Ultraviolet Divergences and Counterterm Structure in Maximal Supergravity

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Are there quantum miracles happening in maximal supergravity?

Outline

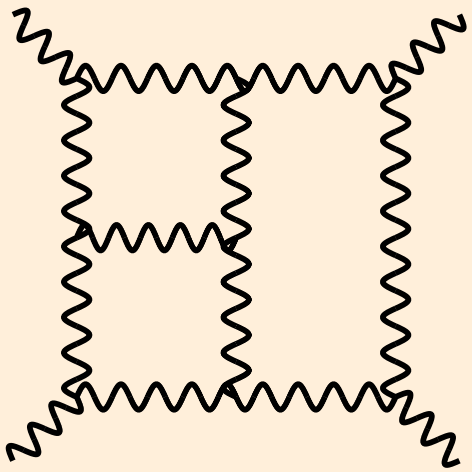
- Nonrenormalization theorems and BPS degree
- Unitarity-based calculations
- Ectoplasm & superspace cohomology
- Duality constraints on counterterms
- The Volume of Superspace
- Current outlook

Ultraviolet Divergences in Gravity

- Simple power counting in gravity and supergravity theories leads to a naïve degree of divergence

$$\Delta = (D-2)L + 2$$

in D spacetime dimensions. So, for D=4, L=3, one expects $\Delta = 8$. In dimensional regularization, only logarithmic divergences are seen ($\frac{1}{\epsilon}$ poles, $\epsilon = D - 4$), so 8 powers of momentum would have to come out onto the external lines of such a diagram.



- Local supersymmetry implies that the pure curvature part of such a D=4, 3-loop divergence candidate must be built from the square of the Bel-Robinson tensor
Deser, Kay & K.S.S 1977

$$\int \sqrt{-g}\, T_{\mu\nu\rho\sigma} T^{\mu\nu\rho\sigma}\ , \qquad T_{\mu\nu\rho\sigma} = R_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta} R_{\rho\alpha\sigma\beta} + {}^{*}R_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta}\, {}^{*}R_{\rho\alpha\sigma\beta}$$

- This is directly related to the α'^3 corrections to the superstring effective action, except that in the string context such contributions occur with finite coefficients. In string theory, the corresponding question is how poles might develop in $(\alpha')^{-1}$ as one takes the zero-slope limit $\alpha' \to 0$ and how this bears on the ultraviolet properties of the corresponding field theory.

Berkovits 2007
Green, Russo & Vanhove 2007, 2010

- The consequences of supersymmetry for the ultraviolet structure are not restricted to the requirement that counterterms be supersymmetric invariants.
- There exist more powerful "nonrenormalization theorems," the most famous of which excludes infinite renormalization of chiral invariants in D=4, N=1 supersymmetry; these are given in N=1 superspace by integrals over just *half* the superspace:

$$\int d^2\theta W(\phi(x,\theta,\bar{\theta}))\ , \quad \bar{D}\phi = 0 \qquad \text{(cf. full superspace } \int d^4\theta L(\phi,\bar{\phi})\text{)}$$

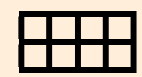
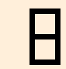
- However, maximally extended SYM and supergravity theories do not have formalisms with all supersymmetries linearly realised "off-shell" in superspace. So the power of such nonrenormalization theorems is restricted to the off-shell linearly realizable subalgebra.

- The degree of "off-shell" supersymmetry is the maximal supersymmetry for which the algebra can close without use of the equations of motion.
- Knowing the extent of this off-shell supersymmetry is tricky, and may involve formulations (*e.g.* harmonic superspace) with infinite numbers of auxiliary fields. Galperin, Ivanov, Kalitsin, Ogievetsky & Sokatchev
- For maximal N=4 Super Yang-Mills and maximal N=8 supergravity, the linearly realizable supersymmetry has been known since the 1980's to be at least *half* the full supersymmetry of the theory. So at that time the first generally allowed counterterms were expected to have "1/2 BPS" structure as compared to the full supersymmetry of the theory.

