A massive S-matrix from massless amplitudes

Michael Kiermaier

Princeton University

NBI Summer Institute, Aug 29, 2011

based on:

N. Craig, H. Elvang, MK, T. Slatyer 1104.2050

MK 1105.5385

Motivation

On-shell methods: Successes

Tremendous conceptual progress in

- massless
- planar
- maximally supersymmetric Yang-Mills theory.

Motivation

On-shell methods: Successes

Tremendous conceptual progress in

- massless
- planar
- maximally supersymmetric Yang-Mills theory.

Challenges

- draw lessons for less "special" theories from massless $\mathcal{N} = 4$ SYM

Motivation

On-shell methods: Successes

Tremendous conceptual progress in

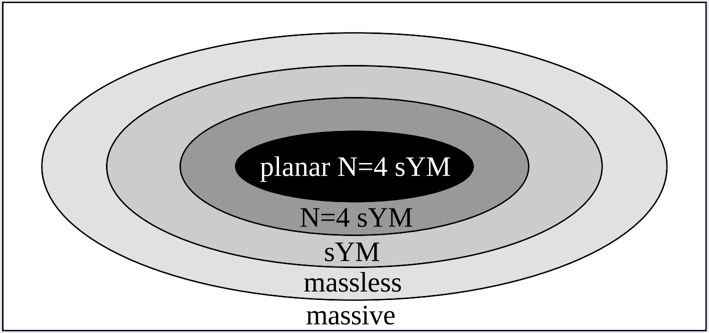
- massless
- planar
- maximally supersymmetric Yang-Mills theory.

Challenges

- draw lessons for less "special" theories from massless $\mathcal{N} = 4$ SYM
- Even better:
  Recycle results in massless $\mathcal{N} = 4$ SYM for other theories?

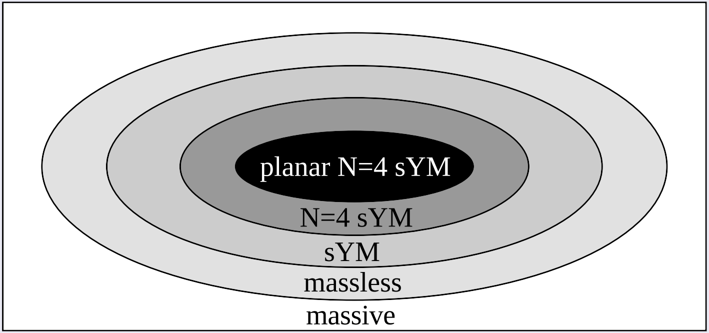
Motivation

Level of Difficulty [L. Dixon, 1105.0771]



Motivation

Level of Difficulty [L. Dixon, 1105.0771]



Can we somehow cheat:
Compute massive amplitudes from massless on-shell amplitudes?

Motivation

QFT phenomena studied using on-shell methods

- on-shell amplitudes
- light-like Wilson loops
- light-like correlation functions

Motivation

QFT phenomena studied using on-shell methods

- on-shell amplitudes
- light-like Wilson loops
- light-like correlation functions
- gauge theory=(gravity)^2 at tree and loop-level [KLT, BCJ]

Motivation

QFT phenomena studied using on-shell methods

- on-shell amplitudes
- light-like Wilson loops
- light-like correlation functions
- gauge theory=(gravity)^2 at tree and loop-level [KLT, BCJ]
- form factors
- β functions from bubble coefficients

Motivation

QFT phenomena studied using on-shell methods

- on-shell amplitudes
- light-like Wilson loops
- light-like correlation functions
- gauge theory=(gravity)^2 at tree and loop-level [KLT, BCJ]
- form factors
- β functions from bubble coefficients

What about spontaneous symmetry breaking?

Motivation

QFT phenomena studied using on-shell methods

- on-shell amplitudes
- light-like Wilson loops
- light-like correlation functions
- gauge theory=(gravity)^2 at tree and loop-level [KLT, BCJ]
- form factors
- β functions from bubble coefficients

What about spontaneous symmetry breaking?
Specifically: Can we compute amplitudes in the spontaneously broken theory from on-shell amplitudes in the unbroken theory?

Spontaneous symmetry breaking in $\mathcal{N} = 4$ SYM

The origin of moduli space

gauge group $U(M+N)$, R-symmetry group $SU(4)_R$

Spontaneous symmetry breaking in $\mathcal{N} = 4$ SYM

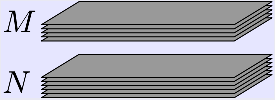
The origin of moduli space

gauge group $U(M+N)$, R-symmetry group $SU(4)_R$

Coulomb branch: the simplest case

scalar vevs: $\big\langle (\phi_{12})_A{}^B \big\rangle = \big\langle (\phi_{34})_A{}^B \big\rangle = m\,\delta_A{}^B \qquad \text{for} \quad 1 \leq A, B \leq M\,.$

brane picture:



Spontaneous symmetry breaking in $\mathcal{N} = 4$ SYM

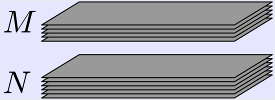
The origin of moduli space

gauge group $U(M+N)$, R-symmetry group $SU(4)_R$

Coulomb branch: the simplest case

scalar vevs: $\langle(\phi_{12})_A{}^B\rangle = \langle(\phi_{34})_A{}^B\rangle = m\,\delta_A{}^B$ for $1 \leq A, B \leq M$.

brane picture:



spontaneously breaks gauge and R-symmetry group:

$$U(M{+}N) \to U(M) \times U(N)\,, \qquad SU(4)_R \to Sp(4) \supset SU(2) \times SU(2)\,.$$

states decompose as: $A_\mu = \begin{pmatrix} (A_\mu)_{N\times N} & (W_\mu)_{N\times M} \\ (\overline{W}_\mu)_{M\times N} & (\tilde{A}_\mu)_{M\times M} \end{pmatrix}$.

Spontaneous symmetry breaking in $\mathcal{N} = 4$ SYM

Recent interest in Coulomb branch amplitudes

- as a regulator of IR divergences in the massless theory
  [Alday, Henn, Naculich, Plefka, Schnitzer, Schuster]
- Coulomb-branch inspired regularization of SUSY Wilson loop?
  [Mason, Skinner; Carot-Huot]

Spontaneous symmetry breaking in $\mathcal{N} = 4$ SYM

Recent interest in Coulomb branch amplitudes

- as a regulator of IR divergences in the massless theory [Alday, Henn, Naculich, Plefka, Schnitzer, Schuster]
- Coulomb-branch inspired regularization of SUSY Wilson loop? [Mason, Skinner; Carot-Huot]
- as the 4d reduction of 6d massless SYM [Bern, Carrasco, Dennen, Huang, Ita; Brandhuber, Korres, Koschade, Travaglini]

Spontaneous symmetry breaking in $\mathcal{N} = 4$ SYM

Recent interest in Coulomb branch amplitudes

- as a regulator of IR divergences in the massless theory [Alday, Henn, Naculich, Plefka, Schnitzer, Schuster]
- Coulomb-branch inspired regularization of SUSY Wilson loop? [Mason, Skinner; Carot-Huot]
- as the 4d reduction of 6d massless SYM [Bern, Carrasco, Dennen, Huang, Ita; Brandhuber, Korres, Koschade, Travaglini]
- interesting in their own right, as the simplest massive theory [Boels; Craig, Elvang, MK, Slatyer]

Spontaneous symmetry breaking in $\mathcal{N} = 4$ SYM

Recent interest in Coulomb branch amplitudes

- as a regulator of IR divergences in the massless theory
  [Alday, Henn, Naculich, Plefka, Schnitzer, Schuster]
- Coulomb-branch inspired regularization of SUSY Wilson loop?
  [Mason, Skinner; Carot-Huot]
- as the 4d reduction of 6d massless SYM [Bern, Carrasco, Dennen, Huang, Ita; Brandhuber, Korres, Koschade, Travaglini]
- interesting in their own right, as the simplest massive theory
  [Boels; Craig, Elvang, MK, Slatyer]

Crucial simple properties

- massive and massless phase connected on moduli space of vacua

Spontaneous symmetry breaking in $\mathcal{N} = 4$ SYM

Recent interest in Coulomb branch amplitudes

- as a regulator of IR divergences in the massless theory [Alday, Henn, Naculich, Plefka, Schnitzer, Schuster]
- Coulomb-branch inspired regularization of SUSY Wilson loop? [Mason, Skinner; Carot-Huot]
- as the 4d reduction of 6d massless SYM [Bern, Carrasco, Dennen, Huang, Ita; Brandhuber, Korres, Koschade, Travaglini]
- interesting in their own right, as the simplest massive theory [Boels; Craig, Elvang, MK, Slatyer]

Crucial simple properties

- massive and massless phase connected on moduli space of vacua
- masses of any amplitude sum to zero: $\sum_i m_i = 0\,.$

  (obvious from 6d momentum conservation)

Explicit amplitudes

Convenient variables to express Coulomb-branch amplitudes?

Explicit amplitudes

Convenient variables to express Coulomb-branch amplitudes?

Review: Massless spinor helicity formalism

$$p_i \leftrightarrow |i\rangle[i| \qquad \Rightarrow \qquad \text{e.g.} \quad \left\langle g_1^- g_2^- g_3^+ \cdots g_n^+ \right\rangle = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}\,.$$

Explicit amplitudes

Convenient variables to express Coulomb-branch amplitudes?

Review: Massless spinor helicity formalism

$$p_i \leftrightarrow |i\rangle[i| \qquad \Rightarrow \qquad \text{e.g. } \left\langle g_1^- g_2^- g_3^+ \cdots g_n^+ \right\rangle = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle \cdots \langle n1\rangle} .$$

Massive spinor helicity formalism [Kleiss, Stirling; Dittmaier]

Pick arbitrary massless reference vector q

Explicit amplitudes

Convenient variables to express Coulomb-branch amplitudes?

Review: Massless spinor helicity formalism

$$p_i \leftrightarrow |i\rangle[i| \qquad \Rightarrow \qquad \text{e.g.}\ \left\langle g_1^- g_2^- g_3^+ \cdots g_n^+ \right\rangle = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle \cdots \langle n1\rangle}\,.$$

Massive spinor helicity formalism [Kleiss, Stirling; Dittmaier]

Pick arbitrary massless reference vector q, then decompose:

$$p_i \leftrightarrow |i^\perp\rangle[i^\perp| - \frac{m^2}{2\,p_i\cdot q}|q\rangle[q|\,,$$

Explicit amplitudes

Convenient variables to express Coulomb-branch amplitudes?

Review: Massless spinor helicity formalism

$$p_i \leftrightarrow |i\rangle[i| \qquad \Rightarrow \qquad \text{e.g.} \;\; \left\langle g_1^- g_2^- g_3^+ \cdots g_n^+ \right\rangle = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle \cdots \langle n1\rangle} \,.$$

Massive spinor helicity formalism [Kleiss, Stirling; Dittmaier]

Pick arbitrary massless reference vector q, then decompose:

$$p_i \leftrightarrow |i^\perp\rangle[i^\perp| - \frac{m^2}{2\, p_i \cdot q}|q\rangle[q| \,, \qquad \epsilon^+ = \frac{\sqrt{2}|q\rangle[i^\perp|}{\langle i^\perp q\rangle}\,, \;\; \epsilon^- = \frac{\sqrt{2}|i^\perp\rangle[q|}{[i^\perp q]}\,, \ldots$$

Explicit amplitudes

Convenient variables to express Coulomb-branch amplitudes?

