Gluons for (almost) nothing, and gravitons for free

(a constrained poem in a graphy S-matrix)

NBIA Summer Institute 2011,
Strings, Gauge Theory, and the LHC
30 August

J. J. M. Carrasco
Stanford Institute for Theoretical Physics

Based on work with:
Zvi Bern, Johannes Broedel, Lance Dixon,
Henrik Johansson, and Radu Roiban

What is the right way to write down gauge and gravity scattering amplitudes?

insightful?

compact?

doable?

Consider a Vilanelle

Do Not Go Gentle Into That Good Night

Do not go gentle into that good night,
Old age should burn and rave at close of day;
Rage, rage against the dying of the light.

Though wise men at their end know dark is right,
Because their words had forked no lightning they
Do not go gentle into that good night.

Good men, the last wave by, crying how bright
Their frail deeds might have danced in a green bay,
Rage, rage against the dying of the light.

Wild men who caught and sang the sun in flight,
And learn, too late, they grieved it on its way,
Do not go gentle into that good night.

Grave men, near death, who see with blinding sight
Blind eyes could blaze like meteors and be gay,
Rage, rage against the dying of the light.

And you, my father, there on that sad height,
Curse, bless, me now with your fierce tears, I pray.
Do not go gentle into that good night.
Rage, rage against the dying of the light.

-Dylan Thomas

Do Not Go Gentle Into That Good Night

Do not go gentle into that good night,
Old age should burn and rave at close of day;
Rage, rage against the dying of the light.

Though wise men at their end know dark is right,
Because their words had forked no lightning they
Do not go gentle into that good night.

Good men, the last wave by, crying how bright
Their frail deeds might have danced in a green bay,
Rage, rage against the dying of the light.

Wild men who caught and sang the sun in flight,
And learn, too late, they grieved it on its way,
Do not go gentle into that good night.

Grave men, near death, who see with blinding sight
Blind eyes could blaze like meteors and be gay,
Rage, rage against the dying of the light.

And you, my father, there on that sad height,
Curse, bless, me now with your fierce tears, I pray.
Do not go gentle into that good night.
Rage, rage against the dying of the light.

-Dylan Thomas

Do Not Go Gentle Into That Good Night

Do not go gentle into that good night,
Old age should burn and rave at close of day;
Rage, rage against the dying of the light.

Though wise men at their end know dark is right,
Because their words had forked no lightning they
Do not go gentle into that good night.

Good men, the last wave by, crying how bright
Their frail deeds might have danced in a green bay,
Rage, rage against the dying of the light.

Wild men who caught and sang the sun in flight,
And learn, too late, they grieved it on its way,
Do not go gentle into that good night.

Grave men, near death, who see with blinding sight
Blind eyes could blaze like meteors and be gay,
Rage, rage against the dying of the light.

And you, my father, there on that sad height,
Curse, bless, me now with your fierce tears, I pray.
Do not go gentle into that good night.
Rage, rage against the dying of the light.

-Dylan Thomas

Do Not Go Gentle Into That Good Night

Do not go gentle into that good night,
Old age should burn and rave at close of day;
Rage, rage against the dying of the light.

Though wise men at their end know dark is right,
Because their words had forked no lightning they
Do not go gentle into that good night.

Good men, the last wave by, crying how bright
Their frail deeds might have danced in a green bay,
Rage, rage against the dying of the light.

Wild men who caught and sang the sun in flight,
And learn, too late, they grieved it on its way,
Do not go gentle into that good night.

Grave men, near death, who see with blinding sight
Blind eyes could blaze like meteors and be gay,
Rage, rage against the dying of the light.

And you, my father, there on that sad height,
Curse, bless, me now with your fierce tears, I pray.
Do not go gentle into that good night.
Rage, rage against the dying of the light.

-Dylan Thomas

Do Not Go Gentle Into That Good Night

Do not go gentle into that good night,
Old age should burn and rave at close of day;
Rage, rage against the dying of the light.

Though wise men at their end know dark is
right,
Because their words had forked no lightning
they
Do not go gentle into that good night.

Good men, the last wave by, crying how bright
Their frail deeds might have danced in a green
bay,
Rage, rage against the dying of the light.

Wild men who caught and sang the sun in
flight,
And learn, too late, they grieved it on its way,
Do not go gentle into that good night.

Grave men, near death, who see with blinding
sight
Blind eyes could blaze like meteors and be gay,
Rage, rage against the dying of the light.

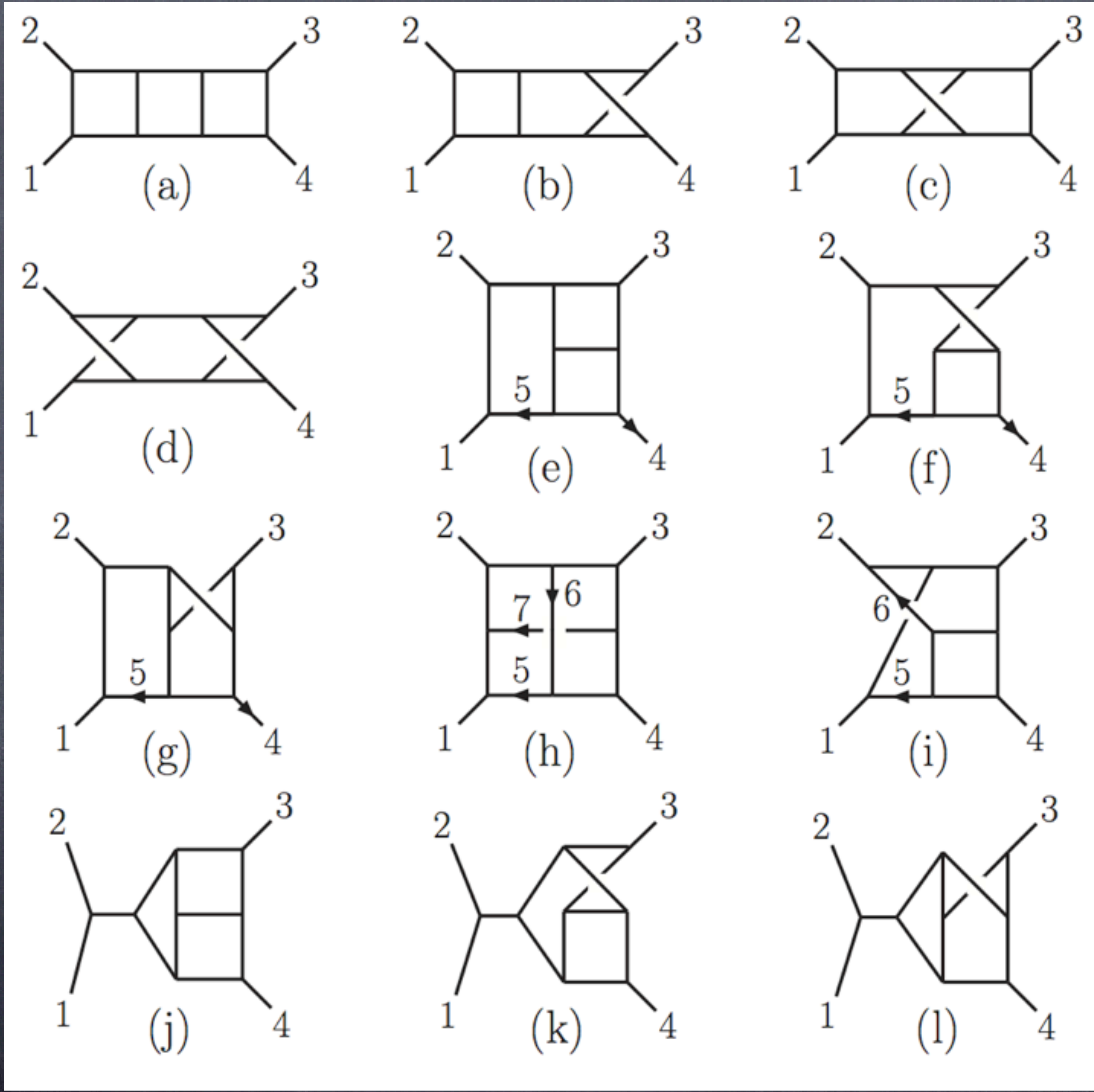
And you, my father, there on that sad height,
Curse, bless, me now with your fierce tears, I
pray.
Do not go gentle into that good night.
Rage, rage against the dying of the light.

-Dylan Thomas

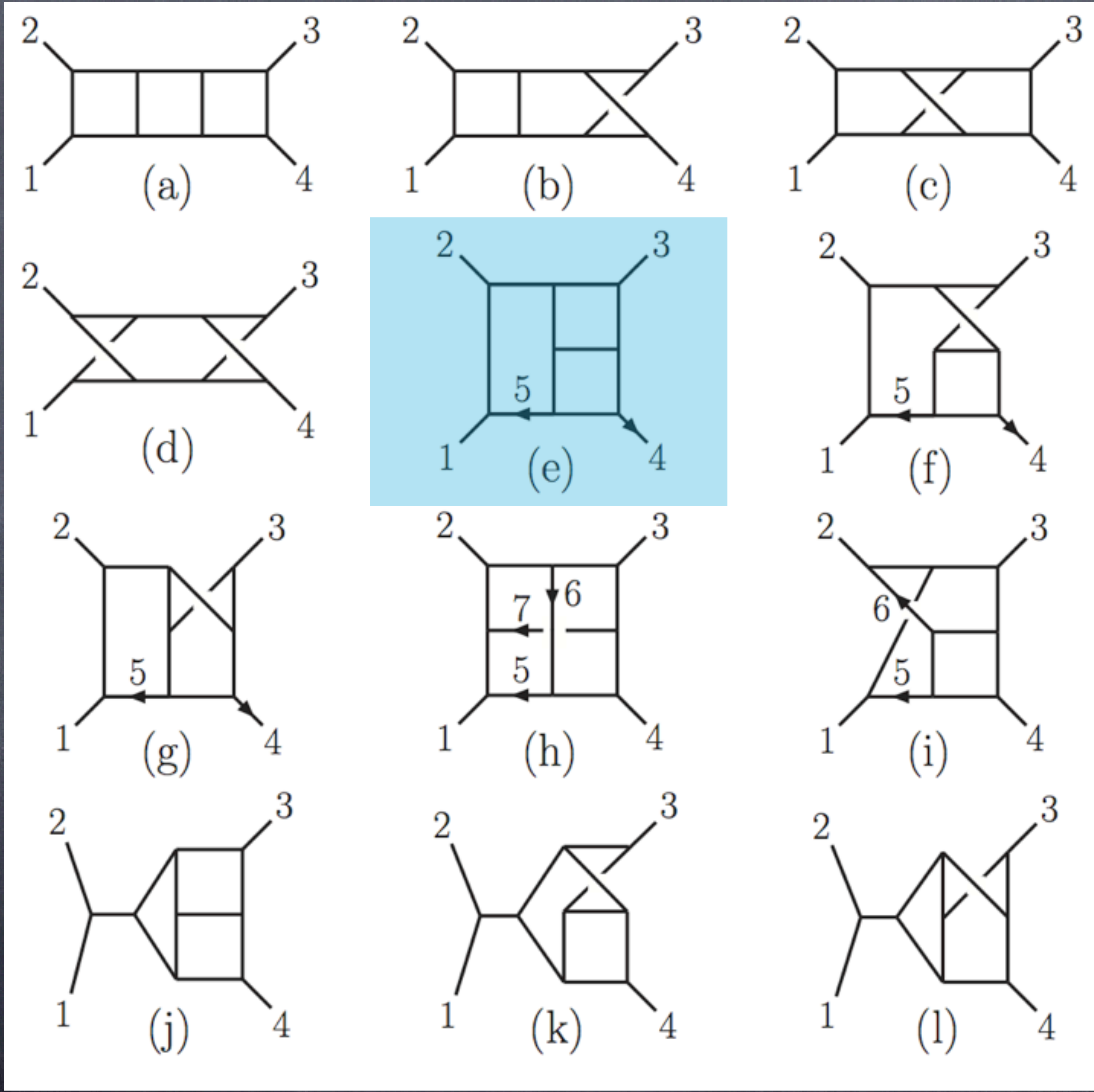
What's going on?

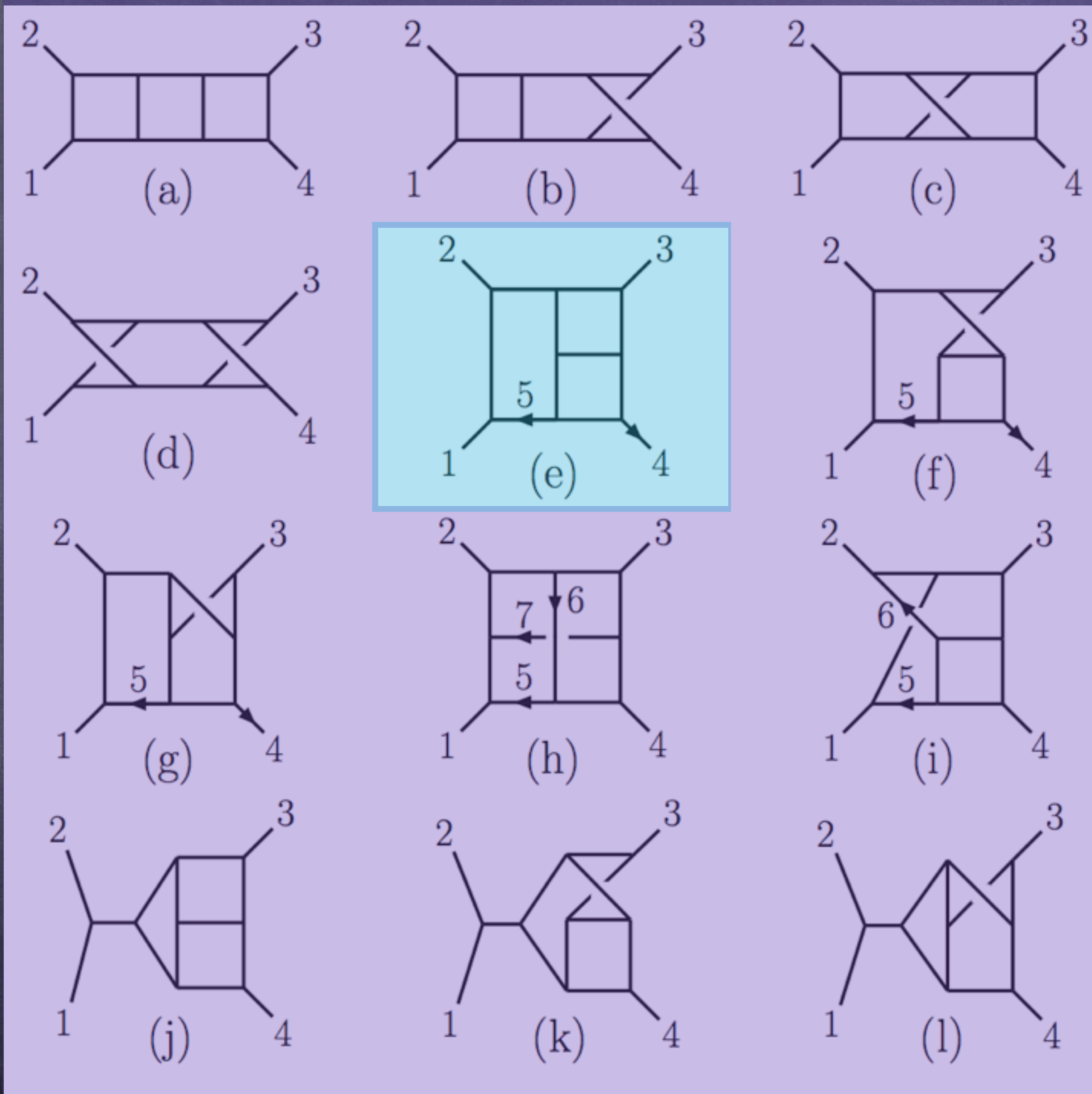
- Minimal information in.
- Relations propagate this information to a full solution.

Consider an Amplitude



(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
(i)
(j)
(k)
(l)


(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
(i)
(j)
(k)
(l)


(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
(i)
(j)
(k)
(l)

So what are these relations for YM?

a duality between color and kinematic numerator factors for gauge theories

$$\frac{(-i)^L}{g^{n-2+2L}}\mathcal{A}^{\text{loop}} = \sum_{\mathcal{G}\in\text{cubic}} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S(\mathcal{G})} \frac{n(\mathcal{G})c(\mathcal{G})}{D(\mathcal{G})}$$

(n=numerator, c=color, S=symmetry, D=denominator)

completely changing our way of calculating

write down gauge theory amplitudes with minimal input from theory

trivially write down related gravity amplitudes

Map of talk

- Tree insights from loop level results

  (sometimes it's easier to discover things at loops!)

- Generalizing duality to loop level

- Current Knowledge/Future outlook

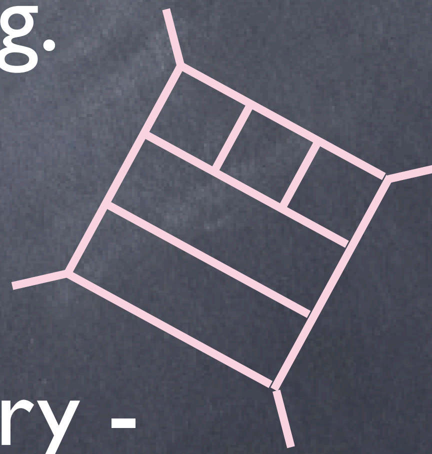
Graphy Thinking!

Take seriously the idea of momentum-flow graphs as a very natural way to organize amplitudes

$$\text{Amplitude} \sim \sum_i f(\text{graph}_i)$$

Conventional wisdom: these sorts of diagrams are a handy trick for calculating.

"Recent" wisdom: these sorts of diagrams are a (occasionally) handy ***old-fashioned*** trick for calculating.
but local representations are having a come-back!



The point: this is more than a trick...

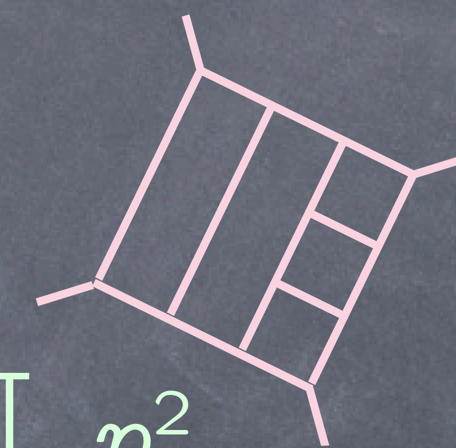
Conservation of momenta is a very **physical** symmetry - representations making this manifest are natural places to hunt for physical **kinematic** structure.

The ability to simultaneously encode **color** information is very special for gauge theory amplitudes.