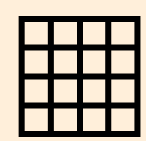
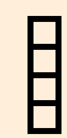
- The 3-loop R^4 candidate maximal supergravity counterterm has a structure very similar to that of an F^4 N=4 super Yang-Mills invariant. Both of these are 1/2 BPS invariants, involving integration over just half the corresponding full superspaces:

Howe, K.S.S. & Townsend 1981
Kallosh 1981

$$\Delta I_{SYM} = \int (d^4\theta d^4\bar{\theta})_{105} \,\mathrm{tr}(\phi^4)_{105}$$

 ***105*** $\quad \phi_{ij}$  ***6*** of SU(4)

$$\Delta I_{SG} = \int (d^8\theta d^8\bar{\theta})_{232848} (W^4)_{232848}$$

 ***232848*** $\quad W_{ijkl}$  ***70*** of SU(8)

- Versions of these supergravity and SYM operators do occur as counterterms at one loop in D=8. However, the one-loop level often has special renormalization features, so one needs to be careful not to make unwarranted conclusions about the general acceptability of these counterterms.

- Of course, there are other symmetries in supergravity beside diffeomorphism invariance and supersymmetry. In particular, D=4, N=8 supergravity also has a rigid nonlinearly realised E_7 symmetry. At leading order, this symmetry is realised by constant shifts of the 70 scalars.

- The R^4 candidate satisfies at least the minimal requirement of invariance under such constant shifts of the 70 scalars because, at the leading 4-particle order, the integrand may be written such that every scalar field is covered by a derivative.

Unitarity-based calculations

Bern, Carrasco, Dixon, Johansson & Roiban 2007 ... 2011

- The calculational front has made impressive progress since the late 1990s.
- These have led to unanticipated and surprising cancellations at the 3- and 4-loop orders, yielding new lowest possible orders for the super Yang-Mills and supergravity divergence onsets:

Max. SYM first divergences, current lowest possible orders.

Dimension D	10	8	7	6	5	4
Loop order L	1	1	2	3	6?	∞
BPS degree	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
Gen. form	$\partial^2 F^4$	F^4	$\partial^2 F^4$	$\partial^2 F^4$	$\partial^2 F^4$	finite

Blue: known divergences

Max. supergravity first divergences, current lowest possible orders.

Dimension D	11	10	8	7	6	5	4
Loop order L	2	2	1	2	3	6?	5?
BPS degree	0	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	0	$\frac{1}{4}$
Gen. form	$\partial^{12} R^4$	$\partial^{10} R^4$	R^4	$\partial^6 R^4$	$\partial^6 R^4$	$\partial^{12} R^4$	$\partial^4 R^4$

Algebraic Renormalization

Dixon; Howe, Lindstrom & White; Piguet & Sorella; Hennaux; Stora; Baulieu & Bossard

- Another approach to analyzing the divergences in supersymmetric gauge theories, using the full supersymmetry, begins with the Callan-Symanzik equation for the renormalization of the Lagrangian as a operator insertion, governing, *e.g.*, mixing with the half-BPS SYM operator $S^{(4)} = \mathrm{tr}(F^4)$. Letting the classical action be $S^{(2)}$, the C-S equation for SYM in dimension D is $\mu\frac{\partial}{\partial\mu}[S^{(2)}\cdot\Gamma] = (4-D)[S^{(2)}\cdot\Gamma] + \gamma_{(4)}g^{2n_{(4)}}[S^{(4)}\cdot\Gamma] + \cdots$, where $n_{(4)} = 4,\ 2,\ 1 \quad$ for $D = 5,\ 6,\ 8$.
- From this one learns that $(n_{(4)}-1)\beta_{(4)} = \gamma_{(4)}$ so the beta function for the $S^{(4)} = \mathrm{tr}(F^4)$ operator is determined by the anomalous dimension $\gamma_{(4)}$.