Review: Massless spinor helicity formalism

$$p_i \leftrightarrow |i\rangle[i| \qquad \Rightarrow \qquad \text{e.g. } \langle g_1^- g_2^- g_3^+ \cdots g_n^+ \rangle = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle \cdots \langle n1\rangle} \, .$$

Massive spinor helicity formalism [Kleiss, Stirling; Dittmaier]

Pick arbitrary massless reference vector q, then decompose:

$$p_i \leftrightarrow |i^\perp\rangle[i^\perp| - \frac{m^2}{2\, p_i \cdot q}|q\rangle[q| \, , \qquad \epsilon^+ = \frac{\sqrt{2}|q\rangle[i^\perp|}{\langle i^\perp q\rangle} \, , \quad \epsilon^- = \frac{\sqrt{2}|i^\perp\rangle[q|}{[i^\perp q]} \, , \ldots$$

Reference vector q

- introduced as a technical tool, defines a basis of “helicity amplitudes”
- BUT: has no direct physical significance or interpretation!

Explicit amplitudes

Example: Ultra-helicity violating (UHV) amplitudes

4-point amplitude: $$\langle W_1^- \overline{W}_2^+ g_3^+ g_4^+ \rangle = -\frac{m^2 \langle q1^\perp \rangle^2 [34]}{\langle q2^\perp \rangle^2 \langle 34 \rangle (P_{23}^2 + m^2)} \,.$$

Explicit amplitudes

Example: Ultra-helicity violating (UHV) amplitudes

4-point amplitude: $$\left\langle W_1^- \overline{W}_2^+ \, g_3^+ \, g_4^+ \right\rangle = -\frac{m^2 \langle q1^\perp\rangle^2 [34]}{\langle q2^\perp\rangle^2 \langle 34\rangle (P_{23}^2 + m^2)} \, .$$

n-point amplitude: `[see also: Forde, Kosower; Ferrario et al]`

$$\left\langle W_1^- \overline{W}_2^+ \, g_3^+ \ldots g_n^+ \right\rangle$$

$$= -\frac{m^2 \langle q\,1^\perp\rangle^2}{\langle q\,2^\perp\rangle^2 \, \langle 34\rangle\langle 45\rangle \cdots \langle n{-}1, n\rangle (P_{n1}^2 + m^2)} \times [3| \prod_{j=3}^{n-2} \left[1 + \frac{P_J |j{+}1\rangle [j{+}1|}{P_J^2 + m^2} \right] |n] \, .$$

Explicit amplitudes

Example: Ultra-helicity violating (UHV) amplitudes

4-point amplitude: $$\left\langle W_1^- \overline{W}_2^+ \, g_3^+ \, g_4^+ \right\rangle = -\frac{m^2 \langle q1^\perp \rangle^2 [34]}{\langle q2^\perp \rangle^2 \langle 34 \rangle (P_{23}^2 + m^2)} \,.$$

n-point amplitude: `[see also: Forde, Kosower; Ferrario et al]`

$$\left\langle W_1^- \overline{W}_2^+ \, g_3^+ \ldots g_n^+ \right\rangle$$

$$= -\frac{m^2 \langle q\, 1^\perp \rangle^2}{\langle q\, 2^\perp \rangle^2 \, \langle 34 \rangle \langle 45 \rangle \cdots \langle n{-}1, n \rangle (P_{n1}^2 + m^2)} \times [3| \prod_{j=3}^{n-2} \left[1 + \frac{P_J |j{+}1\rangle [j{+}1|}{P_J^2 + m^2} \right] |n] \,.$$

Example: Maximally $SU(4)$-violating amplitudes

4-point amplitude: $$\langle W^- \overline{W}^+ \phi^{34} \phi^{34} \rangle = -\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] (P_{23}^2 + m^2)} \,.$$

Explicit amplitudes

Example: Ultra-helicity violating (UHV) amplitudes

4-point amplitude: $$\left\langle W_1^- \overline{W}_2^+ \, g_3^+ \, g_4^+ \right\rangle \;=\; -\,\frac{m^2 \langle q1^\perp\rangle^2 [34]}{\langle q2^\perp\rangle^2 \langle 34\rangle (P_{23}^2 + m^2)} \, .$$

n-point amplitude: [see also: Forde, Kosower; Ferrario et al]

$$\left\langle W_1^- \overline{W}_2^+ \, g_3^+ \ldots g_n^+ \right\rangle$$
$$= \; -\,\frac{m^2 \langle q\, 1^\perp\rangle^2}{\langle q\, 2^\perp\rangle^2 \, \langle 34\rangle\langle 45\rangle \cdots \langle n{-}1, n\rangle (P_{n1}^2 + m^2)} \times [3| \prod_{j=3}^{n-2} \left[1 + \frac{P_J |j{+}1\rangle [j{+}1|}{P_J^2 + m^2} \right] |n] \, .$$

Example: Maximally $SU(4)$-violating amplitudes

4-point amplitude: $$\langle W^- \overline{W}^+ \phi^{34} \phi^{34} \rangle = -\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp](P_{23}^2 + m^2)} \, .$$

n-point amplitude:

$$\left\langle W_1^- \overline{W}_2^+ \, \phi_3^{34} \ldots \phi_n^{34} \right\rangle \;=\; \frac{-m^{n-2} \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp](P_{23}^2 + m^2)(P_{234}^2 + m^2) \cdots (P_{23\ldots n\text{-}1}^2 + m^2)} \, .$$

Massive amplitudes from massless amplitudes

Goal

Coulomb branch S-matrix $= \sum$ massless on-shell amplitudes

Massive amplitudes from massless amplitudes

Goal

Coulomb branch S-matrix $= \sum$ massless on-shell amplitudes

Idea

Probe the moduli space through scalar soft limits!

Massive amplitudes from massless amplitudes

Goal

Coulomb branch S-matrix $=$ $\sum$ massless on-shell amplitudes

Idea

Probe the moduli space through scalar soft limits!

Compare: Adler zeros, Soft-pion theorems

- Goldstone bosons: soft-limits probe vacuum manifold
- BUT: Goldstone bosons $\Leftrightarrow$ global symmetry $\Leftrightarrow$ all vacua equivalent
- soft scalar limits vanish!

$$\lim_{\epsilon \to 0} \langle \phi_{\epsilon q} \ldots \rangle = 0 \,.$$

Massive amplitudes from massless amplitudes

Goal

Coulomb branch S-matrix $=$ $\sum$ massless on-shell amplitudes

Idea

Probe the moduli space through scalar soft limits!

Compare: Adler zeros, Soft-pion theorems

- Goldstone bosons: soft-limits probe vacuum manifold
- BUT: Goldstone bosons $\Leftrightarrow$ global symmetry $\Leftrightarrow$ all vacua equivalent
- soft scalar limits vanish!

$$\lim_{\epsilon\to 0}\langle \phi_{\epsilon q}\ldots\rangle \;=\; 0\,.$$

- old theorem with modern applications:
  finiteness of $\mathcal{N}=8$ supergravity for $L<7$ loops!

Massive amplitudes from massless amplitudes

Goal

Coulomb branch S-matrix $=$ $\sum$ massless on-shell amplitudes

Naive ansatz:

$$\langle\, W_1 \overline{W}_2 \ldots \rangle \;\stackrel{?}{=}\; \langle\, g_1\, g_2 \ldots \rangle \;+\; O(m) \qquad .$$

Massive amplitudes from massless amplitudes

Goal

Coulomb branch S-matrix $=$ $\sum$ massless on-shell amplitudes

Naive ansatz:

$$\langle\, W_1 \overline{W}_2 \ldots \rangle \stackrel{?}{=} \langle\, g_1\, g_2 \ldots \rangle + \langle\, g_1 \phi_q^{\mathrm{vev}} g_2 \ldots \rangle + O(m^2)$$

Massive amplitudes from massless amplitudes

Goal

Coulomb branch S-matrix $=$ $\sum$ massless on-shell amplitudes

Naive ansatz:

$$\langle\, W_1 \overline{W}_2 \,\ldots \rangle \;\stackrel{?}{=}\; \langle\, g_1\, g_2 \,\ldots \rangle \;+\; \lim_{\varepsilon\to 0} \langle\, g_1 \phi^{\mathrm{vev}}_{\varepsilon q}\, g_2 \,\ldots \rangle \;+\; O(m^2)$$

Massive amplitudes from massless amplitudes

Goal

Coulomb branch S-matrix $\;=\; \sum$ massless on-shell amplitudes

Naive ansatz:

$$\left\langle\, W_1 \overline{W}_2 \,\ldots \right\rangle \;\stackrel{?}{=}\; \lim_{\varepsilon\to 0} \sum_{s=0}^{\infty} \left\langle\, g_1\, \phi^{\text{vev}}_{\varepsilon q_1} \phi^{\text{vev}}_{\varepsilon q_2} \ldots \phi^{\text{vev}}_{\varepsilon q_s}\, g_2\, \ldots \right\rangle .$$

Massive amplitudes from massless amplitudes

Goal

Coulomb branch S-matrix $\quad = \quad \sum$ massless on-shell amplitudes

Naive ansatz:

$$\left\langle\, W_1 \overline{W}_2 \,\ldots \right\rangle \;\stackrel{?}{=}\; \lim_{\varepsilon\to 0} \sum_{s=0}^{\infty} \left\langle\, g_1\, \phi^{\text{vev}}_{\varepsilon q_1} \phi^{\text{vev}}_{\varepsilon q_2} \ldots \phi^{\text{vev}}_{\varepsilon q_s}\, g_2 \,\ldots \right\rangle .$$

Puzzles

Massive amplitudes from massless amplitudes

Goal

Coulomb branch S-matrix $=$ $\sum$ massless on-shell amplitudes

Naive ansatz:

$$\big\langle\, W_1 \overline{W}_2 \,\ldots \big\rangle \;\stackrel{?}{=}\; \lim_{\varepsilon\to 0} \sum_{s=0}^{\infty} \big\langle\, g_1\, \phi^{\mathrm{vev}}_{\varepsilon q_1} \phi^{\mathrm{vev}}_{\varepsilon q_2} \ldots \phi^{\mathrm{vev}}_{\varepsilon q_s}\, g_2 \,\ldots \big\rangle\,.$$

Puzzles

1. massive and massless amplitude are on-shell, $p_W^2 = -m^2$, but $p_g^2 = 0$! How are momenta related?

Massive amplitudes from massless amplitudes

Goal

Coulomb branch S-matrix $=$ $\sum$ massless on-shell amplitudes

Naive ansatz:

$$\Big\langle\, W_1 \overline{W}_2 \,\ldots \Big\rangle \;\stackrel{?}{=}\; \lim_{\varepsilon\to 0} \sum_{s=0}^{\infty} \Big\langle\, g_1\, \phi^{\mathrm{vev}}_{\varepsilon q_1} \phi^{\mathrm{vev}}_{\varepsilon q_2} \ldots \phi^{\mathrm{vev}}_{\varepsilon q_s}\, g_2 \,\ldots \Big\rangle\,.$$

Puzzles

1. massive and massless amplitude are on-shell, $p_W^2 = -m^2$, but $p_g^2 = 0$! How are momenta related?
2. soft-limit q_i dependent! How to choose q_i?

