QCD string: from Tevatron to LHC

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Based on:

Y. M. Phys. Rev. D **83**, 026007 (2011) [arXiv:1012.0708 [hep-th]]

Phys. Lett. B **699**, 199 (2011) [arXiv:1103.2269 [hep-th]]

Extending: Y. M., Poul Olesen

- Phys. Rev. Lett. **102**, 071602 (2009) [arXiv:0810.4778 [hep-th]]
- Phys. Rev. D **80**, 026002 (2009) [arXiv:0903.4114 [hep-th]]
- Phys. Rev. D **82**, 045025 (2010) [arXiv:1002.0055 [hep-th]]
- JHEP **08**, 095 (2010) [arXiv:1006.0078 [hep-th]]

P. Buividovich, Y.M. Nucl. Phys. B **834**, 453 (2010) [arXiv:0911.1083 [hep-th]]

Contents of the talk

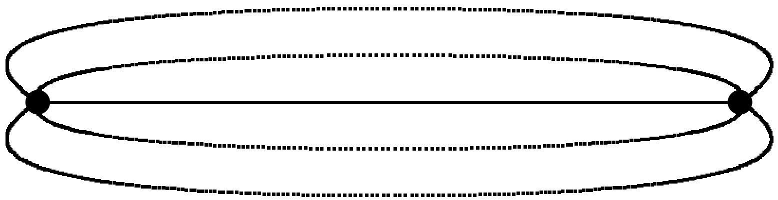
- QCD string as an effective long string:
  consistent quantization in $d = 4$ and the mean-field analysis

- description of Monte Carlo data for QCD spectrum by
  Nambu–Goto $+$ extrinsic curvature

- meson scattering amplitudes in the Regge regime and
  effective Reggeon trajectory in QCD

QCD string as such

QCD string is formed by fluxes of gluon field at distances larger than confinement scale $1/\Lambda_{QCD}$



Lines of force between static quarks for distances $> 1/\Lambda_{QCD}$ (confinement)

Figure 1: Lines of force between static quarks.

Monte Carlo and linear hadron Regge trajectories support this picture.

Perturbative QCD works at small distances (asymptotic freedom), while effective string theory works at large distances.

QCD string as such (continued)

QCD string is not pure bosonic Nambu–Goto Y.M., Migdal (1979)
Extra (fermionic) degrees of freedom are required at string world-sheet.

But the asymptote of large Wilson loops is universal:

classical string

$$W(C) \overset{\text{large } C}{\propto} \mathrm{e}^{-K S_{\text{min}}(C)} \quad \Longrightarrow \quad \text{the area law = confinement}$$

semiclassical correction Lüscher, Symanzik, Weisz (1980)

$$W(C) \overset{\text{rectangle}}{\propto} \mathrm{e}^{-KRT + \frac{\#}{24}\frac{\pi T}{R}} \quad \Longrightarrow \quad \text{the Lüscher term}$$

for rectangle with $T \gg R$

$\# = d - 2$ (as for bosonic string) Ambjorn, Olesen, Peterson (1984)
De Forcrand, Schierholz, Schneider, Teper (1985)

Universality of the $1/R$ term owing to Lüscher's roughening

$$\left\langle x_\perp^2 \right\rangle \propto \alpha' \log(R^2/\alpha') \gg \alpha' \qquad \alpha' = 1/2\pi K$$

Next orders in $1/R$ are *not* universal.

Example of QCD_2

QCD is solvable in d=2 as $N \to \infty$ 't Hooft (1974)

Free theory (no interaction) in axial gauge $A_1 = 0$: $[A_1, A_2] = 0$.

Propagator in the axial gauge

$$D_{\mu\nu}(x-y) = -\frac{1}{2}\delta_{\mu 2}\delta_{\nu 2}\,|x_1 - y_1|\,\delta^{(1)}(x_2 - y_2)$$

results in the area law for Wilson loops without self-intersections

$$W(C) = \mathrm{e}^{-g^2 N \oint_C \mathrm{d}x^\mu \mathrm{d}y^\nu\, D_{\mu\nu}(x-y)/2} = \mathrm{e}^{-A} \qquad A = \frac{g^2 N}{2}\text{Area}$$

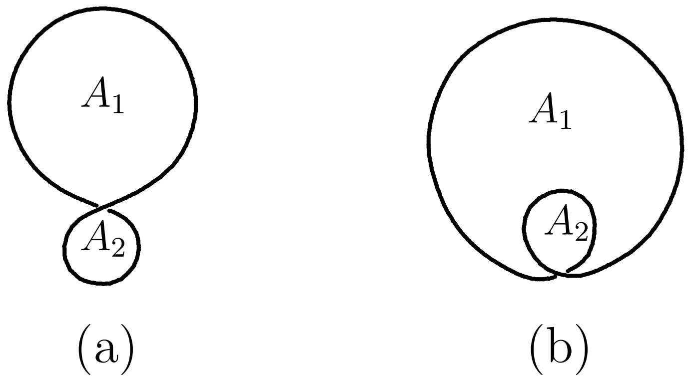
because

$$\int x_1 \mathrm{d}x_2 = \text{Area}$$

This looks like bosonic string in d=2, but this is not the case for self-intersecting contours Kazakov, Kostov (1980), Bralić (1980)

Example of QCD$_2$ (continued)

Simplest contours with self-intersections:



$$W(\mathrm{a}) = \mathrm{e}^{-A_1 - A_2}$$
$$W(\mathrm{b}) = (1 - 2A_2)\,\mathrm{e}^{-A_1 - 2A_2}$$

$A_1 + 2A_2 =$ total (folded) area

Figure 3: Contours with one self-intersection.

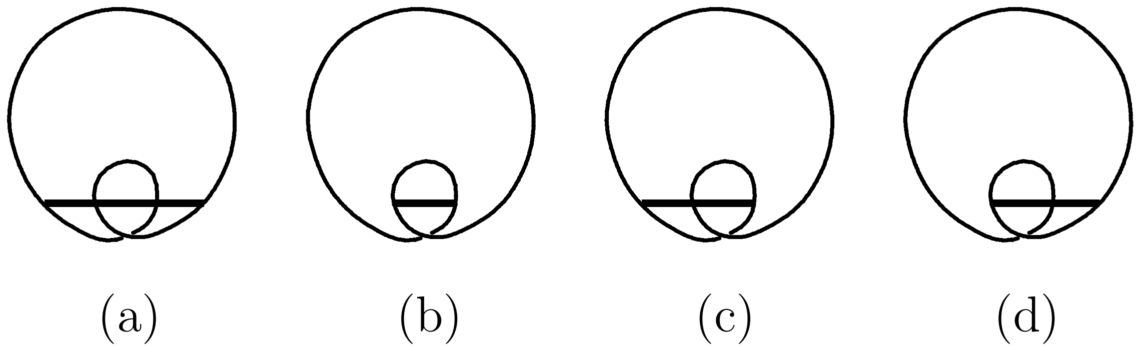


Figure 3: Four types of diagrams of order $g^2 N$.

W(b) differs (same to order $g^2 N$) from the Abelian $W(\mathrm{b}) = \mathrm{e}^{-A_1 - 4A_2}$.
Pre-exponentials ($\Longrightarrow$ self-intersections) are not essential for large C.

Monte-Carlo data for energy levels

Athenodorou, Bringoltz, Teper (2010)

3+1 SU(3) LGT: closed winding string (flux tube) of length circumstance R. Spectrum of Nambu–Goto string Arvis (1983)

$$E_n = \sqrt{(KR)^2 + \left(n - \frac{d-2}{24}\right) 8\pi K} \qquad 8 \Longrightarrow 2 \quad \text{for open string}$$

absolutely beautifully describes the lattice calculations

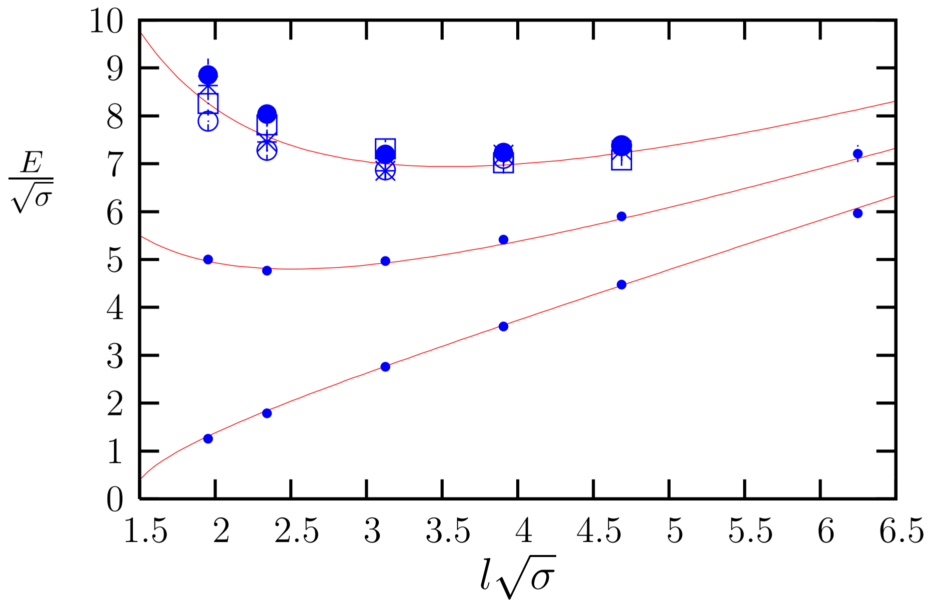


Figure 4: Lightest flux tube energies for longitudinal momenta $q = 0$, $\bullet$, $q = 1$, $\bullet$, and $q = 2$ in SU(3) at $\beta = 6.0625$. The four $q = 2$ states are $J^{P_t} = 0^+(\star)$, $1^{\pm}(\circ)$, $2^+(\square)$, $2^-(\bullet)$. Lines are Nambu-Goto predictions.

$$\sqrt{K}R_c = \sqrt{\frac{d-2}{3}\pi} \approx 1.44$$

$$R/R_c \geq 1.4$$

Polyakov string formulation

Polyakov (1981)

Independent intrinsic metric g_{ab} at string worldsheet.
Quadratic (in $X^\mu(z)$) action in conformal gauge $g_{ab} = \mathrm{e}^{\varphi}\eta_{ab}$

$$S = 2K \int \mathrm{d}^2 z\, \partial X \cdot \bar{\partial} X + \frac{26-d}{24\pi} \int \mathrm{d}^2 z\, (\partial\varphi\bar{\partial}\varphi + \mu^2\, \mathrm{e}^{\varphi}) + \text{ ghosts}$$

2d conformal theory:

$$c\,(\text{central charge}) = d\,(\text{matter}) + 26 - d\,(\text{Liouville}) + 26\,(\text{ghosts}) = 0$$

$d = 1$ barrier in Knizhnik-Polyakov-Zamolodchikov–David-Distler-Kawai treatment of the (nonlinear) measure $\mathcal{D}\varphi$:

$$\text{string susceptibility } = \frac{d - 1 - \sqrt{(25-d)(1-d)}}{12}$$

is well-defined only for $d \leq 1$ or $d \geq 25$ otherwise it becomes complex.