- Combining the supersymmetry generator with a commuting spinor parameter to make a scalar operator $\mathcal{Q} = \bar{\varepsilon} Q$, the expression of SUSY invariance for a D-form density in D-dimensions is $\mathcal{Q}\mathcal{L}_D + d\mathcal{L}_{D-1} = 0$. Combining this with the SUSY algebra $\mathcal{Q}^2 = -i(\bar{\varepsilon}\gamma^\mu\varepsilon)\partial_\mu$ and using the Poincaré Lemma, one finds

$$i_{i(\bar{\varepsilon}\gamma\varepsilon)}\mathcal{L}_D + S_{(\mathcal{Q})|\Sigma}\mathcal{L}_{D-1} + d\mathcal{L}_{D-2} = 0 \ .$$

- Hence, one can consider cocycles of the extended nilpotent differential $d + S_{(\mathcal{Q})|\Sigma} + i_{i(\bar{\varepsilon}\gamma\varepsilon)}$ acting on formal form-sums $\mathcal{L}_D + \mathcal{L}_{D-1} + \mathcal{L}_{D-2} + \cdots$.

- The supersymmetry Ward identities then imply that the whole cocycle must be renormalized in a coherent way. In order for an operator like $S^{(4)}$ to mix with the classical action $S^{(2)}$, their cocycles need to have the same structure.

Ectoplasm

Voronov 1992; Gates, Grisaru, Knut-Whelau, & Siegel 1998
Berkovits and Howe 2008; Bossard, Howe & K.S.S. 2009

- The construction of supersymmetric invariants is isomorphic to the construction of cohomologically nontrivial closed forms in superspace: $I = \int_{M_0} \sigma^* \mathcal{L}_D$ is invariant (where σ^* is a pull-back to a section of the projection map down to the purely bosonic "body" subspace M_0) if $\mathcal{L}_D$ is a closed form in superspace, and it is nonvanishing only if $\mathcal{L}_D$ is nontrivial.
- Using the BRST formalism, handle all gauge symmetries including space-time diffeomorphisms by the nilpotent BRST operator s. The invariance condition for $\mathcal{L}_D$ is $s\mathcal{L}_D + d_0\mathcal{L}_{D-1} = 0$, where d_0 is the usual bosonic exterior derivative. Since $s^2 = 0$ and s anticommutes with d_0, one obtains $s\mathcal{L}_{D-1} + d_0\mathcal{L}_{D-2} = 0$, etc.

- Solving the BRST Ward identities thus becomes a cohomological problem. Note that the supersymmetry ghost is a commuting field. One needs to study the cohomology of the nilpotent operator $\delta = s + d_0$, whose components $\mathcal{L}_{D-q,q}$ are (D-q) forms with ghost number q, *i.e.* (D-q) forms with q spinor indices. The spinor indices are totally *symmetric* since the supersymmetry ghost is *commuting*.

- For gauge-invariant supersymmetric integrands, this establishes an isomorphism between the cohomology of closed forms in superspace (aka "ectoplasm") and the construction of BRST-invariant counterterms.

Superspace cohomology

Bonora, Pasti & Tonin 1987

- Flat superspace has a standard basis of invariant 1-forms

$$\begin{aligned} E^a &= dx^a - \frac{i}{2} d\theta^\alpha (\Gamma^a)_{\alpha\beta}\theta^\beta \\ E^\alpha &= d\theta^\alpha \end{aligned}$$

dual to which are the superspace covariant derivatives (∂_a, D_α)

- There is a natural bi-grading of superspace forms into even and odd parts:

$$\Omega^n = \oplus_{n=p+q}\Omega^{p,q}$$

- Correspondingly, the flat superspace exterior derivative splits into three parts with bi-gradings (1,0), (0,1) & (-1,2):

$$d = d_0(1,0) + d_1(0,1) + t_0(-1,2)$$

bosonic der. fermionic der. torsion

$$d_0 \leftrightarrow \partial_a \quad d_1 \leftrightarrow \partial_\alpha$$

where for a (p,q) form in flat superspace, one has

$$(t_o\omega)_{a_2\cdots a_p\beta_1\cdots\beta_{q+2}} \sim (\Gamma^{a_1})_{(\beta_1\beta_2}\omega_{a_1\cdots a_p\beta_3\cdots\beta_{q+2})}$$

