3}) = (1,1,1,1,1)$

1. In the $\lambda' \to 0$ limit the values for the auxiliary roots are 0 and ∞.
2. A solution for an auxiliary root which is 0 in the $\lambda' \to 0$ limit is rather a power series in λ' whose limit is 0 when $\lambda' \to 0$.
3. A solution for an infinite auxiliary root will be a power series containing negative powers of λ', such that in the $\lambda' \to 0$ it will diverge.
4. Request that the Bethe equations should contain only integer powers of λ' in their expansion.
5. $\Longrightarrow$ Bethe roots must contain only half integer powers of λ'. This can be translated in an expansion for the Zhukovsky variable x. Denoting with a subscript ∞ or 0 a Zhukovsky variable associated to a root which tends to ∞ or 0 respectively, we write

$$x_\infty = \sum_{k=1}^{\infty} c_k \left(\sqrt{\lambda'}\right)^{k+\frac{1}{2}} \quad , \quad x_0 = i + \sum_{k=1}^{\infty} d_k \left(\sqrt{\lambda'}\right)^{k-\frac{1}{2}}$$

- consider the expansion for x_∞. If we want the Bethe equations to be satisfied at the order λ'^j, then the first $j-1$ coefficients in the expansion must be vanishing, in order for the expansion for x to start at least at $\lambda'^{j+\frac{1}{2}}$.
- This expansion for u_1 and u_3 around infinity allows to fulfill all the Bethe equations and is therefore a meaningful solution.

The Bethe equations for this case read

$$\frac{1}{J}\left[A - 2\pi n + \lambda' B + \lambda'^2\left(C + 16 n^5 \pi^5\right)\right] = 0\,,$$

which gives

$$A = 2\pi n, \quad B = 0, \quad C = -16 n^5 \pi^5\,,$$

and therefore

$$E_{11111} = 4n^2\pi^2\lambda' - 8n^4\pi^4\lambda'^2 + 32n^6\pi^6\lambda'^3 - \frac{1}{J}\left(-8n^2\pi^2\lambda' + 32n^4\pi^4\lambda'^2 - 256n^6\pi^6\lambda'^3\right) + \ldots\,.$$

In the multiplet of string states having plane wave degenerate energy equal to $2\omega_n$, this finite size correction exists and has a high multiplicity.

details

Boson-boson and fermion-fermion sectors are compared in the Tables

Table: Boson-boson spectrum comparison

Expansion coefficient					Multiplicity	Corresponding BA states				
a_1	a_2	a_3	a_4	a_5		K_4	$K_{\bar{4}}$	K_3	K_2	K_1
0	−4	32	−256	2048	2	2	0	1	1	$1_{\text{branch 1}}$
−1	4	−32	256	−2048	4	2	0	1	1	$1_{\text{branch 2}}$
						1	1	1	1	$1_{\text{branch 1}}$
−2	12	−96	768	−6144	2	1	1	1	1	$1_{\text{branch 2}}$

Table: Fermion-fermion spectrum comparison

Expansion coefficient					Multiplicity	Corresponding BA states				
a_1	a_2	a_3	a_4	a_5		K_4	$K_{\bar{4}}$	K_3	K_2	K_1
1	−12	96	−768	6144	2	2	0	2	2	0
0	−4	32	−256	2048	8	1	1	2	2	0
						2	0	2	1	$0_{\text{branch 1}}$
						2	0	2	1	$0_{\text{branch 2}}$
						2	0	2	0	0
−1	4	−32	256	−2048	6	1	1	2	1	$0_{\text{branch 1}}$
						1	1	2	1	$0_{\text{branch 2}}$
						1	1	2	0	0

Outline

Summary and Conclusions

- We derived the finite size corrections (the $1/J$-corrections) in the near pp-wave limit to the dispersion relations for both light and heavy magnons for type II A string on $AdS_4 \times \mathbb{CP}_3$.
- The strong-weak coupling interpolating function $h(\lambda)$, entering the magnon dispersion relation, does not receive a one-loop correction, in agreement with the algebraic curve spectrum.
- Divergences cancel, the interacting Hamiltonian we derived is **correct!**
- The finite-size corrections to two oscillator state energies and the corresponding solutions of the Bethe equations are completely **identical** up to the fourth order in $\lambda' \equiv \frac{\lambda}{J^2}$. The multiplicity is the **same**.
- The check is highly non-trivial and heavily points towards the **integrability of the theory**.

Open problems and Outlook

- Fermionic two oscillator states of the type $|s\rangle = a_i^\dagger d_\alpha^\dagger |0\rangle$.
- **What about the 4 heavy magnons?**

Type IIA on $AdS_4 \times \mathbb{C}P^3$

- The ABJM theory is conjectured to be the world-volume effective theory of a stack of $N' = kN$ M2-branes probing $\mathbb{C}^4/\mathbb{Z}_k$ singularity
- The near horizon limit of the geometry of N' M2-branes on $\mathbb{C}^4/\mathbb{Z}_k$ gives the $AdS_4 \times S^7/\mathbb{Z}_k$ geometry
- In the large N limit the gravitational dual of N' M2 branes in flat space is $AdS_4 \times S^7$, orbifolding:

$$ds^2_{11} = \frac{\hat{R}^2}{4} ds^2_{AdS_4} + \hat{R}^2 ds^2_{S^7/\mathbb{Z}_k}$$

where

$$\hat{R}^2 = (2^5 \pi^2 N')^{1/3} l_p^2 \qquad \text{and} \qquad F_{(4)} = \frac{3\hat{R}^3}{8} \epsilon_{AdS_4}$$

where ϵ_{AdS_4} is the unit volum form of AdS_4

- S^7: $z_1, z_2, z_3, z_4 \in \mathbb{C}^4$ orbifolding $\mathbb{Z}_k$: $z_i \to e^{\frac{2\pi\imath}{k}} z_i$

$$ds^2_{S^7/\mathbb{Z}_k} = \sum_{a=1}^{4} dz_a d\bar{z}_a \qquad \sum_{a=1}^{4} z_a \bar{z}_a = 1$$