Cubic Organization:

Theory dependent

$$\text{Amplitude} \sim \sum_{i \in \text{cubic}} \frac{h(\text{graph}_i)}{D(\text{graph}_i)}$$



$$D(\text{graph}_i) = \prod_{p \in \text{internal edges}} p^2$$

Gauge theory:

$$h(\text{graph}_i) \propto n(\text{graph}_i)c(\text{graph}_i)\cdots$$

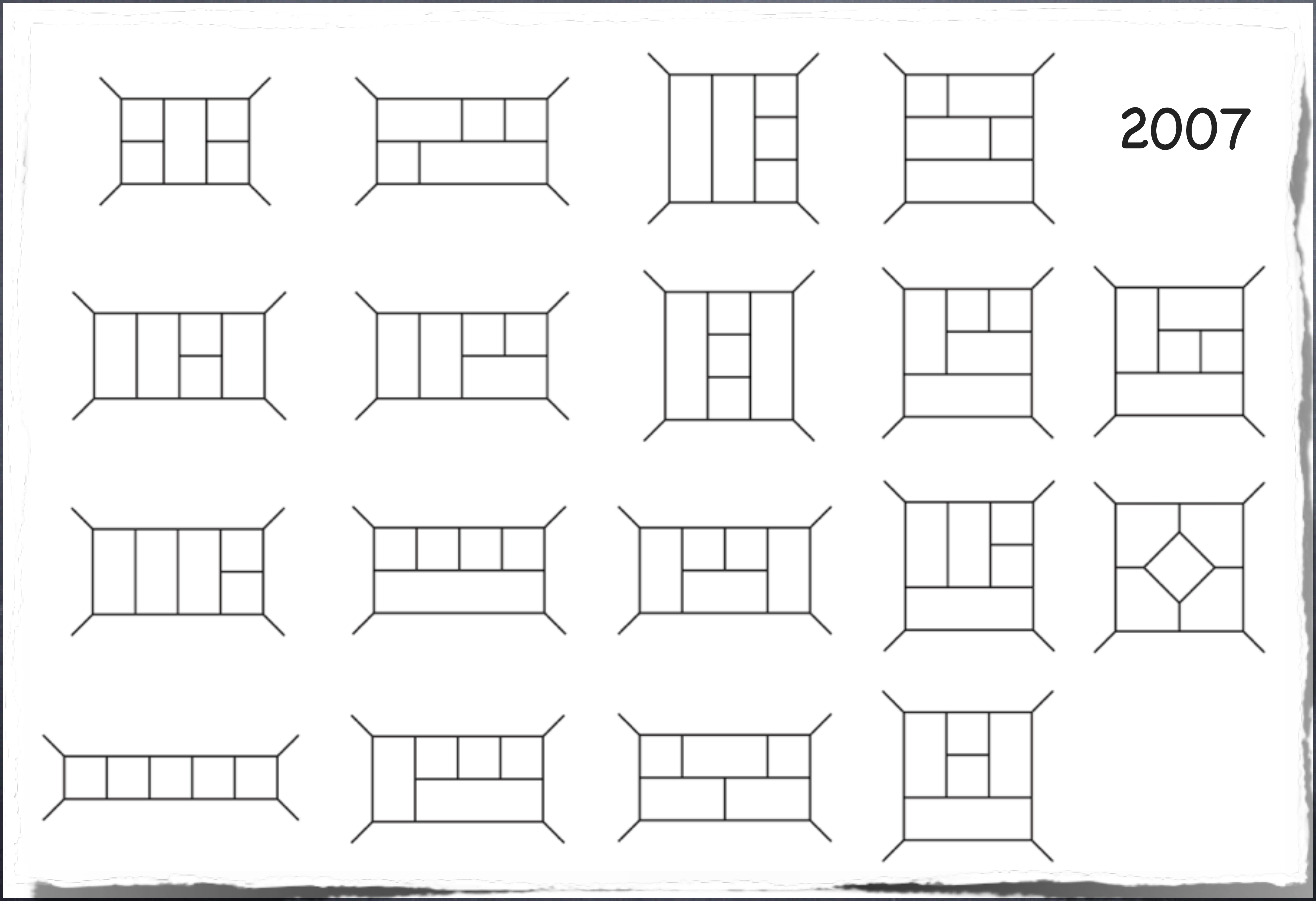
n(.) kinematic numerator "dressing"(antisymmetric)

c(.) group theoretic color factor:

Dress vertices of diagram (i) with
the structure constants $f^{abc} = \text{Tr}([T^a, T^b]T^c)$

Bern, JJMC, Johansson, Kosower

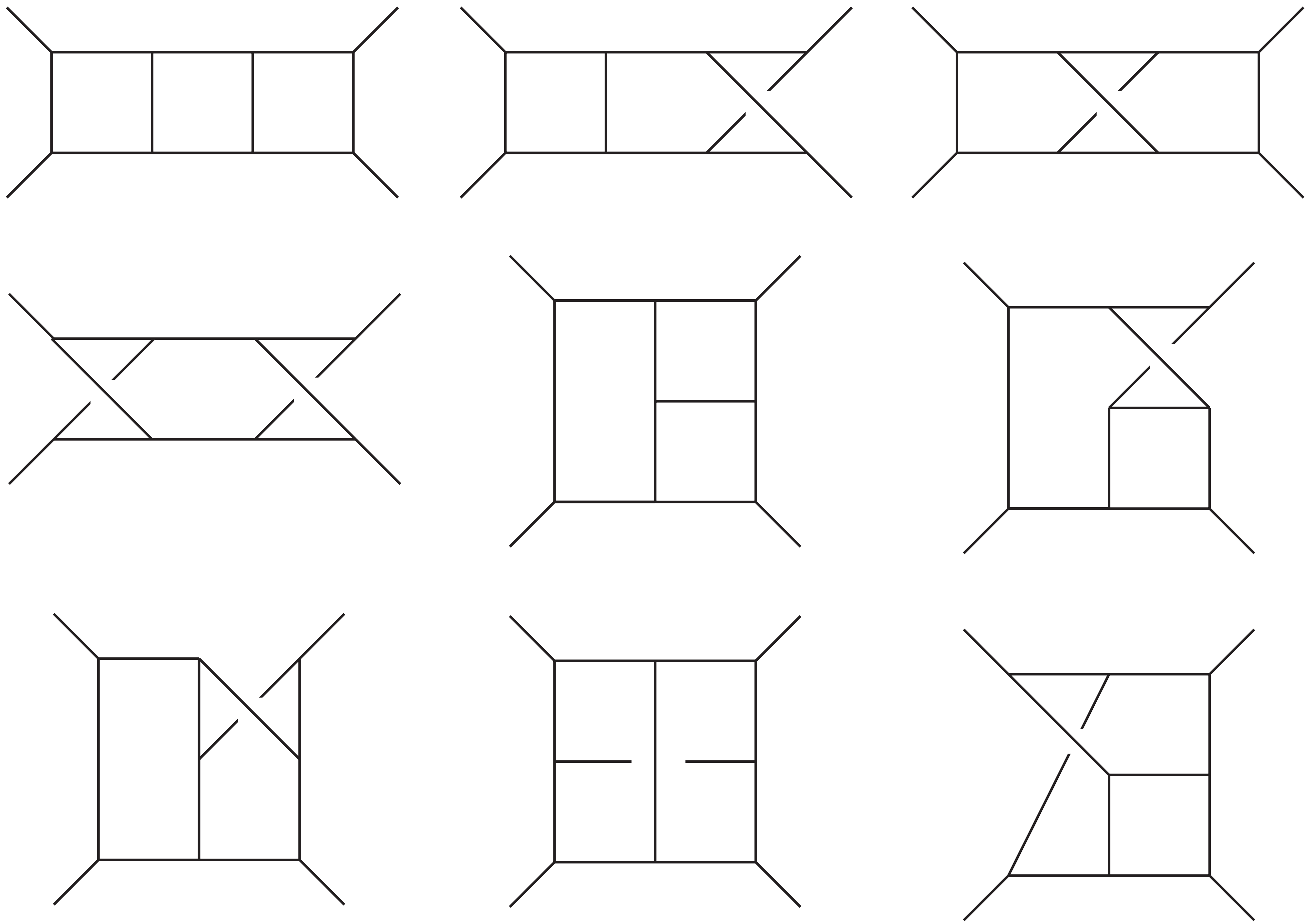
5 loop, 4pt, planar N=4 sYM



3 loop, 4-pt full N=4 sYM
3 loop, 4-pt full N=8 SUGRA

Bern, JJMC, Dixon, Johansson, Kosower, Roiban

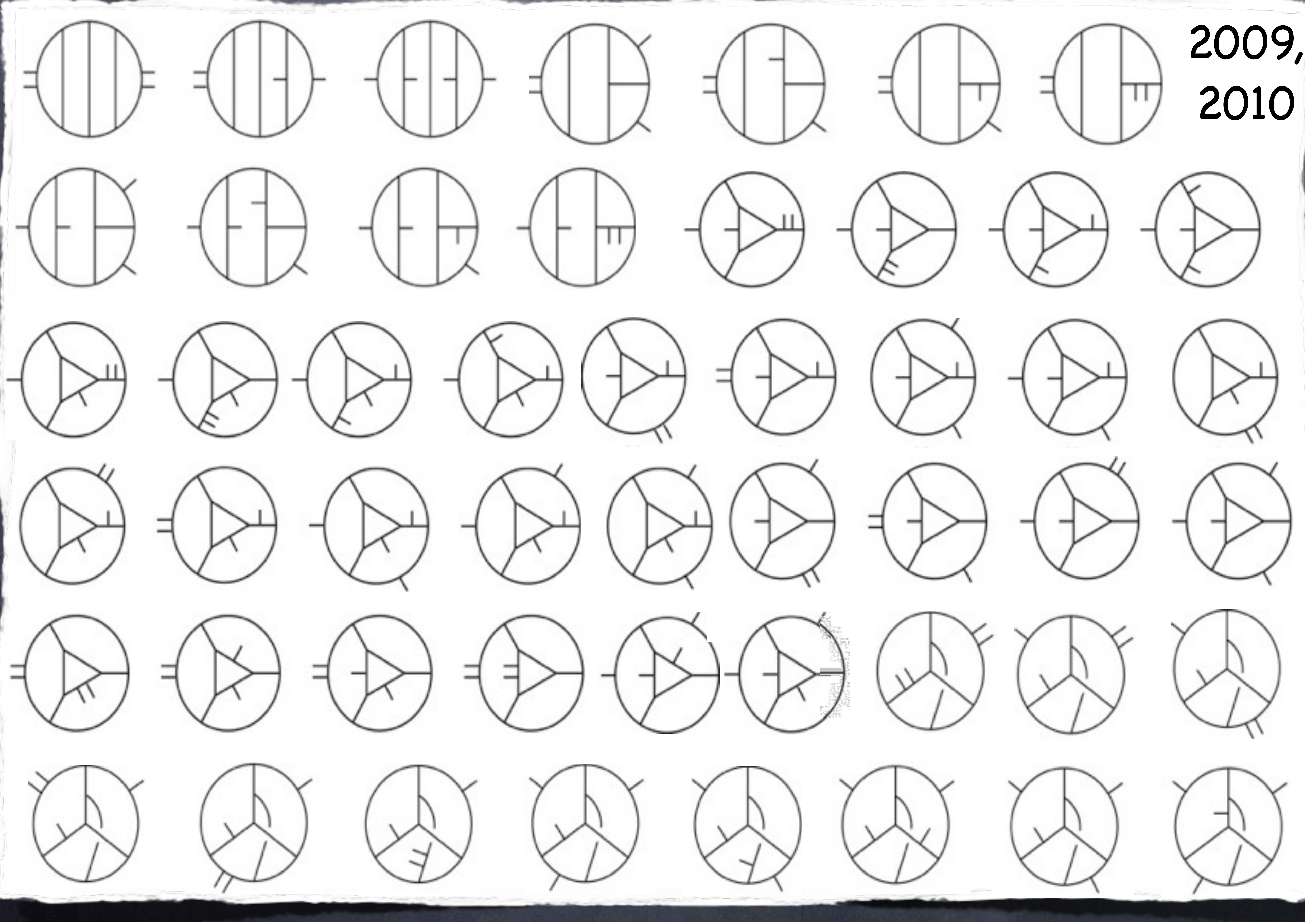
2007, 2008, 2010



Bern, JJMC, Dixon, Johansson, Roiban

4 loop, 4pt full N=4 sYM and N=8 SUGRA

2009, 2010



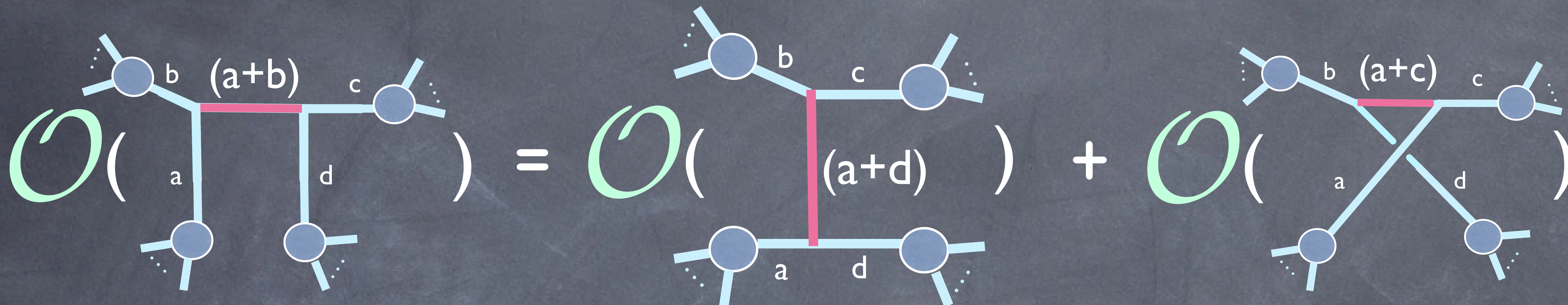
Why need anything more?

- Go beyond four-loops (five-loop N=8 SUGRA critical test for question of finiteness)
- Go beyond four-point -- there are entire theories to understand, and more to a theory than its UV behavior
- Scattering is very physical way at getting at the information in a QFT -- discovering structures in scattering (even perturbative) ==> discoveries about the language of the theory

Surprise at tree-level!

BCJ (2008)

Can always find a representation, so for every int. edge:



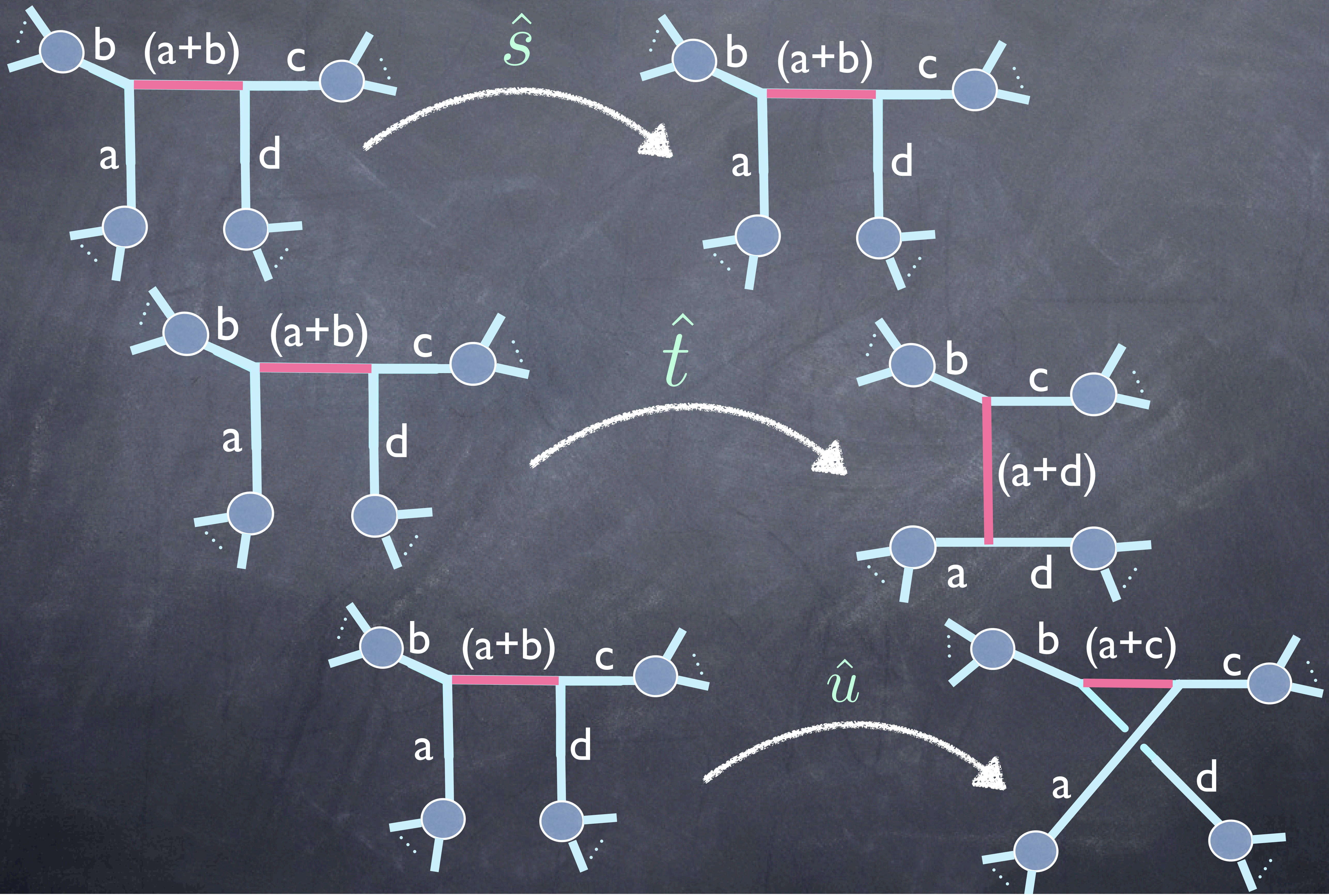
(Graph statement of Jacobi Relation)

$$\mathcal{O}(.) = c(.) \longleftrightarrow \mathcal{O}(.) = n(.)$$

$$\mathcal{A}_m^{\text{tree}} = g^{(m-2)} \sum_{\mathcal{G} \in \text{cubic}} \left(\frac{c(\mathcal{G})\, n(\mathcal{G})}{D(\mathcal{G})} \right)$$

(originally verified thru 8pt, now we know it's true)

Introduce 3 graph operators taking graph & edge -> graph



Look at N=4 SYM, 2-loops

(surpressing prefactor)

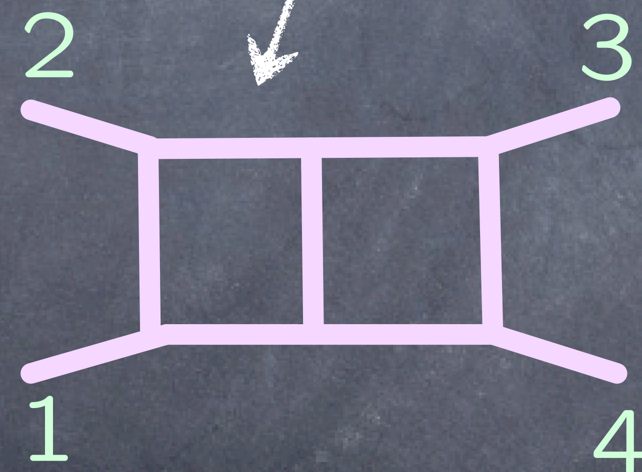
Bern, Dixon, Dunbar, Perelstein, Rozowsky

$$\mathcal{A}_4^{(2)} \propto \sum_{\text{ext. leg perms.}} \left[C^{(a)} I^{(a)} + C^{(b)} I^{(b)} \right]$$

Scalar integrals with diagrams representing denominators, encoding conservation of momenta

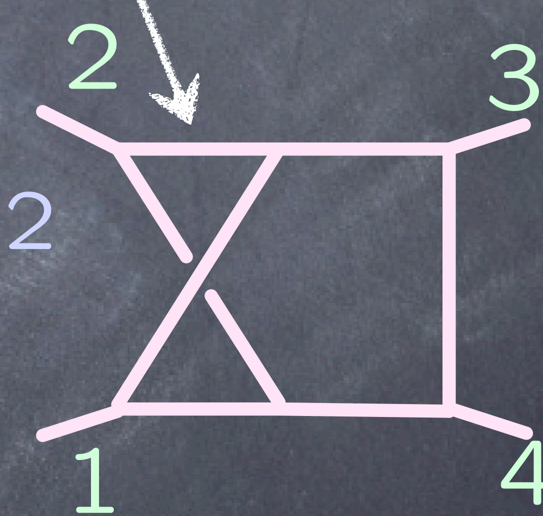
(a)

$(k_1 + k_2)^2$



(b)

$(k_1 + k_2)^2$

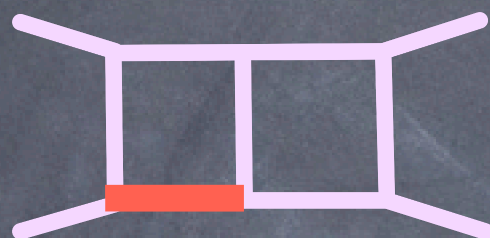
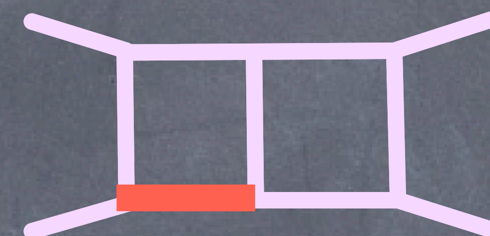