- The nilpotence of the total exterior derivative d implies the relations

$$\begin{aligned} t_0^2 &= 0 \\ t_0 d_1 + d_1 t_0 &= 0 \\ d_1^2 + t_0 d_0 + d_0 t_0 &= 0 \end{aligned}$$

- Then, since $d\mathcal{L}_D = 0$, the lowest dimension nonvanishing component (or "generator") $\mathcal{L}_{D-q,q}$ must satisfy $t_0 \mathcal{L}_{D-q,q} = 0$ so $\mathcal{L}_{D-q,q}$ belongs to the t_0 cohomology group $H_t^{D-q,q}$.

- Starting with the t_0 cohomology groups $H_t^{p,q}$, one then defines a spinorial exterior derivative $d_s \ : \ H_t^{p,q} \to H_t^{p,q+1}$ by $d_s[\omega] = [d_1 \omega]$, where the [ ] brackets denote H_t classes.

Cederwall, Nilsson & Tsimpis 2002
Howe & Tsimpis 2003

- One finds that d_s is nilpotent, $d_s^2 = 0$, and so one can define spinorial cohomology groups $H_s^{p,q} = H_{d_s}(H_t^{p,q})$.

The groups $H_s^{0,q}$ give multi *pure spinors.*

- This formalism gives a way to reformulate BRST cohomology in terms of spinorial cohomology. The lowest dimension component, or *generator*, of a counterterm's superform must be d_s closed, *i.e.* it must be an element of $H_s^{D-q,q}$.

- Solving $d_s[\mathcal{L}_{D-q,q}] = 0$ allows one to solve for all the higher components of $\mathcal{L}_D$ in terms of $\mathcal{L}_{D-q,q}$ for normal cocyles.

- To illustrate how this formalism works, consider N=1 supersymmetry in D=10. Corresponding to the $\varkappa$ symmetries of strings and 5-branes, we have the D=10 Gamma matrix identities $t_0\Gamma_{1,2} = 0 \qquad t_0\Gamma_{5,2} = 0$.

- The second of these is relevant to the construction of *d*-closed forms in D=10. One may have a generator
$$L_{5,5} = \Gamma_{5,2}M_{0,3}$$
where $d_s[M_{0,3}] = 0$. The simplest example of such a form corresponds to a full superspace integral over *S*:
$$M_{\alpha\beta\gamma} = T_{\alpha\beta\gamma,\delta_1\cdots\delta_5}(D^{11})^{\delta_1\cdots\delta_5}S$$
where $T_{\alpha\beta\gamma,\delta_1\cdots\delta_5}$ is constructed from the D=10 Gamma matrices; it is totally symmetric in $\alpha\beta\gamma$ and totally antisymmetric in $\delta_1\cdots\delta_5$.

Cohomological non-renormalization

- Spinorial cohomology then allows one to derive non-renormalization theorems for counterterms: the cocycle structure of candidate counterterms must match that of the classical action.
  - For example, in maximal SYM, this leads to non-renormalization theorems ruling out the F^4 counterterm that was otherwise expected at L=4 in D=5.
  - Similar non-renormalization theorems exist in supergravity, but their study is complicated by local supersymmetry and the density character of counterterm integrands.

Duality invariance constraints

cf also Broedel & Dixon 2010

- Maximal supergravity has a series of duality symmetries which extend the automatic GL(11-D) symmetry obtained upon dimensional reduction from D=11, e.g. E_7 in the N=8, D=4 theory, with the 70 scalars taking their values in an E_7/SU(8) coset target space.

Bossard, Hillman & Nicolai 2010

- The N=8, D=4 theory can be formulated in a manifestly E_7 covariant (but non-manifestly Lorentz covariant) formalism. Anomalies for SU(8), and hence E_7, cancel. Marcus 1985

- Combining the requirement of continuous duality invariance with the spinorial cohomology requirements gives further restrictions on counterterms.