- The coordinates z_a can be associated with the 4 scalars Y^A of the ABJM theory

- The orbifolding is implemented as follows write

$$z_a = \mu_a e^{i\phi_a} \quad \text{we span} \quad S^7 \quad \text{if} \quad \sum_{a=1}^{4} \mu_a^2 = 1$$

we associate to each angle ϕ_a the angular momentum $J_a = -i\partial_{\phi_a}$. Introduce the angle

$$\gamma = \frac{1}{4}(\phi_1 + \phi_2 + \phi_3 + \phi_4)$$

- The orbifold $S^7/\mathbb{Z}_k$ is now implemented as the identification

$$\boxed{\gamma \equiv \gamma + \frac{2\pi}{k}}$$

We have that

$$J_1 + J_2 + J_3 + J_4 = -i\partial_\gamma$$

Thus, we see that the orbifolding is equivalent to the quantization condition on the angular momenta

$$\boxed{J_1 + J_2 + J_3 + J_4 \in k\mathbb{Z}}$$

- We can write

$$ds_{11}^2 = \frac{\hat{R}^2}{4}\left(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\hat{\Omega}_2^2\right) + \overbrace{\hat{R}^2 ds^2_{\mathbb{C}P^3} + \hat{R}^2(d\gamma + A)^2}^{ds^2_{S^7/\mathbb{Z}_k}}$$

Compare with $\boxed{ds_{11}^2 = e^{-2\phi/3} ds^2_{IIA} + e^{4\phi/3}(d\gamma + A)^2}$

using the standard relation between M-theory and type IIA, we get the following background of type IIA supergravity

$$\boxed{ds^2 = \frac{R^2}{4}\left(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\hat{\Omega}_2^2\right) + R^2 ds^2_{\mathbb{C}P^3}}$$

with $R^2 = \frac{\hat{R}^3}{k}$, $\frac{R^2}{l_s^2} = \frac{\sqrt{2^5\pi^2 N'}}{k} = \sqrt{\frac{2^5\pi^2 N}{k}} = \sqrt{2^5\pi^2\lambda}$ and

$$g_s^2 = e^{2\phi} = \frac{R^2}{k^2} = \left(\frac{2^5\pi^2\lambda}{k^4}\right)^{\frac{1}{2}}$$

- By demanding a small curvature and a small string coupling $\Rightarrow$ this background is a valid background for type IIA string theory when $\lambda \gg 1$ and $N \ll k^5$
- in the type II A description we should require that the dependence on γ is absent, nothing should depend on the direction that shrinks to zero

$$\boxed{J_1 + J_2 + J_3 + J_4 = 0}$$

The $\mathrm{SU}(2) \times \mathrm{SU}(2)$ sector

- We take a limit of type II A string theory on $\mathrm{AdS}_4 \times \mathbb{C}P^3$ which corresponds to zooming in to the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ sector
- Consider the M-theory background $\mathrm{AdS}_4 \times S^7$ corresponding to M2 branes on $\mathbb{C}^4$
- The 2 $\mathrm{SU}(2)$'s are gotten from splitting $\mathbb{C}^4 \to \mathbb{C}^2 \times \mathbb{C}^2$
- the first $\mathrm{SU}(2)$, corresponding to $A_{1,2}$ is associated to $z_{1,2}$. The second $\mathrm{SU}(2)$ corresponding to $B_{1,2}$ is associated to $z_{3,4}$.
- Split the S^7 into 2 S^3 one for each $\mathbb{C}^2$

$$ds^2_{S^7} = d\theta^2 + \cos^2\theta d\Omega_3^2 + \sin^2\theta d\Omega_3'^2$$

Introduce now the angles

$$\varphi_1 = \phi_1 - \phi_2 \ , \quad \varphi_2 = \phi_4 - \phi_3$$

$$\gamma = \frac{1}{4}(\phi_1 + \phi_2 + \phi_3 + \phi_4) \ , \quad \delta = \frac{1}{4}(\phi_1 + \phi_2 - \phi_3 - \phi_4)$$

thus

$$J_1 + J_2 + J_3 + J_4 = -i\partial_\gamma \quad , \text{ and we define } \quad 2J = J_1 + J_2 - J_3 - J_4 = -i\partial_\delta$$

In these coordinates the 2 3-spheres become

$$d\Omega_3^2 = \frac{1}{4} d\Omega_2^2 + \left(d\gamma + d\delta + \frac{1}{2} \cos\theta_1 d\varphi_1 \right)^2 \qquad d{\Omega'_3}^2 = \frac{1}{4} d{\Omega'_2}^2 + \left(d\gamma - d\delta + \frac{1}{2} cos\theta_2 d\varphi_2 \right)^2$$

$$\boxed{d\Omega_2^2 = d\theta_1^2 + \sin^2\theta_1 d\varphi_1^2 \qquad d{\Omega'_2}^2 = d\theta_2^2 + \sin^2\theta_2 d\varphi_2^2}$$

- **the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ sector is the subspace $\mathbb{R} \times S^2 \times \ S^2 \sim \mathbb{R}P^2$ of AdS$_4 \times \mathbb{C}P^3$**

  The two 2-sphere correspond to the two $\mathrm{SU}(2)$'s

- On the string theory side, the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ symmetry of the two S^2 is a subgroup of the $\mathrm{SU}(4)$ symmetry of $\mathbb{C}P^3$. Cartan generators of the $\mathrm{SU}(4)$

  $$S_z^{(1)} \equiv \frac{J_1 - J_2}{2} = -i\partial_{\varphi_1} \ , \quad S_z^{(2)} \equiv \frac{J_4 - J_3}{2} = -i\partial_{\varphi_2} \ , \quad J = -\frac{i}{2}\partial_\delta$$