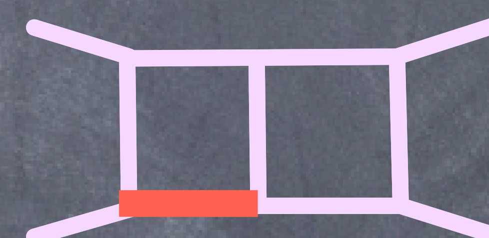
Numerator "dressings" of integrals (n_i)

color: $C^{(i)} \equiv$ Dress vertices with the structure constants f^{abc}

Hint of a new duality:

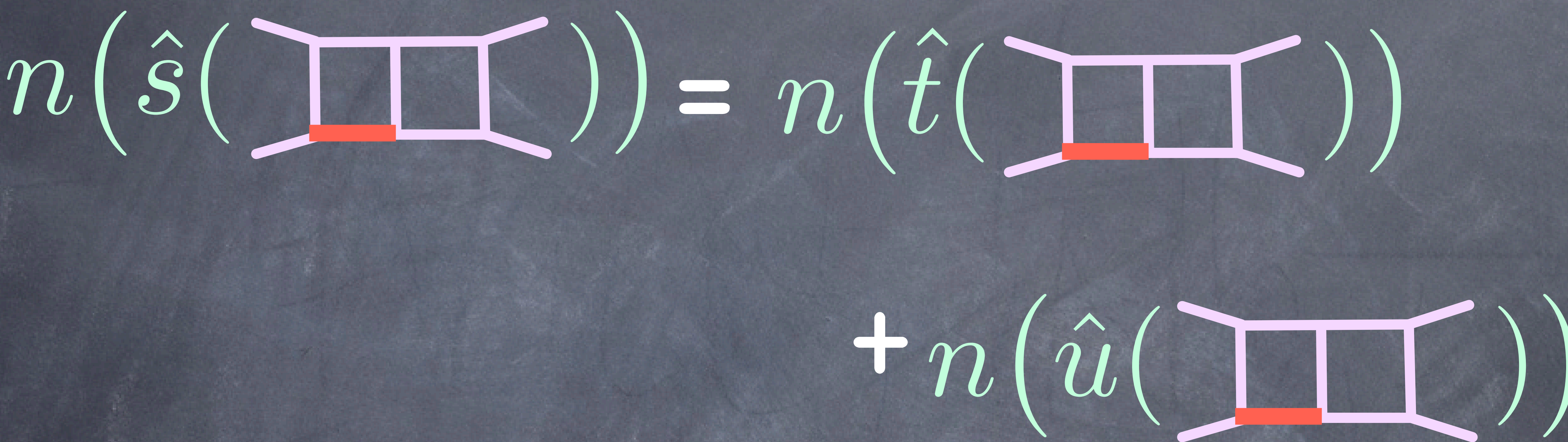
The numerator dressings n(graph) obey the graphical Jacobi relation:

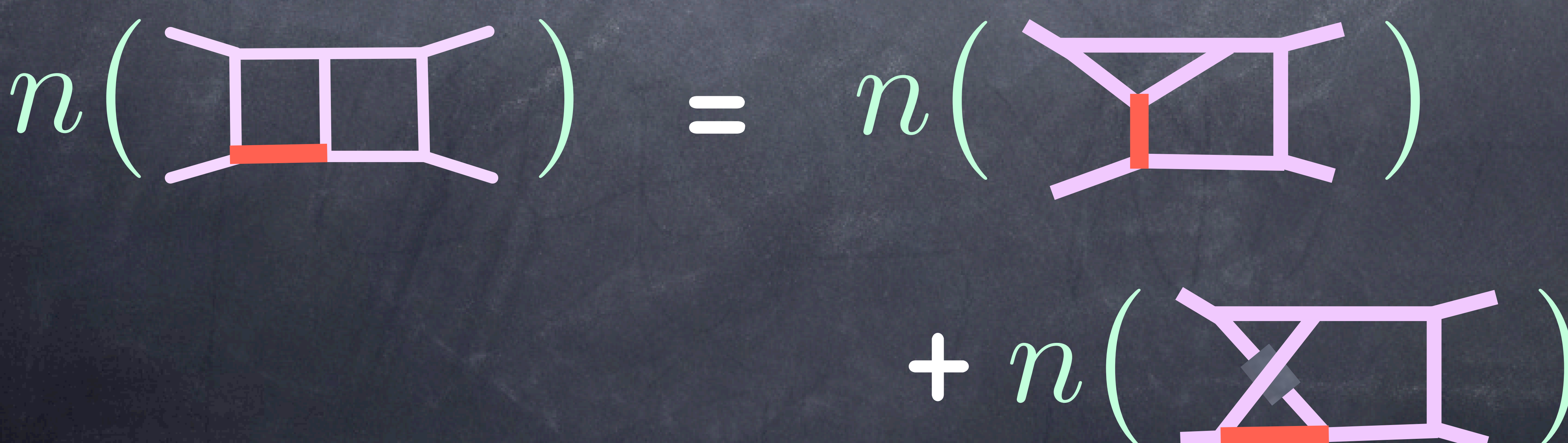
$n\Big(\hat{s}\big($  $\big)\Big)$ = $n\Big(\hat{t}\big($  $\big)\Big)$

\+ $n\Big(\hat{u}\big($  $\big)\Big)$

Hint of a new duality:

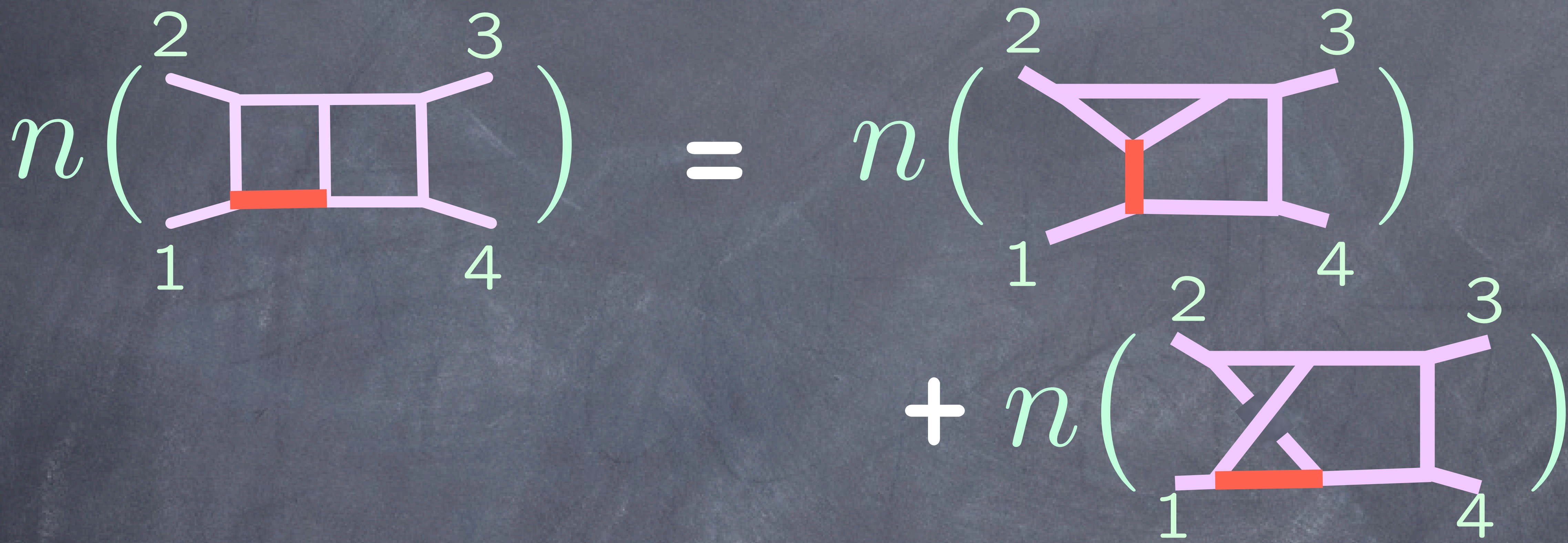
The numerator dressings n(graph) obey the graphical Jacobi relation:





Hint of a new duality:

The numerator dressings n(graph) obey the graphical Jacobi relation:



$$(k_1 + k_2)^2 = 0 + (k_1 + k_2)^2$$

N=4 SYM, 3-loops

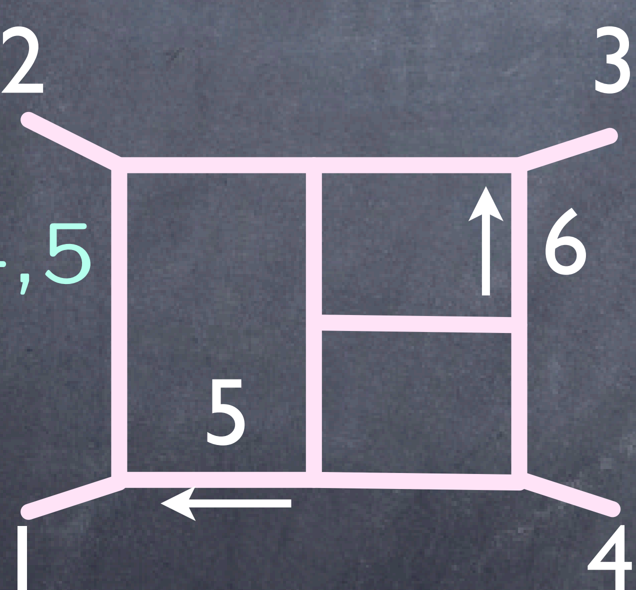
Bern, JJMC, Dixon, Johansson, Kosower, Roiban

$$\mathcal{A}_4^{(3)} \propto \sum_{\text{ext. leg perms}} 9 \text{ integrals}$$

Numerator "dressings" of integrals n(graphs)

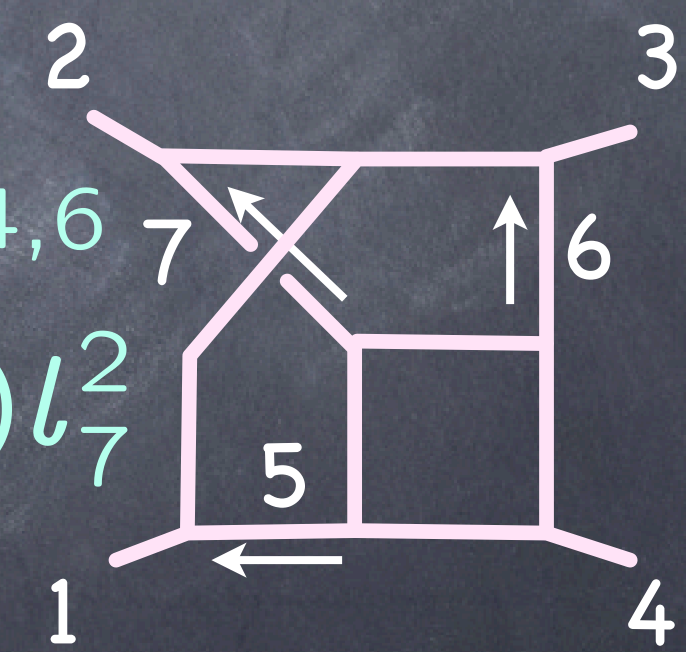
(e).

$s_{1,2}s_{4,5}$



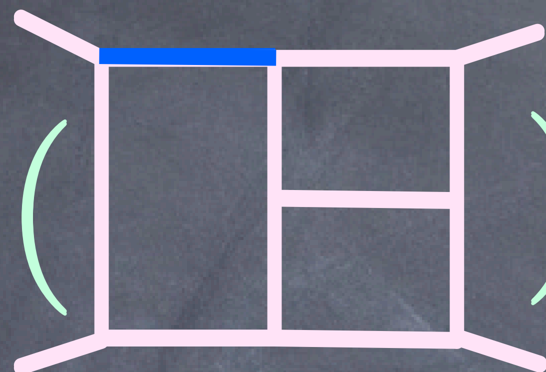
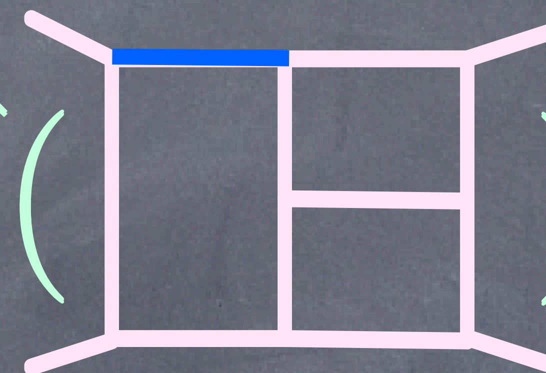
(i).

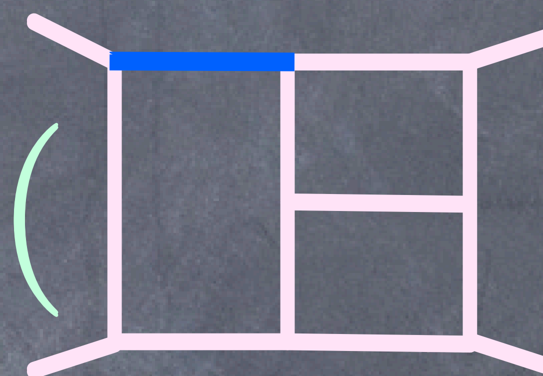
$$s_{1,2}s_{4,5} - s_{1,2}s_{4,6} - \frac{1}{3}(s_{1,2} - s_{1,4})l_7^2$$



$$s_{a,b} = (k_a + k_b)^2$$

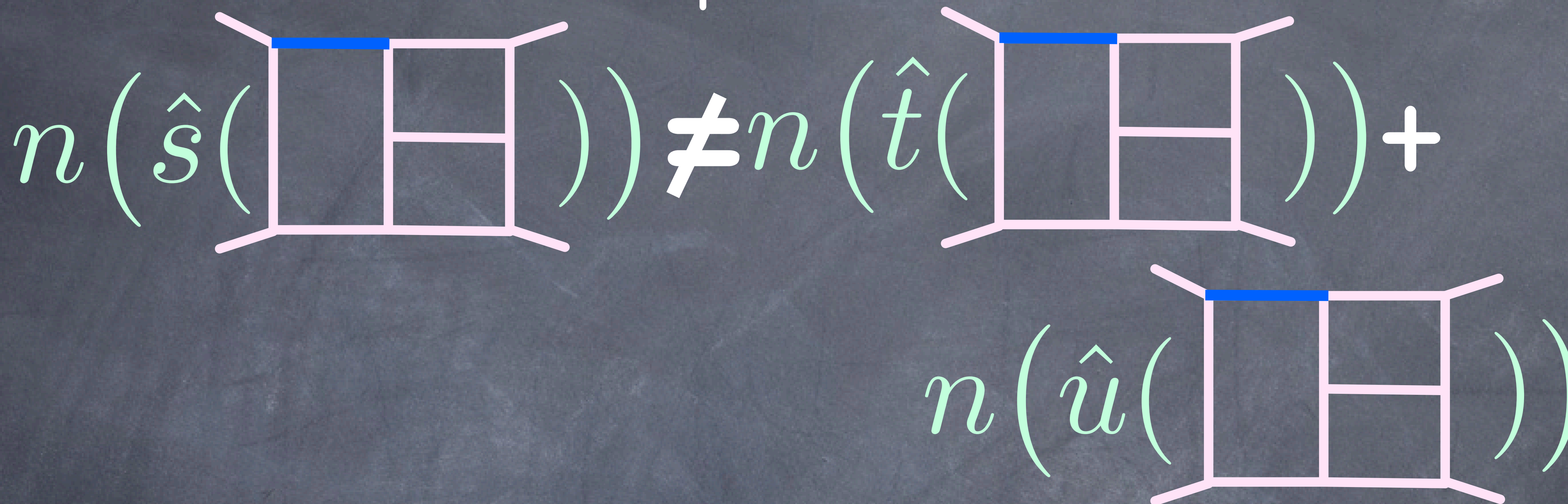
Off-shell, doesn't (automatically) work at 3-loops!

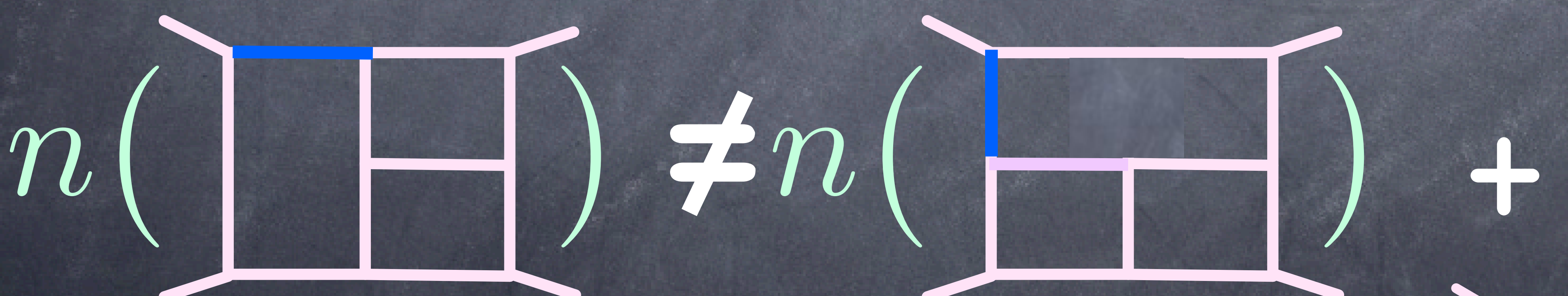
$n(\hat{s}($  $)) \neq n(\hat{t}($  $)) +$

$n(\hat{u}($  $))$

n(graph) = numerator kinematic dressing

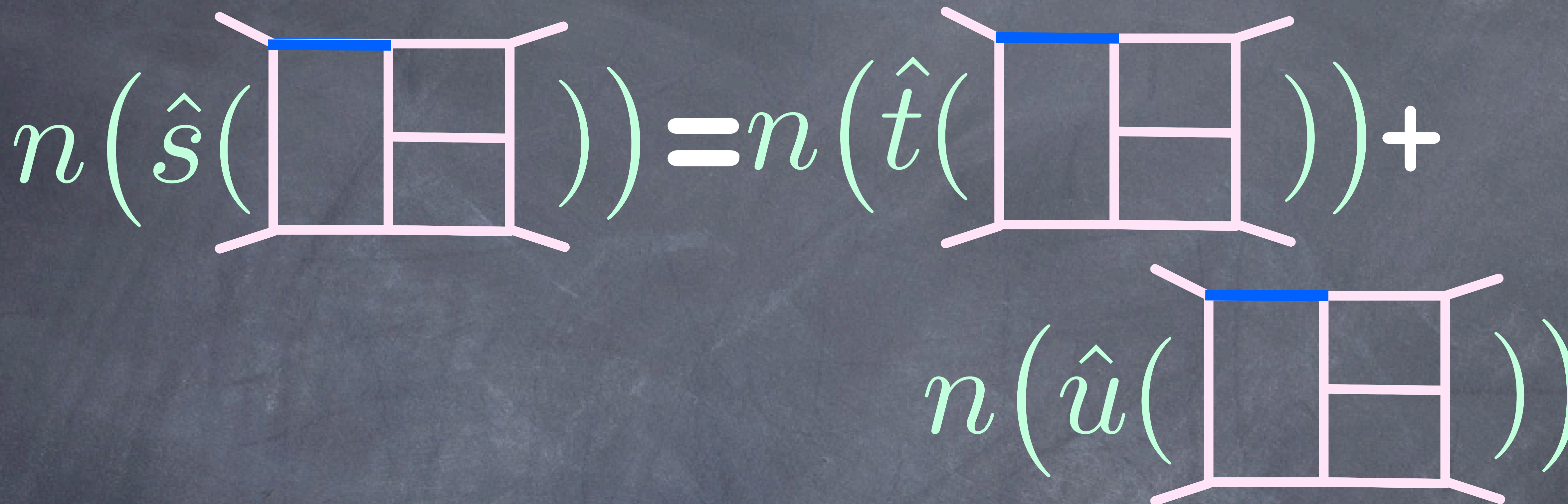
Off-shell, doesn't (automatically) work at 3-loops!