- Supergravity Duality Groups and String Theory discretisations:

D	$E_{11-D(11-D)}(\mathbb{R})$	K_D	$E_{11-D(11-D)}(\mathbb{Z})$
10A	$\mathbb{R}^+$	1	1
10B	$Sl(2,\mathbb{R})$	$SO(2)$	$Sl(2,\mathbb{Z})$
9	$Sl(2,\mathbb{R}) \times \mathbb{R}^+$	$SO(2)$	$Sl(2,\mathbb{Z})$
8	$Sl(3,\mathbb{R}) \times Sl(2,\mathbb{R})$	$SO(3) \times SO(2)$	$Sl(3,\mathbb{Z}) \times Sl(2,\mathbb{Z})$
7	$Sl(5,\mathbb{R})$	$SO(5)$	$Sl(5,\mathbb{Z})$
6	$SO(5,5,\mathbb{R})$	$SO(5) \times SO(5)$	$SO(5,5,\mathbb{Z})$
5	$E_{6(6)}(\mathbb{R})$	$USp(8)$	$E_{6(6)}(\mathbb{Z})$
4	$E_{7(7)}(\mathbb{R})$	$SU(8)/\mathbb{Z}_2$	$E_{7(7)}(\mathbb{Z})$
3	$E_{8(8)}(\mathbb{R})$	$SO(16)$	$E_{8(8)}(\mathbb{Z})$

- The scalar target-space manifold is G_D/K_D. In string theory, the duality group becomes discretised to $G_D(\mathbb{Z})$, but this discretisation occurs due to nonperturbative effects outside the context of field-theoretic supergravity.

- Densities: In a curved superspace, an invariant is constructed from the top (pure "body") component in a coordinate basis:

$$I = \frac{1}{D!}\int d^Dx\, \varepsilon^{m_D \ldots m_1}\; E_{m_D}{}^{A_D} \cdots E_{m_1}{}^{A_1}\; L_{A_1 \ldots A_D}(x, \theta = 0)$$

- Referring this to a preferred "flat" basis and identifying $E_M{}^A$ components with vielbeins and gravitinos, one has in D=4

$$I = \frac{1}{24}\int \Big(e^a_\wedge e^b_\wedge e^c_\wedge e^d\; L_{abcd} + 4 e^a_\wedge e^b_\wedge e^c_\wedge \psi^{\underline{\alpha}} L_{abc\underline{\alpha}} + 6 e^a_\wedge e^b_\wedge \psi^{\underline{\alpha}}_\wedge \psi^{\underline{\beta}}\; L_{ab\underline{\alpha}\,\underline{\beta}}$$
$$+ 4 e^a_\wedge \psi^{\underline{\alpha}}_\wedge \psi^{\underline{\beta}}_\wedge \psi^{\underline{\gamma}} L_{a\underline{\alpha}\,\underline{\beta}\,\underline{\gamma}} + \psi^{\underline{\alpha}}_\wedge \psi^{\underline{\beta}}_\wedge \psi^{\underline{\gamma}}_\wedge \psi^{\underline{\delta}} L_{\underline{\alpha}\,\underline{\beta}\,\underline{\gamma}\,\underline{\delta}}\Big)$$

  - Thus the "soul" components of the cocycle also contribute to the local supersymmetric covariantization.

- Since the gravitinos do not transform under the D=4 E_7 duality, the L_{ABCD} form components have to be *separately* duality invariant.

- At leading order, the $E_7/SU(8)$ coset generators of E_7 simply produce *constant shifts* in the 70 scalar fields, as we have seen. This leads to a much easier check of invariance than analysing the full spinorial cohomology problem.

Howe, K.S.S. & Townsend 1981

- Although the pure-body (4,0) component L_{abcd} of the R^4 counterterm has long been known to be shift-invariant at lowest order (since all 70 scalar fields are covered by derivatives), it is harder for the fermionic "soul" components to be so, since they are of lower dimension.

- Thus, one finds that the maxi-soul (0,4) $L_{\alpha\beta\gamma\delta}$ component is *not* invariant under constant shifts of the 70 scalars. Hence the D=4, N=8, 3 -loop R^4 1/2 BPS counterterm is not E_7 duality invariant, so it is ruled out as an allowed counterterm.