  On the gauge theory side, $S_z^{(1)}$ counts the total spin for the $A^{1,2}$ scalars and $S_z^{(2)}$ for the $B_{1,2}$ scalars, in operators of the form

  $$\boxed{\mathcal{O} = \chi^{b_1 \dots b_J}_{a_1 \dots a_J} \mathrm{Tr}(A^{a_1} B_{b_1} \dots . A^{a_J} B_{b_J})}.$$

- The bare scaling dimension of each scalar is $\frac{1}{2}$. Total conformal dimension of $\mathcal{O}$

  $$\boxed{\Delta_0 = J}$$

  The $\mathrm{SU}(2) \times \mathrm{SU}(2)$ is defined as the sector consisting of operators with $\Delta_0 = J$

back

Fermions: General GS superstring action for type IIA

The quadratic fermionic Lagrangian is given by **(Cvetic, Lu, Pope, Stelle, 2000)**

$$\mathcal{L}_F = -i\bar{\theta}M^{ij}\Gamma_\mu \tilde{D}_j\theta\partial_i X^\mu - \frac{i}{16}\partial_i X^\mu \partial_j X^\nu e^\phi \bar{\theta} M^{ij}\left(\Gamma_{11}\Gamma_\mu \Gamma^{\lambda\sigma}\Gamma_\nu F_{\lambda\sigma} + \frac{1}{12}\Gamma_\mu \Gamma^{\lambda\sigma\tau\rho}\Gamma_\nu F_{\lambda\sigma\tau\rho}\right)\theta$$

with $M^{ij} \equiv \left(H^{ij} - \epsilon^{ij}\Gamma_{11}\right)$, $H^{ij} \equiv \sqrt{-h}h^{ij}$ and $\tilde{D}_i\theta \equiv \partial_i\theta + \frac{1}{4}\partial_i X^\mu \omega_\mu{}^{ab}\Gamma_{ab}\theta$

- In type IIA we have two Majorana-Weyl spinors $\theta^{1,2}$ with opposite chirality, *i.e.* $\Gamma_{11}\theta^1 = \theta^1$ and $\Gamma_{11}\theta^2 = -\theta^2$. 32 component real spinor $\theta = \theta^1 + \theta^2$.
- it can be rewritten as

$$\mathcal{L} = \mathcal{L}_{\rm kin} + \mathcal{L}_{\rm WZ}\ , \quad \mathcal{L}_{\rm kin} = -\frac{1}{2}h^{AB}S_{AB}$$

$$S_{AB} = g_{\mu\nu}\partial_A X^\mu \partial_B X^\nu + i\bar{\theta}\Gamma_\mu(\partial_A X^\mu \tilde{D}_B\theta + \partial_B X^\mu \tilde{D}_A\theta) + \frac{i}{8}\partial_A X^\mu \partial_B X^\nu \bar{\theta}(\Gamma_\mu M \Gamma_\nu + \Gamma_\nu M \Gamma_\mu)\theta$$

and the Wess-Zumino part of the Lagrangian given by

$$\mathcal{L}_{\rm WZ} = i\epsilon^{AB}\bar{\theta}\Gamma_{11}\Gamma_\mu \partial_A X^\mu \tilde{D}_B\theta + \frac{i}{8}\epsilon^{AB}\partial_A X^\mu \partial_B X^\nu \bar{\theta}\Gamma_{11}\Gamma_\mu M \Gamma_\nu\theta$$

where strengths $F_{(2)}$ and $F_{(4)}$ have been combined into the matrix M as

$$M = -\frac{1}{2}F_{\mu\nu}\Gamma_{11}\Gamma^{\mu\nu} + \frac{1}{24}F_{\mu\nu\rho\sigma}\Gamma^{\mu\nu\rho\sigma}$$

Supersymmetric fermionic directions for type IIA on $AdS_4 \times \mathbb{C}P^3$

Define the quantity (Nilsson and Pope, Class. Quantum Grav.(1984))

$$Q = \Gamma_{0123}\Gamma_{11}F^{ab}\Gamma_{ab}$$

here $\Gamma_{0123}\Gamma_{11}F_{ab} = J_{ab}$ is the Kähler form on $\mathbb{C}P^3$

$$Q = \Gamma_{0123}\Gamma_{11}(-\Gamma_{49} - \Gamma_{56} + \Gamma_{78}) = \Gamma_{5678} - \Gamma_{49}(\Gamma_{56} - \Gamma_{78})$$

Note that $Q^2 = 2Q + 3$. Q has 24 eigenvalues -1 and 8 eigenvalues 3.
Recalling that in the ground state the fermion fields of the $D = 10$ theory are set to zero, the criterion for unbroken supersymmetry of the vacuum is that their supersymmetric variations should also vanish.

$$\delta\theta\big|_{\mathrm{vacuum}} = 0$$

It is satisfied for the 24 eigenvalue

$$Q = -1$$

Then the projector on to the supersymmetric fermionic directions is

$$P = \frac{3 - Q}{4}$$

All supersymmetric fermionic directions are characterized by $P\theta = \theta$ or equivalently $Q\theta = -\theta$

$$Q = \Gamma_{0123}\Gamma_{11}(-\Gamma_{49} - \Gamma_{56} + \Gamma_{78}) = \Gamma_{5678} - \Gamma_{49}(\Gamma_{56} - \Gamma_{78})$$

- $\Gamma_{5678}^2 = \Gamma_{4956}^2 = \Gamma_{4978}^2 = 1$. If $\Gamma_{5678}\theta = -\theta$ we see that $Q\theta = -\theta$ $\Longrightarrow$ 16 supersymmetric directions.
- If $\Gamma_{5678}\theta = \theta$ then we need in addition that $\Gamma_{4956}\theta = \theta$ to get $Q = -1$ $\Longrightarrow$ 8 supersymmetric directions

$\Longrightarrow$ total of 24 supersymmetric directions.