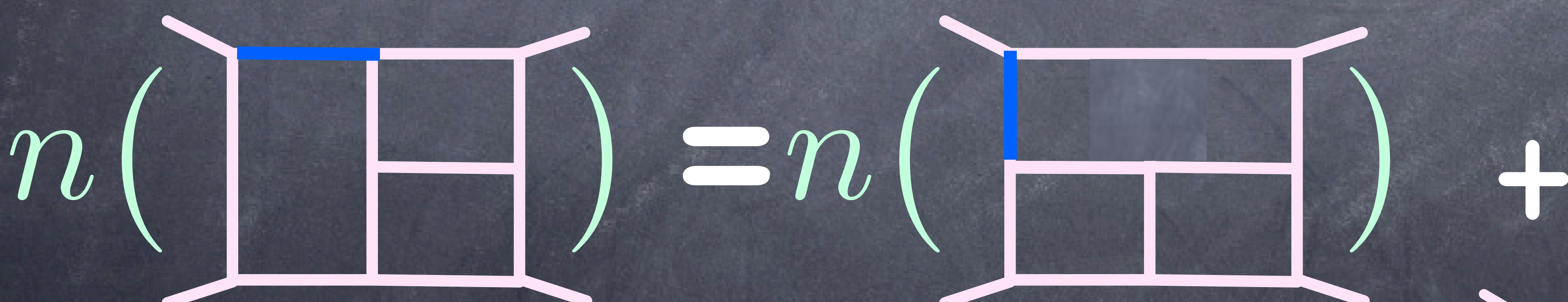




n(graph) = numerator kinematic dressing

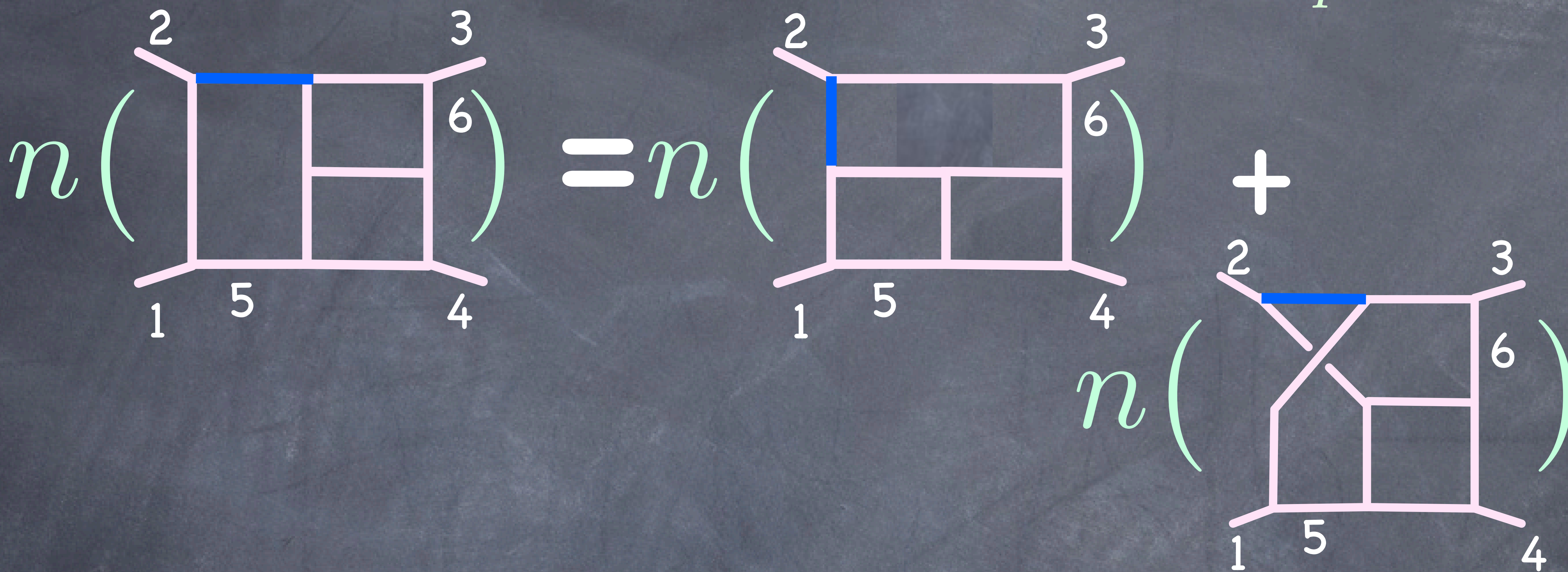
With all but **indicated** momenta on shell: $p^2 = 0$





n(graph) = numerator kinematic dressing

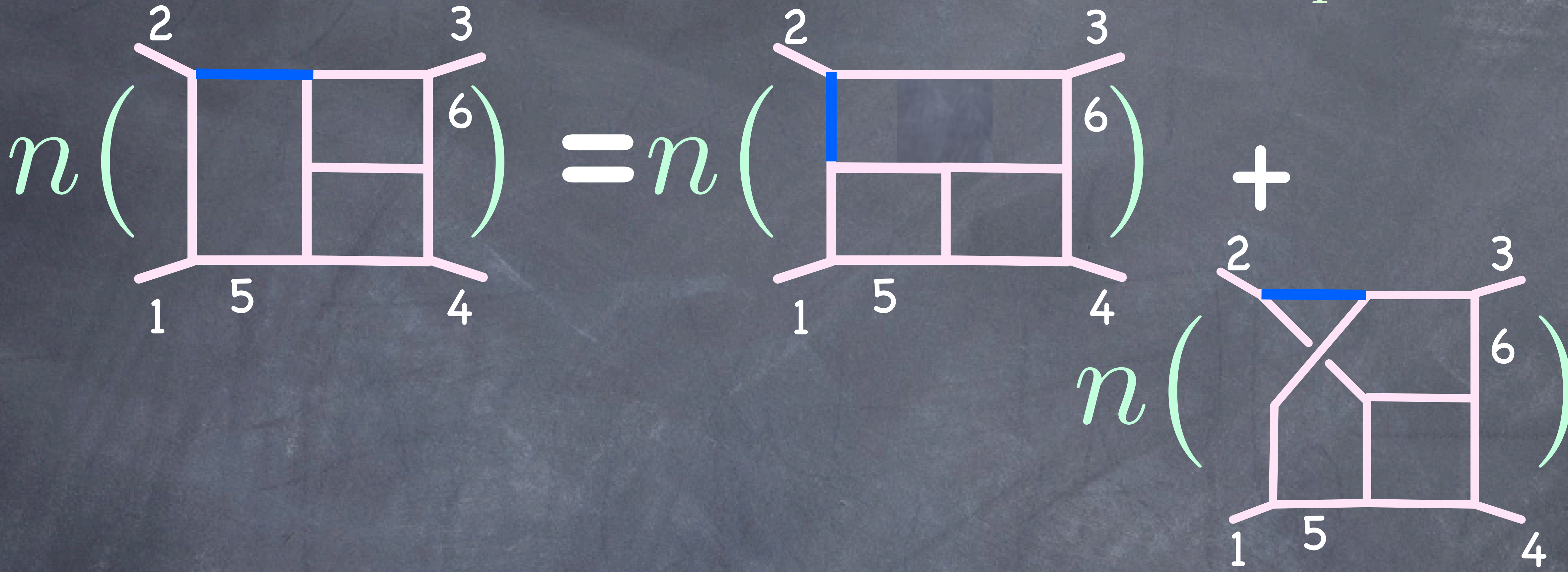
With all but **indicated** momenta on shell: $p^2 = 0$



n(graph) = numerator kinematic dressing

$$s_{a,b} = (k_a + k_b)^2$$

With all but **indicated** momenta on shell: $p^2 = 0$

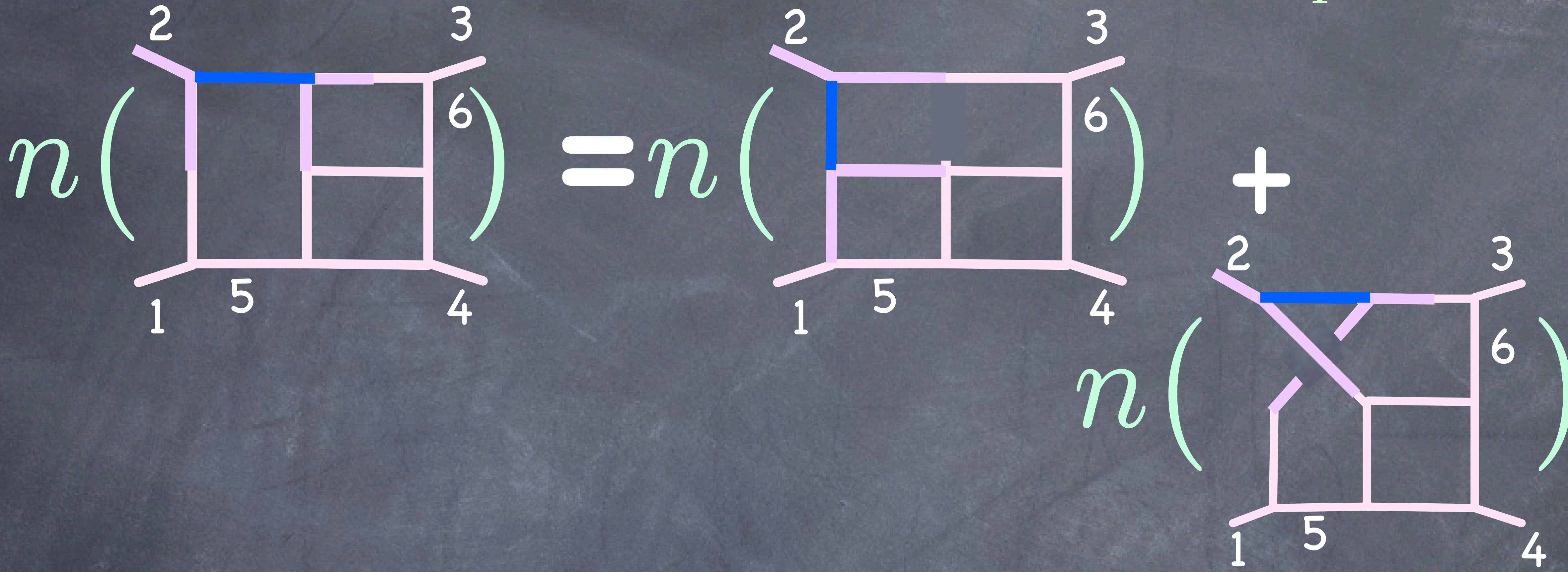


$$s_{12}s_{45} = s_{14}s_{46} + (s_{12}s_{45} - s_{14}s_{46})$$

n(graph) = numerator kinematic dressing

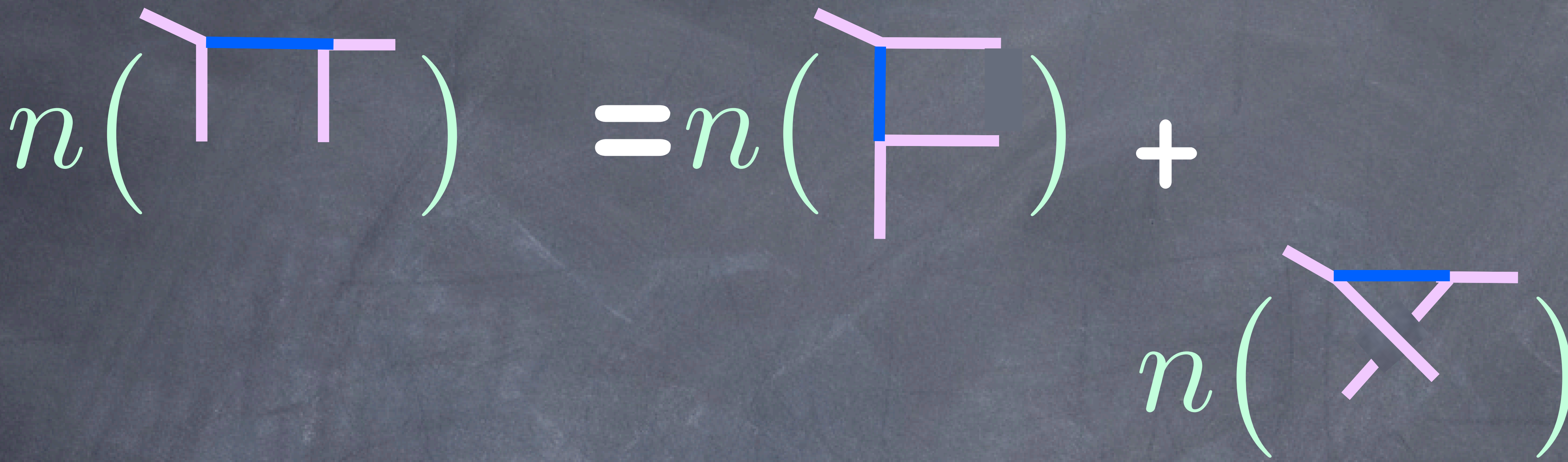
$$s_{a,b} = (k_a + k_b)^2$$

With all but **indicated** momenta on shell: $p^2 = 0$



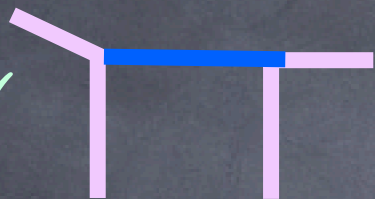
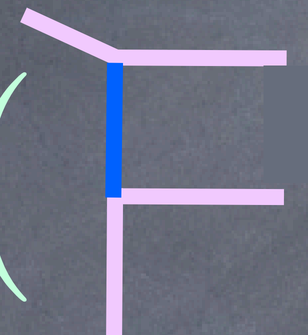
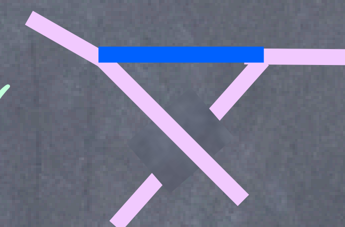
n(graph) = numerator kinematic dressing

With all but **indicated** momenta on shell: $p^2 = 0$

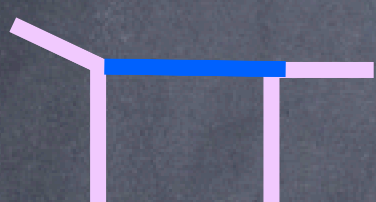
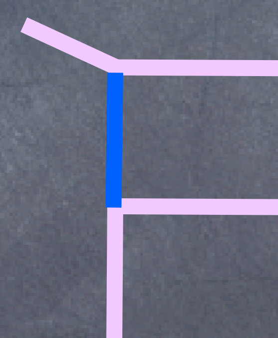
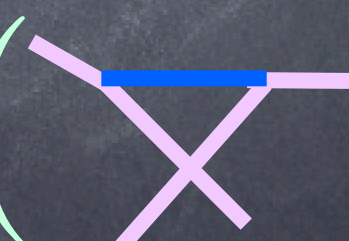


n(graph) = numerator kinematic dressing

With all but **indicated** momenta on shell: $p^2 = 0$

$n($  $)$ $= n($  $)$ $+$ $n($  $)$

examine color factors of 4-pt
uncut gluonic tree:

$c($  $)$ $= c($  $)$ $+$ $c($  $)$

true by Color Jacobi identity!

n(graph) = numerator kinematic dressing
c(graph) = color factor

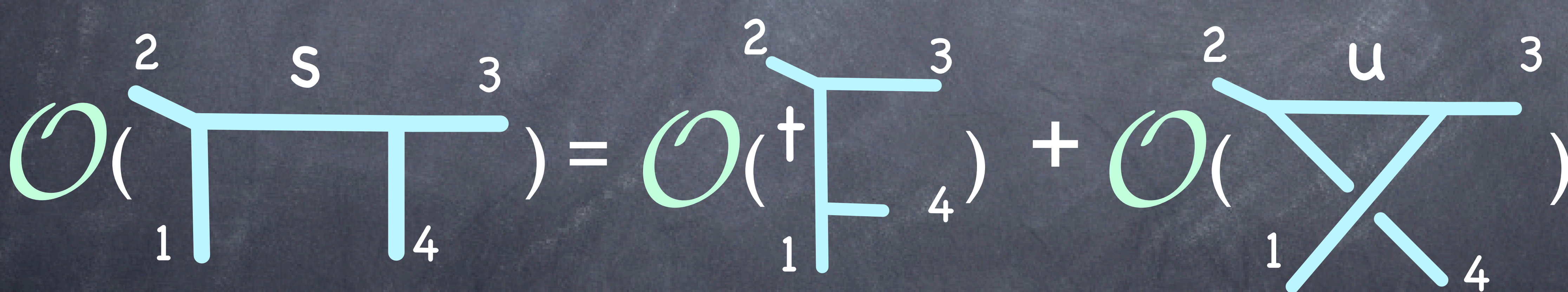
So what's going on? Let's get graphy!

Four-point tree amplitude: $g^2\left(\frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}\right)$

Of course there's a freedom ("generalized gauge invariance"):

$n_i \rightarrow n_i + \Delta_i$ as long as $\frac{c_s \Delta_s}{s} + \frac{c_t \Delta_t}{t} + \frac{c_u \Delta_u}{u} = 0$

Turns out that all Δ choices satisfy a **duality between color and kinematics**:



$\mathcal{O}(.) = n(.)$ kinematic "dressing"

$\mathcal{O}(.) = c(.)$ color factor

Can this be generalized?

m-point gauge tree amplitude:

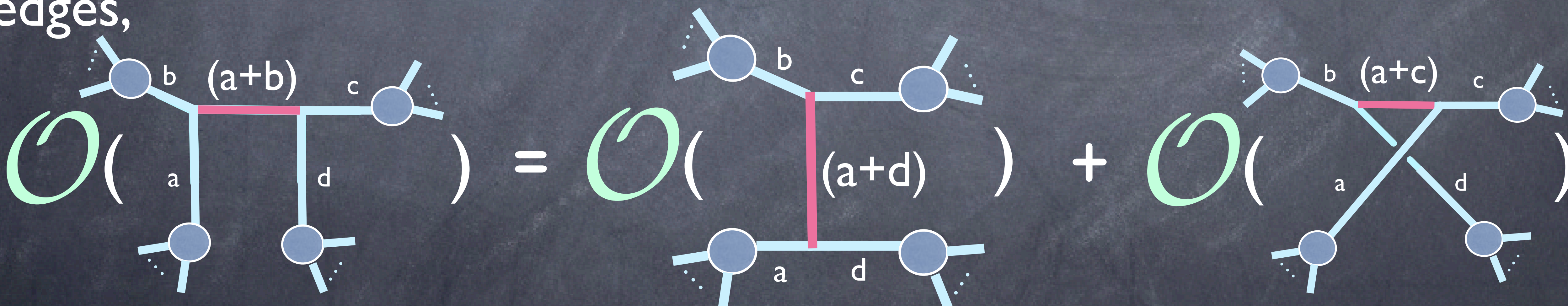
$$\mathcal{A}_m^{\text{tree}} = g^{(m-2)} \sum_{\mathcal{G}\in \text{cubic}} \left(\frac{c(\mathcal{G}) n(\mathcal{G})}{D(\mathcal{G})} \right)$$

Hypothesize to all points: **Color ⟷ Kinematic Duality**

General freedom:

$$n(\mathcal{G}) \to n(\mathcal{G}) + \Delta(\mathcal{G}), \quad \sum_{\mathcal{G}\in \text{cubic}} \left(\frac{c(\mathcal{G}) \Delta(\mathcal{G})}{D(\mathcal{G})} \right) = 0$$

Conjectured can always find a choice of Δ such that for all graphs & edges,



$$\mathcal{O}(.) = n(.) \qquad\qquad \mathcal{O}(.) = c(.)$$

kinematic “dressing” color factor

(originally verified thru 8pt, now we know it’s true)

Interesting tree-level Jacobi-satisfying numerator representations!