Bossard, Howe & K.S.S. 2010

N=5, N=6

- Similar analysis of the D=4 3-loop R^4 invariants in N=5 and N=6 supergravities shows them to be likewise ruled out by the analogous requirements of SU(5,1) and SO*(12) duality invariances.
- In N=6 supergravity, there is a 4-loop $\partial^2 R^4$ type invariant. Similar analysis indicates that this also is ruled out.
    - In maximal supergravity, such a $\Delta = 10$ invariant might have been expected at one loop in D=10. However, in maximal supergravity this invariant vanishes subject to the classical field equations. But in D=4, N=6 it does not vanish, so it could have been a threatening counterterm.

Infinities versus infinities: dimensional reduction versus duality

Elvang & Kiermaier 2010 (from IIA string theory)
Bossard, Howe & K.S.S. 2010 (from supergravity)
Beisert, Elvang, Freedman, Kiermaier, Morales & Stiebeger 2010

- Left out of control so far are some of the most interesting cases: L=5,6 in D=4 maximal supergravity, corresponding to the 1/4 BPS $\partial^4 R^4$ and 1/8 BPS $\partial^6 R^4$ type counterterms.
  - Here, a different kind of duality-based argument comes into play.
- In fact, the *existence* of the 1/2 BPS L=1, D=8 R^4, the 1/4 BPS L=2, D=7 $\partial^4 R^4$ and the 1/8 BPS L=3, D=6 $\partial^6 R^4$ types of divergences together with the *uniqueness* of the corresponding D=4 counterterm structures allows one to rule out the corresponding D=4 candidates.

Drummond, Heslop, Howe & Kerstan 2003

- The existence of these D=8, 7 & 6 divergences indicate that the corresponding forms of the $R^4,\ \partial^4 R^4$ & $\partial^6 R^4$ counterterms have to be such that the purely gravitational parts of these invariants are not dressed by e^ϕ scalar prefactors – otherwise, they would violate the corresponding

$$SL(3,\mathbb{R})\times SL(2,\mathbb{R}),\ SL(5,\mathbb{R})\ \&\ SO(5,5)$$

duality symmetries: lowest-order shift symmetries would then be violated.

- Upon dimensional reduction to D=4, the Einstein-frame classical N=8 action $\int d^4x(R\sqrt{-g}\ +\ \dots)$ is arranged to have no scalar prefactors. But then dimensional reduction of the $R^4,\ \partial^4 R^4$ & $\partial^6 R^4$ counterterms in general causes such prefactors to appear.

- These dimensional reductions from D=8, 7 & 6 don't have even the requisite SU(8) symmetry. But they can be rendered SU(8) invariant by averaging, *i.e.* by integrating the dimensionally reduced counterterms over

Elvang & Kiermaier 2010
Elvang, Freedman & Kiermaier 2010

$$SU(8)/(SO(3) \times SO(2)) \ , \ SU(8)/SO(5) \ \text{ or } \ SU(8)/(SO(5) \times SO(5)) \ .$$

  - The action of SU(8) on evident scalar combinations such as the compactification volume modulus $\phi = \vec{\alpha} \cdot \vec{\phi}$ is highly nonlinear, so SU(8) averaging is difficult to do explicitly.
  - However, some ideas from string theory come to the rescue: scalar prefactors need to satisfy certain Laplace equations, even in the pure supergravity limit.

Green & Sethi 1999; Sinha 2002;
Green & Vanhove 2005; Green, Russo & Vanhove 2010

- Starting from a known duality invariant in some higher dimension D, the dimensional reduction to D=4 giving the n-loop candidate $\partial^{2(n-3)}R^4$ counterterm has a scalar prefactor $f_n(\phi)$ satisfying

Bossard, Howe & K.S.S. 2010

$$\left(\nabla^2_\phi + \frac{D-4}{D-2}n(32-D-n)\right) f_n(\phi) = 0$$

- This Laplace equation is SU(8) covariant, and must be satisfied equally by the dimensional reduction of the D-dimensional counterterm and by the SU(8) averaged version of this counterterm.