The 8 remaining non-supersymmetric directions are characterized by $\Gamma_{5678}\theta = \theta$ and $\Gamma_{4956}\theta = -\theta$ corresponding to $Q\theta = 3\theta$.

Coset model supersymmetric Lagrangian

- Aim: $\Longrightarrow$ construct a supersymmetric Lagrangian with full $Osp(6|4)$ symmetry
- Problem: $\Longrightarrow$ it is not possible to make the whole symmetry manifest
- Solution: $\Longrightarrow$ the target space can be realized as a coset $\boxed{\frac{OSp(6|4)}{U(3)\times SO(3,1)}}$

(Metsaev, Tseytlin, 9805028 for $AdS_5 \times S^5$, Arutyunov and Frolov, 0806.4940 for $AdS_4 \times \mathbb{C}P^3$)

The superconformal algebra $OSp(6|4)$ has schematically the form

$$[B_i, B_j] = f_{ij}^k B_k \ , \quad [F_\alpha, B_i] = f_{\alpha i}^\beta F_\beta \ , \quad \{F_\alpha, F_\beta\} = f_{\alpha\beta}^i B_i$$

- B_i denotes all the bosonic generators (i.e. the generators of $SO(3,2) \times SU(4)$) and F_α all the fermionic generators (i.e. the 32 supercharges). f_{ij}^k, $f_{\alpha i}^\beta$ and $f_{\alpha\beta}^i$ $\Longrightarrow$ structure constants of $OSp(6|4)$.
- Generalize the worldsheet derivative to include all the generators of $OSp(6|4)$ $\Longrightarrow$ introduce the left-invariant Maurer-Cartan forms L^i and L^α

$$D = P(d + L^i B_i + L^\alpha F_\alpha)$$

d is the worldsheet derivative and L^i and L^α are one-forms on the worldsheet.

Demanding that $D^2 = 0$ leads to the Maurer-Cartan equations

$$dL^i + f^i_{jk} L^j \wedge L^k - f^i_{\alpha\beta} L^\alpha \wedge L^\beta = 0 \ , \quad dL^\alpha - f^\alpha_{\beta i} L^\beta \wedge L^i = 0$$

Solved by the generalized Maurer-Cartan forms L^i_s and L^α_s for any real number $s \in \mathbb{R}$

$$L^i = L^i_s|_{s=1} \ , \quad L^\alpha = L^\alpha_s|_{s=1}$$

along with the differential equations

$$\partial_s \tilde{L}^t_s = \theta^\alpha f^i_{\alpha\beta} \tilde{L}^i_s \ , \quad \partial_s L^\alpha_s = D\theta^\alpha - \theta^\beta f^\alpha_{\beta i} \tilde{L}^i_s \ , \quad \text{where} \quad L^i_s = (L_0)^i + \tilde{L}^i_s$$

$(L_0)^i$ is L^i when setting the fermionic coordinates to zero $\theta^\alpha = 0$.
The solution **Kallosh, Rahmfeld and Rajaraman (9805217) for $AdS_5 \times S^5$ and Gomis, Sorokin and Wulff (0811.1566), Grassi, Sorokin, Wulff, (0903.5407) for $AdS_4 \times \mathbb{C}P^3$**

$$L^i_s = (L_0)^i + 2\theta^\alpha f^i_{\alpha\beta} \left(\frac{\sinh^2(\frac{s}{2}\mathcal{M})}{\mathcal{M}^2} \right)^\beta_\gamma (D\theta)^\gamma$$

$$L^\alpha_s = \left(\frac{\sinh(s\mathcal{M})}{\mathcal{M}} \right)^\alpha_\beta (D\theta)^\beta \qquad \text{with} \qquad (\mathcal{M}^2)^\alpha_\beta = -\theta^\gamma f^\alpha_{\gamma i} \theta^\delta f^i_{\delta\beta}$$

the covariant derivative of θ being of the form $(D\theta)^\alpha = P(d\theta^\alpha - (L_0)^i f^\alpha_{\beta i} \theta^\beta)$

$$D\theta = P(d - \tfrac{1}{R}\Gamma_{0123}\Gamma_a e^a + \tfrac{1}{4}\omega^{ab}\Gamma_{ab})\theta$$

and the supervielbeins being given by

$$L(s)^a_A = E(s)^a_\mu \partial_A X^\mu + E(s)^a_\alpha \partial_A \theta^\alpha \ , \quad L(s)^\alpha_A = E(s)^\alpha_\mu \partial_A X^\mu + E(s)^\alpha_\beta \partial_A \theta^\beta$$

with $L^a_A = L(s=1)^a_A$ and $L^\alpha_A = L(s=1)^\alpha_A$.

