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Is there any **torsion** in your future?

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DD, Alexander Tumanov and Alexey Vladimirov, arXiv:1104.2432
and in preparation

General Relativity in terms of the metric tensor:

$$S = \frac{M_P^2}{16\pi} \int d^4x \sqrt{g} \left(-2\Lambda - \bar{R}\right)$$

Planck mass **cosmological term** **Eistein – Hilbert action**

Christoffel symbol or, better, Levi – Civita connection:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa} \left(\partial_\mu g_{\kappa\nu} + \partial_\nu g_{\kappa\mu} - \partial_\kappa g_{\mu\nu}\right)$$

Riemann tensor:

$$R^{\kappa}{}_{\lambda,\mu\nu} = \partial_\mu \Gamma^{\kappa}_{\nu\lambda} - \partial_\nu \Gamma^{\kappa}_{\mu\lambda} + \Gamma^{\kappa}_{\mu\rho}\Gamma^{\rho}_{\nu\lambda} - \Gamma^{\kappa}_{\nu\rho}\Gamma^{\rho}_{\mu\lambda}$$

Scalar curvature:

$$\bar{R} = \bar{R}^{\mu}_{\lambda,\mu\nu}\, g^{\lambda\nu}$$

Cartan's formulation of general relativity (1920's):

Independent variables, instead of the metric tensor, are

1) *vierbein* or frame field e_μ^A, $g_{\mu\nu} = e_\mu^A e_\nu^A$, $A = 1, 2, 3, 4.$

2) spin connection $\omega_\mu^{AB} = -\omega_\mu^{BA}$ Yang – Mills potential of the Lorentz SO(4) group

SO(4) Yang – Mills field strength or curvature [historically, first example of "Yang-Mills"] :

$$F_{\mu\nu}^{AB} = \partial_\mu \omega_\nu^{AB} - \partial_\nu \omega_\mu^{AB} + \omega_\mu^{AC} \omega_\nu^{CB} - \omega_\nu^{AC} \omega_\mu^{CB}$$

Gravity action:

$$S = \frac{M_P^2}{16\pi} \int d^4x \left(-2\Lambda \det(e) - \frac{1}{4} \epsilon^{\kappa\lambda\mu\nu} \epsilon_{ABCD} F_{\kappa\lambda}^{AB} e_\mu^C e_\nu^D \right)$$

Classically, and with no sources, it is equivalent to the Einstein – Hilbert action.
Proof: The action in quadratic in ω_μ, so saddle point integration in ω_μ is exact.

Saddle-point equation for ω_μ :

$$D_\mu^{AB} e_\nu^B - D_\nu^{AB} e_\mu^B \equiv 2T_{\mu\nu}^A = 0, \qquad D_\mu^{AB} = \partial_\mu \delta^{AB} + \omega_\mu^{AB}$$

this combination is called **torsion**

24 algebraic equation on 24 components of ω_μ^{AB} determine the saddle-point uniquely as

$$\bar{\omega}_\mu^{AB}(e) = \frac{1}{2} e^{A\kappa}(\partial_\mu e_\kappa^B - \partial_\kappa e_\mu^B) - \frac{1}{2} e^{B\kappa}(\partial_\mu e_\kappa^A - \partial_\kappa e_\mu^A) - \frac{1}{2} e^{A\kappa} e^{B\lambda} e_\mu^C (\partial_\kappa e_\lambda^C - \partial_\lambda e_\kappa^C)$$

Substituting the saddle-point value back into the action, one recovers identically the Einstein – Hilbert action written in terms of $g_{\mu\nu}$. **Torsion appears to be zero dynamically**, even if one allows it, as in Cartan formulation.

However ! There are fermions in Nature. Fermions `gravitate', and are sources of torsion.