Massive amplitudes from massless amplitudes

Goal

Coulomb branch S-matrix $=$ $\sum$ massless on-shell amplitudes

Naive ansatz:

$$\left\langle W_1 \overline{W}_2 \ldots \right\rangle \overset{?}{=} \lim_{\varepsilon\to 0} \sum_{s=0}^{\infty} \left\langle g_1\, \phi^{\mathrm{vev}}_{\varepsilon q_1} \phi^{\mathrm{vev}}_{\varepsilon q_2} \ldots \phi^{\mathrm{vev}}_{\varepsilon q_s}\, g_2 \ldots \right\rangle .$$

Puzzles

1. massive and massless amplitude are on-shell, $p_W^2 = -m^2$, but $p_g^2 = 0$! How are momenta related?
2. soft-limit q_i dependent! How to choose q_i?
3. massive "helicity" frame-dependent! How to choose W-polarizations?

Massive amplitudes from massless amplitudes

Goal

Coulomb branch S-matrix $=$ $\sum$ massless on-shell amplitudes

Naive ansatz:

$$\langle\, W_1 \overline{W}_2 \,\ldots \rangle \;\stackrel{?}{=}\; \lim_{\varepsilon\to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi^{\text{vev}}_{\varepsilon q_1} \phi^{\text{vev}}_{\varepsilon q_2} \ldots \phi^{\text{vev}}_{\varepsilon q_s}\, g_2 \,\ldots \rangle\,.$$

Puzzles

1. massive and massless amplitude are on-shell, $p_W^2 = -m^2$, but $p_g^2 = 0$! How are momenta related?
2. soft-limit q_i dependent! How to choose q_i?
3. massive "helicity" frame-dependent! How to choose W-polarizations?
4. RHS generically ill-defined due to soft divergences!

Massive amplitudes from massless amplitudes

$$\langle\, W_1 \overline{W}_2 \ldots \rangle \quad \stackrel{?}{=} \quad \lim_{\varepsilon \to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi^{\text{vev}}_{\varepsilon q_1} \phi^{\text{vev}}_{\varepsilon q_2} \ldots \phi^{\text{vev}}_{\varepsilon q_s}\, g_2 \ \ldots \rangle \quad .$$

Puzzle 1: How are momenta related?

Massive amplitudes from massless amplitudes

$$\left\langle\, W_1 \overline{W}_2 \,\ldots \right\rangle \quad \stackrel{?}{=} \quad \lim_{\varepsilon\to 0} \sum_{s=0}^{\infty} \left\langle\, g_1\, \phi^{\mathrm{vev}}_{\varepsilon q_1} \phi^{\mathrm{vev}}_{\varepsilon q_2} \ldots \phi^{\mathrm{vev}}_{\varepsilon q_s}\, g_2 \,\ldots \right\rangle \quad .$$

Puzzle 1: How are momenta related?

decompose massive W momenta as: $p_i \;=\; p_i^{\perp} - \frac{m_i^2}{2\, q\cdot p_i} q\,.$

choose $p_i^{\perp}$ as massless gluon momentum on RHS!

Massive amplitudes from massless amplitudes

$$\langle\, W_1 \overline{W}_2 \ldots \rangle \quad \overset{?}{=} \quad \lim_{\varepsilon \to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi^{\text{vev}}_{\varepsilon q_1} \phi^{\text{vev}}_{\varepsilon q_2} \ldots \phi^{\text{vev}}_{\varepsilon q_s}\, g_2 \ldots \rangle \quad .$$

Puzzle 1: How are momenta related?

decompose massive W momenta as: $p_i \;=\; p_i^{\perp} - \frac{m_i^2}{2\, q \cdot p_i} q\,.$

choose $p_i^{\perp}$ as massless gluon momentum on RHS!

Puzzle 2: How to map W-polarizations to massless states?

Massive amplitudes from massless amplitudes

$$\langle\, W_1 \overline{W}_2 \ldots \rangle \quad \stackrel{?}{=} \quad \lim_{\varepsilon\to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi^{\mathrm{vev}}_{\varepsilon q_1} \phi^{\mathrm{vev}}_{\varepsilon q_2} \ldots \phi^{\mathrm{vev}}_{\varepsilon q_s}\, g_2 \ldots \rangle \quad .$$

Puzzle 1: How are momenta related?

decompose massive W momenta as: $p_i \;=\; p_i^{\perp} - \frac{m_i^2}{2\, q\cdot p_i} q\,.$

choose $p_i^{\perp}$ as massless gluon momentum on RHS!

Puzzle 2: How to map W-polarizations to massless states?

- massive W polarizations $\epsilon^{+} = \frac{\sqrt{2}|q\rangle[i^{\perp}|}{\langle i^{\perp} q\rangle}\,, \quad \epsilon^{-} = \frac{\sqrt{2}|i^{\perp}\rangle[q|}{[i^{\perp} q]}$
  $\Leftrightarrow$ massless $\pm$-helicity gluons.

Massive amplitudes from massless amplitudes

$$\langle\, W_1 \overline{W}_2 \ldots \rangle \quad \stackrel{?}{=} \quad \lim_{\varepsilon\to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi^{\rm vev}_{\varepsilon q_1} \phi^{\rm vev}_{\varepsilon q_2} \ldots \phi^{\rm vev}_{\varepsilon q_s}\, g_2 \,\ldots \rangle \quad .$$

Puzzle 1: How are momenta related?

decompose massive W momenta as: $p_i \;=\; p_i^{\perp} - \frac{m_i^2}{2\,q\cdot p_i} q\,.$

choose $p_i^{\perp}$ as massless gluon momentum on RHS!

Puzzle 2: How to map W-polarizations to massless states?

- massive W polarizations $\epsilon^{+} = \frac{\sqrt{2}|q\rangle[i^{\perp}|}{\langle i^{\perp} q\rangle}\,, \quad \epsilon^{-} = \frac{\sqrt{2}|i^{\perp}\rangle[q|}{[i^{\perp} q]}$
  $\Leftrightarrow$ massless $\pm$-helicity gluons.
- longitudinal W boson with polarization $\not{\epsilon}^{L} = \frac{1}{m_i}\left(\not{p}_i^{\perp} + \frac{m_i^2}{2q\cdot p_i}\not{q}\right)$
  $\Leftrightarrow$ massless scalar $\frac{1}{\sqrt{2}}(\phi^{12}+\phi^{34})\,.$

Massive amplitudes from massless amplitudes

$$\langle\, W_1 \overline{W}_2 \ldots \rangle \quad \stackrel{?}{=} \quad \lim_{\varepsilon \to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi^{\mathrm{vev}}_{\varepsilon q_1} \phi^{\mathrm{vev}}_{\varepsilon q_2} \ldots \phi^{\mathrm{vev}}_{\varepsilon q_s}\, g_2 \ldots \rangle \quad .$$

Puzzle 1: How are momenta related?

decompose massive W momenta as: $p_i = p_i^{\perp} - \frac{m_i^2}{2\, q \cdot p_i} q$.

choose $p_i^{\perp}$ as massless gluon momentum on RHS!

Puzzle 2: How to map W-polarizations to massless states?

- massive W polarizations $\epsilon^{+} = \frac{\sqrt{2}|q\rangle[i^{\perp}|}{\langle i^{\perp} q\rangle}$, $\epsilon^{-} = \frac{\sqrt{2}|i^{\perp}\rangle[q|}{[i^{\perp} q]}$
  $\Leftrightarrow$ massless $\pm$-helicity gluons.
- longitudinal W boson with polarization $\not{\epsilon}^{L} = \frac{1}{m_i}\left(\not{p}_i^{\perp} + \frac{m_i^2}{2q \cdot p_i} \not{q}\right)$
  $\Leftrightarrow$ massless scalar $\frac{1}{\sqrt{2}}(\phi^{12} + \phi^{34})$.

LHS now depends on one arbitrary massless reference spinor $q \Leftrightarrow$ RHS?

Massive amplitudes from massless amplitudes

$$\left\langle W_1 \overline{W}_2 \ldots \right\rangle \quad \stackrel{?}{=} \quad \lim_{\varepsilon \to 0} \sum_{s=0}^{\infty} \left\langle g_1 \, \phi^{\text{vev}}_{\varepsilon q_1} \phi^{\text{vev}}_{\varepsilon q_2} \ldots \phi^{\text{vev}}_{\varepsilon q_s} \, g_2 \, \ldots \right\rangle \quad .$$

Puzzle 3: How to choose soft-scalar momenta q_i?

Massive amplitudes from massless amplitudes

$$\langle\, W_1 \overline{W}_2 \ldots \rangle \quad \stackrel{?}{=} \quad \lim_{\varepsilon\to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi^{\mathrm{vev}}_{\varepsilon q_1} \phi^{\mathrm{vev}}_{\varepsilon q_2} \ldots \phi^{\mathrm{vev}}_{\varepsilon q_s}\, g_2\, \ldots \rangle \quad .$$

Puzzle 3: How to choose soft-scalar momenta q_i?

LHS depends only on one reference null momentum q

Massive amplitudes from massless amplitudes

$$\langle\, W_1 \overline{W}_2 \ldots \rangle \quad \stackrel{?}{=} \quad \lim_{\varepsilon \to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi^{\text{vev}}_{\varepsilon q_1} \phi^{\text{vev}}_{\varepsilon q_2} \ldots \phi^{\text{vev}}_{\varepsilon q_s}\, g_2 \ldots \rangle \quad .$$

Puzzle 3: How to choose soft-scalar momenta q_i?

LHS depends only on one reference null momentum q

$\Rightarrow$ need to set all $q_i = q$ to match parameters!

Massive amplitudes from massless amplitudes

$$\langle\, W_1 \overline{W}_2 \ldots \rangle \quad \stackrel{?}{=} \quad \lim_{\varepsilon \to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi^{\rm vev}_{\varepsilon q_1} \phi^{\rm vev}_{\varepsilon q_2} \ldots \phi^{\rm vev}_{\varepsilon q_s}\, g_2\, \ldots \rangle \quad .$$

Puzzle 3: How to choose soft-scalar momenta q_i?

LHS depends only on one reference null momentum q

$\Rightarrow$ need to set all $q_i = q$ to match parameters!

New Puzzle: collinear divergences as $q_i \to q$!

Massive amplitudes from massless amplitudes

$$\langle\, W_1 \overline{W}_2 \,\ldots \rangle \quad \stackrel{?}{=} \quad \lim_{\varepsilon\to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi^{\mathrm{vev}}_{\varepsilon q_1} \phi^{\mathrm{vev}}_{\varepsilon q_2} \,..\, \phi^{\mathrm{vev}}_{\varepsilon q_s}\, g_2 \,\ldots \rangle \quad .$$

Puzzle 3: How to choose soft-scalar momenta q_i?

LHS depends only on one reference null momentum q
$\Rightarrow$ need to set all $q_i = q$ to match parameters!

New Puzzle: collinear divergences as $q_i \to q$!

collinear divergences are anti-symmetric in momenta.