Lüscher term can be reproduced á la Durhuus, Olesen, Petersen (1984)
by conformally mapping $T \gg R$ rectangle onto the upper half-plane.

Effective string theory

Polchinski, Strominger (1991)

Closed string winding along a compact direction of large radius R is described by nonpolynomial action

$$S_{\text{eff}} = 2K \int \mathrm{d}^2 z\, \partial X \cdot \bar{\partial} X - \frac{\beta}{2\pi} \int \mathrm{d}^2 z\, \frac{\partial^2 X \cdot \bar{\partial}^2 X}{\partial X \cdot \bar{\partial} X} + \dots \qquad \beta = \frac{26-d}{12}$$

It can be analyzed order by order in $1/R$ by expanding around the classical solution

$$X^\mu_{\text{cl}} = (e^\mu z + \bar{e}^\mu \bar{z})\, R \qquad e \cdot e = \bar{e} \cdot \bar{e} = 0 \quad e \cdot \bar{e} = -1/2$$

It looks like the Liouville action in the Polyakov formulation expressed (modulo total derivatives and the constraints) via an induced metric

$$\mathrm{e}^{\varphi_{\text{ind}}} = 2\, \partial X \cdot \bar{\partial} X$$

(in the conformal gauge), which is not treated independently.

This effective string theory has been analyzed using the conformal field theory technique order by order in $1/R$, revealing the Arvis spectrum of the Nambu–Goto string in d-dimensions.

Effective string theory (cont.)

Conformal symmetry is maintained in $d \neq 26$ order by order in $1/R$:

$$\delta X^\mu = \epsilon(z)\partial X^\mu - \frac{\beta\alpha'}{4}\partial^2\epsilon(z)\frac{\bar{\partial}X^\mu}{\partial X\cdot\bar{\partial}X} + \text{c.c.}$$

It transforms X^μ nonlinearly and the corresponding conserved energy-momentum tensor is

$$T_{zz} = -\frac{1}{\alpha'}\partial X\cdot\partial X + \frac{\beta}{2}\frac{\partial^3 X\cdot\bar{\partial}X}{\partial X\cdot\bar{\partial}X} + \mathcal{O}(R^{-2})$$

Expanding around the classical solution $X^\mu = X^\mu_{\text{cl}} + Y^\mu_{\text{q}}$, we obtain

$$T_{zz} = -\frac{2R}{\alpha'}e\cdot\partial Y_{\text{q}} - \frac{1}{\alpha'}\partial Y_{\text{q}}\cdot\partial Y_{\text{q}} - \frac{\beta}{R}\bar{e}\cdot\partial^3 Y_{\text{q}} + \mathcal{O}(R^{-2})$$

The central charge is determined by

$$\langle T_{zz}(z_1)T_{zz}(z_2)\rangle = \frac{d+12\beta}{2(z_1-z_2)^4} + \mathcal{O}\left((z_1-z_2)^{-2}\right)$$

to be $d+12\beta = 26$ and is cancelled by ghosts at any d.

Mean-field approximation for bosonic string

Reproducing the saddle point in the large-d limit by O. Alvarez (1981)

Mean-field + fluctuations (the ansatz) in world-sheet parametrization:

$$X^1_{\rm mf}(\omega) = \frac{\omega_1}{\omega_R}R+\delta X^1(\omega) \quad X^2_{\rm mf}(\omega) = \frac{\omega_2}{\omega_T}T+\delta X^2(\omega) \quad X^\perp(\omega) = \delta X^\perp(\omega)$$

$\omega_1, \omega_2 \in \omega_R \times \omega_T$ rectangle, ω_R, ω_T change under reparametrizations.

The mean-field action (with account for the Lüscher term)

$$S_{\rm mf} = \frac{1}{4\pi\alpha'}\left(R^2\frac{\omega_T}{\omega_R} + T^2\frac{\omega_R}{\omega_T}\right) - \frac{\pi(d-2)}{24}\frac{\omega_T}{\omega_R}$$

Minimization with respect to ω_T/ω_R reproduces the square root

$$\left(\frac{\omega_T}{\omega_R}\right)_* = \frac{T}{\sqrt{R^2 - R_c^2}} \qquad S_{\rm mf*} = \frac{T}{2\pi\alpha'}\sqrt{R^2 - R_c^2} \qquad R_c^2 = \pi^2\frac{(d-2)}{6}\alpha'$$

For upper half-plane parametrization

$$\left(\frac{\omega_T}{\omega_R}\right) = \frac{K\left(\sqrt{r}\right)}{K\left(\sqrt{1-r}\right)} \qquad r = \frac{s_{43}s_{21}}{s_{42}s_{31}} \qquad s_{ij} \equiv s_i - s_j$$

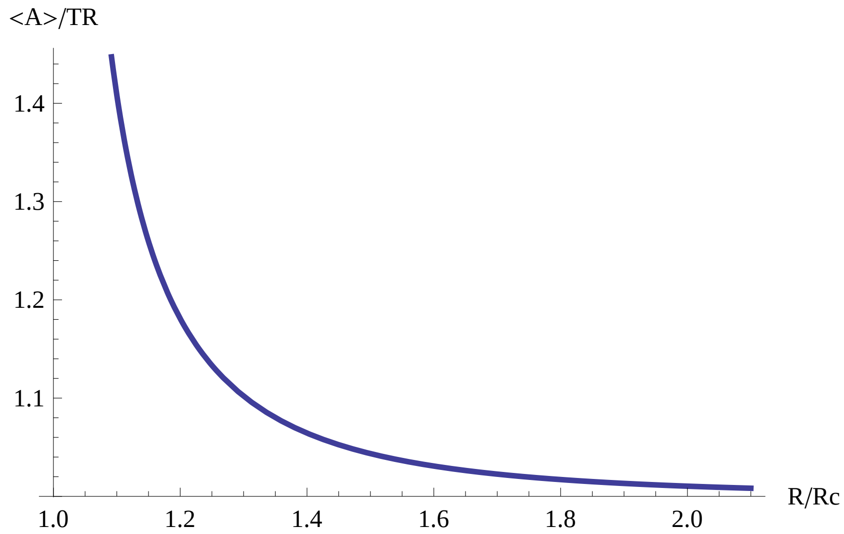
= the Grötzsch modulus which is monotonic in r.

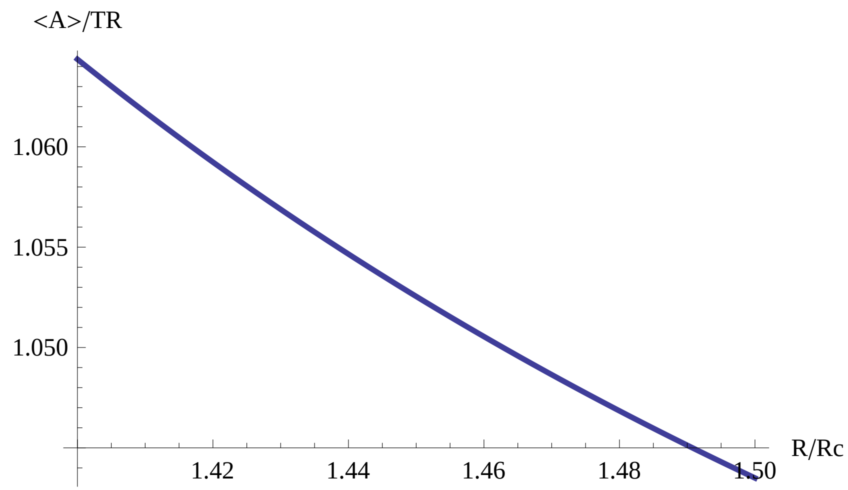
Mean-field approximation for bosonic string (cont.)

Mean-field is applicable if fluctuations around the minimal surface are small. The ratio the dominant area to the minimal area:

$$\frac{\langle \text{Area} \rangle}{A_{\text{min}}} = \frac{1}{RT} \frac{\text{d}}{\text{d}K} S_{\text{mf}} = \frac{1 - R_c^2/2R^2}{\sqrt{1 - R_c^2/R^2}}$$

It diverges when $R \to R_c$ from above $\Longrightarrow$ typical surfaces becomes very large and the mean-field approximation ceases to be applicable.





Mean-field is exact for bosonic string and it is approached by QCD string with an exponential accuracy $\exp(-C\,R/R_c)$

Why mean-field is exact for bosonic string?

Mean-field works generically at large d but is expected to be exact at any d for bosonic string. The arguments are:

1) it is true in the semiclassical approximation;

2) it reproduces an exact result in $d = 26$;

3) it agrees with the existence 't Hooft (1974) of a massless bound state in $N = \infty$ QCD_2 for massless quarks.

In static gauge $d - 2$ fluctuating (transverse) degrees of freedom.

In conformal gauge:

path integral over reparametrizations of the boundary contributes 24,

ghosts contribute 26,

fluctuations of X^μ contribute d.

All together: $d + 24 - 26 = d - 2$

QCD string as rigid string

Polyakov (1986), Kleinert (1986)

Adding extrinsic curvature term to Nambu–Goto

$$S_{\text{rigid string}} = \frac{K}{2} \int \mathrm{d}^2\omega \, \partial_a X \cdot \partial_a X + \frac{1}{2\alpha} \int \mathrm{d}^2\omega \, \frac{1}{\sqrt{g}} \Delta X \cdot \Delta X$$

dimensionless α

to be distinguished from intrinsic (or scalar) curvature
(Gauss–Bonnet in 2d $\Longrightarrow$ genus)

$$R = D^2 X \cdot D^2 X - D^a D^b X \cdot D_a D_b X$$

Original expectation was that rigidity smoothen crumpling of surfaces (which is related to tachyonic instability).