The standard (minimal) way to include fermions into General Relativity: [V. Fock, H. Weyl (1929)]

$$S_{\mathrm{f}} = i \int d^4x \, \det(e) \, \frac{1}{2} \left(\overline{\Psi} \, e^{A\mu} \gamma_A \mathcal{D}_\mu \Psi - \overline{\mathcal{D}_\mu \Psi} \, e^{A\mu} \gamma_A \Psi \right), \qquad \mathcal{D}_\mu = \partial_\mu + \frac{1}{8} \omega_\mu^{BC} [\gamma_B \gamma_C]$$

$$\det(e) \, e^{D\nu} = \frac{1}{6} \epsilon^{\kappa\lambda\mu\nu} \, \epsilon^{ABCD} \, e_\kappa^A e_\lambda^B e_\mu^C, \qquad e^{D\nu} e_{E\nu} = \delta_E^D.$$

the contravariant tetrad is the inverse matrix

This action is invariant under

i) general coordinate transformations (diffeomorphisms) $x^\mu \to x'^\mu(x)$

ii) local Lorentz rotations $\Psi(x) \to L(x) \Psi(x), \qquad L(x) \in SO(4) \simeq SU(2)_{L} \times SU(2)_{R}.$

just as is the bosonic Einstein – Cartan action.

With fermions switched in, the saddle-point variation with respect to the connection ω_μ^{AB} gives, generally, a **nonzero torsion**: $T_{\mu\nu}^{A} \sim (\bar{\Psi} \ldots \Psi)$ bilinear fermion current

Standard Einstein's gravity with matter, as a quantum theory, is **not renormlizable**; moreover it is not well-defined as a path integral since the action is **not positive-definite**. Therefore, at best it should be viewed as an **effective low-energy quantum theory**, like the effective chiral Lagrangian in Quantum Chromodynamics, or σ-models for solid states.

The best one can do in the absence of a well-defined microscopic theory, is to treat General Relativity as a gradient expansion, with ∂^2 / M_P^2 being the expansion parameter.
The expansion coefficients should be in principle determined from observations.

All possible leading terms in torsion, curvature, gradients have been systematically written down by Tumanov, Vladimirov and D.D. [arXiv:1104.2432] . There are in general

- 4 fermion couplings to torsion
- 5 terms ~ T T
- 10 terms ~ R R
- 4 terms ~ ∇ T R, …

Torsion propagates in higher orders $(\nabla T \cdot \nabla T)$

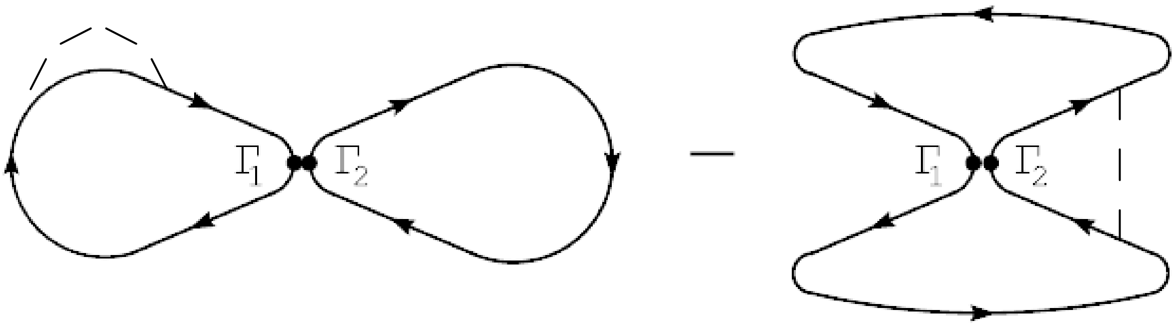
The net result is that in the leading order there is a local 4-fermion addition to the standard gravity:

$$\Delta S = \frac{const.}{M_P^2}\left(< V \cdot V > + < A \cdot A > + < V \cdot A > \right), \qquad V = \bar{\Psi}\gamma_A \Psi, \qquad A = \bar{\Psi}\gamma_A \gamma_5 \Psi$$

This correction is, normally, very small and hardly detectable today. However, there have been speculations that close to the Big Bang when fermion densities were presumably large, the 4-fermion interaction induced by nonzero torsion could be very important and even replace the \`inflation’ and solve the \`horizon problem’!