Massive amplitudes from massless amplitudes

$$\langle W_1 \overline{W}_2 \ldots \rangle \overset{?}{=} \lim_{\varepsilon\to 0} \sum_{s=0}^{\infty} \langle g_1 \, \phi^{\mathrm{vev}}_{\varepsilon q} \phi^{\mathrm{vev}}_{\varepsilon q} \,..\, \phi^{\mathrm{vev}}_{\varepsilon q} \, g_2 \ldots \rangle_{sym} .$$

Puzzle 3: How to choose soft-scalar momenta q_i?

LHS depends only on one reference null momentum q
$\Rightarrow$ need to set all $q_i = q$ to match parameters!

New Puzzle: collinear divergences as $q_i \to q$!

collinear divergences are anti-symmetric in momenta.

Resolution: Symmetrize in q_i before taking the limit $q_i \to q$.

Makes sense, because vev scalars should not be color-ordered!

Massive amplitudes from massless amplitudes

$$\langle\, W_1 \overline{W}_2 \, \ldots \rangle \quad \stackrel{?}{=} \quad \lim_{\varepsilon\to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi^{\mathrm{vev}}_{\varepsilon q} \phi^{\mathrm{vev}}_{\varepsilon q} \, .. \, \phi^{\mathrm{vev}}_{\varepsilon q} \, g_2 \, \ldots \rangle_{sym} \, .$$

Puzzle 4: soft-divergences in the limit $\varepsilon \to 0$

Massive amplitudes from massless amplitudes

$$\langle\, W_1 \overline{W}_2 \ldots \rangle \quad \stackrel{?}{=} \quad \lim_{\varepsilon\to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi^{\text{vev}}_{\varepsilon q}\, \phi^{\text{vev}}_{\varepsilon q} \,..\, \phi^{\text{vev}}_{\varepsilon q}\, g_2 \ldots \rangle_{sym} .$$

Puzzle 4: soft-divergences in the limit $\varepsilon \to 0$

Finite soft limits at leading non-vanishing order!

Massive amplitudes from massless amplitudes

$$\langle\, W_1 \overline{W}_2 \ldots \rangle \quad \stackrel{?}{=} \quad \lim_{\varepsilon\to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi^{\mathrm{vev}}_{\varepsilon q} \phi^{\mathrm{vev}}_{\varepsilon q} \,..\, \phi^{\mathrm{vev}}_{\varepsilon q}\, g_2 \,\ldots \rangle_{sym} \,.$$

Puzzle 4: soft-divergences in the limit $\varepsilon \to 0$

Finite soft limits at leading non-vanishing order!

$O(m^2)$ term in $\langle W^- \overline{W}^+ \phi^{34} \phi^{34} \rangle$:

$$\langle\, g_1^- \, \phi^{\mathrm{vev}}_{\varepsilon q} \phi^{\mathrm{vev}}_{\varepsilon q} \, g_2^+ \, \phi_3^{34} \phi_4^{34} \,\rangle_{\mathrm{sym}} \;=\; -\,\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] (P_{23}^\perp)^2} + O(\varepsilon^1)\,. \quad \text{(correct!)}\,.$$

Divergent soft limits at subleading orders!

Massive amplitudes from massless amplitudes

$$\left\langle\, W_1 \overline{W}_2 \ldots \right\rangle \quad \overset{?}{=} \quad \lim_{\varepsilon\to 0} \sum_{s=0}^{\infty} \left\langle\, g_1\, \phi^{\text{vev}}_{\varepsilon q}\, \phi^{\text{vev}}_{\varepsilon q} \,..\, \phi^{\text{vev}}_{\varepsilon q}\, g_2 \,\ldots \right\rangle_{sym} .$$

Puzzle 4: soft-divergences in the limit $\varepsilon \to 0$

Finite soft limits at leading non-vanishing order!

$O(m^2)$ term in $\langle W^- \overline{W}^+ \phi^{34} \phi^{34} \rangle$:

$$\left\langle\, g_1^-\, \phi^{\text{vev}}_{\varepsilon q}\, \phi^{\text{vev}}_{\varepsilon q}\, g_2^+\, \phi_3^{34} \phi_4^{34} \,\right\rangle_{\text{sym}} \;=\; -\,\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] (P_{23}^\perp)^2} + O(\varepsilon^1)\,. \quad \text{(correct!)}\,.$$

Divergent soft limits at subleading orders!

$O(m^4)$ term in $\langle W^- \overline{W}^+ \phi^{34} \phi^{34} \rangle$:

$$\left\langle\, g_1^-\, \phi^{\text{vev}}_{\varepsilon q}\, \phi^{\text{vev}}_{\varepsilon q}\, \phi^{\text{vev}}_{\varepsilon q}\, \phi^{\text{vev}}_{\varepsilon q}\, g_2^+\, \phi_3^{34} \phi_4^{34} \,\right\rangle_{\text{sym}} = \frac{1}{\varepsilon} \times \frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] (P_{23}^\perp)^2} \left(\frac{m^2}{4\, q\cdot p_1} + \frac{m^2}{4\, q\cdot p_2} \right) + O(\varepsilon^0)\,.$$

Massive amplitudes from massless amplitudes

$$\langle\, W_1 \overline{W}_2 \ldots \rangle \quad \overset{?}{=} \quad \lim_{\varepsilon\to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi^{\mathrm{vev}}_{\varepsilon q} \phi^{\mathrm{vev}}_{\varepsilon q} \ldots \phi^{\mathrm{vev}}_{\varepsilon q}\, g_2 \ldots \rangle_{sym} \,.$$

Puzzle 4: soft-divergences in the limit $\varepsilon \to 0$

Finite soft limits at leading non-vanishing order!

$O(m^2)$ term in $\langle W^- \overline{W}^+ \phi^{34}\phi^{34}\rangle$:

$$\langle\, g_1^-\, \phi^{\mathrm{vev}}_{\varepsilon q} \phi^{\mathrm{vev}}_{\varepsilon q}\, g_2^+\, \phi_3^{34} \phi_4^{34}\, \rangle_{\mathrm{sym}} \;=\; -\,\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] (P_{23}^\perp)^2} + O(\varepsilon^1)\,. \quad \text{(correct!)}\,.$$

Divergent soft limits at subleading orders!

$O(m^4)$ term in $\langle W^- \overline{W}^+ \phi^{34}\phi^{34}\rangle$:

$$\langle\, g_1^-\, \phi^{\mathrm{vev}}_{\varepsilon q} \phi^{\mathrm{vev}}_{\varepsilon q} \phi^{\mathrm{vev}}_{\varepsilon q} \phi^{\mathrm{vev}}_{\varepsilon q}\, g_2^+\, \phi_3^{34} \phi_4^{34}\, \rangle_{\mathrm{sym}} = \frac{1}{\varepsilon} \times \frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] (P_{23}^\perp)^2} \left(\frac{m^2}{4\, q \cdot p_1} + \frac{m^2}{4\, q \cdot p_2} \right) + O(\varepsilon^0)\,.$$

Related problem: $O(m^2)$ violation of momentum conservation on RHS

$$\sum_i p_i^\perp \;=\; \sum_i \left(p_i + \frac{m_i^2}{2\, q\cdot p_i} q \right) \;=\; \left[\sum_{i=1}^{n} \frac{m_i^2}{2\, q \cdot p_i} \right] q \;\neq\; 0\,.$$

Massive amplitudes from massless amplitudes

$$\langle W_1 \overline{W}_2 \ldots \rangle \quad \stackrel{?}{=} \quad \lim_{\varepsilon \to 0} \sum_{s=0}^{\infty} \langle g_1 \, \phi_{\varepsilon q}^{\mathrm{vev}} \phi_{\varepsilon q}^{\mathrm{vev}} \ldots \phi_{\varepsilon q}^{\mathrm{vev}} \, g_2 \, \ldots \rangle_{sym} \, .$$

Special choice of q

We must impose

$$\sum_{i=1}^{n} \frac{m_i^2}{2\, q \cdot p_i} \; = \; 0$$

Massive amplitudes from massless amplitudes

$$\langle\, W_1 \overline{W}_2 \ldots \rangle \quad \stackrel{?}{=} \quad \lim_{\varepsilon \to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi^{\mathrm{vev}}_{\varepsilon q}\, \phi^{\mathrm{vev}}_{\varepsilon q} \,..\, \phi^{\mathrm{vev}}_{\varepsilon q}\; g_2\, \ldots \rangle_{sym} \,.$$

Special choice of q

We must impose

$$\sum_{i=1}^{n} \frac{m_i^2}{2\, q \cdot p_i} \;=\; 0$$

to

- prevent $1/\varepsilon$ divergence at first subleading order on RHS.
- guarantees momentum conservation on RHS: $\sum_i p_i^{\perp} = 0$

Massive amplitudes from massless amplitudes

$$\langle\, W_1 \overline{W}_2 \ldots \rangle \quad \stackrel{?}{=} \quad \lim_{\varepsilon\to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi^{\text{vev}}_{\varepsilon q}\, \phi^{\text{vev}}_{\varepsilon q} \,..\, \phi^{\text{vev}}_{\varepsilon q}\; g_2\, \ldots \rangle_{sym} \,.$$

Special choice of q

We must impose

$$\sum_{i=1}^{n} \frac{m_i^2}{2\, q\cdot p_i} \;=\; 0$$

to

- prevent $1/\varepsilon$ divergence at first subleading order on RHS .
- guarantees momentum conservation on RHS: $\sum_i p_i^{\perp} = 0$

only constrains choice of helicity basis!

any Coulomb-branch amplitude expressible in any q-helicity basis!

Massive amplitudes from massless amplitudes

$$\langle W_1 \overline{W}_2 \ldots \rangle \quad \stackrel{?}{=} \quad \lim_{\varepsilon \to 0} \sum_{s=0}^{\infty} \langle g_1 \, \phi^{\rm vev}_{\varepsilon q} \phi^{\rm vev}_{\varepsilon q} \ldots \phi^{\rm vev}_{\varepsilon q} \, g_2 \ldots \rangle_{sym} \, .$$

Special choice of q

We must impose

$$\sum_{i=1}^{n} \frac{m_i^2}{2\, q \cdot p_i} \; = \; 0$$

to

- prevent $1/\varepsilon$ divergence at first subleading order on RHS .
- guarantees momentum conservation on RHS: $\sum_i p_i^{\perp} = 0$

only constrains choice of helicity basis!

any Coulomb-branch amplitude expressible in any q-helicity basis!