This is partially true!!!

Mean-field for rigid string

Introducing $\rho = \sqrt{g}$ and Lagrange multipliers λ^{ab}:

$$S_{\text{r.s.}} = K \int \mathrm{d}^2\omega\, \rho + \frac{1}{2\alpha}\int \mathrm{d}^2\omega \frac{1}{\rho}\Delta X \cdot \Delta X + \frac{1}{2}\int \mathrm{d}^2\omega\, \lambda^{ab}\left(\partial_a X \cdot \partial_b X - \rho\delta_{ab}\right)$$

Mean-field (variational) ansatz (only $X^{\perp}$ fluctuates): exact at large d but approximate at finite d (summing bubble graphs for $O(d)$-vector field).

$$X^1_{\text{mf}}(\omega) = \frac{\omega_1}{\omega_R} R \quad X^2_{\text{mf}}(\omega) = \omega_2 \;\; (\omega_T = T) \quad X^{\perp}(\omega) = \delta X^{\perp}(\omega)$$

$$\rho_{\text{mf}}(\omega) = \rho \quad \lambda^{11}_{\text{mf}}(\omega) = \lambda^{11} \quad \lambda^{22}_{\text{mf}}(\omega) = \lambda^{22} \quad \lambda^{12}_{\text{mf}}(\omega) = \lambda^{21}_{\text{mf}}(\omega) = 0$$

$$\begin{aligned} \frac{1}{T} S_{\text{mf}} &= \frac{1}{2}\left(\lambda_{11}\omega_R + \lambda_{22}\frac{R^2}{\omega_R}\right) + \rho\left(K - \frac{\lambda^{11}}{2} - \frac{\lambda^{22}}{2}\right)\omega_R \\ &\quad + \frac{d}{2T}\,\mathrm{tr}\,\ln\left(-\lambda^{11}\partial_1^2 - \lambda^{22}\partial_2^2 + \frac{1}{\alpha\rho}(\partial_1^2 + \partial_2^2)^2\right) \end{aligned}$$

$$\frac{d}{2T}\,\mathrm{tr}\,\ln(\ldots) \longrightarrow \begin{cases} 1) \quad -\frac{\pi d}{6\omega_R}\sqrt{\frac{\lambda^{22}}{\lambda^{11}}} & \alpha \to \infty \\ 2) \quad -\frac{\pi d}{3\omega_R} + \frac{d}{2}\sqrt{\alpha\rho\lambda^{11}} & \alpha \to 0 \end{cases} \qquad \text{(closed string)}$$

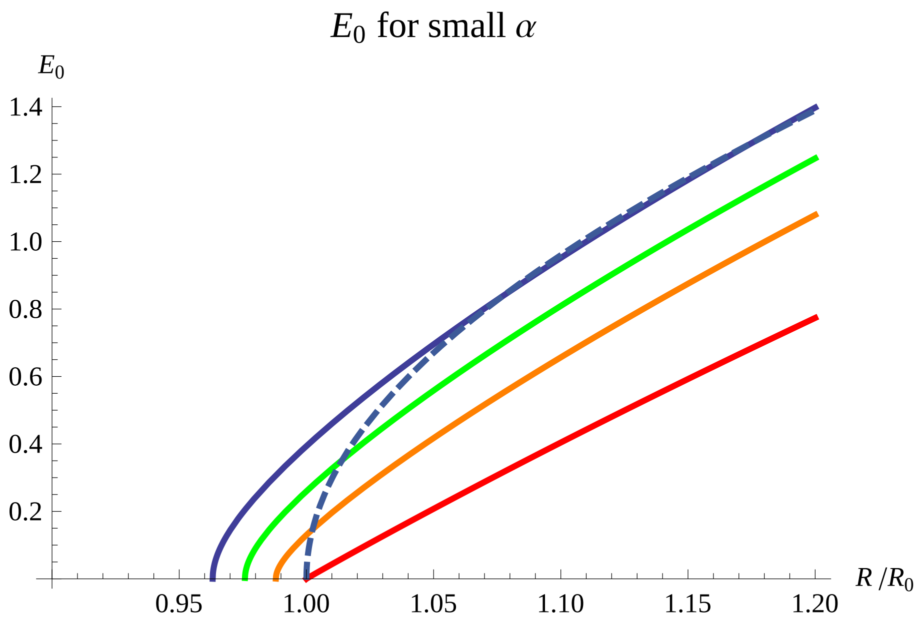
Mean-field for rigid string (cont.)

Both limiting cases can be analyzed analytically.

1) same mean-field as above (large α) Alvarez (1981)

2) solvable in square roots (small α) Polchinski, Yang (1992)

$$E_0 = \lambda^{11}\omega_R \quad \sqrt{\lambda^{11}} = \frac{3}{8}\frac{d\sqrt{\alpha}}{R} + \sqrt{\frac{9}{16}\frac{d^2\alpha}{R^2} + K - \frac{\pi d}{3R^2}} \quad \omega_R = \sqrt{R^2 - \frac{dR}{2}\sqrt{\frac{\alpha}{\lambda^{11}}}}$$



$\alpha = 0.3$

$\alpha = 0.2$

$\alpha = 0.1$

$\alpha = 0$

Tachyonic singularity moves left to smaller values of R/R_0.

Exact determinant

Olesen, Yang (1987); Braaten, Pisarski, Tse (1987); Germán, Kleinert (1988)

Integrating over dk_2 (as $T \to \infty$) and regularizing via zeta function

and introducing $\Lambda = \dfrac{\sqrt{\alpha\rho\lambda^{11}}\,\omega_R}{2\pi}$ instead of ρ

$$\begin{aligned}
\frac{1}{T}S_{\text{mf}} &= \frac{1}{2}\left(\lambda_{11}\omega_R + \lambda_{22}\frac{R^2}{\omega_R}\right) + \left(\frac{2K}{\lambda^{11}} - 1 - \frac{\lambda^{22}}{\lambda^{11}}\right)\frac{2\pi^2\Lambda^2}{\alpha\omega_R} \\
&+ \frac{2\pi d}{\omega_R}\left[-\frac{1}{6} + \frac{\Lambda}{2} + \frac{\Lambda^2}{4}\left(1 + \frac{\lambda^{22}}{\lambda^{11}}\right)\ln\frac{1}{\mu a_{\text{UV}}}\right] \\
&+ \frac{2\pi d}{\omega_R}\sum_{n\geq 1}\left[\sqrt{\frac{\Lambda^2}{2} + n^2 + \Lambda\sqrt{\frac{\Lambda^2}{4} + \left(1 - \frac{\lambda^{22}}{\lambda^{11}}\right)n^2}}\right. \\
&\left. + \sqrt{\frac{\Lambda^2}{2} + n^2 - \Lambda\sqrt{\frac{\Lambda^2}{4} + \left(1 - \frac{\lambda^{22}}{\lambda^{11}}\right)n^2}} - 2n - \frac{\Lambda^2}{4n}\left(1 + \frac{\lambda^{22}}{\lambda^{11}}\right)\right]
\end{aligned}$$

renormalization (A.F.) $\alpha(\mu) = \dfrac{\alpha}{1 - \dfrac{\alpha d}{4\pi}\ln\dfrac{1}{\mu a}}$ $K(\mu) = \dfrac{K}{1 - \dfrac{\alpha d}{4\pi}\ln\dfrac{1}{\mu a}}$

Numerical analysis

Mathematica says (as expected):

- Λ is large as $R \to \infty$ and/or $\alpha \to \infty$
  (Nambu–Goto with exponential corrections)
- Λ is small for $R \lesssim R_0$
  (analytic Polchinski–Yang solution approximately works)

For $\alpha = 2.65$: $E_0 > 0$ when $R/R_0 > 0.6$

Operators of lower dimension (not relevant in infrared) have to be added within the effective string theory description of QCD string. They can be systematically induced by internal degrees of freedom of QCD string: *e.g.* massive fermions or higher dimensions *á la* Sakharov's induced gravity

Remember $$R_0 = \sqrt{\frac{(d-2)\pi}{3K}} \approx 0.7 \text{ fermi}$$ already small

Pertubative QCD is alternatively applicable at small distances.

Induced extrinsic curvature

Determinant of 2d Laplacian (or Dirac operator squared) in conformal gauge:

$$\mathrm{tr}\ln\Delta = \frac{1}{12\pi}\int \mathrm{d}^2 z\left(\mu_0\,\mathrm{e}^{\varphi} \mp \partial\varphi\bar{\partial}\varphi\right)$$

For the induced metric $\mathrm{e}^{\varphi} = 2\partial X \cdot \bar{\partial} X$

$$\int \mathrm{d}^2 z\,\partial\varphi\bar{\partial}\varphi = \frac{1}{4}\int \mathrm{d}^2 z\ \mathrm{e}^{-\varphi}\Delta X \cdot \Delta X$$

— no other operators of same dimension than extrinsic curvature.