- Infinitesimal shift invariance for the 70 scalars, and hence E_7 invariance, can only be realised if $f_n(\phi) = 1$.

- Starting from the known infinities at L=1,2&3 loops in D=8,7&6, one thus learns the impossibility of E_7 invariance in D=4 for all the corresponding dimensionally reduced & SU(8) averaged D=4 operators: the 1/2 BPS R^4 candidate, the 1/4 BPS $\partial^4 R^4$ candidate and the 1/8 BPS $\partial^6 R^4$ candidate.

Drummond, Heslop, Howe & Kerstan 2003

- Since these D=4 counterterm candidates are *unique* (as shown by conformal multiplet decomposition), just based on supersymmetry together with the linearly realised SU(8) symmetry, their failure to be E_7 invariant completely rules out the corresponding candidate counterterms. Thus the 1/2, 1/4 and 1/8 BPS R^4, $\partial^4 R^4$ and $\partial^6 R^4$ N=8 counterterms are *not allowed* as counterterms.

The Volume of Superspace

E_7 invariant counterterms long known to exist for L>7:
Howe & Lindstrom 1981
Kallosh 1981

- It had long been anticipated that a manifestly E_7 invariant counterterm of D=4, N=8 supergravity would occur at weight $\Delta = 16$ corresponding to to the 7-loop order:
  $\int d^4x d^{32}\theta E(x,\theta)$ - the full volume of N=8 superspace.
- Left unresolved, however, was just what this invariant looks like in ordinary component-field terms.
  - As with the other candidate initial counterterms we have considered, we are interested in its on-shell expression in terms of curvatures, etc.
  - Natural guess for the general structure: $\partial^8 R^4$

Vanishing Volume

Bossard, Howe, K.S.S. & Vanhove 1105.6087

- The 7-loop situation, however, turns out to be more complex: the superspace volume actually *vanishes* on-shell.
- Simply integrating out the volume $\int d^4xd^{32}\theta E(x,\theta)$ using the superspace constraints implying the classical field equations would be an ugly task.
- However, using an on-shell implementation of harmonic superspace together with a superspace implementation of the normal-coordinate expansion, one can nonetheless see that it vanishes on-shell for all supersymmetry extensions N.

- N=8 supergravity has a natural *SU(8)* R-symmetry group under which the 8 gravitini transform in the **8** representation. In (8,1,1) harmonic superspace, one augments the normal (x^μ, θ^i_α) superspace coordinates by an additional set of bosonic corrdinates $u^I{}_j \quad I = 1; r = 2, \ldots, 7; 8$ parametrising the flag manifold $(U(1) \times U(6) \times U(1))\backslash SU(8)$

Hartwell & Howe 1994

- Contracting the usual superspace basis vectors with these and their inverses, one has $\begin{array}{c}\tilde{E}^\alpha_I = u^i{}_I \tilde{E}^\alpha_i \\ \tilde{E}^{\dot{\alpha} I} = u^I{}_i \tilde{E}^{\dot{\alpha} i}\end{array}$
  - Then work just with manifest *U(1)xU(6)xU(1)* covariance.

- Combining these with the $d^J{}_I$ vector fields on the harmonic flag manifold, one finds that the subset

$$\hat{E}_{\hat{A}} := \{\tilde{E}^1_\alpha, \tilde{E}_{\dot{\alpha}\,8}, d^1{}_r, d^r{}_8, d^1{}_8\}\ , \quad 2 \leq r \leq 7$$

is in involution:

$$\{\hat{E}_{\hat{A}}, \hat{E}_{\hat{B}}\} = C_{\hat{A}\hat{B}}{}^{\hat{C}}\,\hat{E}_{\hat{C}}$$

- One can then define Grassman-analytic superfields annihilated by the dual superspace derivatives $D_{\alpha\,1}\,, \bar{D}^8_{\dot{\alpha}}$