BCJ

Bern, Dennen, Huang, Kiermaier

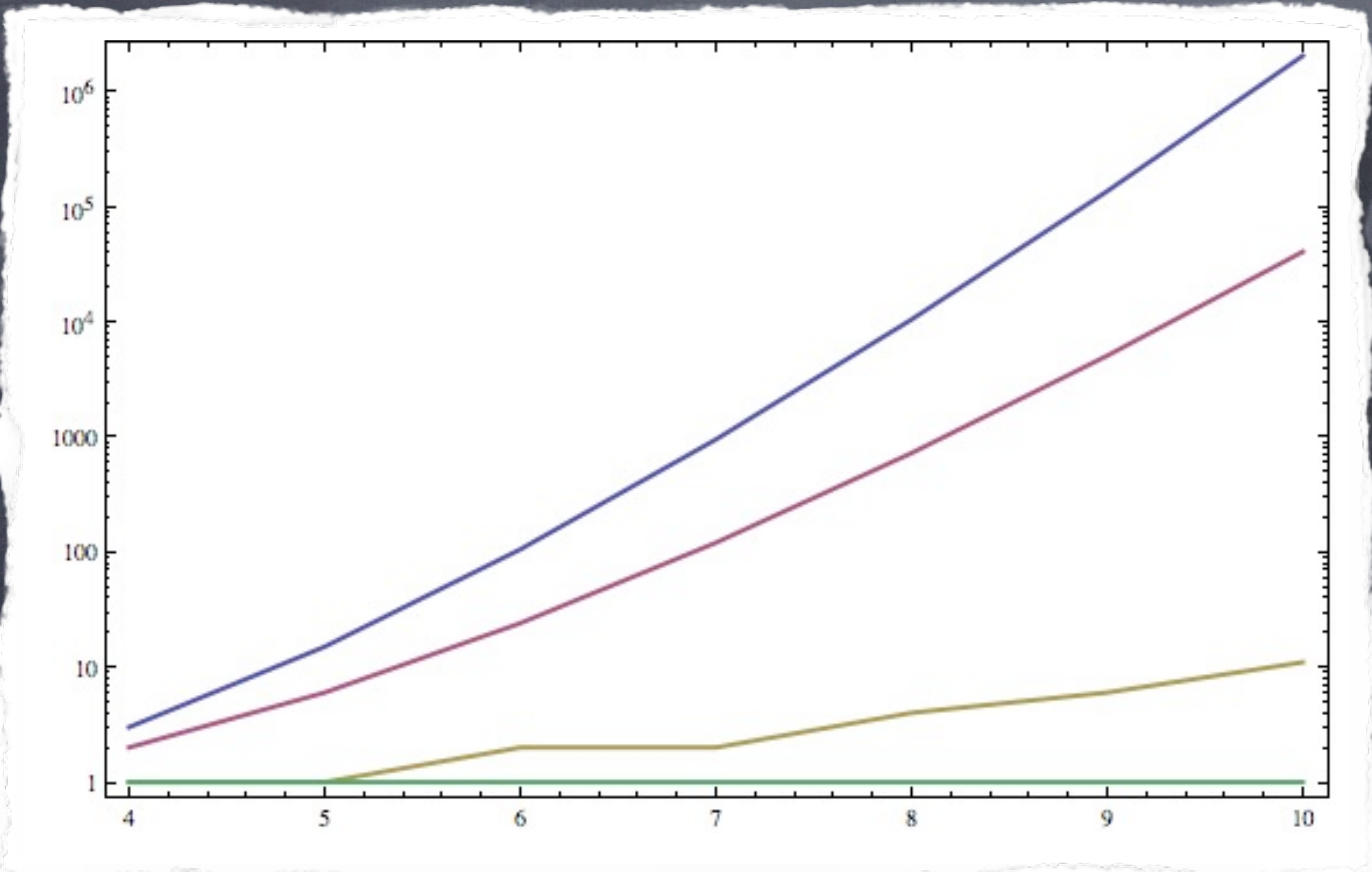
Kiermaier

Bjerrum-Bohr, Damgaard, Sondegaard, Vanhove

Mafra, Schlotterer, Stieberger

Broedel, JJMC

What have we gained?

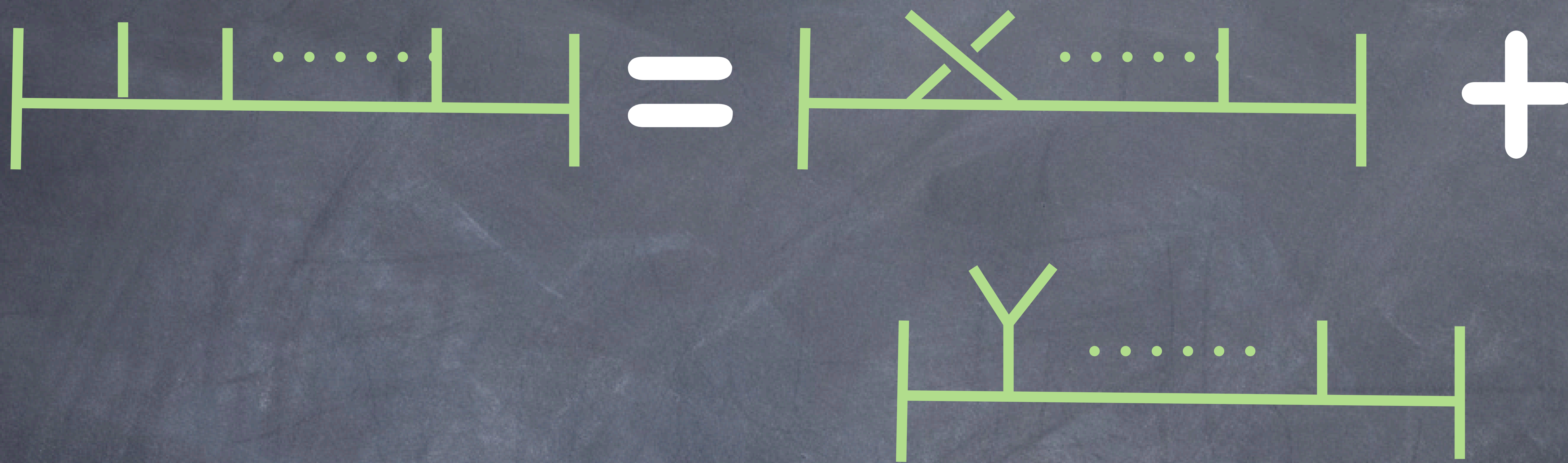


(2m-5)!! diags

(m-2)! numerators unconstrained by dual kinematic Jacobi

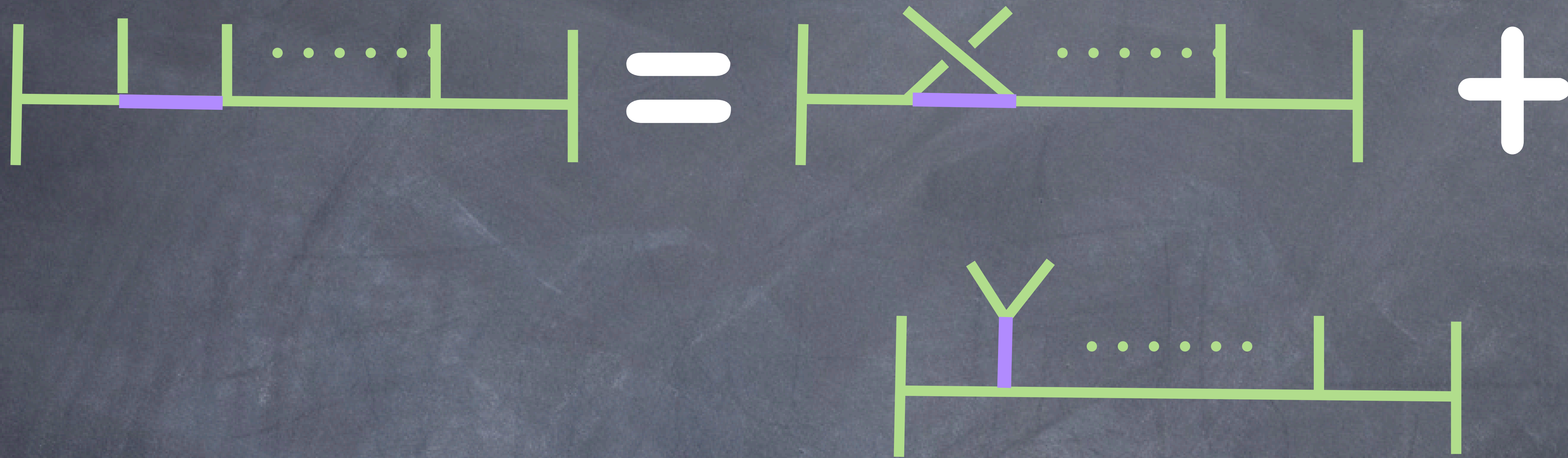
unique topologies
http://oeis.org/A000672

Multiplicity: (m)



All cubic trees in terms of 1 topology for each multiplicity

Symmetric numerator functions => only one numerator for each mulitplicity



All cubic trees in terms of 1 topology for each multiplicity

Symmetric numerator functions => only one numerator for each mulitplicity

Gravity?

$$A_m^{\text{tree}} \propto \sum_{\mathcal{G} \in \text{cubic}} \left(\frac{n(\mathcal{G}) c(\mathcal{G})}{D(\mathcal{G})} \right)$$

color factors just sitting there obeying antisymmetry and Jacobi relations.

Gravity?

$$\mathcal{A}_m^{\text{tree}} \propto \sum_{\mathcal{G}\in\text{cubic}} \left(\frac{n(\mathcal{G})c(\mathcal{G})}{D(\mathcal{G})} \right)$$

color factors just sitting there obeying antisymmetry and Jacobi relations.

$$\sum_{\mathcal{G}\in\text{cubic}} \frac{n(\mathcal{G})\tilde{n}(\mathcal{G})}{D(\mathcal{G})}$$ = Gravity amplitude in a related theory

How to find duality-satisfying numerators?

Easy way at tree-level is to involve

color-ordered partial amplitudes

- With particles all in the adjoint representation of $SU(N_c)$, the full tree amplitude can be decomposed:

(color group generators)

$$\mathcal{A}_n^{\text{tree}}(1,\ldots,n) = g^{n-2} \sum_{P(2,\ldots,n)} Tr[T^{a_1}\ldots T^{a_n}] \times \mathrm{A}_n^{\text{tree}}(1,\ldots,n)$$

color ordered (stripped) `partial' amplitude annotated with roman A

Full gauge theory amplitudes given with calligraphic $\mathcal{A}$

Structure constants: $f^{abc} = \mathrm{Tr}([T^a, T^b]T^c)$

How to find duality-satisfying numerators?

BCJ

m-point

Easy way at tree-level is to involve color-ordered partial amplitudes

- Write all m-point graphs and all independent Jacobi relations between their numerators
- Solve linear equations in terms of (m-2)! Jacobi-independent numerators (e.g. can let them all be half-ladders)
- Expand all color-ordered amplitudes in terms of their constituent graphs:

$$A_m^{\text{tree}}(1,2,3,\ldots,m) = \sum_{g \in \text{cyclic}} \frac{n(g)}{\prod_{l \in p(g)} l^2}$$

- Write the graphs in the (m-2)! graph basis, and solve the linear relations in terms of the color-ordered amplitudes.
- **This is it--you have a duality-satisfying representation.**

(symmetric is trickier)

Features:

- Completely straightforward solution of linear relations (trickiest bit is drawing graphs)
- Makes all residual gauge-freedom manifest: gauge freedom = (m-3)x(m-3)! completely unconstrained numerator functions. (can use to, e.g. make symmetric numerator functions)
- Independent of dimension and helicity structure
- Interesting consequence for gauge-independent quantities: fewer independent color-ordered scattering amplitudes

"Observable" implications:

Only **(n-3)!** independent color-ordered tree partial-amplitudes for n-point interaction. (c.f. (n-2)! from Kleis-Kuijf)

e.g. 5 pt has 2 indep. color-ordered amps not 6:

$$A_5^{\text{tree}}(12345) \qquad A_5^{\text{tree}}(12354)$$

6 pt has 6 indep. color-ordered amps not 12:

$$\mathrm{A}_6^{\text{tree}}(123456) \quad \mathrm{A}_6^{\text{tree}}(123564) \quad \mathrm{A}_6^{\text{tree}}(123645)$$

$$\mathrm{A}_6^{\text{tree}}(123546) \quad \mathrm{A}_6^{\text{tree}}(123465) \quad \mathrm{A}_6^{\text{tree}}(123654)$$

We found a general formula expressing any n-point color ordered amplitude in terms of chosen **(n-3)!** basis for SYM.

since proved! Bjerrum-Bohr, Damgaard, Vanhove; Stieberger
Feng, He, (R.) Huang, Jia

KLT field expressions:

Kawai, Lewellen, Tye

Bern, Dixon, Perelstein, Rozowsky

Gravity tree amplitudes

$i \in \{2, \ldots, n/2\}$

$j \in \{n/2+2, \ldots, n-2\}$

$$M_n^{\text{tree}}(\underline{1}, \ldots, \underline{n-1}, \underline{n}) =$$

$$i(-1)^{n+1} \sum_{perms(2,\ldots,n-2)} \Big[A_n^{\text{tree}}(1,\ldots,n-1,n) \sum_{perms(i,j)} f(i_1,\ldots,i_j)$$

$$\times \overline{f}(l_1,\ldots,l_{j'}) \widetilde{A}_n^{\text{tree}}(i_1,\ldots,i_j,1,n-1,l_1,\ldots,l_{j'},n) \Big]$$

Color-ordered gauge tree amplitudes

$$f(i_1,\ldots,i_j) = s_{1,i_j} \prod_{m=1}^{j-1} \left(s_{1,i_m} + \sum_{k=m+1}^{j} g(i_m,i_k) \right),$$

$$\overline{f}(l_1,\ldots,l_{j'}) = s_{l_1,n-1} \prod_{m=2}^{j'} \left(s_{l_m,n-1} + \sum_{k=1}^{m-1} g(l_k,l_m) \right)$$

$$g(i,j) = \left\{ \begin{array}{ll} s_{i,j} & \text{if } i > j \\ 0 & \text{else} \end{array} \right\}$$

$$s_{a,b} = (k_a + k_b)^2$$

KLT field expressions:

Kawai, Lewellen, Tye

Bern, Dixon, Perelstein, Rozowsky

Gravity tree amplitudes

$$i \in \{2, \ldots, n/2\}$$
$$j \in \{n/2+2, \ldots, n-2\}$$

$$M_n^{\rm tree}(\underline{1}, \ldots, \underline{n-1}, \underline{n}) = i(-1)^{n+1} \sum_{perms(2,\ldots,n-2)} \Big[A_n^{\rm tree}(\underline{1}, \ldots, \underline{n-1}, \underline{n}) \sum_{perms(i,j)} f(i_1, \ldots, i_j) \times \overline{f}(l_1, \ldots, l_{j'}) \widetilde{A}_n^{\rm tree}(i_1, \ldots, i_j, \underline{1}, \underline{n-1}, l_1, \ldots, l_{j'}, \underline{n}) \Big]$$

Color-ordered gauge tree amplitudes

$$f(i_1, \ldots, i_j) = s_{1,i_j} \prod_{m=1}^{j-1} \left(s_{1,i_m} + \sum_{k=m+1}^{j} g(i_m, i_k) \right),$$

$$\overline{f}(l_1, \ldots, l_{j'}) = s_{l_1,n-1} \prod_{m=2}^{j'} \left(s_{l_m,n-1} + \sum_{k=1}^{m-1} g(l_k, l_m) \right)$$

$$g(i,j) = \left\{ \begin{array}{ll} s_{i,j} & \text{if } i > j \\ 0 & \text{else} \end{array} \right\}$$

$$s_{a,b} = (k_a + k_b)^2$$

KLT field expressions:

Kawai, Lewellen, Tye

Bern, Dixon, Perelstein, Rozowsky

Gravity tree amplitudes

$$M_n^{\text{tree}}(\underline{1}, \ldots, \underline{n-1}, \underline{n}) =$$

$i \in \{2, \ldots, n/2\}$
$j \in \{n/2+2, \ldots, n-2\}$

$$i(-1)^{n+1} \sum_{perms(2,\ldots,n-2)} \Big[A_n^{\text{tree}}(\underline{1}, \ldots, \underline{n-1}, \underline{n}) \sum_{perms(i,j)} f(i_1, \ldots, i_j)$$
$$\times \overline{f}(l_1, \ldots, l_{j'}) \widetilde{A}_n^{\text{tree}}(i_1, \ldots, i_j, \underline{1}, \underline{n-1}, l_1, \ldots, l_{j'}, \underline{n}) \Big]$$

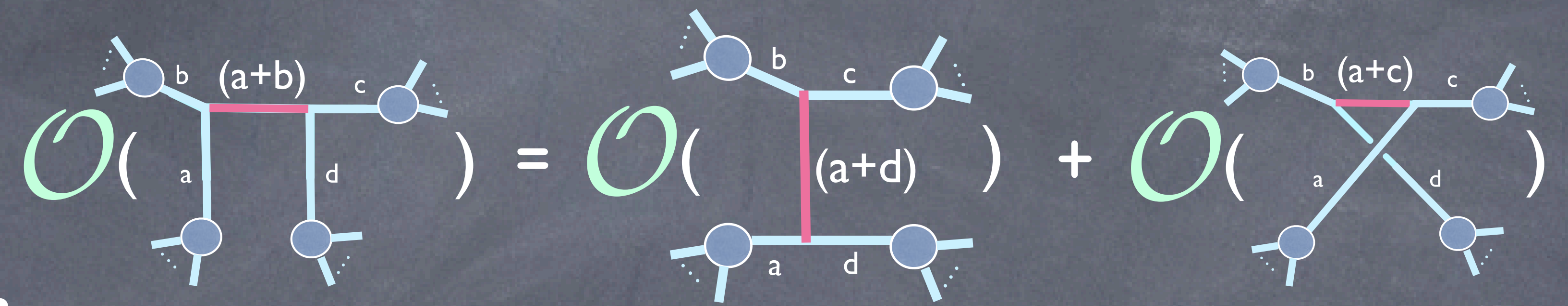
Color-ordered gauge tree amplitudes

New "observable" relations allow re-expression of KLT in terms of different "basis" amplitudes: Left-right symmetric, etc.

But we can do better...