A logarithmically divergent coefficient appears when induced by 4d fermions pulled back to the worldsheet Wiegmann (1989)

Parthasarathy, Viswanathan (1999)

Likewise, extrinsic curvature is induced by higher dimensions in AdS/CFT with confining background if # massless= # modes acquired mass

Greensite, Olesen (1999)

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- QCD string as an effective long string:
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- description of Monte Carlo data for QCD spectrum by
  Nambu–Goto + extrinsic curvature

- meson scattering amplitudes in the Regge regime and
  effective Reggeon trajectory in QCD:
  – momentum Wilson loops
  – reparametrization path integral
  – relative strength perturbative-QCD/QCD-string

Momentum loops

Migdal (1986)

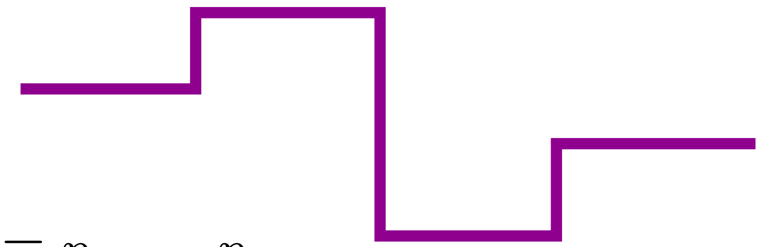
Scattering amplitudes are given by a reparametrization-invariant functional Fourier transformation

$$A[p(\cdot)] = \int \mathcal{D}x \, \mathrm{e}^{\mathrm{i}\int p\cdot \mathrm{d}x} \, J[x(\cdot)] \, W[x(\cdot)]$$

$J[x(\cdot)]$ is process-dependent

of the Wilson loop (to be identified with string disk amplitude) for piecewise constant momentum loop $p^\mu(t)$:

$$p(t) = p_i \qquad \text{for } t_i < t < t_{i+1}$$



$$\dot{p}(t) = -\sum_i \Delta p_i \delta(t - t_i) \qquad \text{with} \quad \Delta p_i \equiv p_{i-1} - p_i$$

representing M momenta of (all incoming) particles.
Then momentum conservation is automatic while an (infinite) volume V is produced, say, by integration over $x_0 = x_M$.

The Fourier transformation of string vertex operators is reproduced:

$$\int \mathrm{d}t \, p(t) \cdot \dot{x}(t) = -\int \mathrm{d}t \, \dot{p}(t) \cdot x(t) = \sum_i \Delta p_i \cdot x_i$$

Reparametrization path integral

Polyakov (1997)

Wilson loop of large size in large-N QCD = string disk amplitude integrated over reparametrizations of the boundary contour:

$$W[x(\cdot)] = \int \mathcal{D}_{\text{diff}} t(s)\, \mathrm{e}^{-K S[x(t)]}$$

i.e. over functions $t(s)$ with $t'(s) \geq 0$ (string tension $K = 1/2\pi\alpha'$) first appeared in off-shell propagator Cohen, Moore, Nelson, Polchinski (1986)

Douglas algorithm for solving the Plateau problem Douglas (1931) (finding the minimal surface) is to minimize the boundary functional

$$S[x(t)] = \frac{1}{4\pi} \int_{-\infty}^{+\infty} \frac{\mathrm{d}s_1 \mathrm{d}s_2}{(s_1 - s_2)^2} \left[x(t(s_1)) - x(t(s_2))\right]^2$$

with respect to reparametrizations $t(s)$ ($\dot{t}(s) \geq 0$).

This representation can be derived for critical strings (bosonic string in $d = 26$ or superstring in $d = 10$).

Area law for asymptotically large C (or very large K) $\Longrightarrow$ a saddle point in the reparametrization path integral at $t(s) = t_*(s)$.
Zig-zag or backtracking symmetry holds for the minimal area.

Nontrivial example: ellipse

Unit-disk parametrization of an ellipse: $z = r\,\mathrm{e}^{\mathrm{i}\phi}$:

$x^1 = a\cos\theta(\phi),\; x^2 = b\sin\theta(\phi)$ a and b are major and minor radii

Suppose $\theta * (\phi) = \phi$, then Douglas' integral

$$S[x(\theta)] = \pi\frac{a^2+b^2}{2} \qquad \text{rather than} \qquad \pi ab$$

Equality is only for a circle $a = b$, when the unit-disk coordinates are conformal.

Douglas' minimization for an ellipse gives incomplete elliptic integral:

$$\theta'_*(\phi) = \frac{\pi}{2K(\nu)}\frac{1}{\sqrt{(1-\nu)^2 + 4\nu\sin^2\phi}} \qquad \frac{\pi K\left(\sqrt{1-\nu^2}\right)}{2K(\nu)} = \log\frac{a+b}{a-b}$$

where $K(\nu)$ is the complete elliptic integral of the first kind.
(This can be obtained by conformal mapping of unit disk onto interior of an ellipse Schwarz (1869)).

Large loops and minimal area

Gaussian fluctuations around the saddle-point $t_*(s)$ result in a pre-exponential factor

$$W\left[x(\cdot)\right] \stackrel{\text{large loops}}{=} F\left[\sqrt{K}x(\cdot)\right] \mathrm{e}^{-KS_{\text{min}}[x(\cdot)]}\left[1+\mathcal{O}\left((KS_{\text{min}})^{-1}\right)\right],$$

which is contour dependent

Asymptotic area law is recovered modulo the pre-exponential which is not essential for large loops.

More subtle effects (such as the Lüscher term) resides in the pre-exponential factor, coming from fluctuations around $t_*(s)$

$$t(s) = t_*(s) + \frac{\beta(s)}{\sqrt{KS_{\text{min}}}}$$

For a $R \times T$ rectangle Y.M., Olesen (2010)

$$F\left[\text{rectangle}\right] \propto \mathrm{e}^{\pi T/R} \qquad \text{for } T \gg R$$

reproducing the Lüscher term for bosonic string in $d = 26$.

Momentum disk amplitude

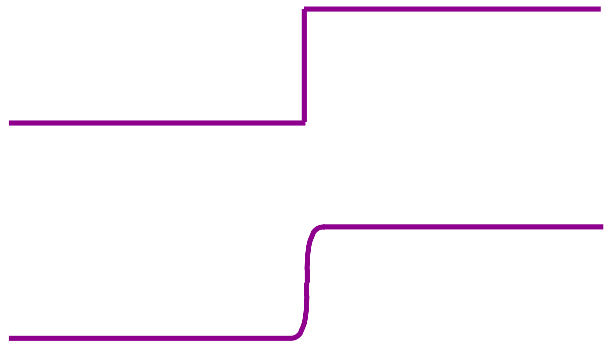
After the Gaussian path integration over $\mathcal{D}x^\mu(t)$ the amplitude

$$A[p(\cdot)] = \int \mathcal{D}s(t) \exp\left(\alpha' \int_{-\infty}^{+\infty} \mathrm{d}t_1 \int_{-\infty}^{+\infty} \mathrm{d}t_2\, \dot{p}(t_1)\cdot\dot{p}(t_2) \ln|s(t_1)-s(t_2)|\right)$$

looks like the disk amplitude

$=$ Wilson loop with

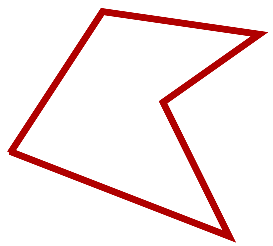
$$x^\mu(t) = \frac{1}{K} p^\mu(t)$$



Smeared step-wise (with boundary Liouville field $\varphi(t_i)$ for covariance)

$$p^\mu(t) = \frac{1}{\pi} \sum_i \Delta p_i^\mu \arctan\frac{(t-t_i)}{\varepsilon_i} \xrightarrow{\varepsilon_i \to 0} \frac{1}{2} \sum_i \Delta p_i^\mu \,\mathrm{sign}\,(t-t_i) \qquad \varepsilon_i = \varepsilon\, \mathrm{e}^{-\varphi(t_i)}$$

$\Longrightarrow$ polygon with vertices



$$x_i^\mu = \frac{1}{K} p_i^\mu \qquad x_i^\mu - x_{i-1}^\mu = \frac{1}{K} \Delta p_i^\mu$$

Similar to Wilson-loop/scattering-amplitude duality in $\mathcal{N}=4$ SYM

Alday, Maldacena (2007), Drummond, Korchemsky, Sokatchev (2008)

Invariant regularization and Liouville field

The Gaussian exponent

$$-\pi \int_{-\infty}^{+\infty} \mathrm{d}t_1\, \mathrm{d}t_2\, \dot{p}(t_1)\cdot\dot{p}(t_2)\, G\left(s(t_1), s(t_2)\right)$$
$$= \sum_{k\neq l} \Delta p_k \cdot \Delta p_l \log|s_k - s_l| - \pi \sum_j \Delta p_j^2 G(s_j, s_j)$$

For an invariant regularization $G(s_j, s_j)$ involves the Liouville field

Polyakov (1981)

$$G(s_i, s_j) = -\frac{1}{\pi} \ln|s_i - s_j| \qquad \text{for } |s_i - s_j| \gg \varepsilon_i, \varepsilon_j$$
$$G(s_j, s_j) \longrightarrow G_\varepsilon(s_j, s_j) = \frac{1}{\pi}\log\frac{1}{\varepsilon} + \frac{1}{2\pi}\varphi(s_j)$$

For critical bosonic string (in $d = 26$): Aoyama, Dhar, Namazie (1986)

$$A = \int \mathcal{D}\varphi(s) \int \prod_m \mathrm{d}s_m\, \mathrm{e}^{\varphi(s_m)/2 - \pi\alpha'\Delta p_m^2 G(s_m, s_m)} \prod_{j\neq m} |s_j - s_m|^{\alpha'\Delta p_j\cdot\Delta p_m}$$