Structure constants and $\mathcal{M}^2$

$(\mathcal{M}^2)^\alpha_\beta$ can be generally written as

$$(\mathcal{M}^2)^\alpha_\beta = -\theta^\gamma f^\alpha_{\gamma i}\theta^\delta f^i_{\delta\beta} = -\theta^\gamma \tilde{f}^\alpha_{\gamma a}\theta^\delta \hat{f}^a_{\delta\beta} - \theta^\gamma \tilde{f}^\alpha_{\gamma\ ab}\theta^\delta \hat{f}^{ab}_{\delta\beta}$$

where the fermions are 32-dimensional. The structure constants $\tilde{f}$ can be read from the covariant derivative of θ, $(D\theta)^\alpha = d\theta^\alpha - \tilde{f}^\alpha_{\beta\ a} e^a\theta^\beta - \tilde{f}^\alpha_{\beta\ ab}\omega^{ab}$. The $\hat{f}$ can be derived from the supervielbeins. We have

$$\tilde{f}^\alpha_{\beta a} = \frac{1}{R}(\Gamma_{0123}P\Gamma_a P)^\alpha_{\ \beta}\ , \quad \tilde{f}^\alpha_{\beta\ ab} = -\frac{1}{4}(P\Gamma_{ab}P)^\alpha_{\ \beta}$$

where, here, $a, b = 0, \ldots, 9$.
For $\hat{f}^a_{\alpha\beta}$ we get

$$\hat{f}^a_{\alpha\beta} = 2i(P\Gamma^0\Gamma^a P)_{\alpha\beta}$$

The non-vanishing structure constants $\hat{f}^{ab}_{\alpha\beta}$ are

$$\hat{f}^{\hat{a}\hat{b}}_{\alpha\beta} = -\frac{4i}{R}(P\Gamma^0\Gamma_{0123}\Gamma^{\hat{a}\hat{b}}P)_{\alpha\beta}\ , \quad \hat{f}^{a'b'}_{\alpha\beta} = \frac{2i}{R}\left(P\Gamma^0\left(\Gamma_{0123}\Gamma^{a'b'} - J^{a'b'}\Gamma_{11}\right)P\right)_{\alpha\beta}$$

where $\hat{a} = 0, 1, 2, 3; \quad a' = 4, \cdots, 9$

Supercoset Lagrangian for type IIA on $AdS_4 \times \mathbb{C}P^3$

The Lagrangian is

$$\mathcal{L} = -\frac{1}{2} h^{AB} \eta_{ab} L_A^a L_B^b - 2i\epsilon^{AB} \int_0^1 ds L(s)_A^a (\bar{\theta}\Gamma_a \Gamma_{11})_\alpha L(s)_B^\alpha$$

We write the total Lagrangian as

$$\mathcal{L} = \mathcal{L}_{\rm kin} + \mathcal{L}_{\rm WZ} \qquad \mathcal{L}_{\rm kin} = -\frac{1}{2} h^{AB} S_{AB} \qquad S_{AB} \equiv \eta_{ab} L_A^a L_B^b$$

Dividing S_{AB} according to the number of fermions

$$S_{AB} = S_{AB}^{(0\rm f)} + S_{AB}^{(2\rm f)} + S_{AB}^{(4\rm f)}$$

we found for the kinetic term

$$S_{AB}^{(0\rm f)} = g_{\mu\nu} \partial_A X^\mu \partial_B X^\nu \quad , \qquad S_{AB}^{(2\rm f)} = i\bar{\theta}\Gamma_\mu (\partial_A X^\mu D_B \theta + \partial_B X^\mu D_A \theta)$$

quartic term $\Longrightarrow$ $S_{AB}^{(4\rm f)} = -(\bar{\theta}\Gamma^a D_A \theta)(\bar{\theta}\Gamma_a D_B \theta) + \frac{i}{12} \bar{\theta}\Gamma_\mu \mathcal{M}^2 (\partial_A X^\mu D_B \theta + \partial_B X^\mu D_A \theta)$

The WZ term

$$\mathcal{L}_{\rm WZ} = \mathcal{L}_{\rm WZ}^{(2\rm f)} + \mathcal{L}_{\rm WZ}^{(4\rm f)} \qquad \mathcal{L}_{\rm WZ}^{(2\rm f)} = -i\epsilon^{AB} \partial_A X^\mu \bar{\theta}\Gamma_\mu \Gamma_{11} D_B \theta$$

quartic term $\Longrightarrow$ $\mathcal{L}_{\rm WZ}^{(4\rm f)} = \frac{1}{2}\epsilon^{AB} (\bar{\theta}\Gamma^a D_A \theta)(\bar{\theta}\Gamma_a \Gamma_{11} D_B \theta) - \frac{i}{12} \epsilon^{AB} \partial_A X^\mu \bar{\theta}\Gamma_\mu \Gamma_{11} \mathcal{M}^2 D_B \theta$

Equivalence of supercoset and Cvetic et al. Lagrangian

Using

$$F_{(2)} = \frac{2}{R}(-e^4 \wedge e^9 - e^5 \wedge e^6 + e^7 \wedge e^8) \qquad F_{(4)} = \frac{6}{R} e^0 \wedge e^1 \wedge e^2 \wedge e^3$$

we compute

$$\boxed{M = -\tfrac{1}{2} F_{\mu\nu} \Gamma_{11} \Gamma^{\mu\nu} + \tfrac{1}{24} F_{\mu\nu\rho\sigma} \Gamma^{\mu\nu\rho\sigma} = -\tfrac{8}{R} \Gamma_{0123} P}$$

Inserting this into the Cvetic et al. Lagrangian we see that this is equivalent to the Lagrangian derived from the coset model up to the second order in the fermions provided we have that

$$\tilde{D}_A \theta = P \tilde{D}_A \theta$$

for any spinor with $P\theta = \theta$ on the $\mathrm{AdS}_4 \times \mathbb{C}P^3$ background. This is true if

$$\omega_\mu^{ab} [P, \Gamma_{ab}] = 0 \qquad (1)$$

For nearly all non-zero components of ω_μ^{ab} you have that a and b are such that $[P, \Gamma_{ab}] = 0$. The only 8 components for which this is not the case are $\omega_{x_1}^{45}$, $\omega_{x_1}^{69}$, $\omega_{y_1}^{46}$, $\omega_{y_1}^{59}$, $\omega_{x_2}^{47}$, $\omega_{x_2}^{89}$, $\omega_{y_2}^{48}$ and $\omega_{y_2}^{79}$. However, if we consider $\omega_{x_1}^{45}$ and $\omega_{x_1}^{69}$ we see that $\omega_{x_1}^{45} = -\omega_{x_1}^{69}$. For (1) to hold for $\mu = x_1$ it is therefore sufficient that $[P, \Gamma_{45} - \Gamma_{69}] = 0$ which indeed is the case, . It works similar for $\mu = y_1, x_2, y_2$ hence we have checked explicitly that Eq.(1) holds.