- Some non-vanishing curvatures are

$$R^1_{\alpha\dot{\beta}8,}{}^1{}_1 = R^1_{\alpha\dot{\beta}8,}{}^8{}_8 = -B_{\alpha\dot{\beta}}$$

where $B_{\alpha\dot{\beta}} = \bar{\chi}^{1ij}_{\dot{\beta}}\chi_{\alpha\,8ij}$ is Grassman-analytic

Normal coordinates for a 28+4 split

- One can define normal coordinates Kuzenko & Tartaglino-Mazzucchelli 2008

$$\zeta^{\hat{A}} = \{\zeta^{\alpha} = \delta^{\alpha}_{\mu}\theta^{\mu}_{i}u^{i}{}_{1}\,, \bar{\zeta}^{\dot{\alpha}} = \delta^{\dot{\alpha}}_{\dot{\mu}}u^{8}{}_{i}\,\bar{\theta}^{\dot{\mu}\,i}, z^{r}{}_{1}, z^{8}{}_{r}, z^{8}{}_{1}\}$$

associated to the vector fields $\hat{E}_{\hat{A}}$.

- Expanding the superspace Berezinian determinant in these, one finds the flow equation

$$\zeta^{\hat{\alpha}}\partial_{\hat{\alpha}}\ln E = -\frac{1}{3}B_{\alpha\dot{\beta}}\zeta^{\alpha}\bar{\zeta}^{\dot{\beta}} + \frac{1}{18}B_{\alpha\dot{\beta}}B_{\alpha\dot{\alpha}}\zeta^{\alpha}\zeta^{\beta}\bar{\zeta}^{\dot{\alpha}}\bar{\zeta}^{\dot{\beta}}$$

- Integrating, one finds the expansion of the determinant in the four fermionic coordinates $\zeta^{\hat{\alpha}} = (\zeta^{\alpha}\,, \zeta^{\dot{\alpha}})$:

$$E(\hat{x},\zeta,\bar{\zeta}) = \mathcal{E}(\hat{x})\left(1 - \frac{1}{6}B_{\alpha\dot{\beta}}\zeta^{\alpha}\zeta^{\dot{\beta}}\right)$$

  - However, since this has only ζ^2 terms, integration over the four $\zeta^{\hat{\alpha}}$ vanishes.

1/8 BPS E_7 invariant candidate notwithstanding

- Despite the vanishing of the full N=8 superspace volume, one can nonetheless use the harmonic superspace formalism to construct a different manifestly E_7 -invariant candidate:

$$I^8 := \int d\mu_{(8,1,1)}\, B_{\alpha\dot\beta}\, B^{\alpha\dot\beta}$$

- At the leading 4-point level, this can be written as a full superspace integral with respect to the linearised N=8 supersymmetry. It cannot, however, be rewritten as a full-superspace integral at the nonlinear level.
- Full-superspace manifestly E_7 -invariant candidates exist in any case from 8 loops onwards.

Current outlook

- As far as one knows, the first acceptable D=4 counterterm for maximal supergravity still occurs at L=7 loops ($\Delta = 16$).

- Current divergence expectations for maximal supergravity are consequently:

Dimension D	11	10	8	7	6	5	4
Loop order L	2	2	1	2	3	6	7
BPS degree	0	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	0	0
Gen. form	$\partial^{12}R^4$	$\partial^{10}R^4$	R^4	$\partial^6 R^4$	$\partial^6 R^4$	$\partial^{12}R^4$	$\partial^8 R^4$

Blue: known divergences

Green: anticipated divergences

Possible 5-loop test?

- At the 4-point level, the unitarity-based calculations of Bern, Carrasco, Dixon, Johansson & Roiban simultaneously give results for all dimensions D at a given loop order L. So one can contemplate *fractional dimensions* when necessary.

- A key clue would be a parting of the ways of maximal SYM and maximal supergravity at L=5 in their divergence behaviour. From what we now know, L=5 max SYM should show a divergence ($\frac{1}{\epsilon}$ pole) at D=26/5 while L=5 max supergravity could diverge earlier, at D=24/5 with a generic $\partial^8 R^4$ structure. This would be a clear indication that D=4 supergravity should first diverge at L=7.

Accordingly, another bet was made with Zvi Bern