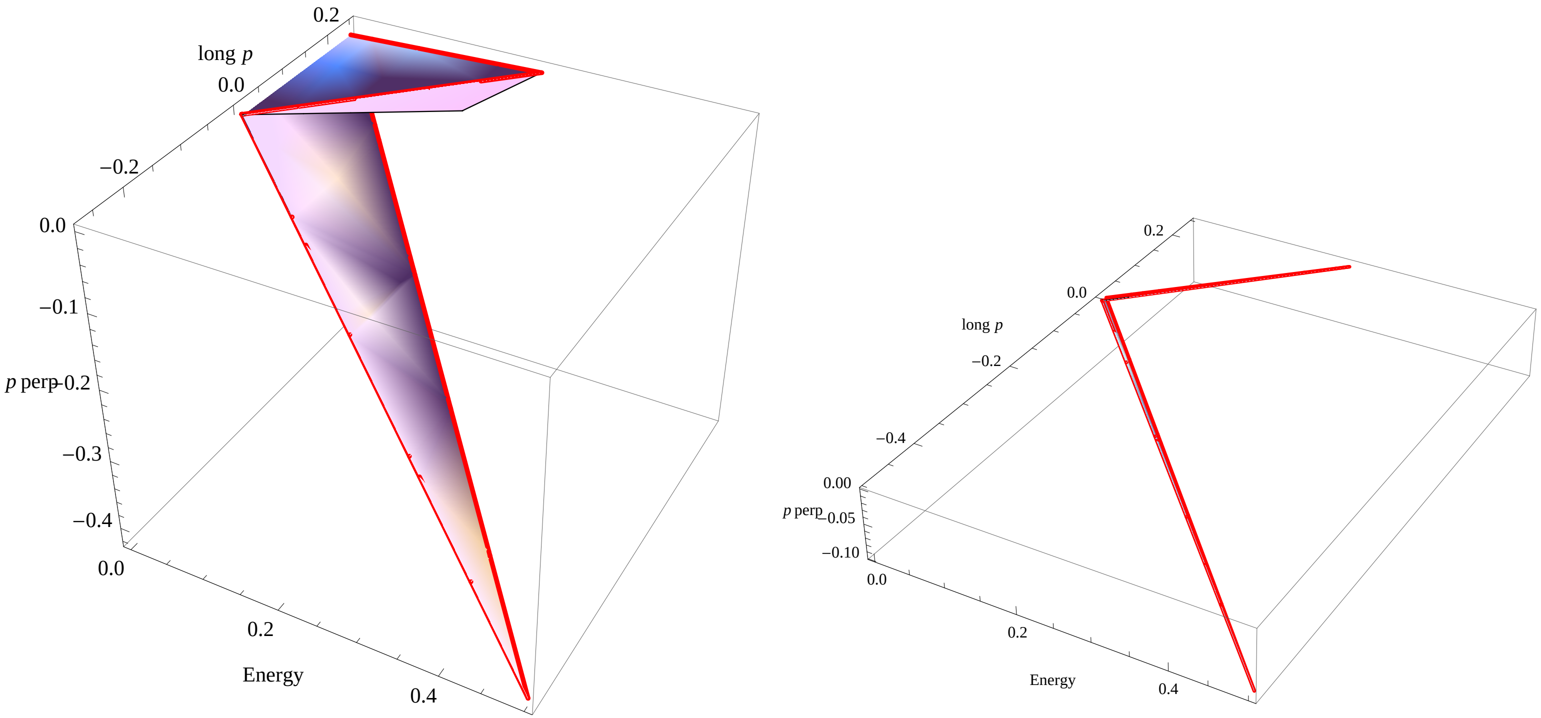
the path integration over $\varphi(s)$ — boundary Liouville field decouples only for tachyonic scalar, massless vector, etc.

Minimal surface

$2 \to 2$ kinematics (u-channel scattering with both $s < 0$ and $t < 0$)

Minimal surface spanned by rectangle with stepwise $p^\mu(t)$:

$$X^\mu(x,y) = \frac{1}{\pi K} \sum_i \Delta p_i^\mu \arctan \frac{(x - s_i)}{y} \qquad s_i = s(t_i)$$



$t/s = 0.46$ (left) and $t/s = 0.02$ (right)

Douglas' minimization

Douglas' minimization results in

$$\sum_{j\neq i} \frac{2\Delta p_i \cdot \Delta p_j}{s_i - s_j} + \pi \sum_j \Delta p_j^2 \left\langle \frac{\partial G\left(s_j, s_j\right)}{\partial s_i} \right\rangle_{\text{reparametrizations}} = 0$$

$$\left\langle G\left(s_j, s_j\right)\right\rangle = \frac{\int \mathcal{D}_{\text{diff}} s \, G\left(s_j, s_j\right)}{\int \mathcal{D}_{\text{diff}} s} \overset{\text{Lovelace}}{=} \frac{1}{\pi} \ln \frac{(s_{j+1} - s_{j-1})}{(s_{j+1} - s_j)(s_j - s_{j-1})\varepsilon}$$

Reparametrization path integral goes over functions obeying $s(t_i) = s_i$ which are zero modes of the Douglas minimization.

Only $M - 3$ independent equations because of projective invariance.

For $M = 4$ we set $s_1 = 0$, $s_3 = 1$, $s_4 = \infty$ in the usual way $\Longrightarrow$

$$s_{2\,*} = \frac{s}{s+t} \qquad \text{otherwise} \qquad \left(\frac{s_{21}s_{43}}{s_{31}s_{42}}\right)_* = \frac{s}{s+t}$$

— the saddle point of the Veneziano amplitude at large $-s, -t$.

Douglas' minimization (cont.)

The polygon bounds the minimal surface of the area

$$K S_{\text{min}} = \alpha' s \ln \frac{s}{s+t} + \alpha' t \ln \frac{t}{s+t} \stackrel{s \gg t}{\to} -\alpha' t \ln \frac{s}{t}$$

whose exponential reproduces the classical Regge behavior:

$$A(s,t) = \mathrm{e}^{-K S_{\text{min}}} \propto s^{\alpha' t}$$

Momentum Lüscher term

Schwarz–Christoffel map of the upper half-plane onto a rectangle:

$$\begin{aligned}\omega(z) &= \sqrt{s_{42}s_{31}}\int_{s_2}^{z}\frac{\mathrm{d}x}{\sqrt{(s_4-x)(s_3-x)(x-s_2)(x-s_1)}}\\ &= 2F\left(\sqrt{\frac{s_{31}(z-s_2)}{s_{32}(z-s_1)}},\sqrt{\frac{s_{32}s_{41}}{s_{42}s_{31}}}\right)\end{aligned}$$

$$R = 2K\left(\sqrt{1-r}\right), \qquad T = 2K\left(\sqrt{r}\right)$$

where

$$r = \frac{s_{43}s_{21}}{s_{42}s_{31}} \qquad s_{ij} = s_i - s_j$$

is the projective-invariant ratio. Therefore,

$$\frac{T}{R} = \frac{K\left(\sqrt{r}\right)}{K\left(\sqrt{1-r}\right)}$$

is projective invariant.

Momentum Lüscher term (cont.)

The calculation of momentum Lüscher term is just as for the static potential.

Using the asymptotes for $-s \ll -t \implies T \gg R$:

$$K\left(\sqrt{r}\right) \stackrel{r \to 1}{\longrightarrow} \frac{1}{2} \ln \frac{16}{1-r}, \qquad K\left(\sqrt{1-r}\right) \stackrel{r \to 1}{\longrightarrow} \frac{\pi}{2},$$

it is clear that each degree of freedom results in the Lüscher term

$$\frac{\pi T}{24 R} = \frac{1}{24} \ln \frac{16 s}{t} \qquad r = r_* = 1 - \frac{t}{s}$$

Semiclassical Reggeon intercept

There are $(d-2)$ such sets for bosonic string, so the linear Regge trajectory is

$$\alpha(t) = \frac{d-2}{24} + \alpha' t$$

In effective string theory the parameter

$$\ln \frac{1}{1-r} = \ln \frac{s}{t}$$

for scattering amplitudes is like T for the static potential. Therefore, the Regge behavior is like the area law:

$$A \propto \mathrm{e}^{\alpha(t)\ln(s/t)} \qquad \text{is similar to} \qquad W \propto \mathrm{e}^{-TV(R)}$$

The semiclassical Regge trajectory of the effective string theory in $d < 26$ can be computed for UHP like in Durhuus, Olesen, Petersen (1984) for the Polyakov string. Now the same result emerges as

$$\alpha(0) = 1 + \frac{d-26}{24} = \frac{d-2}{24}$$

Mean-field approximation for scattering amplitude

For the scattering amplitude the Mandelstam variables s and t play the role of T and R, while r is like ω_R:

$$S_{\text{mf}} = \alpha' s \ln r + \alpha' t \ln(1-r) + \frac{(d-2)}{24} \ln(1-r) \qquad \text{valid as } r \to 1$$

with the momentum Lüscher term included. Minimizing, we have

Janik (2001), Y.M. (2011)

$$r_* = 1 - \frac{\alpha' t + (d-2)/24}{\alpha' s}$$

which results in the linear Regge trajectory

$$\alpha(t) = \frac{(d-2)}{24} + \alpha' t.$$

It is obtained for large d but again is expected to be exact for any d.

Quadratic fluctuations around this mean field are stable for $\alpha(t) < 0$:

$$\alpha' t < -\frac{d-2}{24}$$

Path integrals over reparametrizations

The measure on $Diff(\mathbb{R})$

$$\int\limits_{\substack{s(\tau_0)=s_0 \\ s(\tau_f)=s_f}} \mathcal{D}_{\text{diff}} s(\tau) \cdots = \lim_{N\to\infty} \int_{s_0}^{s_f} \prod_{j=1}^{N-1} \int_{s_0}^{s_{j+1}} \mathrm{d}s_j \frac{1}{(s_{j+1}-s_j)} \frac{1}{(s_1-s_0)} \cdots$$

is invariant under reparametrizations

$$s \to t(s)\,, \quad t(s_0)=s_0\,, \quad t(s_f)=s_f\,, \quad \frac{\mathrm{d}t}{\mathrm{d}s} \geq 0$$

Integration goes over $(N-1)$ subordinated values

$$s_0 \leq \cdots \leq s_{i-1} \leq s_i \leq \cdots \leq s_N = s_f$$

Discretizing $s' = \exp[-\varphi]$ that relates reparametrizations to the boundary value of the Liouville field φ by $s_i - s_{i-1} = \exp[-\varphi_i]$ $\Longrightarrow$

$$\int_{s_0}^{s_f} \mathcal{D}_{\text{diff}} s \cdots = \lim_{N\to\infty} \prod_{i=1}^{N} \int_{-\infty}^{+\infty} \mathrm{d}\varphi_i \, \delta^{(1)}(s_f - s_0 - \sum_{j=1}^{N} \mathrm{e}^{-\varphi_j}) \cdots$$

with the only restriction on φ_i's given by the delta-function.

Path integrals over reparametrizations (cont.)

Regularization of (logarithmically) divergent integral

$$\frac{1}{(s_i - s_{i-1})} \longrightarrow \frac{1}{\Gamma(\delta_i)(s_i - s_{i-1})^{1-\delta_i}} \qquad \text{all } \delta_i = \delta$$

Main integral for the integration at the intermediate point s_i

$$\int_{s_{i-1}}^{s_{i+1}} \mathrm{d}s_i \, \frac{\Gamma^{-1}(\delta_i)\Gamma^{-1}(\delta_{i+1})}{(s_{i+1} - s_i)^{1-\delta_{i+1}}(s_i - s_{i-1})^{1-\delta_i}} = \frac{\Gamma^{-1}(\delta_i + \delta_{i+1})}{(s_{i+1} - s_{i-1})^{1-\delta_i-\delta_{i+1}}}$$

This is an analogue of the well-known formula

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}s_i}{\sqrt{2\pi}} \frac{\mathrm{e}^{-(s_f - s_i)^2/2\nu_1}}{\sqrt{\nu_1}} \frac{\mathrm{e}^{-(s_i - s_0)^2/2\nu_2}}{\sqrt{\nu_2}} = \frac{\mathrm{e}^{-(s_f - s_0)^2/2(\nu_1+\nu_2)}}{\sqrt{(\nu_1 + \nu_2)}}$$

used for calculations with the usual Wiener measure.