Fixing the κ symmetry

1. κ symmetry transformation on the supersymmetric fermionic directions $P\theta = \theta$. We have already imposed a partial κ symmetry gauge choice by demanding $P\theta = \theta$ thus reducing the number of fermionic directions from 32 to 24
2. Fix the remaining 8 directions in the κ symmetry by a gauge choice that follows from our light cone gauge. κ symmetry variations
$$E^\alpha_\mu \delta X^\mu + E^\alpha_\beta \delta\theta^\beta = [(1+\Gamma)\kappa]^\alpha \ , \qquad E^a_\mu \delta X^\mu + E^a_\alpha \delta\theta^\alpha = 0$$
where $(1+\Gamma)/2$ is a spinor projection matrix.
3. Analyze the κ symmetry in the Penrose limit. In this limit we have the super vielbeins
$$E^a = e^a + i\bar{\theta}\Gamma^a D\theta \ , \quad E^\alpha = (D\theta)^\alpha$$
and the κ-symmetry transformations
$$\delta X^\mu (D_\mu\theta)^\alpha + \delta\theta^\alpha = [(1+\Gamma)\kappa]^\alpha \ , \quad \delta t + 2i\bar{\theta}\Gamma^+ D_\mu\theta\delta X^\mu + 2i\bar{\theta}\Gamma^+\delta\theta = 0$$
using $e^+ = \frac{1}{2}dX^+$ and assuming $P\theta = \theta$ and $P\delta\theta = \delta\theta$.

Combining we get

$$\delta X^+ = 2i\bar{\theta}\Gamma^+(1+\Gamma)\kappa$$

For the light cone gauge to be consistent we need $\delta X^+ = 0$ under variations of κ symmetry

$$Q = \Gamma_{5678} - \Gamma_{49}(\Gamma_{56} - \Gamma_{78})$$

- Recall: 16 supersymmetric directions with $\Gamma_{5678}\theta = -\theta$ and 8 with $\Gamma_{5678}\theta = \theta$
- Among the first 16 we can choose that either with $P\Gamma^- P\theta = 0$ or $P\Gamma^+ P\theta = 0$
- Take a supersymmetric fermionic direction with $P\Gamma^- P\theta = 0$. Then $\delta t = -2i\theta^T(1+\Gamma)\kappa \Longrightarrow$ such fermionic directions are not consistent with the light cone gauge.
- The matrix $P\Gamma^- P$ has 8 supersymmetric fermionic directions with eigenvalue zero, characterized by $\Gamma_{5678}\theta = -\theta$ and $\Gamma^-\theta = 0$. Thus, we fix the remaining κ symmetry gauge freedom by demanding that these directions are put to zero. This leaves the following 16 physical fermionic directions in the light cone gauge

8 fermionic directions defined by $\Gamma_{5678}\theta = -\theta, \; \Gamma^+\theta = 0$
8 fermionic directions defined by $\Gamma_{5678}\theta = \theta, \; \Gamma_{4956}\theta = \theta$

This is thus our condition for physical fermionic modes.

It is useful to parameterize this by introducing the projectors

$$\mathcal{P}_+ = \frac{I + \Gamma_{5678}}{2}\frac{I + \Gamma_{4956}}{2} \ , \ \ \mathcal{P}_- = \frac{I - \Gamma_{5678}}{2}\frac{I - \Gamma_{09}}{2}$$
$$\mathcal{P}'_+ = \frac{I + \Gamma_{5678}}{2}\frac{I - \Gamma_{4956}}{2} \ , \ \ \mathcal{P}'_- = \frac{I - \Gamma_{5678}}{2}\frac{I + \Gamma_{09}}{2}$$

named after the eigenvalue of Γ_{5678}. 4,5,6,7 and 8 are transverse directions in $\mathbb{C}P^3$. These projectors are mutually orthogonal to each other and they are all idempotent and symmetric. We have

$$P = \mathcal{P}_+ + \mathcal{P}_- + \mathcal{P}'_- \ , \ \ I = \mathcal{P}_+ + \mathcal{P}_- + \mathcal{P}'_+ + \mathcal{P}'_-$$

We are thus imposing that our spinor $\theta = \theta^1 + \theta^2$ obeys

$$(\mathcal{P}_+ + \mathcal{P}_-)\theta = \theta$$

or, equivalently, $(\mathcal{P}'_+ + \mathcal{P}'_-)\theta = 0$. This is our condition for physical fermionic modes.

back

The computation of the finite part of the mixing matrix is done by

$$R^2 H^{(one-loop)} = H_{4B} + H_{4F} + H_{2B2F} + H_{3B}^2 + H_{B2F}^2.$$

- In the H_{4B} $\sum \mathfrak{a}_p \mathfrak{a}_q \mathfrak{a}_r \mathfrak{a}_s$ terms we take only the matrix elements of the type

$$\langle 0| \mathfrak{a}_n^- \mathfrak{a}_{-n}^- \quad \mathfrak{a}_p^+ \mathfrak{a}_q^+ \; \mathfrak{a}_r^- \mathfrak{a}_s^- \quad \mathfrak{a}_n^+ \mathfrak{a}_{-n}^+ |0\rangle$$

  (and all possible permutations) not involving the sums, i.e. not having contractions inside the Hamiltonian itself, since all other elements contribute to the superficially divergent part already taken into account