Clarifying Gravity Amplitudes

Writing color-ordered gauge tree amplitudes in representation of **duality satisfying cubic-diagrams:**



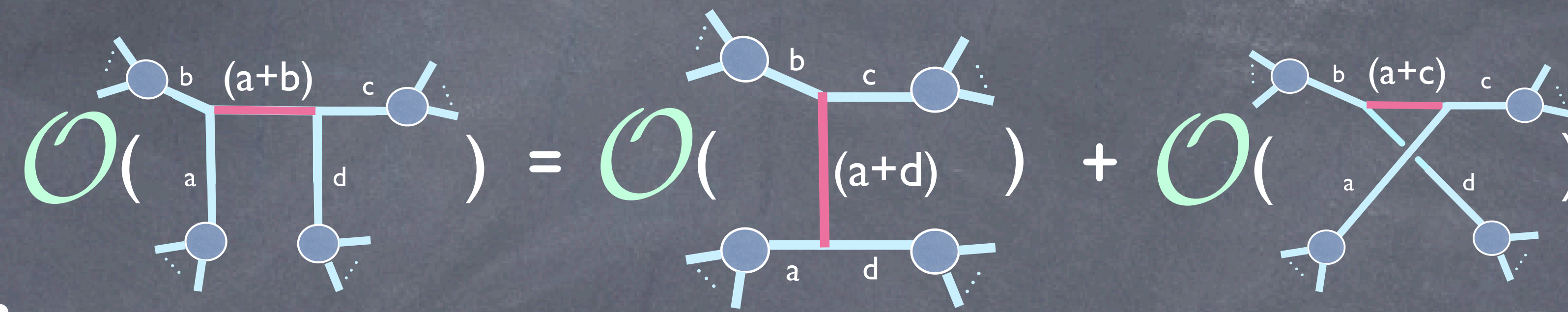
$$\mathrm{A}^{\mathrm{tree}}(\mathrm{perm}) = \sum_{\mathcal{G}\in \mathrm{graphs(perm)}} \frac{n(\mathcal{G})}{D(\mathcal{G})}$$

$$M_n^{\mathrm{tree}}(1,\ldots,n-1,n) = i(-1)^{n+1} \sum_{perms(2,\ldots,n-2)} \Big[A_n^{\mathrm{tree}}(1,\ldots,n-1,n) \sum_{perms(i,j)} f(i_1,\ldots,i_j) \times \overline{f}(l_1,\ldots,l_{j'})\, \widetilde{A}_n^{\mathrm{tree}}(i_1,\ldots,i_j,1,n-1,l_1,\ldots,l_{j'},n) \Big]$$

$$\tilde{\mathrm{A}}^{\mathrm{tree}}(\mathrm{perm}) = \sum_{\mathcal{G}\in \mathrm{graphs(perm)}} \frac{\tilde{n}(\mathcal{G})}{D(\mathcal{G})}$$

Clarifying Gravity Amplitudes

Writing color-ordered gauge tree amplitudes in representation of duality satisfying cubic-diagrams:



Gives gravity tree amplitudes:

$$-iM_n^{\text{tree}} = \sum_{\mathcal{G}\in\text{cubic}} \frac{n(\mathcal{G})\tilde{n}(\mathcal{G})}{D(\mathcal{G})}$$

Gravity as the "double copy" of gauge theory!

$$\mathcal{A}_m^{\text{tree}} \propto \sum_{\mathcal{G}\in\text{cubic}} \left(\frac{n(\mathcal{G})c(\mathcal{G})}{D(\mathcal{G})} \right)$$

$$-iM_n^{\text{tree}} = \sum_{\mathcal{G}\in\text{cubic}} \frac{n(\mathcal{G})\tilde{n}(\mathcal{G})}{D(\mathcal{G})}$$

Note n and $\tilde{n}$ can come from different reps of same theory, or even different theories altogether.

$$\mathcal{N}=4 \text{ sYM} \otimes \mathcal{N}=4 \text{ sYM} \Rightarrow \mathcal{N}=8 \text{ sugra}$$

$$\mathcal{N}=p \text{ sYM} \otimes \mathcal{N}=4 \text{ sYM} \Rightarrow \mathcal{N}=4+p \text{ sugra}$$

(see Zvi's talk)

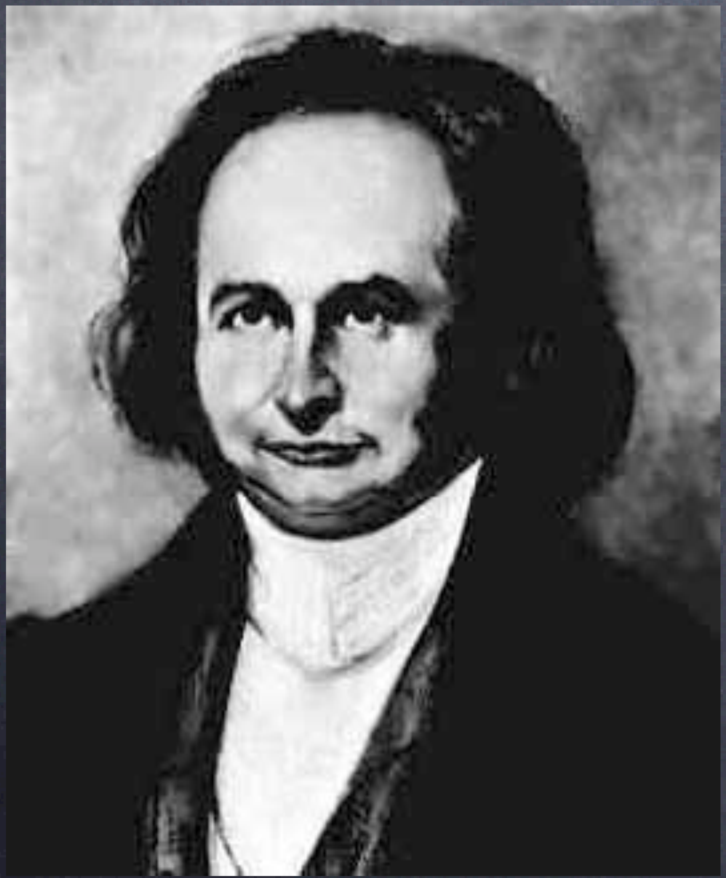
Only one gauge representation need have duality imposed, consequence of general freedom:

$$n(\mathcal{G}) \to n(\mathcal{G}) + \Delta(\mathcal{G}), \quad \sum_{\mathcal{G}\in\text{cubic}} \left(\frac{c(\mathcal{G})\Delta(\mathcal{G})}{D(\mathcal{G})}\right) = 0$$

can only depend on algebraic property of $c(\mathcal{G})$ not numeric values. So as long as $\tilde{n}(\mathcal{G})$ satisfies same algebra (i.e. duality) can shift $n(\mathcal{G})$ as we please.

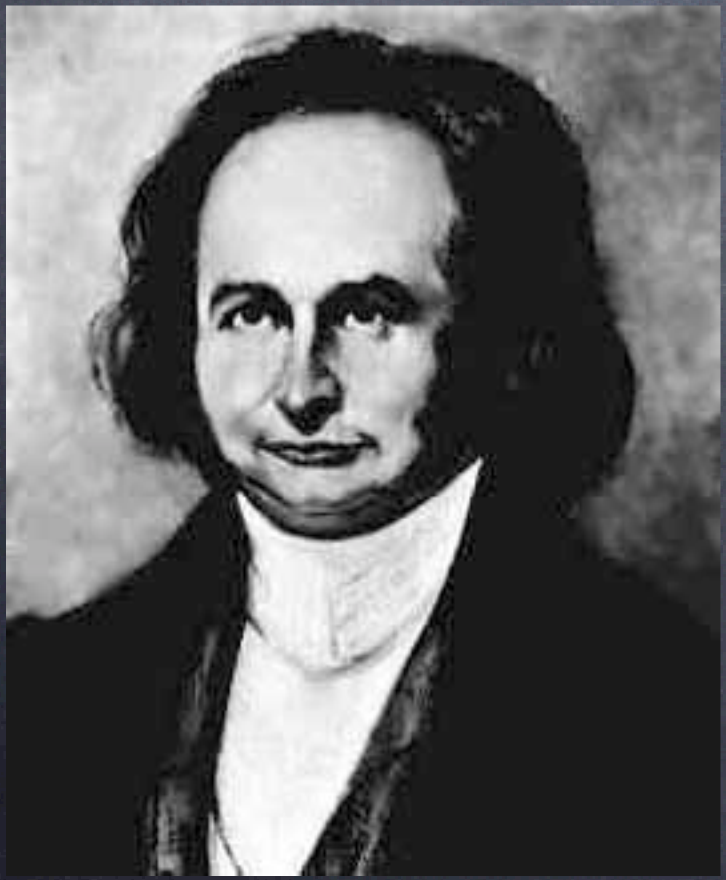
This is all (semi)-classical

- The world is QUANTUM - wouldn't it be great to generalize to loop-order corrections?



This is all (semi)-classical

- The world is QUANTUM - wouldn't it be great to generalize to loop-order corrections?



"One should always generalize." - C. Jacobi

What's the right generalization?

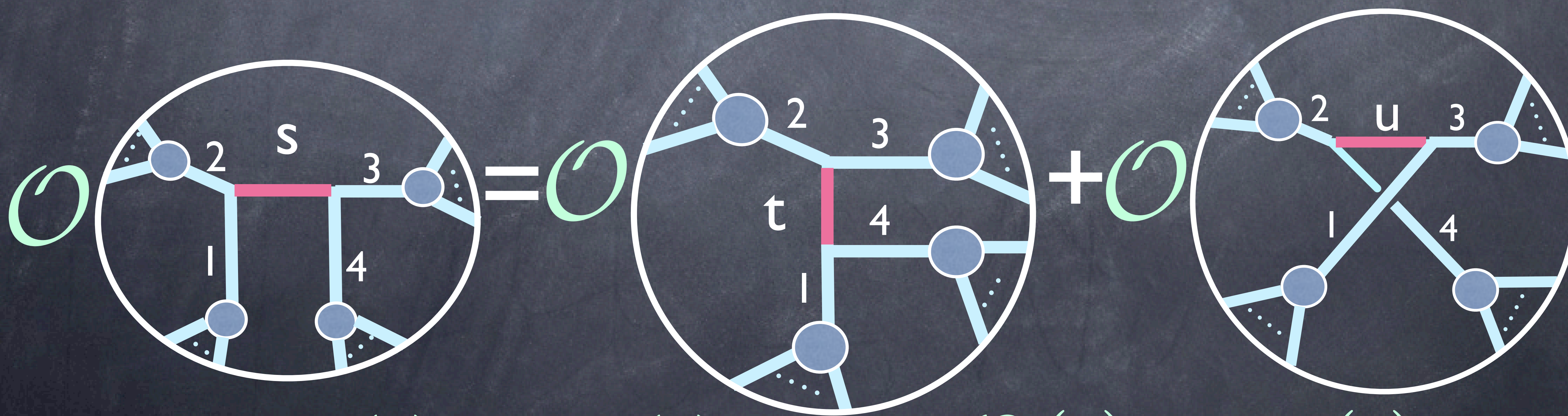
$$\frac{(-i)^L}{g^{n-2+2L}}\mathcal{A}^{\text{loop}} = \sum_{\mathcal{G}\in\text{cubic}} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S(\mathcal{G})} \frac{n(\mathcal{G})c(\mathcal{G})}{D(\mathcal{G})}$$

Hypothesize duality holds unchanged to all loops!

Representation freedom:

$$n(\mathcal{G}) \to n(\mathcal{G}) + \Delta(\mathcal{G}), \quad \sum_{\mathcal{G}\in\text{cubic}} \left(\frac{c(\mathcal{G})\Delta(\mathcal{G})}{D(\mathcal{G})}\right) = 0$$

Conjecture there is always a choice of Δ causing n to satisfy for **all** internal edges from any representation same duality:



$$\mathcal{O}(.) = n(.) \longleftrightarrow \mathcal{O}(.) = c(.)$$

If conjectured duality can be imposed for:

Gauge:

$$\frac{(-i)^L}{g^{n-2+2L}} \mathcal{A}^{\text{loop}} = \sum_{\mathcal{G} \in \text{cubic}} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S(\mathcal{G})} \frac{n(\mathcal{G}) c(\mathcal{G})}{D(\mathcal{G})}$$

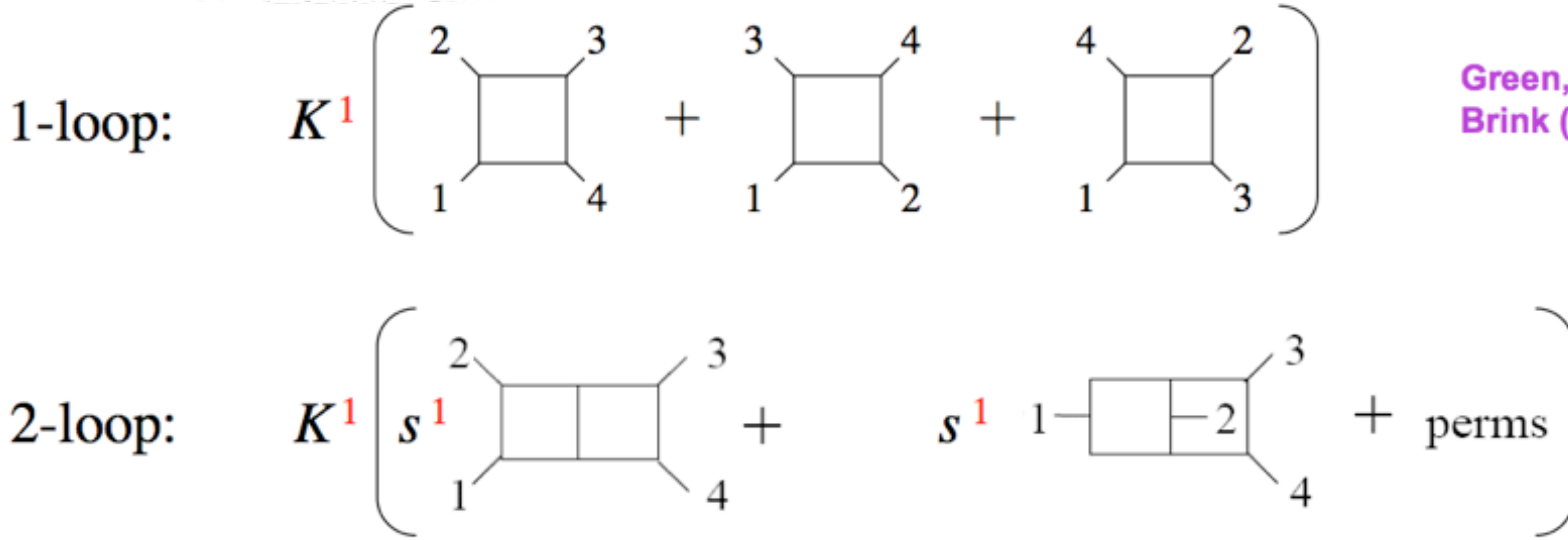
then, through unitarity & tree-level expressions:

Gravity:

$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}} \mathcal{M}^{\text{loop}} = \sum_{\mathcal{G} \in \text{cubic}} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S(\mathcal{G})} \frac{n(\mathcal{G}) \tilde{n}(\mathcal{G})}{D(\mathcal{G})}$$

What we always wanted out of a "loop level" relations!

We know this works beautifully at 1 and 2 loops for N=4 and N=8!



Green, Schwarz, Brink (1982)

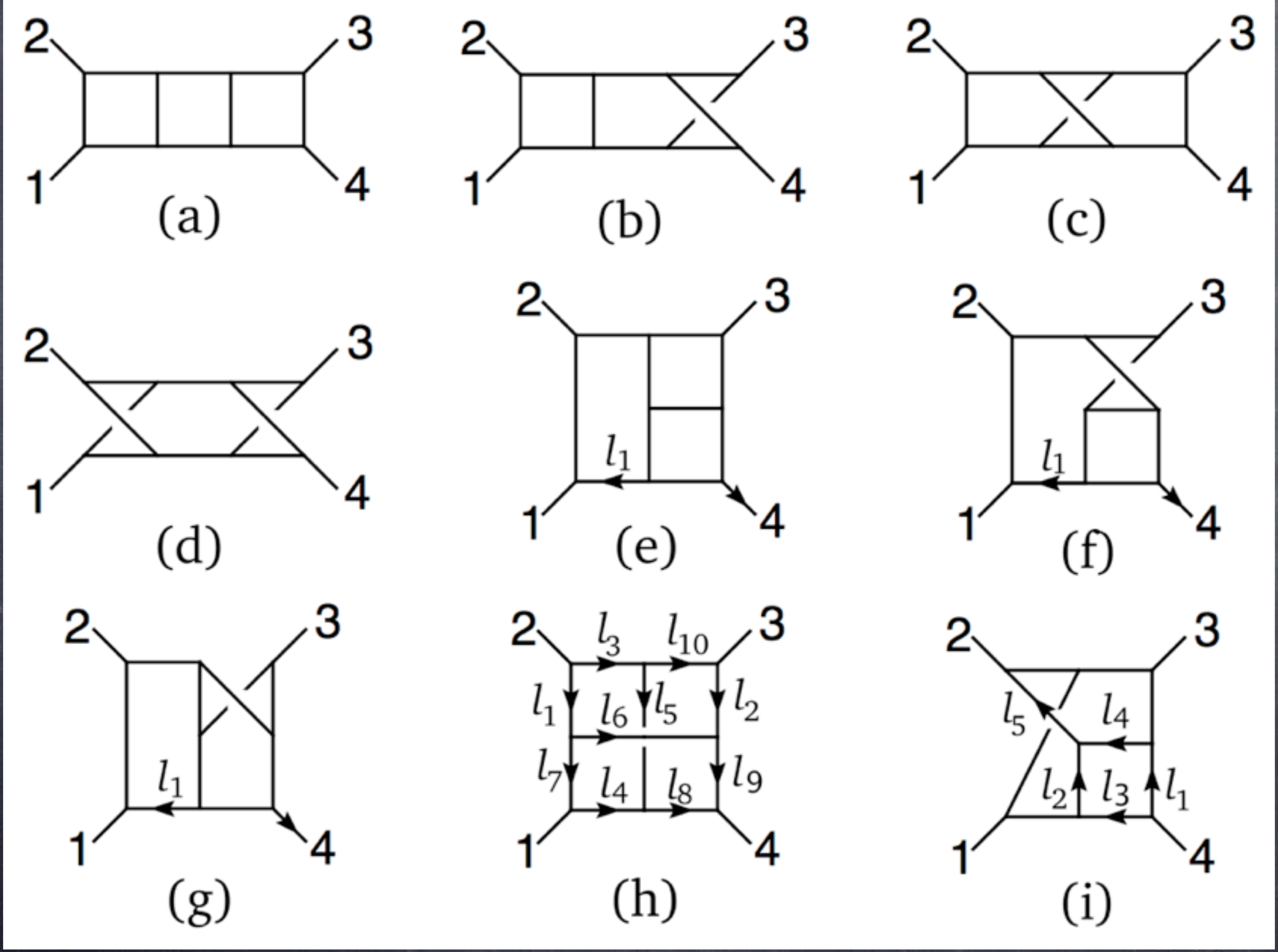
Bern, Dixon, Dunbar, Perelstein and Rozowsky (1998)

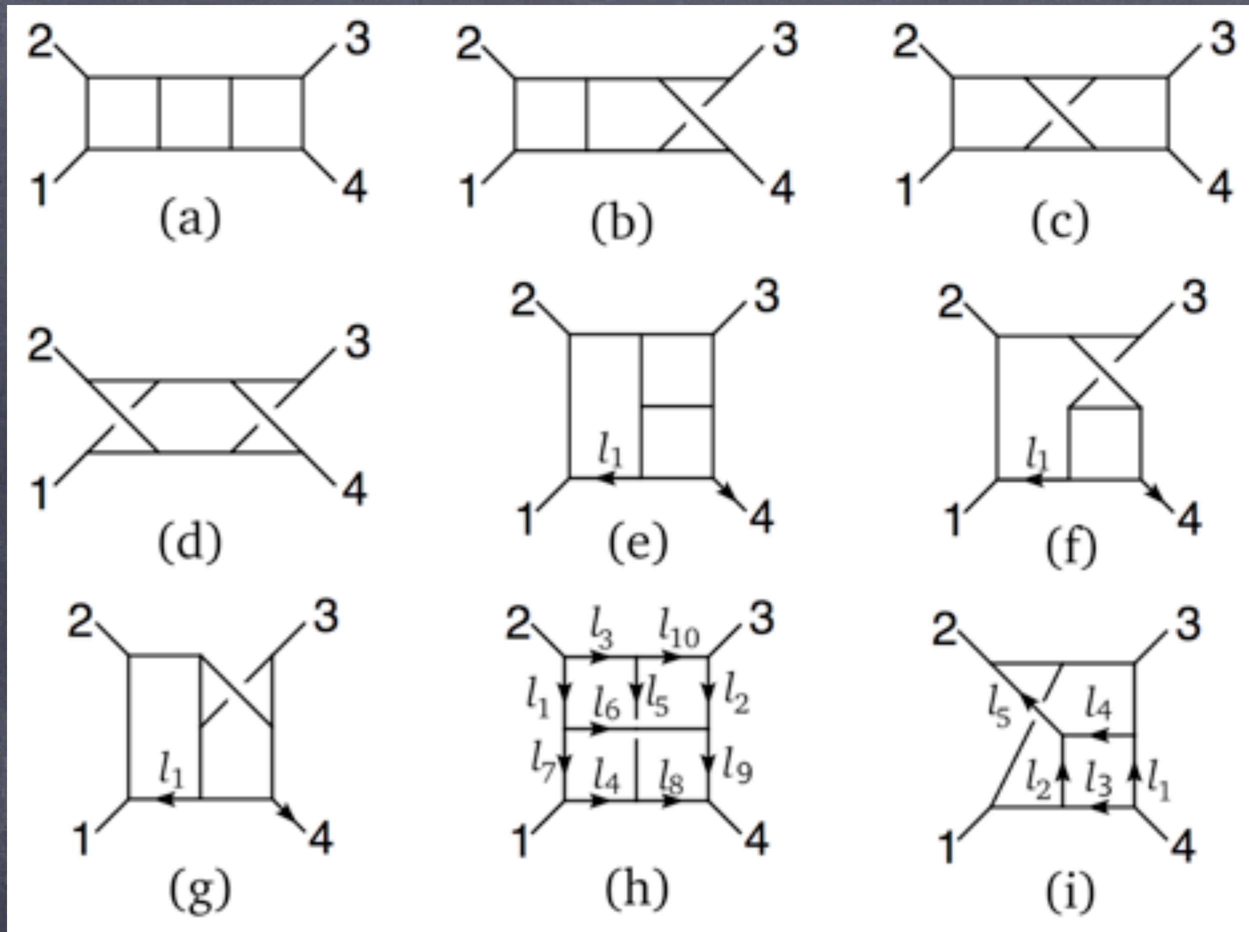
prefactor contains helicity structure: $K = st A_4^{\text{tree}}$