The functional limit is when $N \to \infty$ with $N\delta \to 0$:

$$\int_{s_0}^{s_N = s_f} \mathcal{D}^{(N)}_{\text{diff}} s = \frac{1}{\Gamma(N\delta)} \frac{1}{(s_N - s_0)^{1-N\delta}} \stackrel{N\delta \to 0}{\longrightarrow} N\delta \frac{1}{\left(s_f - s_0\right)}$$

reproducing the projective-invariant result.

Reparametrizations as Lévy stochastic process

Buividivich, Y.M. (2009)

What trajectories are typical in path integral over reparametrizations?

Subordinated stochastic process (gamma-subordinator) with PDF

$$P\left(\Delta s_i\right) = \frac{1}{\Gamma(\delta)\left(\Delta s_i\right)^{1-\delta}} \qquad \delta > 0 \text{ is a time step}$$

$$\mathrm{d}s_f \int_{s_0}^{s_f} \mathcal{D}^{(N)}_{\text{diff}} s \quad \text{— propagator from } s_0 \text{ to } \left[s_f, s_f + \mathrm{d}s_f\right]$$

during the time $\tau = N\delta$

Scaling variable

$$z = \tau \ln \frac{1}{(s_f - s_0)} \Longrightarrow \frac{\tau \mathrm{d}s_f}{\left(s_f - s_0\right)^{1-\tau}} = \mathrm{d}z\, \mathrm{e}^{-z},$$

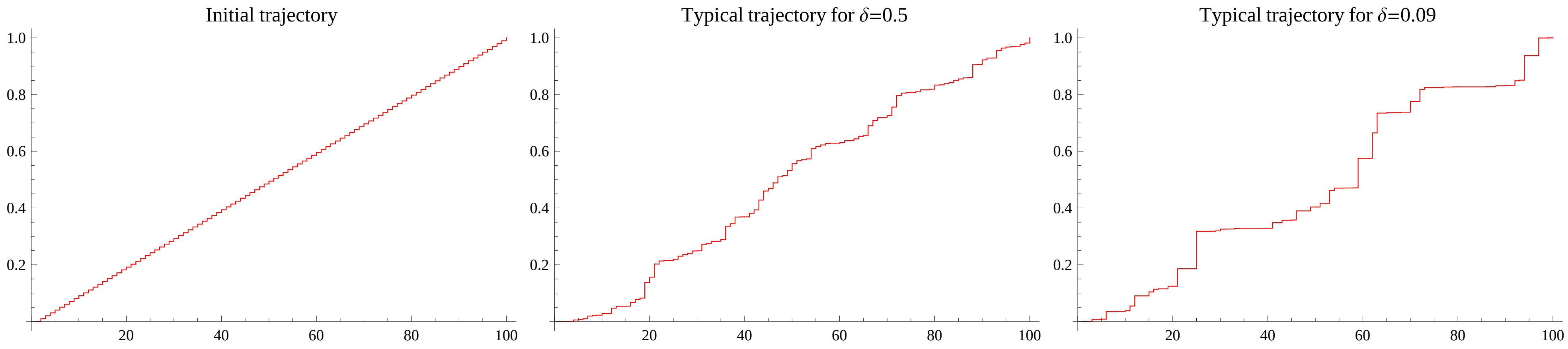
Scaling with

$$(s_f - s_0) \sim \mathrm{e}^{-1/\tau} \Longrightarrow \text{Hausdorff dimension}{=}0$$

supersedes $(s_f - s_0)^2 \sim \tau$ for the Brownian motion (whose $d_H = 2$).

Sample trajectories in path integral

Typical trajectories for the gamma-subordinator (obtained by Metropolis–Hastings algorithm)



Lévy's flights are seen in the right figure.
Their origin is that $P(\Delta s_i)$ is very large at small Δs_i $\Longrightarrow$ most of Δs_i's are small.
Then some of Δs_i has to be large to satisfy the boundary condition.

Hausdorff dimension decreases from 1 to 0 (left to right)

(Horowitz, 1968)

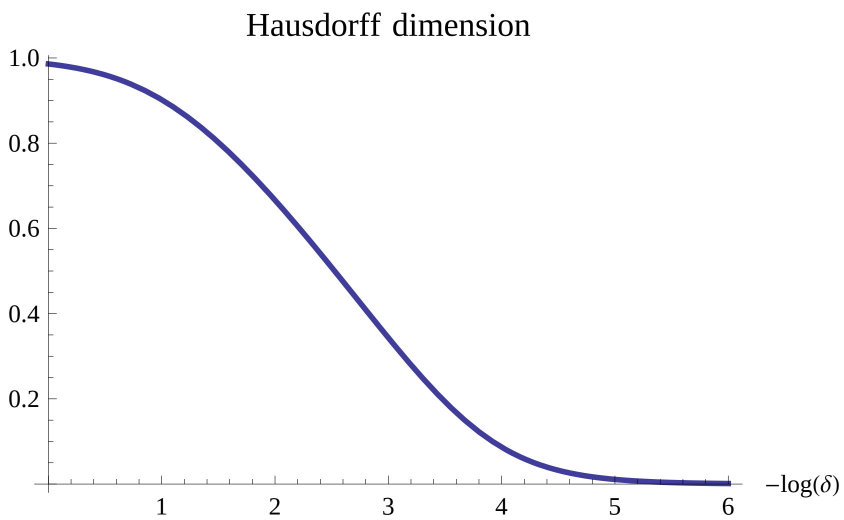
Hausdorff dimension of sample trajectories

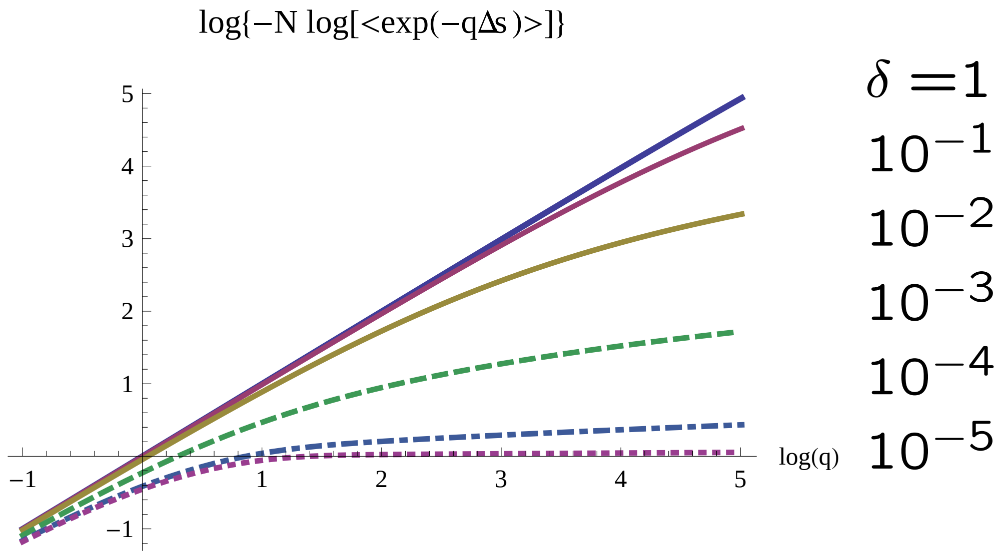
Hausdorff dimension of the discretized process is determined by its characteristic function (Lévy–Khintchine)

$$\left\langle \mathrm{e}^{-q\Delta s_i} \right\rangle = {}_1F_1(\delta, \delta N; -q)$$

as

$$d_H = \lim_{q\to\infty} \frac{\ln\left(-N \ln\left\langle \mathrm{e}^{-q\Delta s_i}\right\rangle\right)}{\ln q}$$





Hausdorff dimension versus $\ln(1/\delta)$ (left) from the slope of the lines (right).

It decreases from 1 for $\delta \gtrsim 1$ to 0 for $\delta N \to 0$

Ambiguities of the measure

More symmetric discretization of the measure

$$\mathcal{D}_{\text{diff}} s = \prod_i \mathrm{d}s_i \frac{(s_{i+1} - s_{i-1})}{(s_{i+1} - s_i)(s_i - s_{i-1})}$$

Lovelace choice

(every multiplier is projective invariant).

It is the one which results in the consistent off-shell (Lovelace) amplitudes with $\alpha(0) = (d-2)/24$.

The results do not change if next-to-neighbor discretization

$$(s_{i+1} - s_i) \Longrightarrow (s_{i+n} - s_i)/n$$

$\Longrightarrow$ a continuum limit in spite of the discontinuities.

It looks like different universality classes (with different $\alpha(0)$).

Consistent off-shell amplitudes

Y.M., Olesen (2010)

Reparametrization path integral results in

$$\left\langle G\left(s_j,s_j\right)\right\rangle = \frac{\int \mathcal{D}_{\text{diff}} s\, G\left(s_j,s_j\right)}{\int \mathcal{D}_{\text{diff}} s} \overset{\text{Lovelace}}{=} \frac{1}{\pi} \ln \frac{(s_{j+1}-s_{j-1})}{(s_{j+1}-s_j)(s_j-s_{j-1})\varepsilon}$$

With the proper choice of discretization of the measure (each multiplier is projective invariant)

$$\mathcal{D}^{(N)}_{\text{diff}} s = \prod_{i=1}^{N} \frac{\mathrm{d} s_i\,(s_{i+1}-s_{i-1})}{(s_{i+1}-s_i)(s_i-s_{i-1})}$$

this gives the scattering amplitude

$$A(\Delta p_1,\ldots,\Delta p_M)$$
$$= \int\limits_{s_{i-1}<s_i} \prod_i \mathrm{d} s_i \prod_{k\neq l} |s_k-s_l|^{\alpha'\Delta p_k\cdot\Delta p_l} \prod_j \left(\frac{|s_j-s_{j-1}|\,|s_{j+1}-s_j|}{|s_{j+1}-s_{j-1}|}\right)^{\alpha'\Delta p_j^2-1}$$

where the integration over s_i (Koba–Nielsen variables) is inherited from the path integral over reparametrizations.