Special choice of q for two massive lines

Simple orthogonality relation: $\quad q \cdot (p_1 + p_2) \; = \; 0 \, .$

Massive amplitudes from massless amplitudes

Summary: Refined proposal

$$\langle\, W_1 \overline{W}_2 \,\ldots \rangle \quad = \quad \lim_{\varepsilon\to 0} \sum_{s=0}^{\infty} \langle\, g_1 \, \phi_{\varepsilon q}^{\rm vev} \phi_{\varepsilon q}^{\rm vev} \,..\, \phi_{\varepsilon q}^{\rm vev} \, g_2 \,\ldots \rangle_{sym} \,.$$

Massive amplitudes from massless amplitudes

Summary: Refined proposal

$$\langle W_1 \overline{W}_2 \dots \rangle \quad = \quad \lim_{\varepsilon \to 0} \sum_{s=0}^{\infty} \langle g_1 \, \phi^{\text{vev}}_{\varepsilon q} \, \phi^{\text{vev}}_{\varepsilon q} \, .\,. \, \phi^{\text{vev}}_{\varepsilon q} \, g_2 \, \dots \rangle_{sym} \, .$$

W-boson, pol $\epsilon_q^{\pm}$		$\pm$-helicity gluon
W-boson, pol ϵ_q^{L}	$\Longleftrightarrow$	scalar $\frac{1}{\sqrt{2}}(\phi^{12}+\phi^{34})$

Massive amplitudes from massless amplitudes

Summary: Refined proposal

$$\langle\, W_1 \overline{W}_2 \,\ldots \rangle \quad = \quad \lim_{\varepsilon\to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi^{\mathrm{vev}}_{\varepsilon q}\, \phi^{\mathrm{vev}}_{\varepsilon q} \,..\, \phi^{\mathrm{vev}}_{\varepsilon q}\, g_2 \,\ldots \rangle_{sym} \,.$$

W-boson, pol $\epsilon_q^{\pm}$		$\pm$-helicity gluon
W-boson, pol ϵ_q^{L}	$\Longleftrightarrow$	scalar $\frac{1}{\sqrt{2}}(\phi^{12}+\phi^{34})$
p_i		$p_i^{\perp} \equiv p_i + \frac{m_i^2}{2\, qp_i} q$

Special choice of q: $\sum_{i=1}^{n} \frac{m_i^2}{2\, q\cdot p_i} \;=\; 0 \quad .$

Massive amplitudes from massless amplitudes

Summary: Refined proposal

$$\langle\, W_1 \overline{W}_2 \,\ldots \rangle \quad = \quad \lim_{\varepsilon \to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi^{\text{vev}}_{\varepsilon q}\, \phi^{\text{vev}}_{\varepsilon q} \,..\, \phi^{\text{vev}}_{\varepsilon q}\, g_2 \,\ldots \rangle_{sym} \,.$$

W-boson, pol $\epsilon_q^{\pm}$		$\pm$-helicity gluon
W-boson, pol ϵ_q^{L}	$\Longleftrightarrow$	scalar $\frac{1}{\sqrt{2}}(\phi^{12}+\phi^{34})$
p_i		$p_i^{\perp} \equiv p_i + \frac{m_i^2}{2\,q p_i} q$

Special choice of q: $\sum_{i=1}^{n} \frac{m_i^2}{2\,q \cdot p_i} \;=\; 0 \quad .$

Open questions:

- show that proposal is free of soft divergences to all orders
- verify proposal for explicit Coulomb-branch amplitudes

Massive amplitudes from massless amplitudes

Summary: Refined proposal

$$\langle\, W_1 \overline{W}_2 \,\ldots \rangle \quad = \quad \lim_{\varepsilon\to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi^{\mathrm{vev}}_{\varepsilon q}\, \phi^{\mathrm{vev}}_{\varepsilon q} \,..\, \phi^{\mathrm{vev}}_{\varepsilon q}\, g_2 \,\ldots \rangle_{sym}\,.$$

W-boson, pol $\epsilon_q^{\pm}$		$\pm$-helicity gluon
W-boson, pol ϵ_q^{L}	$\Longleftrightarrow$	scalar $\frac{1}{\sqrt{2}}(\phi^{12}+\phi^{34})$
p_i		$p_i^{\perp} \equiv p_i + \frac{m_i^2}{2\,qp_i} q$

Special choice of q: $\sum_{i=1}^{n} \frac{m_i^2}{2\, q\cdot p_i} \;=\; 0 \quad .$

Open questions:

- show that proposal is free of soft divergences to all orders
- verify proposal for explicit Coulomb-branch amplitudes

BUT: Infinite sum, vev-scalar symmetrization... daunting task in practice?

Convenient representation for massless amplitudes

$$\langle\, W_1 \overline{W}_2 \,\ldots \rangle \quad = \quad \lim_{\varepsilon\to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi_{\varepsilon q}^{\text{vev}} \phi_{\varepsilon q}^{\text{vev}} \,..\, \phi_{\varepsilon q}^{\text{vev}}\, g_2 \,\ldots \rangle_{sym}\,.$$

- need a convenient representation for massless amplitudes

Convenient representation for massless amplitudes

$$\langle\, W_1 \overline{W}_2 \,\ldots \rangle \quad = \quad \lim_{\varepsilon \to 0} \sum_{s=0}^{\infty} \langle\, g_1 \, \phi^{\rm vev}_{\varepsilon q} \phi^{\rm vev}_{\varepsilon q} \,..\, \phi^{\rm vev}_{\varepsilon q} \, g_2 \,\ldots \rangle_{sym} \,.$$

- need a convenient representation for massless amplitudes
- proposal introduces reference vector $q \Rightarrow$ CSW expansion natural!

Convenient representation for massless amplitudes

$$\langle W_1 \overline{W}_2 \ldots \rangle \quad = \quad \lim_{\varepsilon \to 0} \sum_{s=0}^{\infty} \langle g_1 \, \phi_{\varepsilon q}^{\mathrm{vev}} \phi_{\varepsilon q}^{\mathrm{vev}} \, .. \, \phi_{\varepsilon q}^{\mathrm{vev}} \, g_2 \, \ldots \rangle_{sym} \, .$$

- need a convenient representation for massless amplitudes
- proposal introduces reference vector $q \Rightarrow$ CSW expansion natural!

The CSW expansion for massless amplitudes

MHV vertices, connected by scalar propagators:

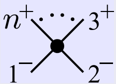
Convenient representation for massless amplitudes

$$\langle\, W_1 \overline{W}_2 \ldots \rangle \quad = \quad \lim_{\varepsilon \to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi^{\mathrm{vev}}_{\varepsilon q}\, \phi^{\mathrm{vev}}_{\varepsilon q} \,..\, \phi^{\mathrm{vev}}_{\varepsilon q}\, g_2 \,\ldots \rangle_{sym} .$$

- need a convenient representation for massless amplitudes
- proposal introduces reference vector $q \Rightarrow$ CSW expansion natural!

The CSW expansion for massless amplitudes

MHV vertices, connected by scalar propagators:



$$\frac{\langle 1^\perp 2^\perp \rangle^4}{\langle 1^\perp 2^\perp \rangle \cdots \langle n^\perp 1^\perp \rangle}$$

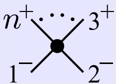
Convenient representation for massless amplitudes

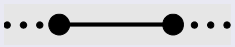
$$\langle\, W_1 \overline{W}_2 \ldots \rangle \quad = \quad \lim_{\varepsilon \to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi^{\mathrm{vev}}_{\varepsilon q} \phi^{\mathrm{vev}}_{\varepsilon q} \ldots \phi^{\mathrm{vev}}_{\varepsilon q}\, g_2 \, \ldots \rangle_{sym} \,.$$

- need a convenient representation for massless amplitudes
- proposal introduces reference vector $q \Rightarrow$ CSW expansion natural!

The CSW expansion for massless amplitudes

MHV vertices, connected by scalar propagators:





$$\frac{\langle 1^\perp 2^\perp \rangle^4}{\langle 1^\perp 2^\perp \rangle \cdots \langle n^\perp 1^\perp \rangle} \qquad\qquad \frac{1}{(P_I^\perp)^2}\,, \qquad P_I^\perp \equiv \sum_{i \in I} p_i^\perp$$

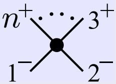
Convenient representation for massless amplitudes

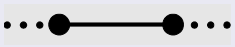
$$\langle\, W_1 \overline{W}_2 \,\ldots\, \rangle \quad = \quad \lim_{\varepsilon\to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi^{\mathrm{vev}}_{\varepsilon q}\, \phi^{\mathrm{vev}}_{\varepsilon q} \,..\, \phi^{\mathrm{vev}}_{\varepsilon q}\, g_2 \,\ldots\, \rangle_{sym}\,.$$

- need a convenient representation for massless amplitudes
- proposal introduces reference vector $q \Rightarrow$ CSW expansion natural!

The CSW expansion for massless amplitudes

MHV vertices, connected by scalar propagators:



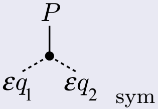


$$\frac{\langle 1^\perp 2^\perp\rangle^4}{\langle 1^\perp 2^\perp\rangle \cdots \langle n^\perp 1^\perp\rangle} \qquad\qquad \frac{1}{(P_I^\perp)^2}\,, \qquad P_I^\perp \equiv \sum_{i\in I} p_i^\perp$$

holomorphic in $|i^\perp\rangle$, CSW prescription $|P_I^\perp\rangle \equiv P_I^\perp|q]$ for internal $P_I^\perp$

Simplification through vev-scalar symmetrization

Vanishing vertices

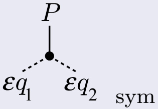


$$\propto \quad \frac{1}{\langle P q_1\rangle\langle q_1 q_2\rangle\langle q_2 P\rangle} + \frac{1}{\langle P q_2\rangle\langle q_2 q_1\rangle\langle q_1 P\rangle} \; = \; 0$$

(obvious from antisymmetry of 3-point ampliutde)

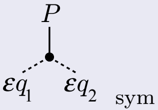
Simplification through vev-scalar symmetrization

Vanishing vertices

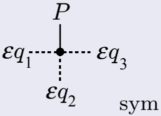


$$\propto \quad \frac{1}{\langle P q_1\rangle\langle q_1 q_2\rangle\langle q_2 P\rangle} + \frac{1}{\langle P q_2\rangle\langle q_2 q_1\rangle\langle q_1 P\rangle} \; = \; 0$$

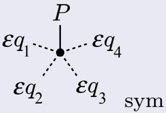
(obvious from antisymmetry of 3-point ampliutde)



$= \; 0 \,,$



$= \; 0 \,,$

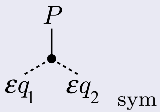


$= \; 0 \,.$

(obvious from $U(1)$-decoupling identity)

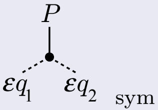
Simplification through vev-scalar symmetrization

Vanishing vertices

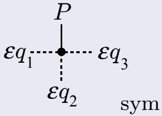


$$\propto \frac{1}{\langle P q_1\rangle\langle q_1 q_2\rangle\langle q_2 P\rangle} + \frac{1}{\langle P q_2\rangle\langle q_2 q_1\rangle\langle q_1 P\rangle} = 0$$

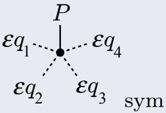
(obvious from antisymmetry of 3-point ampliutde)



$= 0\,,$

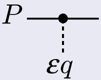


$= 0\,,$

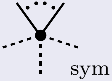


$= 0\,.$

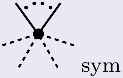
(obvious from $U(1)$-decoupling identity)
vanishing vertices with more non-vev lines:



$= 0\,,$



$= 0\,,$



$= 0\,.$

A four-point tree-level example

The 4-point amplitude $\langle W^- \overline{W}^+ \phi^{34} \phi^{34} \rangle$

$$\langle W^- \overline{W}^+ \phi^{34} \phi^{34} \rangle = -\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] (P_{23}^2 + m^2)} = -\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] \left[(P_{23}^\perp)^2 - m^2 \frac{q \cdot P_{23}}{q \cdot p_2} + m^2 \right]}$$