Duality: $\mathcal{N} = 8$ sugra is obtained if $1 \to 2$ "numerator squaring"

Bern, JJMC, Dixon, Johansson, Kosower, Roiban

Original Palette of Diagrams



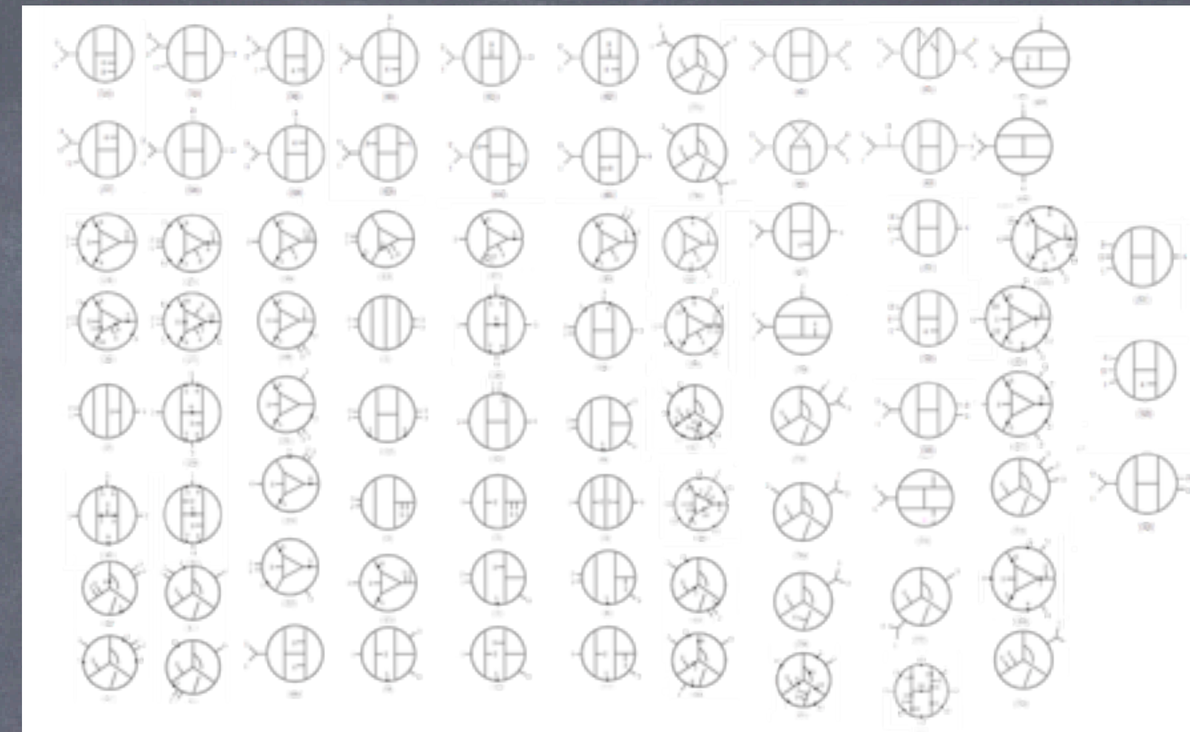


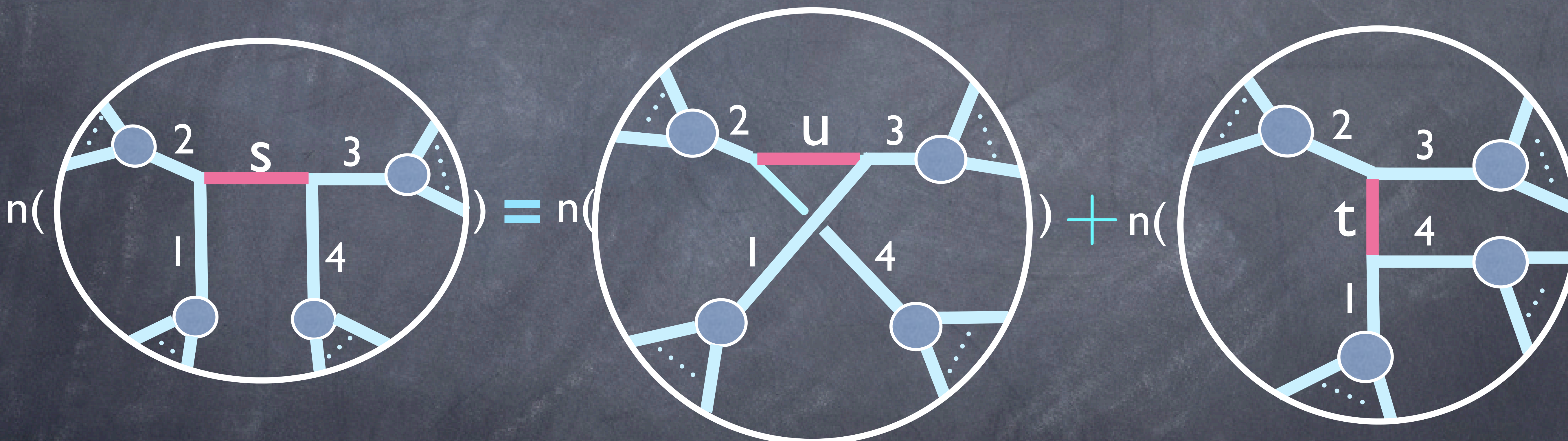
Original solution of three-loop four-point N=4 sYM and N=8 sugra

Integral	$\mathcal{N}=4$ Yang-Mills	$\mathcal{N}=8$ Supergravity
(a)–(d)	s^2	$[s^2]^2$
(e)–(g)	$s(l_1+k_4)^2$	$[s\,(l_1+k_4)^2]^2$
(h)	$s(l_1+l_2)^2+t(l_3+l_4)^2 - sl_5^2 - tl_6^2 - st$	$(s(l_1+l_2)^2+t(l_3+l_4)^2-st)^2 - s^2(2((l_1+l_2)^2-t)+l_5^2)l_5^2 - t^2(2((l_3+l_4)^2-s)+l_6^2)l_6^2 - s^2(2l_7^2l_2^2+2l_1^2l_9^2+l_2^2l_9^2+l_1^2l_7^2) - t^2(2l_3^2l_8^2+2l_{10}^2l_4^2+l_8^2l_4^2+l_3^2l_{10}^2) + 2st\,l_5^2l_6^2$
(i)	$s(l_1+l_2)^2 - t(l_3+l_4)^2 - \frac{1}{3}(s-t)l_5^2$	$(s(l_1+l_2)^2 - t(l_3+l_4)^2)^2 - (s^2(l_1+l_2)^2 + t^2(l_3+l_4)^2 + \frac{1}{3}stu)l_5^2$

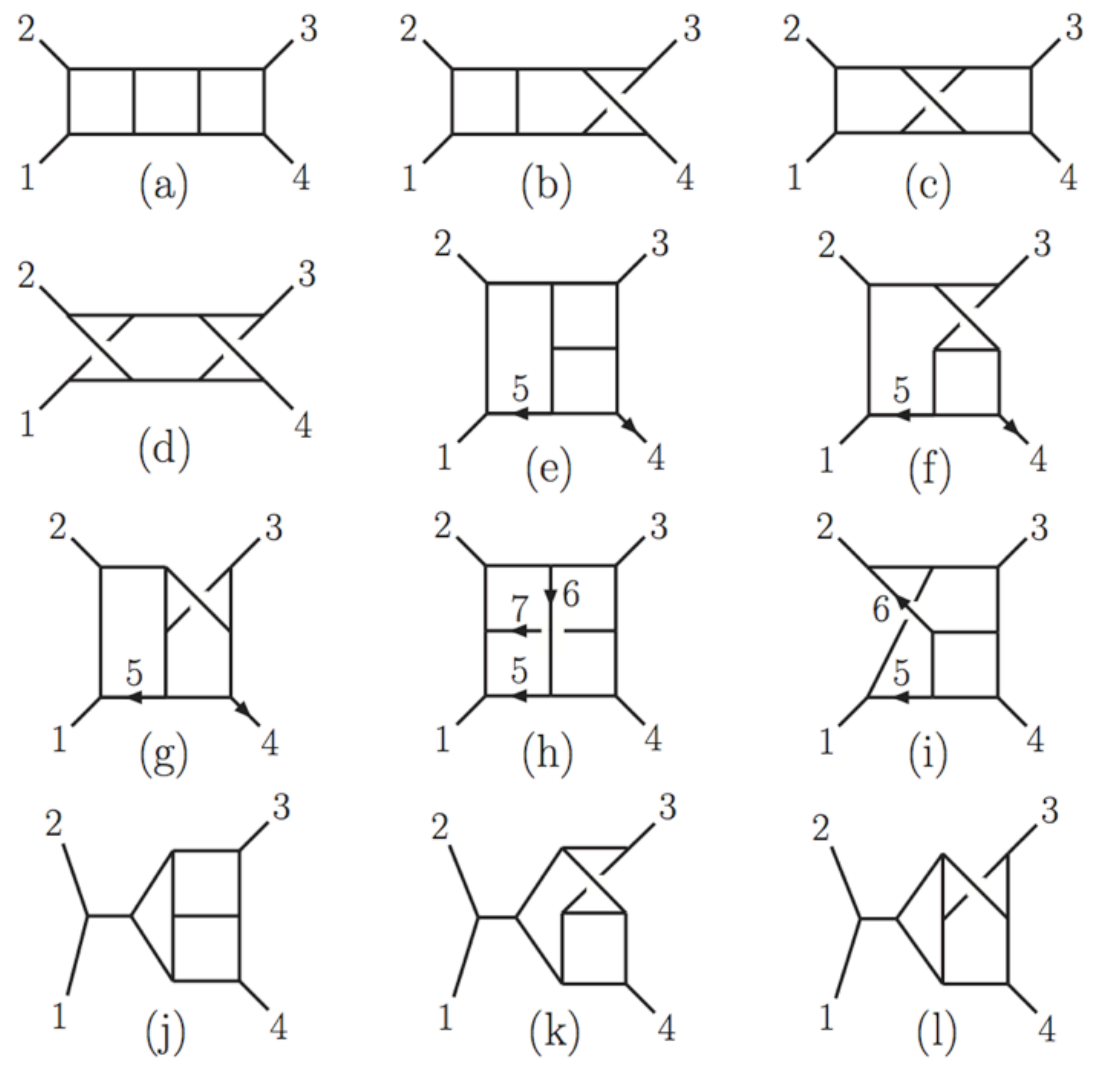
Recipe for finding Δ so dressings satisfy duality:

- **Every edge represents a set of constraints on functional form of the numerators of the graphs. Small fraction needed.**

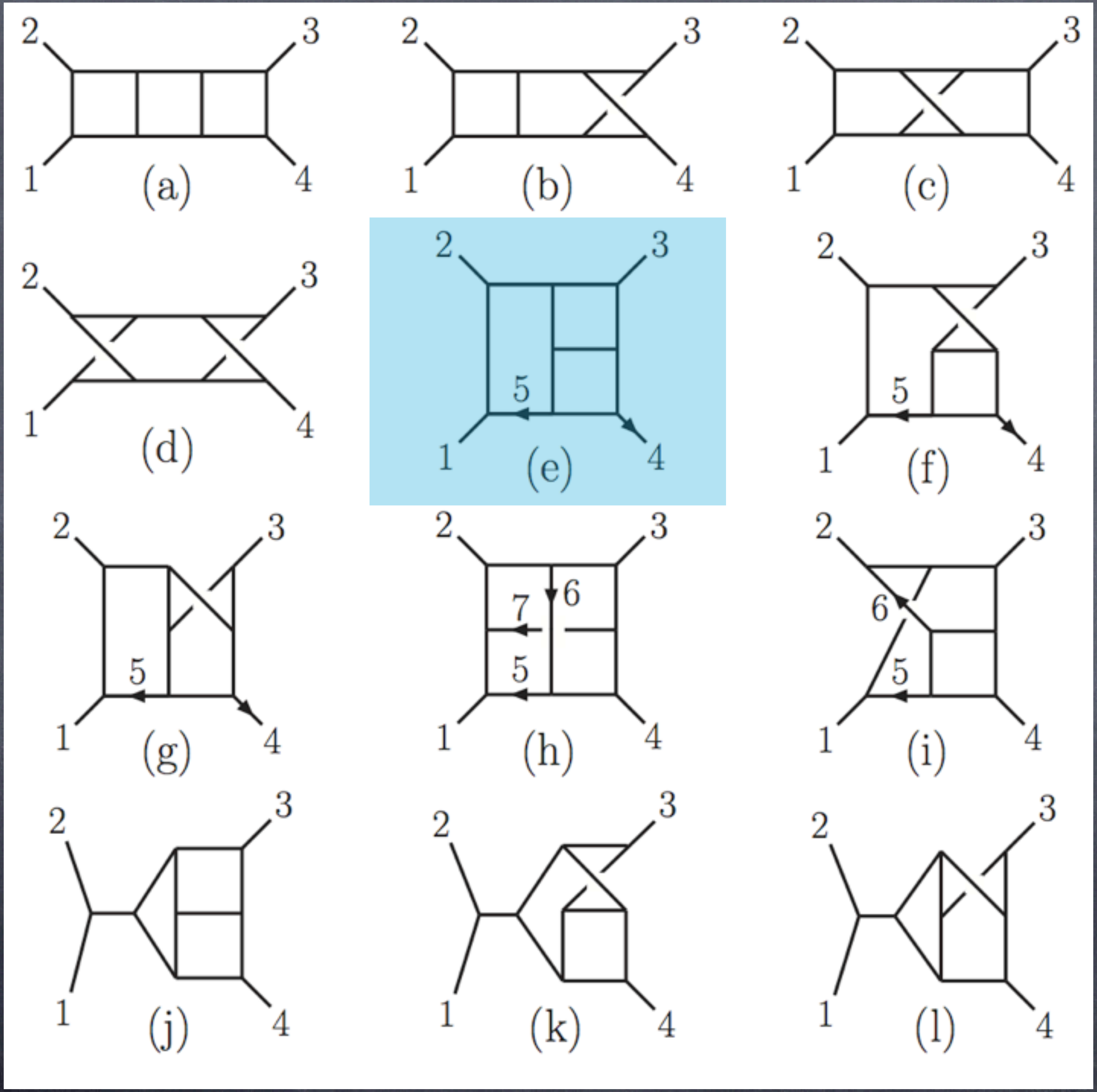


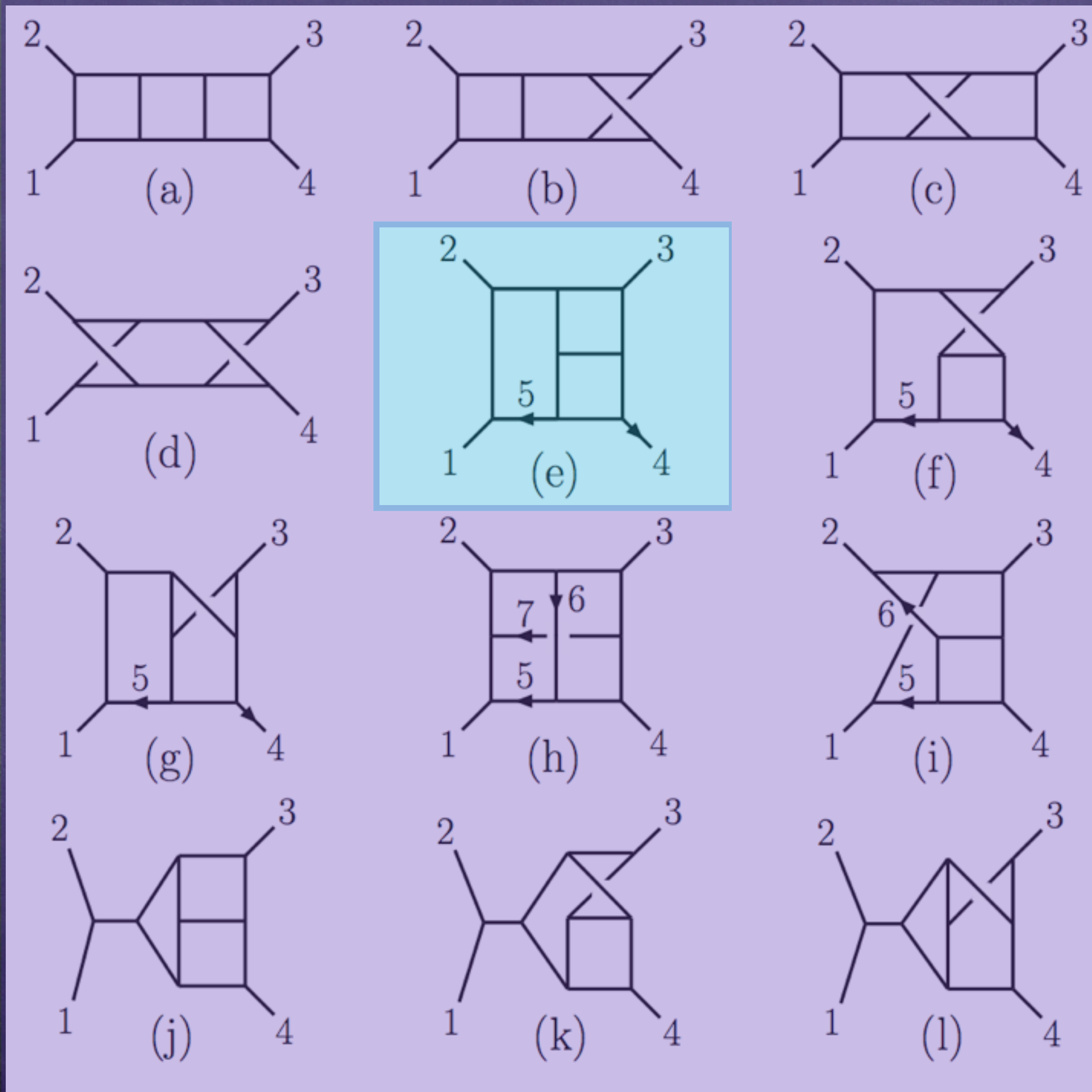


- **Find the independent numerators (solve the linear equations!)**
- **Build ansatze for the masters using functions seen on exploratory cuts**
- **Impose relevant symmetries**
- **Fit to the theory!**