Consistent off-shell amplitudes (cont.)

Di Vecchia, Frau, Lerda, Sciuto (1988)

This is a known off-shell tree string amplitude originally obtained from the Lovelace string vertex operator (instead of the usual one). It is consistent off-shell and invariant under $PSL(2;\mathbb{R})$ projective transformations (subgroup of reparametrizations)

$$s \Rightarrow \frac{as+b}{cs+d} \qquad \text{with} \quad ad-bc=1$$

For 4 scalars this reproduces projective-invariant off-shell amplitude

$$A(\Delta p_1,\Delta p_2,\Delta p_3,\Delta p_4)=\int_0^1 \mathrm{d}x\, x^{-\alpha(s)-1}(1-x)^{-\alpha(t)-1},$$

where $\alpha(t)=1+\alpha' t$ – linear Regge trajectory – and

$$s=-(\Delta p_1+\Delta p_2)^2, \qquad t=-(\Delta p_2+\Delta p_3)^2$$

are usual Mandelstam's variables (for Euclidean metric).

The tachyonic condition $\alpha'\Delta p_j^2=1$ has not to be imposed. The on-shell Veneziano tachyon amplitudes is obtained by setting $\alpha'\Delta p_j^2=1$

Application to QCD

Y.M., Migdal (1981)

Green's functions of M colorless composite quark operators

$$\bar{q}(x_i)q(x_i) \qquad \bar{q}(x_i)\gamma_5 q(x_i) \qquad \bar{q}(x_i)\gamma_\mu q(x_i) \qquad \bar{q}(x_i)\gamma_\mu\gamma_5 q(x_i)$$

are given by the sum over Wilson loops passing via x_i $(i = 1, \ldots, M)$

$$G \equiv \left\langle \prod_{i=1}^{M} \bar{q}(x_i)q(x_i) \right\rangle_{\text{conn}} = \sum_{\text{paths} \ni \{x_1,\ldots,x_M \equiv x_0\}} J[x(\tau)]\, W[x(\tau)]$$

The weight for the path integration is

$$J[x(\tau)] = \int \mathcal{D}k(\tau)\ \text{sp}\ \text{P}\, \text{e}^{\text{i} \int_0^{\mathcal{T}} \text{d}\tau\, [\dot{x}(\tau)\cdot k(\tau) - \gamma \cdot k(\tau)]}$$

for spinor quarks of mass m and scalar operators or

$$J[x(\tau)] = \text{e}^{-\frac{1}{2}\int_0^{\mathcal{T}} \text{d}\tau\, \dot{x}^2(\tau)} = \int \mathcal{D}k(\tau)\ \text{e}^{\int_0^{\mathcal{T}} \text{d}\tau\, [\text{i}\dot{x}(\tau)\cdot k(\tau) - k^2(\tau)/2]}$$

for scalar quarks. τ is the proper time.

The Wilson loop $W(C)$ is in pure Yang–Mills at large N (or quenched). For finite N, correlators of several Wilson loops are present.

Application to QCD (cont.)

Y.M., Olesen (2008)

QCD scattering amplitude = functional Fourier transform

$$A\left(\Delta p_1,\ldots,\Delta p_M\right) = \sum_{\text{paths}} \mathrm{e}^{\mathrm{i}\int_0^{\mathcal{T}} \mathrm{d}\tau\, \dot{x}(\tau)\cdot p(\tau)}\, J[x(\tau)]\, W[x(\tau)]$$

for piecewise constant momentum loop $p(\tau)$ as before.

Substituting the area-law and interchanging the integrals over $x(\tau)$ (Gaussian) and $s(\tau)$, we get

$$A\left(\{\Delta p_m\}\right) \propto \int_0^\infty \mathrm{d}\mathcal{T}\, \mathcal{T}^{M-1}\, \mathrm{e}^{-m\mathcal{T}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}s_{M-1}}{1+s_{M-1}^2} \prod_{i=1}^{M-2} \int_{-\infty}^{s_{i+1}} \frac{\mathrm{d}s_i}{1+s_i^2}$$
$$\times \int \mathcal{D}k(t)\ \mathrm{sp}\ P\, \mathrm{e}^{-\mathrm{i}\mathcal{T}\int \mathrm{d}t\, \gamma\cdot k(t)/(1+t^2)}\, W\left[x(t) = \frac{1}{K}\left(p(t)+k(t)\right)\right]$$

For small m and/or large M, the integral over $\mathcal{T}$ is dominated by large $\mathcal{T} \sim (M-1)/m$ and the path integral over k factorizes:

$$A\left(\{\Delta p_m\}\right) \propto W\left[x(t) = \frac{1}{K} p(t)\right]$$

It is just the same as the Lovelace-type string amplitude!

Justification of large $\mathcal{T}$ as $m \to 0$

Path integral over $x(\tau)$ (for scalar quarks) can be calculated via

mode expansion $$x^\mu(\tau) = x_0^\mu + \sum_{n=1}^{\infty} \left(a_n^\mu \cos \frac{2\pi\tau}{\mathcal{T}} + b_n^\mu \sin \frac{2\pi\tau}{\mathcal{T}} \right) :$$

$$\int_{x(0)=x(\mathcal{T})} \mathcal{D}x(\tau)\, \mathrm{e}^{-\frac{1}{2}\int_0^{\mathcal{T}} \mathrm{d}\tau\, \dot{x}^2(\tau) - \frac{K}{2}\dot{x}*G*\dot{x}} = \prod_{n=1}^{\infty} \left[2\pi \left(\frac{1}{\mathcal{T}} n^2 + K n \right) \right]^{-d}$$

ζ-function regularization gives

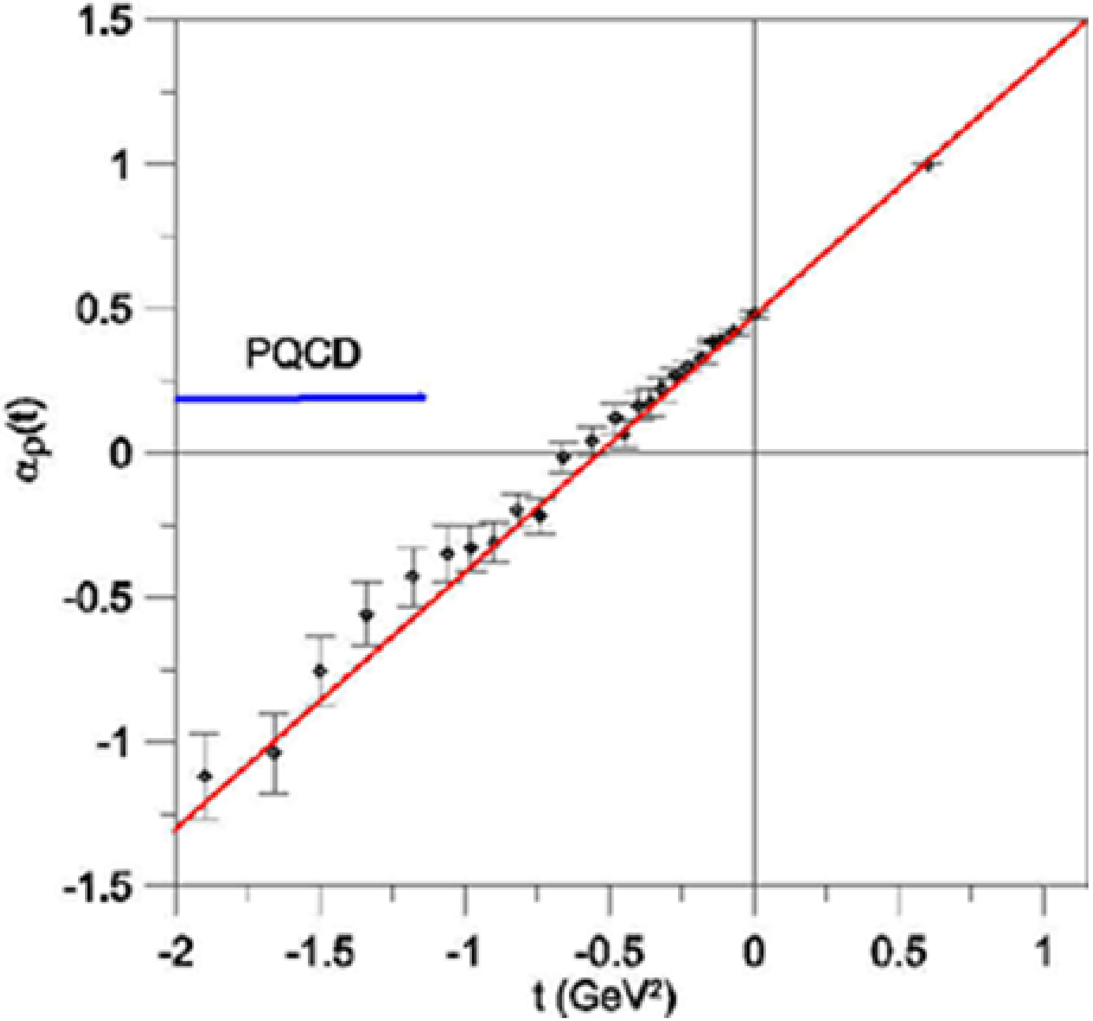
$$\prod_{n=1}^{\infty} A = A^{\zeta(0)} = A^{-1/2} \qquad \prod_{n=1}^{\infty} n = \sqrt{2\pi}$$

$$\prod_{n=1}^{\infty} \left[2\pi \left(\frac{1}{\mathcal{T}} n^2 + K n \right) \right]^{-d} \quad \begin{array}{l} \xrightarrow{\mathcal{T}\to 0} \quad (2\pi\mathcal{T})^{-d/2} \\ \xrightarrow{\mathcal{T}\to \infty} \quad K^{d/2} \end{array}$$

Miniconclusion: large $\mathcal{T}$ are essential in QCD perturbation theory only for $M > 4$, but are essential non-perturbatively: $\int \mathrm{d}\mathcal{T}\, \mathcal{T}^{M-1}\, \mathrm{e}^{-m\mathcal{T}} \ldots$

Effective ρ-trajectory and pQCD prediction

The figure taken from A. B. Kaidalov, hep-ph/0612358



exclusive process

$$\pi^- p \to \pi^0 n$$

It is hard to believe that pQCD reggeization is relevant.