We think it is **a**) doubtful, **b**) in any case beyond the applicability of the gradient expansion.

One has to evaluate the 4-fermion interaction in the hot relativistic Fermi medium.



Two contributions to the average of 4-fermion interaction: Hartree (left) and Fock (right)

$$< (\bar{\Psi}\Gamma_1\Psi)(\bar{\Psi}\Gamma_2\Psi) > = \frac{1}{2}\int\frac{d^4p_1}{(2\pi)^4 i}\mathrm{Tr}(G(p_1)\Gamma_1)\int\frac{d^4p_2}{(2\pi)^4 i}\mathrm{Tr}(G(p_2)\Gamma_2)$$
$$- \frac{1}{2}\int\frac{d^4p_1}{(2\pi)^4 i}\int\frac{d^4p_2}{(2\pi)^4 i}\mathrm{Tr}(G(p_1)\Gamma_1 G(p_2)\Gamma_2),$$

For ideal Fermi gas

$$G(p) = \frac{m + \gamma_0(\mu + i\omega_n + \gamma_i \mathbf{p}_i)}{m^2 - (\mu + i\omega_n)^2 + \mathbf{p}^2}, \qquad \omega_n = 2\pi T\left(n + \frac{1}{2}\right).$$

Matsubara frequency

μ = chemical potential, regulates the difference between # of particles and # of antiparticles

For free (non-interacting) ultra-relativistic fermion gas we obtain

$$< V_B V^B >_{\rm Hart} = \langle (\bar\Psi \gamma_B \Psi)(\bar\Psi \gamma^B \Psi) \rangle_{\rm Hart} = \frac{1}{2}\rho^2,$$

$$< A_B A^B >_{\rm Hart} = 0,$$

$$< A_B V^B >_{\rm Hart} = 0.$$

$$\rho = \frac{\mu T^2}{3} + \frac{\mu^3}{3\pi^2} - m^2 \frac{\mu}{2\pi^2} + \mathcal{O}(m^4)$$

$$< V_B V^B >_{\rm Fock} = \frac{1}{4}\rho^2 - \frac{m^2}{2}\sigma^2,$$

$$\sigma = -\frac{T^2}{6} - \frac{\mu^2}{2\pi^2}$$

$$< A_B A^B >_{\rm Fock} = \frac{1}{4}\rho^2 + \frac{m^2}{2}\sigma^2,$$

$$< A_B V^B >_{\rm Fock} = 0,$$

ρ is the charge density (# of particles - # of antiparticles)

μ is the chemical potential, T is temperature

Corrections to the ideal fermion gas eqs. arise only in the second order in the interaction.

Einstein – Friedman equation for cosmological evolution

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{16\pi}{M_P^2} \Theta_{\mu\nu}$$

Main piece in the fermion stress-energy tensor is the Stefan – Boltzmann law:

$$\Theta_{00} = \frac{7\pi^2}{60} T^4 \qquad\qquad \Theta_{ii} = \frac{1}{3} \Theta_{00}$$

Correction due to torsion:

$$\Theta_{00}^{4-ferm} = \frac{\mu^2}{M_P^2} T^4 .$$

μ is the chemical potential for *conserved* quantum numbers

One can hardly imagine that this correction ever becomes sizable!

Therefore, it seems that there is no torsion in your future.

However…

Main Problem of Quantum Gravity

Standard gravity is not a renormalizable field theory, nor is it even a well-formulated theory, because the requirement of diffeomorphism-invariance makes the action non-positive-definite.