A four-point tree-level example

The 4-point amplitude $\langle W^- \overline{W}^+ \phi^{34} \phi^{34} \rangle$

$$\langle W^- \overline{W}^+ \phi^{34} \phi^{34} \rangle = -\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] (P_{23}^2 + m^2)} = -\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] \left[(P_{23}^\perp)^2 - m^2 \frac{q \cdot P_{23}}{q \cdot p_2} + m^2 \right]}$$

leading order

A four-point tree-level example

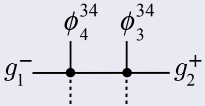
The 4-point amplitude $\langle W^- \overline{W}^+ \phi^{34} \phi^{34} \rangle$

$$\langle W^- \overline{W}^+ \phi^{34} \phi^{34} \rangle = -\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] (P_{23}^2 + m^2)} = -\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] \left[(P_{23}^\perp)^2 - m^2 \frac{q \cdot P_{23}}{q \cdot p_2} + m^2 \right]}$$

leading order

Only one diagram contributes to the massless NMHV amplitude:

$$\lim_{\varepsilon \to 0} \left\langle g_1^- \, \phi_{\varepsilon q}^{\rm vev} \phi_{\varepsilon q}^{\rm vev} \, g_2^+ \, \phi_3^{34} \, \phi_4^{34} \right\rangle_{\rm sym} =$$



$$= -\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] (P_{23}^\perp)^2} .$$

A four-point tree-level example

The 4-point amplitude $\langle W^- \overline{W}^+ \phi^{34} \phi^{34} \rangle$

$$\langle W^- \overline{W}^+ \phi^{34} \phi^{34} \rangle = -\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] (P_{23}^2 + m^2)} = -\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] \left[(P_{23}^\perp)^2 - m^2 \frac{q \cdot P_{23}}{q \cdot p_2} + m^2 \right]}$$

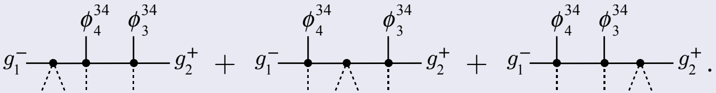
Subleading order: $\left\langle\, g_1^- \, \phi_{\varepsilon q}^{\mathrm{vev}} \phi_{\varepsilon q}^{\mathrm{vev}} \phi_{\varepsilon q}^{\mathrm{vev}} \phi_{\varepsilon q}^{\mathrm{vev}} \, g_2^+ \, \phi_3^{34} \, \phi_4^{34} \,\right\rangle_{\mathrm{sym}}$

A four-point tree-level example

The 4-point amplitude $\langle W^- \overline{W}^+ \phi^{34} \phi^{34} \rangle$

$$\langle W^- \overline{W}^+ \phi^{34} \phi^{34} \rangle = -\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] (P_{23}^2 + m^2)} = -\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] \left[(P_{23}^\perp)^2 - m^2 \frac{q \cdot P_{23}}{q \cdot p_2} + m^2 \right]}$$

Subleading order: $\left\langle \, g_1^- \, \phi_{\varepsilon q}^{\mathrm{vev}} \phi_{\varepsilon q}^{\mathrm{vev}} \phi_{\varepsilon q}^{\mathrm{vev}} \phi_{\varepsilon q}^{\mathrm{vev}} \, g_2^+ \, \phi_3^{34} \, \phi_4^{34} \, \right\rangle_{\mathrm{sym}}$

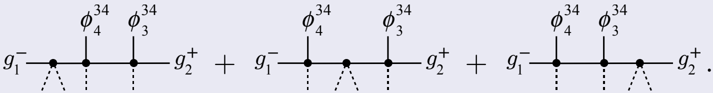


A four-point tree-level example

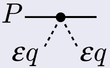
The 4-point amplitude $\langle W^- \overline{W}^+ \phi^{34} \phi^{34} \rangle$

$$\langle W^- \overline{W}^+ \phi^{34} \phi^{34} \rangle = -\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] (P_{23}^2 + m^2)} = -\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] \left[(P_{23}^\perp)^2 - m^2 \frac{q \cdot P_{23}}{q \cdot p_2} + m^2 \right]}$$

Subleading order: $\left\langle\, g_1^- \, \phi^{\mathrm{vev}}_{\varepsilon q} \phi^{\mathrm{vev}}_{\varepsilon q} \phi^{\mathrm{vev}}_{\varepsilon q} \phi^{\mathrm{vev}}_{\varepsilon q} \, g_2^+ \, \phi_3^{34} \, \phi_4^{34} \,\right\rangle_{\mathrm{sym}}$



crucial vertex:



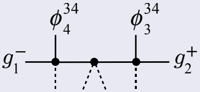
$$\Bigg|_{\mathrm{sym}} = -\frac{1}{2} \langle \phi_{ab} \rangle \langle \phi^{ab} \rangle = -m^2 .$$

A four-point tree-level example

The 4-point amplitude $\langle W^- \overline{W}^+ \phi^{34} \phi^{34} \rangle$

$$\langle W^- \overline{W}^+ \phi^{34} \phi^{34} \rangle = -\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] (P_{23}^2 + m^2)} = -\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] \left[(P_{23}^\perp)^2 - m^2 \frac{q \cdot P_{23}}{q \cdot p_2} + m^2 \right]}$$

Subleading order: $\left\langle\, g_1^- \, \phi^{\mathrm{vev}}_{\varepsilon q} \phi^{\mathrm{vev}}_{\varepsilon q} \phi^{\mathrm{vev}}_{\varepsilon q} \phi^{\mathrm{vev}}_{\varepsilon q} \, g_2^+ \, \phi_3^{34} \, \phi_4^{34} \,\right\rangle_{\mathrm{sym}}$

$\lim_{\varepsilon \to 0}$ 

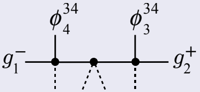
$$= \frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] (P_{23}^\perp)^2} \times \frac{m^2}{(P_{23}^\perp)^2} \, .$$

A four-point tree-level example

The 4-point amplitude $\langle W^- \overline{W}^+ \phi^{34} \phi^{34} \rangle$

$$\langle W^- \overline{W}^+ \phi^{34} \phi^{34} \rangle = -\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] (P_{23}^2 + m^2)} = -\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] \left[(P_{23}^\perp)^2 - m^2 \frac{q \cdot P_{23}}{q \cdot p_2} + m^2 \right]}$$

Subleading order: $\left\langle\, g_1^- \; \phi_{\varepsilon q}^{\text{vev}} \phi_{\varepsilon q}^{\text{vev}} \phi_{\varepsilon q}^{\text{vev}} \phi_{\varepsilon q}^{\text{vev}} \; g_2^+ \; \phi_3^{34} \; \phi_4^{34} \,\right\rangle_{\text{sym}}$

$\lim_{\varepsilon \to 0}$ 

$$= \frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] (P_{23}^\perp)^2} \times \frac{m^2}{(P_{23}^\perp)^2} \,.$$

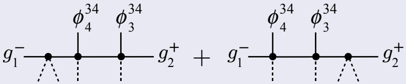
Finite contribution, builds up '$+m^2$' in propagator!

A four-point tree-level example

The 4-point amplitude $\langle W^- \overline{W}^+ \phi^{34} \phi^{34} \rangle$

$$\langle W^- \overline{W}^+ \phi^{34} \phi^{34} \rangle = -\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] (P_{23}^2 + m^2)} = -\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] \left[(P_{23}^\perp)^2 - m^2 \frac{q \cdot P_{23}}{q \cdot p_2} + m^2 \right]}$$

Subleading order: $\left\langle\, g_1^- \, \phi_{\varepsilon q}^{\mathrm{vev}} \phi_{\varepsilon q}^{\mathrm{vev}} \phi_{\varepsilon q}^{\mathrm{vev}} \phi_{\varepsilon q}^{\mathrm{vev}} \; g_2^+ \; \phi_3^{34} \; \phi_4^{34} \,\right\rangle_{\mathrm{sym}}$



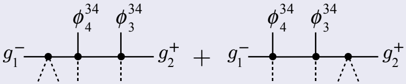
$$= \frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] (P_{23}^\perp)^2} \times \left[\frac{1}{\varepsilon} \left(\frac{m^2}{4\, q \cdot p_1} + \frac{m^2}{4\, q \cdot p_2} \right) - \frac{m^2 \, q \cdot P_{23}}{(P_{23}^\perp)^2 \, q \cdot p_2} + O(\varepsilon) \right].$$

A four-point tree-level example

The 4-point amplitude $\langle W^- \overline{W}^+ \phi^{34} \phi^{34} \rangle$

$$\langle W^- \overline{W}^+ \phi^{34} \phi^{34} \rangle = -\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] (P_{23}^2 + m^2)} = -\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] \left[(P_{23}^\perp)^2 - m^2 \frac{q \cdot P_{23}}{q \cdot p_2} + m^2 \right]}$$

Subleading order: $\left\langle \, g_1^- \; \phi_{\varepsilon q}^{\mathrm{vev}} \phi_{\varepsilon q}^{\mathrm{vev}} \phi_{\varepsilon q}^{\mathrm{vev}} \phi_{\varepsilon q}^{\mathrm{vev}} \; g_2^+ \; \phi_3^{34} \; \phi_4^{34} \, \right\rangle_{\mathrm{sym}}$



$$= \frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] (P_{23}^\perp)^2} \times \left[\frac{1}{\varepsilon} \left(\frac{m^2}{4 \, q \cdot p_1} + \frac{m^2}{4 \, q \cdot p_2} \right) - \frac{m^2 \; q \cdot P_{23}}{(P_{23}^\perp)^2 \; q \cdot p_2} + O(\varepsilon) \right] .$$

Finite sum of divergent diagrams, builds up $p_2^\perp \to p_2$ in propagator!