- This may potentially contribute both to the diagonanl and off-diagonal terms in the bosonic sector. The H_{2B2F} $\sum \mathfrak{d}_p \mathfrak{d}_q \mathfrak{a}_r \mathfrak{a}_s$ part of the Hamiltonian in the finite mixing matrix can contribute only to the mixing between a two-fermion and two-boson state, e.g.

$$\langle 0| \mathfrak{d}_n^- \mathfrak{d}_{-n}^- \quad \mathfrak{d}_p^+ \mathfrak{d}_q^+ \; \mathfrak{a}_r^- \mathfrak{a}_s^- \quad \mathfrak{a}_n^+ \mathfrak{a}_{-n}^+ |0\rangle$$

- The H_{3B}, H_{B2F} terms contribute via virtual particle exchange. Its possible contribution to the matrix element of the finite mixing between degenerate states $|a\rangle$ and $|b\rangle$ is

$$\langle a|H^{(2)}|b\rangle = \sum_i \frac{\langle a|H_3^\dagger|i\rangle\langle i|H_3|b\rangle}{E_a - E_i},$$

  where $H_3 = H_{3B}, H_{3F}$.

back

$(2, 0, 1, 1, 1)$

The only difference with respect to the $(1, 1, 1, 1, 1)$ case is the presence of the S-matrix between the magnons on the right hand side of the Bethe equation. The Bethe equation reads:

$$\frac{1}{J}\left[A + \lambda'\left(B + 8n^3\pi^3\right) + \lambda'^2\left(C - 32n^5\pi^5\right)\right] = 0\,,$$

which gives

$$A = 0, \quad B = -8n^3\pi^3, \quad C = 32n^5\pi^5\,,$$

and therefore

$$E_{20111} = 4n^2\pi^2\lambda' - 8n^4\pi^4\lambda'^2 + 32n^6\pi^6\lambda'^3 - \frac{1}{J}\left(-32n^4\pi^4\lambda'^2 + 256n^6\pi^6\lambda'^3\right) + \ldots\,.$$

We have 10 string states out of the 24 (8 made with 2 bosons + 16 made with 2 fermions) having this correction, then this solution is indeed correct and useful.

$(1,1,2,2,2)$

One can see, similarly to the case with $(.,.,1,1,1)$ auxiliary roots, that the only consistent solution is $u_{1,j}=\infty$ and $u_{3,j}=\infty$. Here also the same subtlety as in the $(.,.,1,1,1)$ case arises: provided that $u_{1,j}=\infty$ and $u_{3,j}=\infty$, the Bethe equations are indeed satisfied whatever are the values of $u_{2,j}$. This creates an issue about the multiplicity of the solution, but I think that the point is that every configuration for $u_{2,j}$ is equivalent and then the solution is physically only one. The Bethe equation reads:

$$\frac{1}{J}\left[A+4n\pi+\lambda' B+\lambda'^2\left(C+16n^5\pi^5\right)\right]=0\,,$$

which gives

$$A=-4n\pi,\quad B=0,\quad C=-16n^5\pi^5\,,$$

and therefore

$$E_{11222}=4n^2\pi^2\lambda'-8n^4\pi^4\lambda'^2+32n^6\pi^6\lambda'^3-\frac{1}{J}\left(-16n^2\pi^2\lambda'+64n^4\pi^4\lambda'^2-448n^6\pi^6\lambda'^3\right)+\dots\,.$$

which is the spectrum of the string states $|\tilde{t}_{1,2}\rangle=\left(\tilde{a}_n^{1,2}\right)^\dagger\left(\tilde{a}_{-n}^{2,1}\right)^\dagger|0\rangle$.

$(2, 0, 2, 2, 2)$ and $(0, 2, 2, 2, 2)$

This case is specular to the the $(1, 1, 2, 2, 2)$ case above, only difference the S-matrix between the magnons on the right hand side of the Bethe equations. The Bethe equation reads:

$$\frac{1}{J}\left[A + 2n\pi + \lambda'\left(B + 8\pi^3 n^3\right) + \lambda'^2\left(C - 32n^5\pi^5\right)\right] = 0\,,$$

which gives

$$A = -2n\pi, \quad B = -8n^3\pi^3, \quad C = 32n^5\pi^5\,,$$

and therefore

$$E_{20222} = 4n^2\pi^2\lambda' - 8n^4\pi^4\lambda'^2 + 32n^6\pi^6\lambda'^3 - \frac{1}{J}\left(-8n^2\pi^2\lambda' + 64n^6\pi^6\lambda'^3\right) + \dots\,.$$

which is the spectrum of the string states $|\tilde{s}_{1,2}\rangle = \left(\tilde{a}_n^{1,2}\right)^\dagger \left(\tilde{a}_{-n}^{1,2}\right)^\dagger |0\rangle$.

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Energies of Bethe states

- Taking into account the exact $\mathbb{Z}_2$ degeneracy due to $p \to -p$ symmetry and the double occurrence of (..210) and (..111) states due to branching of auxiliary roots we obtain 24 states in the tree-level degenerate spectrum, which exactly corresponds to the degenerate string double-magnon spectrum.
- Parameters $\epsilon_{1,2,3}$ are **twists**, introduced in the following way