World scalars $A^\mu B_\mu$, $\quad A^\mu = g^{\mu\nu} A_\nu$ $\quad g^{\mu\nu} = \dfrac{4\,\epsilon^{\alpha_1\alpha_2\alpha_3\mu}\,\epsilon^{\beta_1\beta_2\beta_3\nu}\, g_{\alpha_1\beta_1} g_{\alpha_2\beta_2} g_{\alpha_3\beta_3}}{\epsilon^{\alpha_1\alpha_2\alpha_3\alpha_4}\,\epsilon^{\beta_1\beta_2\beta_3\beta_4}\, g_{\alpha_1\beta_1} g_{\alpha_2\beta_2} g_{\alpha_3\beta_3} g_{\alpha_4\beta_4}}$

are made of **even** number of antisymmetric ***epsilon***'s. But integrands for the general covariant actions are *not* world scalars – they have to transform such that the change in the volume element is compensated:

$$A_\kappa \to A'_\lambda\, \partial x'^\lambda / \partial x^\kappa, \qquad \epsilon^{\kappa\lambda\mu\nu} A_\kappa B_\lambda C_\mu D_\nu \to \left|\frac{dx'}{dx}\right| \epsilon^{\kappa\lambda\mu\nu} A'_\kappa B'_\lambda C'_\mu D'_\nu; \qquad d^4x \left|\frac{dx'}{dx}\right| = d^4x'.$$

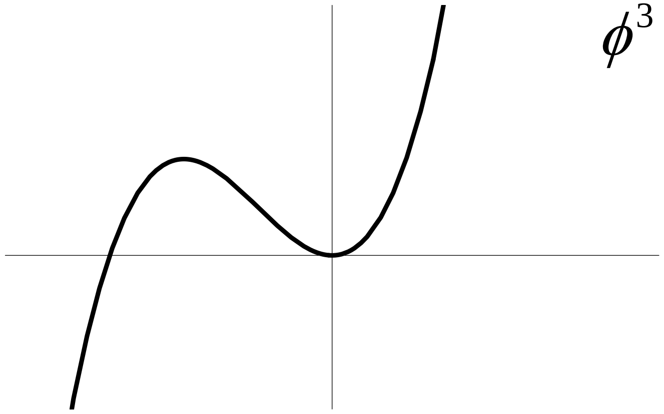
NB: all general covariant (or diffeomorphism-invariant) actions are necessarily *linear* in $\epsilon_{\kappa\lambda\mu\nu}$ and hence are not sign-definite !!

Examples:

$$\det(e) = \frac{1}{4!}\epsilon^{\kappa\lambda\mu\nu}\,\epsilon_{ABCD}\, e^A_\kappa e^B_\lambda e^C_\mu e^D_\nu, \qquad R\sqrt{g} \sim \frac{1}{4}\epsilon^{\kappa\lambda\mu\nu}\,\epsilon_{ABCD}\, F^{AB}_{\kappa\lambda}(\omega)\, e^C_\mu e^D_\nu$$

$$R^2\sqrt{g} \sim \frac{\epsilon^{\alpha\beta\kappa\lambda}\,\epsilon^{\gamma\delta\mu\nu}\, e^A_\alpha e^B_\beta F^{CD}_{\kappa\lambda} e^A_\gamma\, e^B_\delta\, F^{CD}_{\mu\nu}}{\det(e)}$$

General covariance is a “curse” that makes any diffeomorphism-invariant action bottomless!



ϕ^3 theory does not restrict large quantum fluctuations, even though perturbation theory may be well defined.

Minkowski space-time with e^{iS} doesn’t seem to help, if the action can have any sign, and is unbounded: one either cannot define a causal Green’s function, or there is tunneling to a bottomless state!

The Main Problem of Quantum Gravity: Large fluctuations of metric and/or of spin connection are not restricted!

How to define the path integral for Quantum Gravity?