Separation of pQCD and QCD string

Reggeization of $\bar{q}q$ in pQCD is due to double logarithms

Kirschner, Lipatov (1983)

$\mathcal{T}$ is restricted from above by $\tau_{\max} \sim 1/K$ to separate the contribution from small loops associated with pQCD. It plays the role of an infrared cutoff in pQCD, rather than a usual transverse mass μ.

With the double logarithmic accuracy:

$$\text{pQCD ladders} = \frac{2I_1\left(\omega \ln s\tau_{\max}\right)}{\omega \ln s\tau_{\max}} - 1 \qquad \omega = \sqrt{\frac{g^2(t)C_F}{2\pi^2}} \approx .5$$

$g^2 \Rightarrow g^2(t)$? because of charge renormalization. Then asymptotically

$$\text{pQCD ladders} \propto \left(s\tau_{\max}\right)^{\omega(t)}$$

standard pQCD $=$ $\tau_{\max} = \infty$ $\Longrightarrow$ IR regularization by μ.

Wilson loop = either QCD (small loops) or QCD string (large loops) like either $\mathcal{N} = 4$ SYM or IIB superstring in AdS/CFT.
The total amplitude = pQCD (this one) + QCD string (as before).
At finite s the relative coefficient is of most importance.

pQCD + QCD string effective Reggeon trajectory

Simple model of pQCD (small loops) + QCD string (large loops):

$$A = \frac{2I_1\left(0.5\ln(\alpha' s)\right)}{0.5\ln(\alpha' s)} - 1 + R\left(\alpha' s\right)^{0.5+\alpha' t}$$

Plot of the effective Reggeon trajectory (for various s)

$$\alpha_{\mathrm{eff}}(t) = \frac{\ln(A/R)}{\ln(\alpha' s)}$$

$R \approx 20$

$\alpha' s = 10^{40}$

$\alpha' s = 10^{16}$

$\alpha' s = 10^{9}$

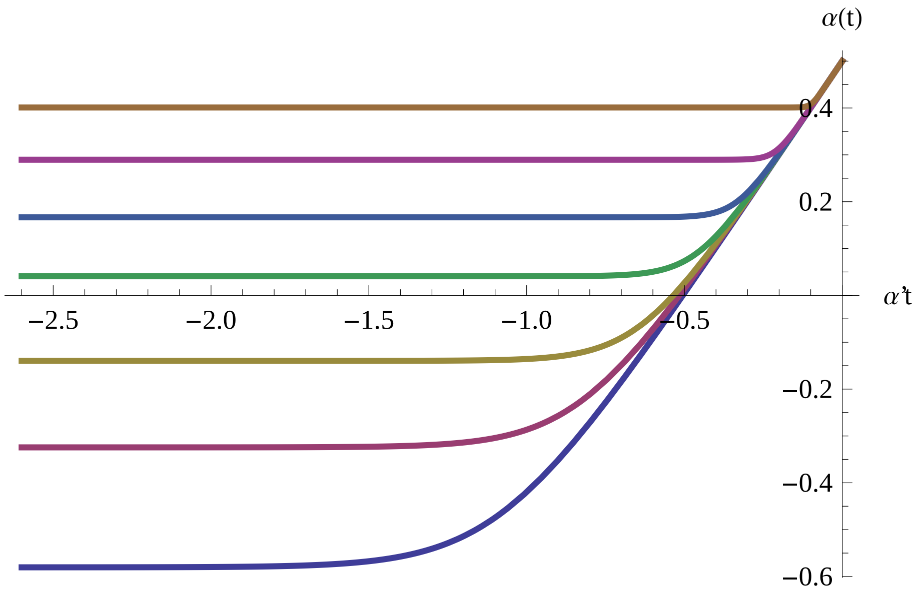
$\alpha' s = 10^{6}$

$\alpha' s = 10^{4}$

$\alpha' s = 10^{3}$

$\alpha' s = 400$

Reggeon trajectory in pQCD + QCD string



Experimental data for $\alpha_\rho(t)$

Kennett *et al.* (1986)

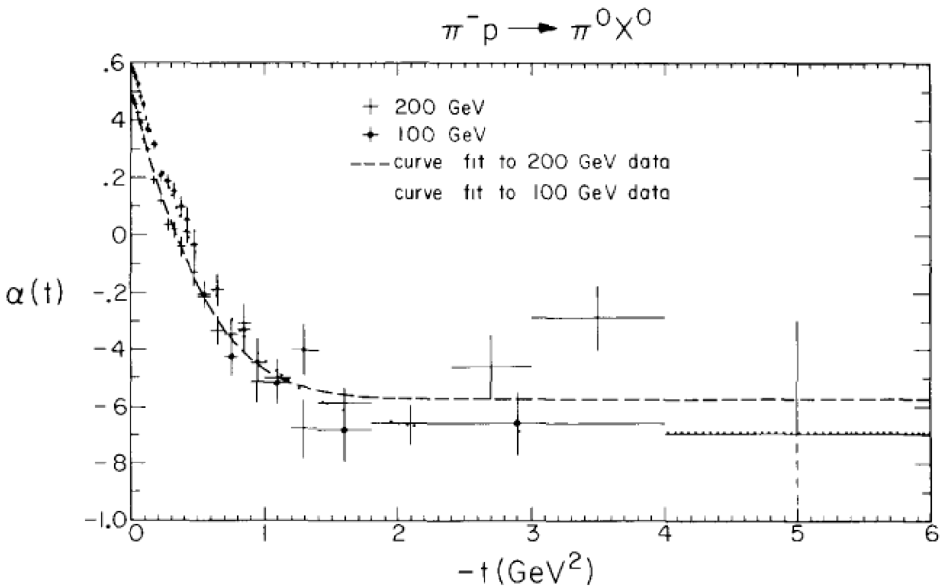
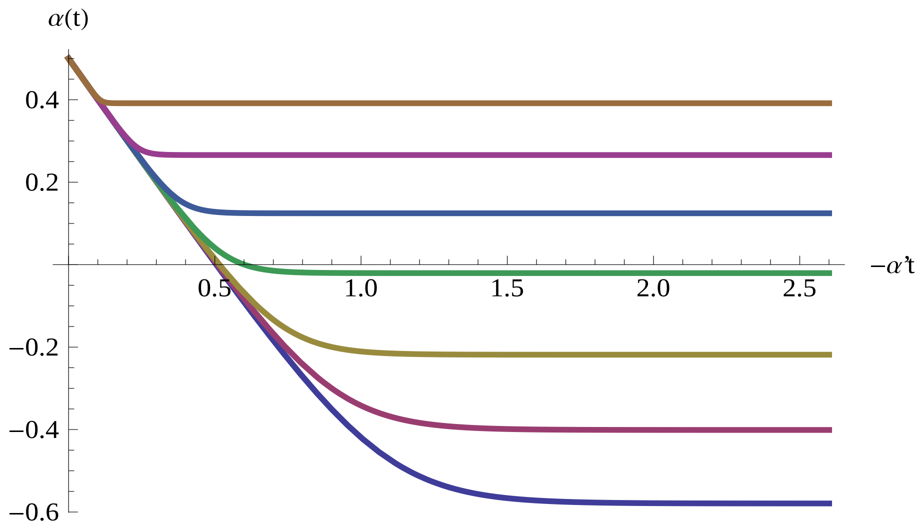


Fig 9 The ρ trajectory, $\alpha(t)$, for $0 < -t < 6$ GeV^2 extracted from fits to the neutral final state reaction, $\pi^- p \to \pi^0 X^0$, with the π^0 energy between 140 and 192 GeV The trajectory derived at 100 GeV is shown by the dotted data points The dashed and dotted curves show the parameterizations of the trajectories from the 200 and 100 GeV beams, respectively (see text for details) The error of ± 0.1 in $\alpha_\rho(t)$ coming from the uncertainty in $\tilde{\alpha}_{NFS}$ is not included in the errors shown in the figure

Reggeon trajectory in pQCD + npQCD



Brodsky, Tang, Thorn (1993)

two regimes as $s \to \infty$

Conclusion (to scattering amplitudes)

- Regge behavior of QCD scattering amplitudes follows from the area law. The only approximation is large N. Great simplification occurs for small m and/or large M (Lovelace-type amplitudes).

- It was crucial for the success of calculations that all integrals are Gaussian except for the one over reparametrizations which reduces to integration over the Koba–Nielsen variables.

- Derivation is legible for those momenta Δp_i for which asymptotically large loops are essential in the sum over C:

$K S_{\text{min}}(C_*) = \alpha' |t| \ln \frac{s}{\max\{|t|, K\}}$ i.e. large s and $|t| \ll s$.

- The classical string has intercept of the Reggeon trajectory $\alpha(0) = 0$ ($\alpha(0) \approx 0.5$ from experiment) but is applicable only for $|t| \gg 1/\alpha'$. The mean-field approximation results in $\alpha(0) = (d-2)/24$.

- Reggeon intercept of $\alpha(0) \approx 0.5$ has to be obtained, most probably by accounting for spontaneous breaking of chiral symmetry

- When $-t \ll s$ becomes large, there are no longer reasons to expect the contribution of large loops to dominate over perturbation theory, which comes from integration over small loops.

QCD string at work

- QCD string can be viewed as an effective long string and analyzed in $d = 4$ by the mean-field method.
- Two important applications of this technique:
  - ground state energy,
  - meson scattering amplitudes in the Regge regime.
- Monte Carlo data for QCD spectrum can be well described by only most relevant operator (Nambu–Goto).
- Extrinsic curvature softens the tachyonic problem and may be induced by additional degrees of freedom of QCD string.
- Reparametrization path integral is most important and results in consistent off-shell amplitudes.
- Polygonal momentum loops are technically very convenient and momentum Lúscher term == intercept.
- Effective Reggeon trajectory changes from QCD string to perturbative QCD with decreasing t at certain s-dependent value of t.