$$
\begin{aligned}
e^{i\epsilon_1} &= \prod_{j=1}^{K_2} \frac{u_{1,k} - u_{2,j} + \frac{i}{2}}{u_{1,k} - u_{2,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{1 - 1/x_{1,k} x^+_{4,j}}{1 - 1/x_{1,k} x^-_{4,j}} \prod_{j=1}^{K_{\bar{4}}} \frac{1 - 1/x_{1,k} x^+_{\bar{4},j}}{1 - 1/x_{1,k} x^-_{\bar{4},j}} , \\
e^{i\epsilon_2} &= \prod_{j\neq k}^{K_2} \frac{u_{2,k} - u_{2,j} - i}{u_{2,k} - u_{2,j} + i} \prod_{j=1}^{K_1} \frac{u_{2,k} - u_{1,j} + \frac{i}{2}}{u_{2,k} - u_{1,j} - \frac{i}{2}} \prod_{j=1}^{K_3} \frac{u_{2,k} - u_{3,j} + \frac{i}{2}}{u_{2,k} - u_{3,j} - \frac{i}{2}} , \\
e^{i\epsilon_3} &= \prod_{j=1}^{K_2} \frac{u_{3,k} - u_{2,j} + \frac{i}{2}}{u_{3,k} - u_{2,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3,k} - x^+_{4,j}}{x_{3,k} - x^-_{4,j}} \prod_{j=1}^{K_{\bar{4}}} \frac{x_{3,k} - x^+_{\bar{4},j}}{x_{3,k} - x^-_{\bar{4},j}} ,
\end{aligned}
$$

to regularize Bethe equations for those solutions which yield $u_i = 0, u + i = \infty$ perturbatively.

The finite-size correction is normalized $E = \frac{8\varepsilon}{J}$, where ε is given as series in powers of λ' in the following Table

Table: Boson-boson spectrum from Bethe Ansatz

state K_4	K_4	K_3	K_2	K_1	Energy correction ε	Auxiliary roots
1	1	1	1	1	$-n^2\pi^2\lambda' + 4n^4\pi^4(\lambda')^2 - 32n^6\pi^6(\lambda')^3 + 256n^8\pi^8(\lambda')^4$	$u_1 = \frac{1}{\epsilon_1}$ $u_2 = \frac{1}{\epsilon_2}$ $u_3 = \frac{1}{\epsilon_3}$
2	0	1	1	1	$-4n^4\pi^4(\lambda')^2 + 32n^6\pi^6(\lambda')^3 - 256n^8\pi^8(\lambda')^4$	———//———
1	1	1	1	1	$-2n^2\pi^2\lambda' + 12n^4\pi^4(\lambda')^2 - 96n^6\pi^6(\lambda')^3 + 768n^8\pi^8(\lambda')^4$	$u_1 = \frac{1}{\epsilon_1}$ $u_2 = \frac{1}{\epsilon_2}$ $x_3 = \to \frac{1}{2\sqrt{2}n\pi\sqrt{\lambda'}}$
2	0	1	1	1	$-n^2\pi^2\lambda' + 4n^4\pi^4(\lambda')^2 - 32n^6\pi^6(\lambda')^3 + 256n^8\pi^8(\lambda')^4$	———//———

Table: Fermion-fermion spectrum from Bethe Ansatz

state K_4	K_4	K_3	K_2	K_1	Energy	Auxiliary roots
1	1	2	2	0	$-4n^4\pi^4\lambda'^2 + 32n^6\pi^6\lambda'^3 - 256n^8\pi^8\lambda'^4$	$u_{21} = \frac{1}{2}\left(u_{31} + u_{32} - i\sqrt{u_{31}^2 - 2u_{32}u_{31} + \ldots}\right.$ $u_{22} = \frac{1}{2}\left(u_{31} + u_{32} + i\sqrt{u_{31}^2 - 2u_{32}u_{31} + \ldots}\right.$ $x_{31} = -2i - \frac{i}{2n^2\pi^2\lambda'} + 2in^2\pi^2\lambda'$ $x_{32} = 2i + \frac{i}{2n^2\pi^2\lambda'} - 2in^2\pi^2\lambda'$
2	0	2	2	0	$n^2\pi^2\lambda' - 12n^4\pi^4\lambda'^2 + 96n^6\pi^6\lambda'^3 - 768n^8\pi^8\lambda'^4$	——//——
1	1	2	1	0	$-n^2\pi^2\lambda' + 4n^4\pi^4\lambda'^2 - 32n^6\pi^6\lambda'^3 + 256n^8\pi^8\lambda'^4$	$u_2 = \frac{1}{2}(u_{31} + u_{32})$ x_{31}, x_{32} solutions of $\frac{(x_3 - x(u+i/2))(x_3 - x(-u+i/2))}{(x_3 - x(u-i/2))(x_3 - x(-u-i/2))} = e^{i\alpha}$
2	0	2	1	0	$-4n^4\pi^4\lambda'^2 + 32n^6\pi^6(\lambda')^3 - 256n^8\pi^8\lambda'^4$	——//——
1	1	2	1	0	$-n^2\pi^2\lambda' + 4n^4\pi^4\lambda'^2 - 32n^6\pi^6\lambda'^3 + 256n^8\pi^8\lambda'^4$	$u_2 = \frac{1}{2}\left(u_{31} + u_{32} + \frac{4}{c_3}\right)$ $x_{31} = \frac{(1+4n^2)c_3}{4\sqrt{2J}\sqrt{\lambda'}}$ $x_{32} = -\frac{(1+4n^2)(-2i+c_3)}{4\sqrt{2J}\sqrt{\lambda'}}$
2	0	2	1	0	$-4n^4\pi^4\lambda'^2 + 32n^6\pi^6(\lambda')^3 - 256n^8\pi^8\lambda'^4$	——//——
1	1	2	0	0	$-n^2\pi^2\lambda' + 4n^4\pi^4\lambda'^2 - 32n^6\pi^6\lambda'^3 + 256n^8\pi^8\lambda'^4$	$x_{31} = \frac{Jc}{4\sqrt{2}n^2\pi^2\sqrt{\lambda'}}$ $x_{32} = \frac{Jc_3}{4\sqrt{2}n^2\pi^2\sqrt{\lambda'}}$
2	0	2	0	0	$-4n^4\pi^4\lambda'^2 + 32n^6\pi^6(\lambda')^3 - 256n^8\pi^8\lambda'^4$	——//——

These tables are remarkable $\Longrightarrow$ **all states presented here are also found on the string side and the energies coincide up to λ'^4!**

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