Use fermionic Grassmannian variables instead of bosonic ones!

$$\int d\psi^\dagger \, d\psi \exp\left(\psi_i^\dagger A_{ij} \psi_j\right) = \epsilon^{i_1 \ldots i_N} \epsilon^{j_1 \ldots j_N} A_{i_1 j_1} \ldots A_{i_N j_N} = \det(A)$$

$$\int d\psi^\dagger \, d\psi \exp\left(\psi_i^\dagger \psi_j^\dagger A_{ij,kl} \psi_k \psi_l\right) = \epsilon^{i_1 \ldots i_N} \epsilon^{j_1 \ldots j_N} A_{i_1 i_2, j_1 j_2} \ldots A_{i_{N-1} i_N, j_{N-1} j_N} \qquad \textit{etc.}$$

The idea is to use composite *vierbeins* or *frame* fields, roughly:

tetrad

$$e_\mu^A = \frac{1}{2}\psi^\dagger \gamma_A D_\mu \psi - \frac{1}{2}(D_\mu \psi)^\dagger \gamma_A \psi$$

$$D_\mu = \partial_\mu + \frac{1}{8}\omega_\mu^{AB}[\gamma_A \gamma_B]$$

spin connection,
gauge field

$$g_{\mu\nu} = e_\mu^A e_\nu^A$$

metric tensor

History of composite frames:

- K. Akama (1978)
- G. Volovik (1990) [superfluid $^3\boldsymbol{He}-B$]
- C. Wetterich (2005, 2011)

use ordinary derivatives,
not covariant $\Rightarrow$
e_μ^A is not a Lorentz vector!

Standard Dirac action in *d*-dim curved space

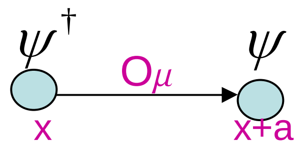
$$S = \int d^d x\, \epsilon^{\mu_1 \dots \mu_d}\, \epsilon^{A_1 \dots A_d}\, e_{\mu_1}^{A_1} \dots e_{\mu_{d-1}}^{A_{d-1}} \left(\psi^\dagger \gamma_{A_d} (D_{\mu_d}\psi) - (D_{\mu_d}\psi)^\dagger \gamma_{A_d} \psi\right)$$

is in fact the cosmological term in disguise:

$$S = \int d^d x\, \epsilon^{\mu_1 \dots \mu_d}\, \epsilon^{A_1 \dots A_d}\, e_{\mu_1}^{A_1} \dots e_{\mu_{d-1}}^{A_{d-1}} e_{\mu_d}^{A_d} = \int d^d x \det(e) = \int d^d x \sqrt{g}$$

All such kind of actions can be easily UV regularized by putting them on a lattice.

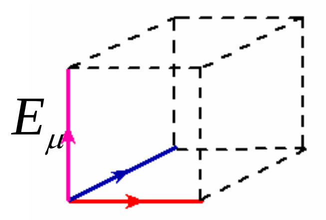
$$e^A{}_\mu \to \psi^\dagger(x)\gamma_A e^{a\omega^{AB}{}_\mu[\gamma_A\gamma_B]/8}\psi(x+a) - \bar{\psi}(x+a)e^{-a\omega^{AB}{}_\mu[\gamma_A\gamma_B]/8}\gamma_A\psi(x)$$



$\mathrm{O}_\mu \in SU(2)_L \times SU(2)_R$

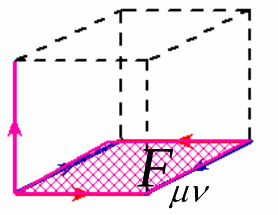
lives on a lattice site
but carries the μ direction

Discretized `cosmological term' action: $\sum_x \epsilon^{\mu_1\ldots\mu_d}\, Tr(e_{\mu_1}\ldots e_{\mu_d})$



Discretized `Einstein – Hilbert' action: $\sum_x \epsilon^{\mu_1\ldots\mu_d}\, Tr(e_{\mu_1}\ldots e_{\mu_{d-2}} F_{\mu_{d-1}\mu_d})$

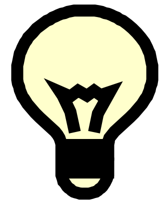
gauge- and diffeomorphism-
invariant!