A four-point tree-level example

The 4-point amplitude $\langle W^- \overline{W}^+ \phi^{34} \phi^{34} \rangle$

$$\langle W^- \overline{W}^+ \phi^{34} \phi^{34} \rangle = -\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] (P_{23}^2 + m^2)} = -\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] \left[(P_{23}^\perp)^2 - m^2 \frac{q \cdot P_{23}}{q \cdot p_2} + m^2 \right]}$$

All orders in m

- Internal 4-point vertices  build up '$+m^2$' in propagator
- External 4-point vertices  build up '$p_2^\perp \to p_2$' in propagator

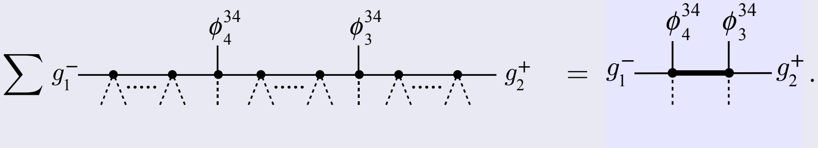
A four-point tree-level example

The 4-point amplitude $\langle W^- \overline{W}^+ \phi^{34} \phi^{34} \rangle$

$$\langle W^- \overline{W}^+ \phi^{34} \phi^{34} \rangle = -\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] (P_{23}^2 + m^2)} = -\frac{m^2 \langle 1^\perp | q | 2^\perp]}{\langle 2^\perp | q | 1^\perp] \left[(P_{23}^\perp)^2 - m^2 \frac{q \cdot P_{23}}{q \cdot p_2} + m^2 \right]}$$

All orders in m

- Internal 4-point vertices build up '$+m^2$' in propagator
- External 4-point vertices build up '$p_2^\perp \to p_2$' in propagator



Checks and Generalizations of the proposal

$$\left\langle\, W_1 \overline{W}_2 \,\ldots \right\rangle \quad = \quad \lim_{\varepsilon\to 0} \sum_{s=0}^{\infty} \left\langle\, g_1\, \phi^{\mathrm{vev}}_{\varepsilon \boldsymbol{q}}\, \phi^{\mathrm{vev}}_{\varepsilon \boldsymbol{q}} \,..\, \phi^{\mathrm{vev}}_{\varepsilon \boldsymbol{q}}\, g_2 \,\ldots \right\rangle_{sym}.$$

Non-trivial Checks

Checks and Generalizations of the proposal

$$\left\langle\, W_1 \overline{W}_2 \,\ldots \right\rangle \quad = \quad \lim_{\varepsilon\to 0} \sum_{s=0}^{\infty} \left\langle\, g_1\, \phi^{\text{vev}}_{\varepsilon q} \phi^{\text{vev}}_{\varepsilon q} \,..\, \phi^{\text{vev}}_{\varepsilon q}\, g_2 \,\ldots \right\rangle_{sym}.$$

Non-trivial Checks

- An all-n amplitude, to all orders in m:

$$\langle W_1^- \overline{W}_2^+ \phi_3^{34} \ldots \phi_n^{34} \rangle \;=$$

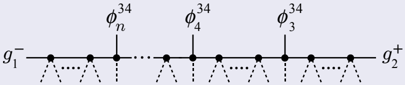
Checks and Generalizations of the proposal

$$\left\langle\, W_1 \overline{W}_2 \,\ldots \right\rangle \quad = \quad \lim_{\varepsilon \to 0} \sum_{s=0}^{\infty} \left\langle\, g_1\, \phi_{\varepsilon q}^{\text{vev}} \phi_{\varepsilon q}^{\text{vev}} \,..\, \phi_{\varepsilon q}^{\text{vev}}\, g_2 \,\ldots \right\rangle_{sym} .$$

Non-trivial Checks

- An all-n amplitude, to all orders in m:

$$\langle W_1^{-} \overline{W}_2^{+} \phi_3^{34} \ldots \phi_n^{34} \rangle \;=\; \sum$$



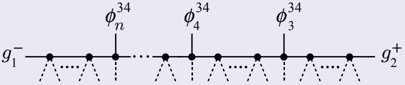
Checks and Generalizations of the proposal

$$\langle\, W_1 \overline{W}_2 \,\ldots \rangle \quad = \quad \lim_{\varepsilon\to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi^{\mathrm{vev}}_{\varepsilon q}\, \phi^{\mathrm{vev}}_{\varepsilon q} \,..\, \phi^{\mathrm{vev}}_{\varepsilon q}\, g_2 \,\ldots \rangle_{sym} \,.$$

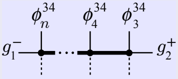
Non-trivial Checks

- An all-n amplitude, to all orders in m:

$$\langle W_1^- \overline{W}_2^+ \phi_3^{34} \ldots \phi_n^{34} \rangle \;=\; \sum$$



$$=$$



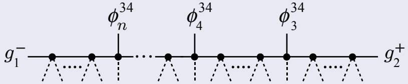
$$= \frac{-m^{n-2}\langle 1^\perp|q|2^\perp]}{\langle 2^\perp|q|1^\perp](P_{23}^2+m^2)\cdots(P_{23\ldots n\text{-}1}^2+m^2)}\,.$$

$$\left\langle\, W_1 \overline{W}_2 \,\ldots \right\rangle \quad = \quad \lim_{\varepsilon\to 0} \sum_{s=0}^{\infty} \left\langle\, g_1\, \phi^{\text{vev}}_{\varepsilon q}\, \phi^{\text{vev}}_{\varepsilon q} \,..\, \phi^{\text{vev}}_{\varepsilon q}\, g_2 \,\ldots \right\rangle_{sym}.$$

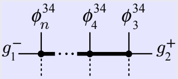
Non-trivial Checks

- An all-n amplitude, to all orders in m:

$$\langle W_1^- \overline{W}_2^+ \phi_3^{34} \ldots \phi_n^{34} \rangle \;=\; \sum$$



$$=$$



$$= \frac{-m^{n-2}\langle 1^\perp|q|2^\perp]}{\langle 2^\perp|q|1^\perp](P_{23}^2+m^2)\cdots(P_{23\ldots n\text{-}1}^2+m^2)}.$$

- Proof of finite soft limit, to all orders in m, for any amplitude!

Generalizations of proposal

- natural proposal for CB amplitudes with arbitrary masses.

  breaking $U(N) \to \prod_k U(M_k) \;\Rightarrow\; \langle\phi\rangle \sim v_k \;\Rightarrow\; m_X = v_{k_1} - v_{k_2}$

  with particle X in bifundamental of $U(M_{k_1}) \times U(M_{k_2})$.

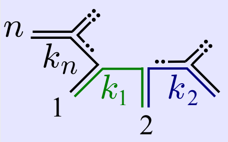
Checks and Generalizations of the proposal

Generalizations of proposal

- natural proposal for CB amplitudes with arbitrary masses.

  breaking $U(N) \to \prod_k U(M_k) \;\Rightarrow\; \langle\phi\rangle \sim v_k \;\Rightarrow\; m_X = v_{k_1} - v_{k_2}$

  with particle X in bifundamental of $U(M_{k_1}) \times U(M_{k_2})$.

  Double-line notation useful: 

Checks and Generalizations of the proposal

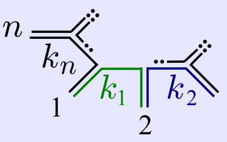
Generalizations of proposal

- natural proposal for CB amplitudes with arbitrary masses.

breaking $U(N) \to \prod_k U(M_k) \;\Rightarrow\; \langle\phi\rangle \sim v_k \;\Rightarrow\; m_X = v_{k_1} - v_{k_2}$

with particle X in bifundamental of $U(M_{k_1}) \times U(M_{k_2})$.

Double-line notation useful:



$$\langle X_1 X_2 \dots X_n \rangle$$

$$= \lim_{\varepsilon \to 0} \sum_{s=0}^{\infty} \sum_{s_1+\ldots+s_n=s} \langle\, Y_1 \underbrace{\phi^{\text{vev}}_{\varepsilon q} \,.\,.\, \phi^{\text{vev}}_{\varepsilon q}}_{s_1 \text{ times}} Y_2 \underbrace{\phi^{\text{vev}}_{\varepsilon q} \,.\,.\, \phi^{\text{vev}}_{\varepsilon q}}_{s_2 \text{ times}} Y_3 \dots Y_n \underbrace{\phi^{\text{vev}}_{\varepsilon q} \,.\,.\, \phi^{\text{vev}}_{\varepsilon q}}_{s_n \text{ times}} \rangle_{\text{sym}} .$$

Checks and Generalizations of the proposal

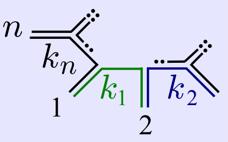
Generalizations of proposal

- natural proposal for CB amplitudes with arbitrary masses.

  breaking $U(N) \to \prod_k U(M_k) \;\Rightarrow\; \langle \phi \rangle \sim v_k \;\Rightarrow\; m_X = v_{k_1} - v_{k_2}$

  with particle X in bifundamental of $U(M_{k_1}) \times U(M_{k_2})$.

  Double-line notation useful:



$$\big\langle X_1 X_2 \ldots X_n \big\rangle$$

$$= \lim_{\varepsilon \to 0} \sum_{s=0}^{\infty} \sum_{s_1+\ldots+s_n=s} \Big\langle Y_1 \underbrace{\phi^{\text{vev}}_{\varepsilon q} \ldots \phi^{\text{vev}}_{\varepsilon q}}_{s_1 \text{ times}} Y_2 \underbrace{\phi^{\text{vev}}_{\varepsilon q} \ldots \phi^{\text{vev}}_{\varepsilon q}}_{s_2 \text{ times}} Y_3 \ldots Y_n \underbrace{\phi^{\text{vev}}_{\varepsilon q} \ldots \phi^{\text{vev}}_{\varepsilon q}}_{s_n \text{ times}} \Big\rangle_{\text{sym}} .$$

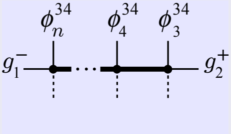
- similar proposal for loop integrand (SUSY important!)

Convenient representation for massless amplitudes

$$\langle\, W_1 \overline{W}_2 \ldots \rangle \quad = \quad \lim_{\varepsilon \to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi^{\mathrm{vev}}_{\varepsilon q}\, \phi^{\mathrm{vev}}_{\varepsilon q} \,..\, \phi^{\mathrm{vev}}_{\varepsilon q}\, g_2\, \ldots \rangle_{sym}\, .$$

- infinite sum over massless amplitudes, complicated symmetrization
- in simple examples:
  infinite sum $\Rightarrow$ single diagram with massive propagators:

$$\langle W_1^- \overline{W}_2^+ \phi_3^{34} \ldots \phi_n^{34} \rangle \;=$$

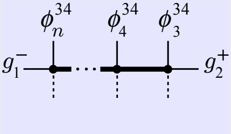


Convenient representation for massless amplitudes

$$\langle\, W_1 \overline{W}_2 \ldots \rangle \quad = \quad \lim_{\varepsilon \to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi^{\text{vev}}_{\varepsilon q}\, \phi^{\text{vev}}_{\varepsilon q} \,..\, \phi^{\text{vev}}_{\varepsilon q}\, g_2 \,\ldots \rangle_{sym} \,.$$

- infinite sum over massless amplitudes, complicated symmetrization
- in simple examples:
  infinite sum $\Rightarrow$ single diagram with massive propagators:

$$\langle W_1^- \overline{W}_2^+ \phi_3^{34} \ldots \phi_n^{34} \rangle \;=$$



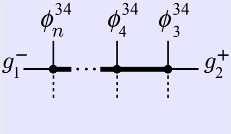
- is this always possible? what are the massive Feynman rules?

Convenient representation for massless amplitudes

$$\langle\, W_1 \overline{W}_2 \ldots \rangle \quad = \quad \lim_{\varepsilon \to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi^{\text{vev}}_{\varepsilon q}\, \phi^{\text{vev}}_{\varepsilon q} \,..\, \phi^{\text{vev}}_{\varepsilon q}\, g_2 \,\ldots \rangle_{sym} \,.$$

- infinite sum over massless amplitudes, complicated symmetrization
- in simple examples:
  infinite sum $\Rightarrow$ single diagram with massive propagators:

$$\langle W_1^- \overline{W}_2^+ \phi_3^{34} \ldots \phi_n^{34} \rangle \;=$$

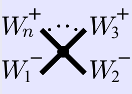


- is this always possible? what are the massive Feynman rules?
  $\Rightarrow$ on-shell derivation of Feynman rules in the broken phase?