Regularized partition function

$$Z = \int \prod_{links} dO_\mu \prod_{sites} d\bar{\psi}\, d\psi\ \exp(S_{cosm} + S_{EH} + \ldots)$$

Haar measure
normalized to unity



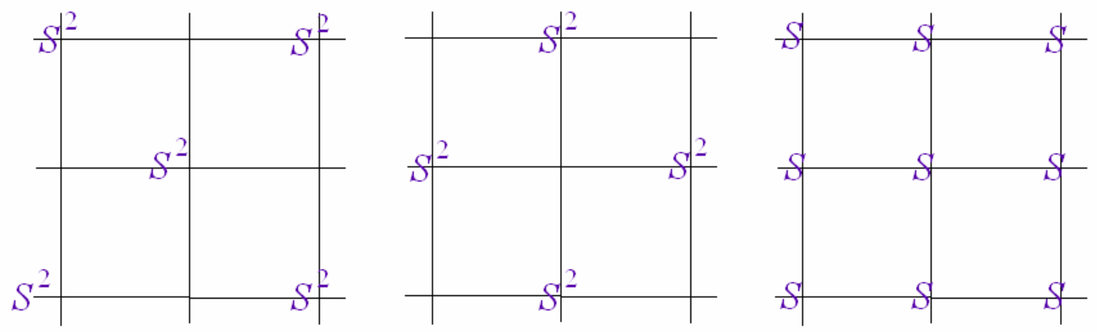
The theory is well-defined, explicitly gauge invariant under local Lorentz group,
and diffeomorphism-invariant in the continuum limit !

How to work with such new kind of lattice gauge theory?

Let's take the simplest variant: **action = cosmological term** in 4d. Since it is simultaneously the Dirac action, it is a propagating theory!

At each lattice one integrates over 8 Grassmann variables $\int d\psi_1^\dagger d\psi_2^\dagger d\psi_3^\dagger d\psi_4^\dagger d\psi_1 d\psi_2 d\psi_3 d\psi_4$
The action has also 8 operators $S(x) \sim \psi^\dagger\psi^\dagger\psi^\dagger\psi^\dagger\psi\psi\psi\psi$ One has to Taylor-expand $e^{\sum_x S(x)}$ such that there are precisely 8 fermion operator per site, otherwise the integral = 0.

There are three possibilities:



Link variables enter quadratically for each link, These contributions are nonzero.

Link variables enter linearly for each link, it is zero.

After integrating over link variables using $\int dU U^\dagger U = \frac{1}{2}\delta\,\delta, \qquad \int dU U U = \frac{1}{2}\epsilon\,\epsilon$
one gets only gauge invariant combinations of fermion operators $(\psi^\dagger\psi), \quad (\psi^\dagger\epsilon\psi^\dagger), \quad (\psi\epsilon\psi)$

There are several ways which term of the type $(\psi^\dagger)^4(\psi)^4$ ascribe to a lattice site. When one term is chosen, it dictates which term is picked at a neighbor site.

The patterns with a given recipe which terms to use, form 1d closed lines and 2d closed surfaces, otherwise the Grassmann integral is zero.

The partition function is a sum over all possible closed lines and all possible closed surfaces!

At the moment it is not clear if they can be large (infrared) or only of the lattice site (ultraviolet), but this is analytically calculable.

Important: $$< \int R\sqrt{g} > = 0$$

It means that, although quantum fluctuations of metrics are fully allowed (and regularized), the true quantum state is not far from flat space!

Therefore, it is not improbable that Newton's law is reproduced, for static sources, but this remains to be checked,

Conclusions

1. When one includes fermions into General Relativity, nonzero torsion is inevitable.

2. However, the ensuing effects in cosmology are so small that they are probably unobservable.

3. All general covariant theories have bottomless actions. To make quantum gravity well defined, one needs to introduce a composite frame built from fermion fields.

4. This is a new type of quantum theory, with multifermion interaction but no propagators! Have to develop anew the saddle-point and mean-field methods.

5. The theory can be easily regularized by putting it on a lattice.

6. At short distances, the spin connection fluctuates wildly, meaning that torsion is strong.

7. Nevertheless, the theory may well possess the classical long-distance limit.