Convenient representation for massless amplitudes

$$\langle\, W_1 \overline{W}_2 \ldots \rangle \quad = \quad \lim_{\varepsilon \to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi^{\mathrm{vev}}_{\varepsilon q}\, \phi^{\mathrm{vev}}_{\varepsilon q} \,..\, \phi^{\mathrm{vev}}_{\varepsilon q}\, g_2 \,\ldots \rangle_{sym} \,.$$

Resummation $\Rightarrow$ Massive CSW-like rules:

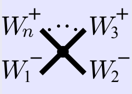
- massive scalar propagators:  $= \frac{1}{P_I^2 + m_I^2}, \quad m_I = \sum_{i \in I} m_i \,.$
- MHV vertex: 

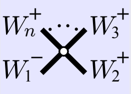
$= \frac{\langle 1^\perp 2^\perp \rangle^4}{\langle 1^\perp 2^\perp \rangle \cdots \langle n^\perp 1^\perp \rangle} \,.$

Convenient representation for massless amplitudes

$$\langle\, W_1 \overline{W}_2 \ldots \rangle \quad = \quad \lim_{\varepsilon \to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi^{\text{vev}}_{\varepsilon q}\, \phi^{\text{vev}}_{\varepsilon q} \,..\, \phi^{\text{vev}}_{\varepsilon q}\, g_2 \,\ldots \rangle_{sym}\,.$$

Resummation ⇒ Massive CSW-like rules:

- massive scalar propagators:  $= \frac{1}{P_I^2 + m_I^2}, \quad m_I = \sum_{i \in I} m_i\,.$
- MHV vertex: 

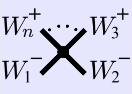
$= \frac{\langle 1^\perp 2^\perp \rangle^4}{\langle 1^\perp 2^\perp \rangle \cdots \langle n^\perp 1^\perp \rangle}\,.$
- UHV vertex: 

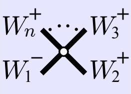
$= K^2 \frac{\langle q^\perp 1^\perp \rangle^4}{\langle 1^\perp 2^\perp \rangle \cdots \langle n^\perp 1^\perp \rangle}, \quad K = \sum_i \frac{m_i \langle 1^\perp i^\perp \rangle}{\langle 1^\perp q \rangle \langle i^\perp q \rangle}\,.$

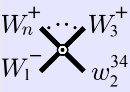
Convenient representation for massless amplitudes

$$\langle\, W_1 \overline{W}_2 \ldots \rangle \quad = \quad \lim_{\varepsilon \to 0} \sum_{s=0}^{\infty} \langle\, g_1\, \phi^{\mathrm{vev}}_{\varepsilon q}\, \phi^{\mathrm{vev}}_{\varepsilon q} \,..\, \phi^{\mathrm{vev}}_{\varepsilon q}\, g_2 \ldots \rangle_{sym} .$$

Resummation $\Rightarrow$ Massive CSW-like rules:

- massive scalar propagators:  $= \frac{1}{P_I^2 + m_I^2}, \quad m_I = \sum_{i \in I} m_i$.
- MHV vertex: 

$= \frac{\langle 1^\perp 2^\perp \rangle^4}{\langle 1^\perp 2^\perp \rangle \cdots \langle n^\perp 1^\perp \rangle}$.
- UHV vertex: 

$= K^2 \frac{\langle q^\perp 1^\perp \rangle^4}{\langle 1^\perp 2^\perp \rangle \cdots \langle n^\perp 1^\perp \rangle}, \quad K = \sum_i \frac{m_i \langle 1^\perp i^\perp \rangle}{\langle 1^\perp q \rangle \langle i^\perp q \rangle}$.
- UHV$\times$MHV vertex: 

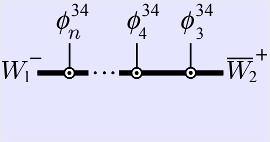
$= K \frac{\langle q^\perp 1^\perp \rangle^2 \langle 1^\perp 2^\perp \rangle^2}{\langle 1^\perp 2^\perp \rangle \cdots \langle n^\perp 1^\perp \rangle}$.

All-n amplitudes from the CSW-like expansion

A simple example: $\langle W_1^- \overline{W}_2^+ \phi_3^{34} \dots \phi_n^{34} \rangle$

All-n amplitudes from the CSW-like expansion

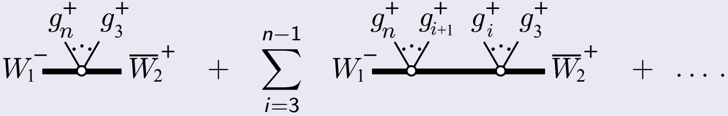
A simple example: $\langle W_1^- \overline{W}_2^+ \phi_3^{34} \ldots \phi_n^{34} \rangle$



$$= \frac{-m^{n-2}\langle 1^\perp|q|2^\perp]}{\langle 2^\perp|q|1^\perp](P_{23}^2+m^2)\cdots(P_{23\ldots n\text{-}1}^2+m^2)}\,.$$

All-n amplitudes from the CSW-like expansion

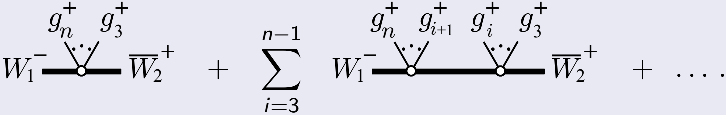
A non-trivial example: $\langle W^- \overline{W}^+ g^+ \dots g^+ \rangle$



- discouraging at first sight: many diagrams!

All-n amplitudes from the CSW-like expansion

A non-trivial example: $\langle W^- \overline{W}^+ g^+ \dots g^+ \rangle$

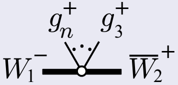


- discouraging at first sight: many diagrams!
- BUT: sum of diagrams can be factorized into one simple term:

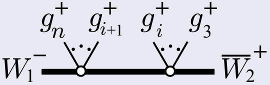
$$-\frac{m^2\langle q1^\perp\rangle^2}{\langle q2^\perp\rangle^2\langle 2^\perp 3\rangle\langle 34\rangle\cdots\langle n1^\perp\rangle} \times \langle 2^\perp| \prod_{j=3}^{n-1}\left[1-\frac{m^2|P_J\rangle\langle j,j+1\rangle\langle P_J|}{(P_J^2+m^2)\langle P_J,j\rangle\langle j+1,P_J\rangle}\right]|1^\perp\rangle\,.$$

All-n amplitudes from the CSW-like expansion

A non-trivial example: $\langle W^- \overline{W}^+ g^+ \dots g^+ \rangle$



$$+ \sum_{i=3}^{n-1}$$



$$+ \quad \dots\,.$$

- discouraging at first sight: many diagrams!
- BUT: sum of diagrams can be factorized into one simple term:

$$-\frac{m^2\langle q1^\perp\rangle^2}{\langle q2^\perp\rangle^2\langle 2^\perp 3\rangle\langle 34\rangle\cdots\langle n1^\perp\rangle} \times \langle 2^\perp| \prod_{j=3}^{n-1}\left[1 - \frac{m^2|P_J\rangle\langle j,j+1\rangle\langle P_J|}{(P_J^2+m^2)\langle P_J,j\rangle\langle j+1,P_J\rangle}\right]|1^\perp\rangle\,.$$

- should be compared to BCFW form of same amplitude:

$$-\frac{m^2\langle q\,1^\perp\rangle^2}{\langle q\,2^\perp\rangle^2\,\langle 34\rangle\langle 45\rangle\cdots\langle n-1,n\rangle(P_{n1}^2+m^2)} \times [3|\prod_{j=3}^{n-2}\left[1 + \frac{P_J|j+1\rangle[j+1|}{P_J^2+m^2}\right]|n]\,.$$

- similar complexity, but no recursion to solve!

Summary and Outlook

Summary

- precise proposal for

$$\boxed{\text{massive on-shell amplitude} \quad = \sum \text{massless on-shell amplitudes}}\,.$$

Summary and Outlook

Summary

- precise proposal for

$$\boxed{\text{massive on-shell amplitude} \quad = \sum \text{massless on-shell amplitudes}}\,.$$

- on-shell description of spontaneous symmetry breaking

Summary and Outlook

Summary

- precise proposal for

$$\boxed{\text{massive on-shell amplitude} \quad = \sum \text{massless on-shell amplitudes}} \, .$$

- on-shell description of spontaneous symmetry breaking
- checked in examples
- cancelation of soft-divergences proven in general

Summary and Outlook

Summary

- precise proposal for

$$\boxed{\text{massive on-shell amplitude} \quad = \sum \text{massless on-shell amplitudes}}\,.$$

- on-shell description of spontaneous symmetry breaking
- checked in examples
- cancelation of soft-divergences proven in general
- at tree-level: specific symmetry-breaking pattern and SUSY irrelevant

Summary and Outlook

Summary

- precise proposal for

$$\boxed{\text{massive on-shell amplitude} \;=\; \sum \text{massless on-shell amplitudes}}\,.$$

- on-shell description of spontaneous symmetry breaking
- checked in examples
- cancelation of soft-divergences proven in general
- at tree-level: specific symmetry-breaking pattern and SUSY irrelevant
- with SUSY $\Rightarrow$ proposal for loop integrand

Summary and Outlook

Summary

- precise proposal for

$$\boxed{\text{massive on-shell amplitude} \quad = \quad \sum \text{massless on-shell amplitudes}}\,.$$

- on-shell description of spontaneous symmetry breaking
- checked in examples
- cancelation of soft-divergences proven in general
- at tree-level: specific symmetry-breaking pattern and SUSY irrelevant
- with SUSY $\Rightarrow$ proposal for loop integrand
- infinte sum over diagrams can be resummed
  $\Rightarrow$ CSW-like expansion for Coulomb-branch amplitudes

Summary and Outlook

Open problems

- prove proposal!

Summary and Outlook

Open problems

- prove proposal!
- applied proposal to particular massless amplitude representation:

$$\text{massless CSW} \longrightarrow \boxed{\textit{proposal}} \xrightarrow{\text{resum}} \text{massive CSW-like expansion}\,.$$

Use proposal to get other convenient expressions for CB amplitudes?

$$\text{other massless representations} \longrightarrow \boxed{\textit{proposal}} \xrightarrow{\text{resum?}} ???\,.$$

For example, to make $6d$ dual conformal invariance manifest?

Summary and Outlook

Open problems

- prove proposal!
- applied proposal to particular massless amplitude representation:

$$\text{massless CSW} \longrightarrow \boxed{\textit{proposal}} \xrightarrow{\text{resum}} \text{massive CSW-like expansion}\,.$$

  Use proposal to get other convenient expressions for CB amplitudes?

$$\text{other massless representations} \longrightarrow \boxed{\textit{proposal}} \xrightarrow{\text{resum?}} ???\,.$$

  For example, to make $6d$ dual conformal invariance manifest?

- apply CSW-like expansion at tree and loop level

Summary and Outlook

Open problems

- prove proposal!
- applied proposal to particular massless amplitude representation:

$$\text{massless CSW} \quad \longrightarrow \quad \boxed{\textit{proposal}} \quad \xrightarrow{\text{resum}} \quad \text{massive CSW-like expansion}\,.$$

  Use proposal to get other convenient expressions for CB amplitudes?

$$\text{other massless representations} \quad \longrightarrow \quad \boxed{\textit{proposal}} \quad \xrightarrow{\text{resum?}} \quad ???\,.$$

  For example, to make $6d$ dual conformal invariance manifest?
- apply CSW-like expansion at tree and loop level
- massive amplitudes useful for rational terms in QCD [Badger, Boels]