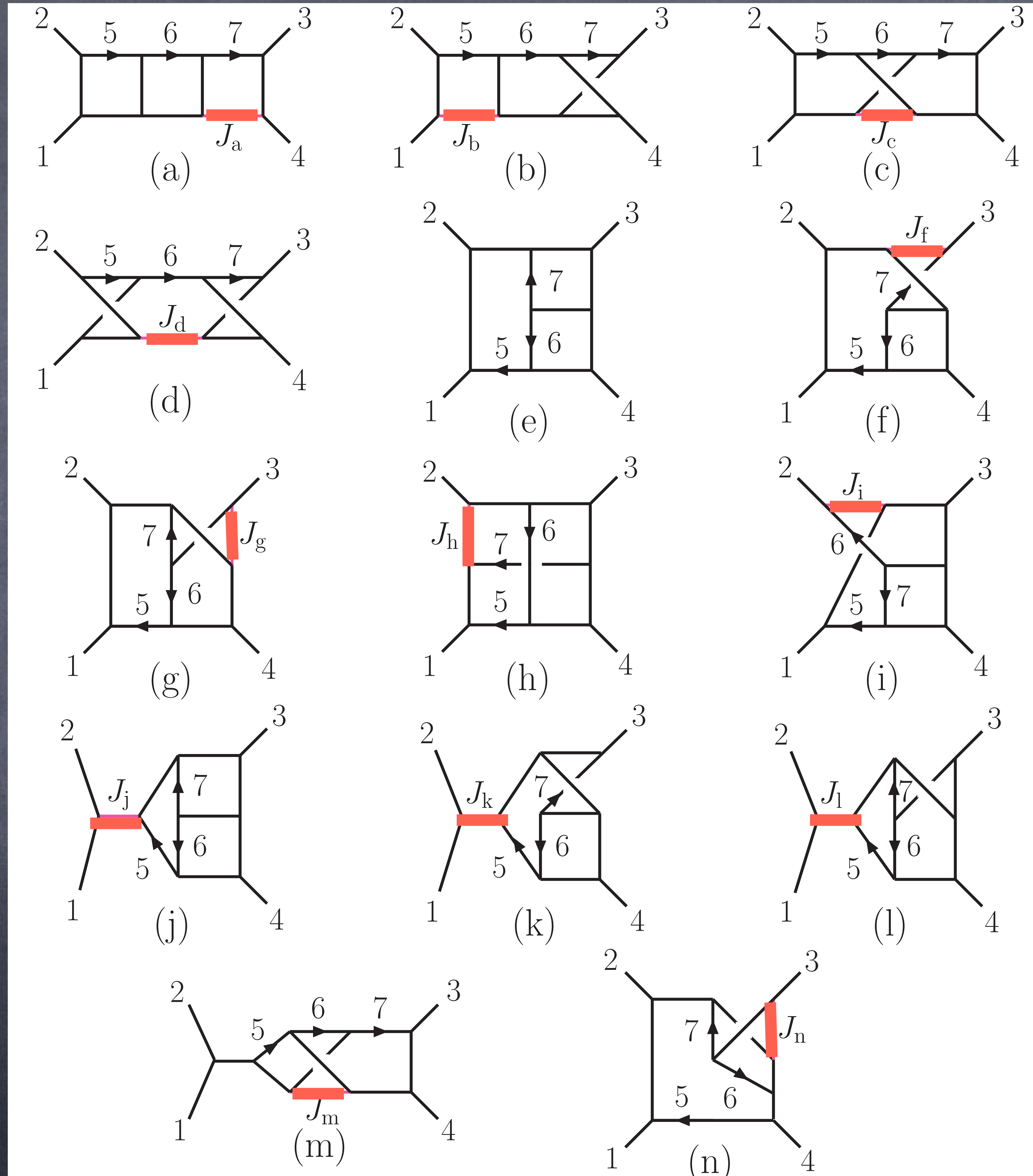

(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
(i)
(j)
(k)
(l)

BCJ (2010)





(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
(i)
(j)
(k)
(l)


(a) J_a
(b) J_b
(c) J_c
(d) J_d
(e)
(f) J_f
(g) J_g
(h) J_h
(i) J_i
(j) J_j
(k) J_k
(l) J_l
(m) J_m
(n) J_n

$$N^{(a)} = N^{(b)}(k_1, k_2, k_3, l_5, l_6, l_7), \qquad (J_a)$$

$$N^{(b)} = N^{(d)}(k_1, k_2, k_3, l_5, l_6, l_7), \qquad (J_b)$$

$$N^{(c)} = N^{(a)}(k_1, k_2, k_3, l_5, l_6, l_7), \qquad (J_c)$$

$$N^{(d)} = N^{(h)}(k_3, k_1, k_2, l_7, l_6, k_{1,3} - l_5 + l_6 - l_7)$$
$$+ N^{(h)}(k_3, k_2, k_1, l_7, l_6, k_{2,3} + l_5 - l_7), \qquad (J_d)$$

$$N^{(f)} = N^{(e)}(k_1, k_2, k_3, l_5, l_6, l_7), \qquad (J_f)$$

$$N^{(g)} = N^{(e)}(k_1, k_2, k_3, l_5, l_6, l_7), \qquad (J_g)$$

$$N^{(h)} = -N^{(g)}(k_1, k_2, k_3, l_5, l_6, k_{1,2} - l_5 - l_7)$$
$$- N^{(i)}(k_4, k_3, k_2, l_6 - l_5, l_5 - l_6 + l_7 - k_{1,2}, l_6), \qquad (J_h)$$

$$N^{(i)} = N^{(e)}(k_1, k_2, k_3, l_5, l_7, l_6)$$
$$- N^{(e)}(k_3, k_2, k_1, -k_4 - l_5 - l_6, -l_6 - l_7, l_6), \qquad (J_i)$$

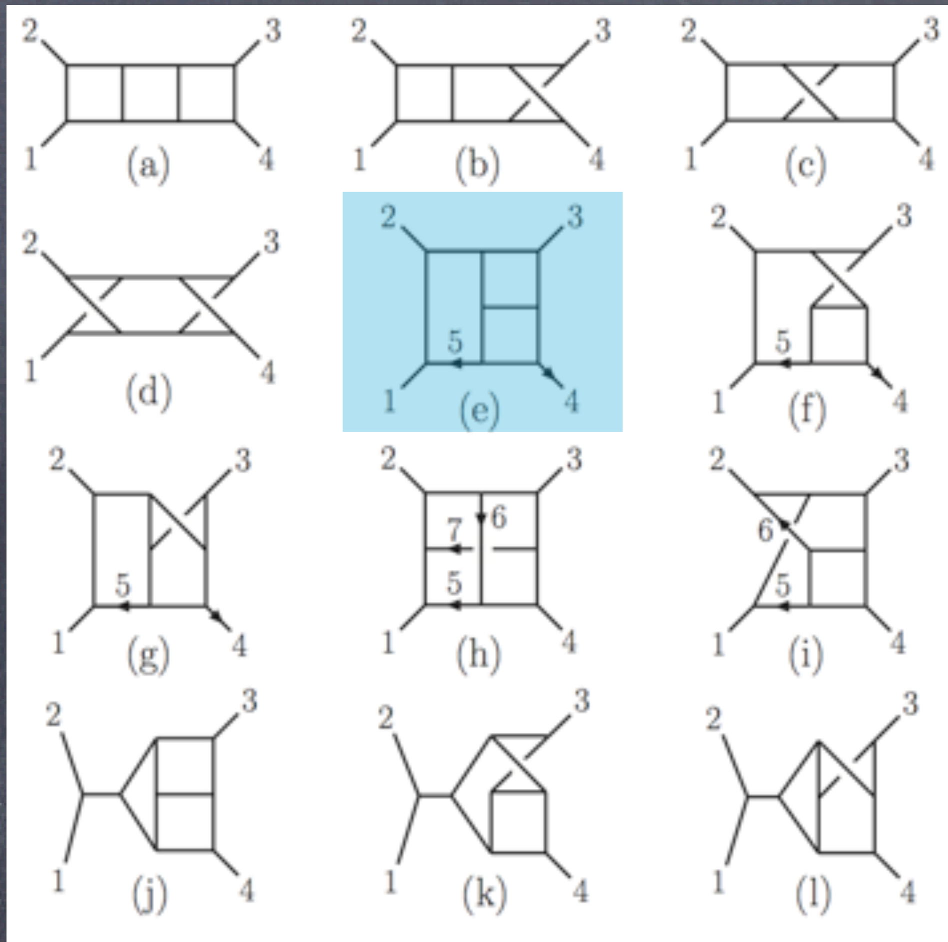
$$N^{(j)} = N^{(e)}(k_1, k_2, k_3, l_5, l_6, l_7) - N^{(e)}(k_2, k_1, k_3, l_5, l_6, l_7), \qquad (J_j)$$

$$N^{(k)} = N^{(f)}(k_1, k_2, k_3, l_5, l_6, l_7) - N^{(f)}(k_2, k_1, k_3, l_5, l_6, l_7), \qquad (J_k)$$

$$N^{(l)} = N^{(g)}(k_1, k_2, k_3, l_5, l_6, l_7) - N^{(g)}(k_2, k_1, k_3, l_5, l_6, l_7), \qquad (J_l)$$

$$N^{(m)} = 0, \qquad (J_m)$$

$$N^{(n)} = 0, \qquad (J_n)$$



BCJ (2010)

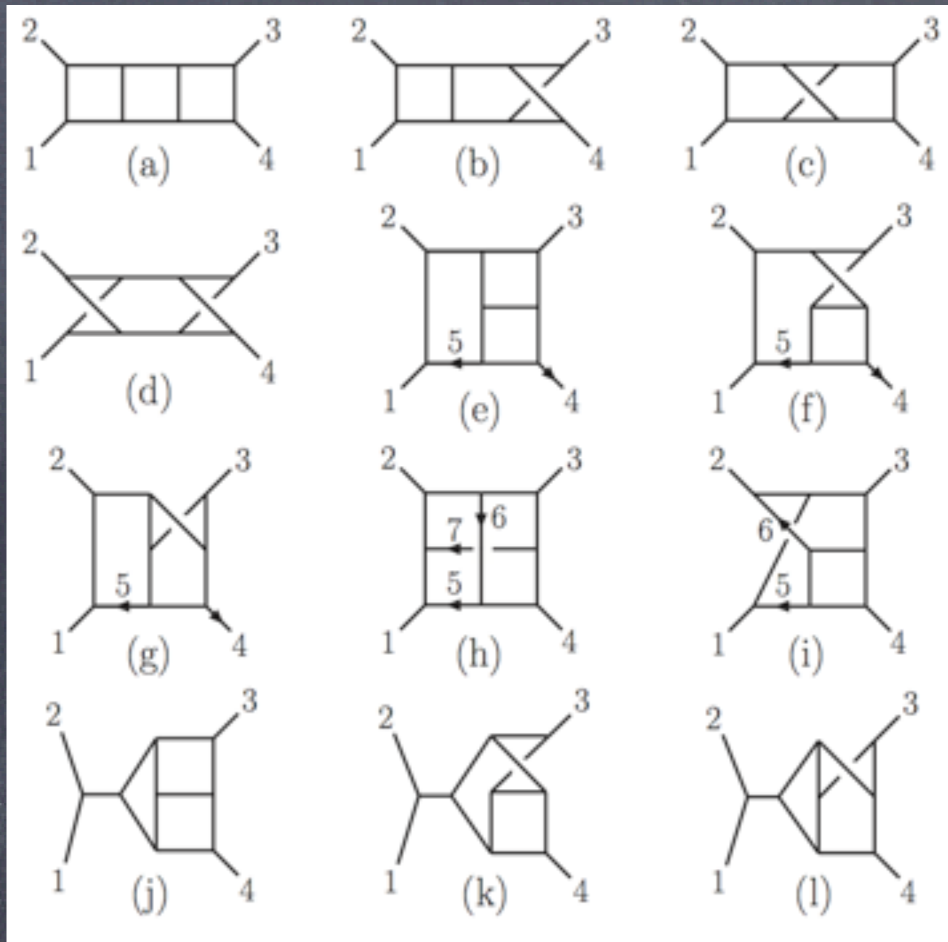
Solution is unique!

Only, e.g., require maximal cut information of (e) graph to build full amplitude!

Squaring numerators gives N=8 supergravity!

$$s = (k_1 + k_2)^2 \quad t = (k_1 + k_4)^2 \quad u = (k_1 + k_3)^2 \quad \tau_{i,j} = 2k_i \cdot l_j$$

Integral $I^{(x)}$	$\mathcal{N} = 4$ Super-Yang-Mills ($\sqrt{\mathcal{N} = 8 \text{ supergravity}}$) numerator
(a)–(d)	s^2
(e)–(g)	$\big(s\,(-\tau_{35} + \tau_{45} + t) - t\,(\tau_{25} + \tau_{45}) + u\,(\tau_{25} + \tau_{35}) - s^2\big)/3$
(h)	$\big(s\,(2\tau_{15} - \tau_{16} + 2\tau_{26} - \tau_{27} + 2\tau_{35} + \tau_{36} + \tau_{37} - u)$ $+t\,(\tau_{16} + \tau_{26} - \tau_{37} + 2\tau_{36} - 2\tau_{15} - 2\tau_{27} - 2\tau_{35} - 3\tau_{17}) + s^2\big)/3$
(i)	$\big(s\,(-\tau_{25} - \tau_{26} - \tau_{35} + \tau_{36} + \tau_{45} + 2t)$ $+t\,(\tau_{26} + \tau_{35} + 2\tau_{36} + 2\tau_{45} + 3\tau_{46}) + u\,\tau_{25} + s^2\big)/3$
(j)-(l)	$s(t-u)/3$



Note:

BOTH N=4 sYM and N=8 sugra

manifestly have same overall powercounting!

$s = (k_1 + k_2)^2 \quad t = (k_1 + k_4)^2 \quad u = (k_1 + k_3)^2 \quad \tau_{i,j} = 2k_i \cdot l_j$

Integral $I^{(x)}$	$\mathcal{N} = 4$ Super-Yang-Mills ($\sqrt{\mathcal{N} = 8 \text{ supergravity}}$) numerator
(a)–(d)	s^2
(e)–(g)	$\left(s\,(-\tau_{35} + \tau_{45} + t) - t\,(\tau_{25} + \tau_{45}) + u\,(\tau_{25} + \tau_{35}) - s^2\right)/3$
(h)	$\left(s\,(2\tau_{15} - \tau_{16} + 2\tau_{26} - \tau_{27} + 2\tau_{35} + \tau_{36} + \tau_{37} - u)\right.$ $\left.+t\,(\tau_{16} + \tau_{26} - \tau_{37} + 2\tau_{36} - 2\tau_{15} - 2\tau_{27} - 2\tau_{35} - 3\tau_{17}) + s^2\right)/3$
(i)	$\left(s\,(-\tau_{25} - \tau_{26} - \tau_{35} + \tau_{36} + \tau_{45} + 2t)\right.$ $\left.+t\,(\tau_{26} + \tau_{35} + 2\tau_{36} + 2\tau_{45} + 3\tau_{46}) + u\,\tau_{25} + s^2\right)/3$
(j)-(l)	$s(t-u)/3$

BCJ (2010)

Integral	$\mathcal{N}=4$ Yang-Mills
(a)–(d)	s^2
(e)–(g)	$s(l_1+k_4)^2$
(h)	$s(l_1+l_2)^2+t(l_3+l_4)^2 - sl_5^2 - tl_6^2 - st$
(i)	$s(l_1+l_2)^2-t(l_3+l_4)^2 -\frac{1}{3}(s-t)l_5^2$

This works too!
(non-trivial check)

$$\sum_{\mathcal{G}\in\text{cubic}}\int\prod_{l=1}^{L}\frac{d^Dp_l}{(2\pi)^D}\frac{1}{S(\mathcal{G})}\frac{n(\mathcal{G})\tilde{n}(\mathcal{G})}{D(\mathcal{G})}$$

$$\tau_{i,j}=2k_i\cdot l_j$$

Integral $I^{(x)}$	$\mathcal{N}=4$ Super-Yang-Mills ($\sqrt{\mathcal{N}=8\text{ supergravity}}$) numerator
(a)–(d)	s^2
(e)–(g)	$\big(s(-\tau_{35}+\tau_{45}+t)-t(\tau_{25}+\tau_{45})+u(\tau_{25}+\tau_{35})-s^2\big)/3$
(h)	$\big(s(2\tau_{15}-\tau_{16}+2\tau_{26}-\tau_{27}+2\tau_{35}+\tau_{36}+\tau_{37}-u) +t(\tau_{16}+\tau_{26}-\tau_{37}+2\tau_{36}-2\tau_{15}-2\tau_{27}-2\tau_{35}-3\tau_{17})+s^2\big)/3$
(i)	$\big(s(-\tau_{25}-\tau_{26}-\tau_{35}+\tau_{36}+\tau_{45}+2t) +t(\tau_{26}+\tau_{35}+2\tau_{36}+2\tau_{45}+3\tau_{46})+u\,\tau_{25}+s^2\big)/3$
(j)-(l)	$s(t-u)/3$

Integral	$\mathcal{N}=4$ Yang-Mills
(a)–(d)	s^2
(e)–(g)	$s(l_1+k_4)^2$
(h)	$s(l_1+l_2)^2+t(l_3+l_4)^2 - sl_5^2 - tl_6^2 - st$
(i)	$s(l_1+l_2)^2-t(l_3+l_4)^2 -\frac{1}{3}(s-t)l_5^2$

This works too!
(non-trivial check)

$$\sum_{\mathcal{G}\in \text{cubic}} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S(\mathcal{G})} \frac{n(\mathcal{G})\tilde{n}(\mathcal{G})}{D(\mathcal{G})}$$

$$\tau_{i,j} = 2k_i \cdot l_j$$

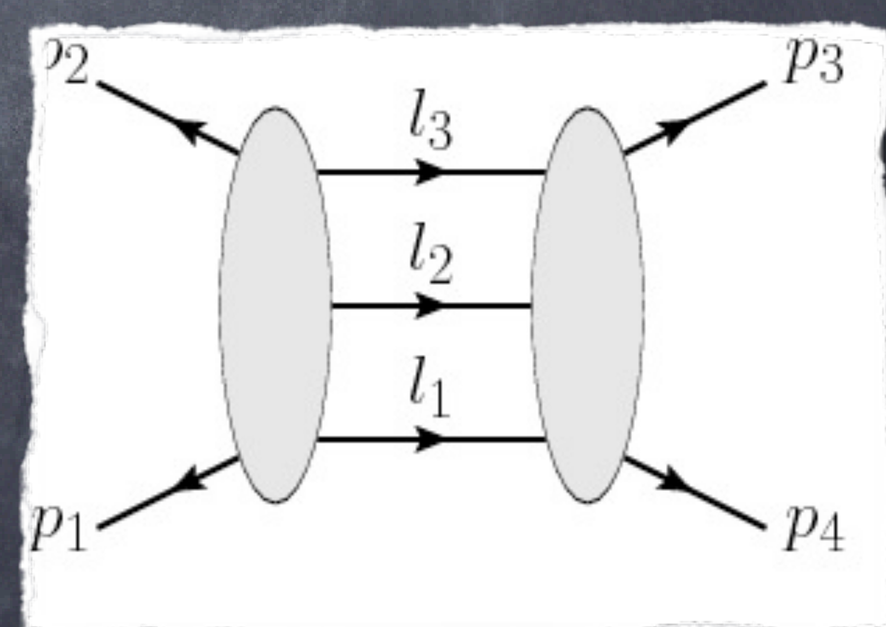
Integral $I^{(x)}$	$\mathcal{N}=4$ Super-Yang-Mills ($\sqrt{\mathcal{N}=8 \text{ supergravity}}$) numerator
(a)–(d)	s^2
(e)–(g)	$\left(s\left(-\tau_{35}+\tau_{45}+t\right)-t\left(\tau_{25}+\tau_{45}\right)+u\left(\tau_{25}+\tau_{35}\right)-s^2\right)/3$
(h)	$\left(s\left(2\tau_{15}-\tau_{16}+2\tau_{26}-\tau_{27}+2\tau_{35}+\tau_{36}+\tau_{37}-u\right) +t\left(\tau_{16}+\tau_{26}-\tau_{37}+2\tau_{36}-2\tau_{15}-2\tau_{27}-2\tau_{35}-3\tau_{17}\right)+s^2\right)/3$
(i)	$\left(s\left(-\tau_{25}-\tau_{26}-\tau_{35}+\tau_{36}+\tau_{45}+2t\right) +t\left(\tau_{26}+\tau_{35}+2\tau_{36}+2\tau_{45}+3\tau_{46}\right)+u\,\tau_{25}+s^2\right)/3$
(j)-(l)	$s(t-u)/3$

Intermezzo: How do we know of amplitude is correct?

ANSWER:

- Integrand satisfies **all** **D-dimensiona** generalized unitarity cuts.

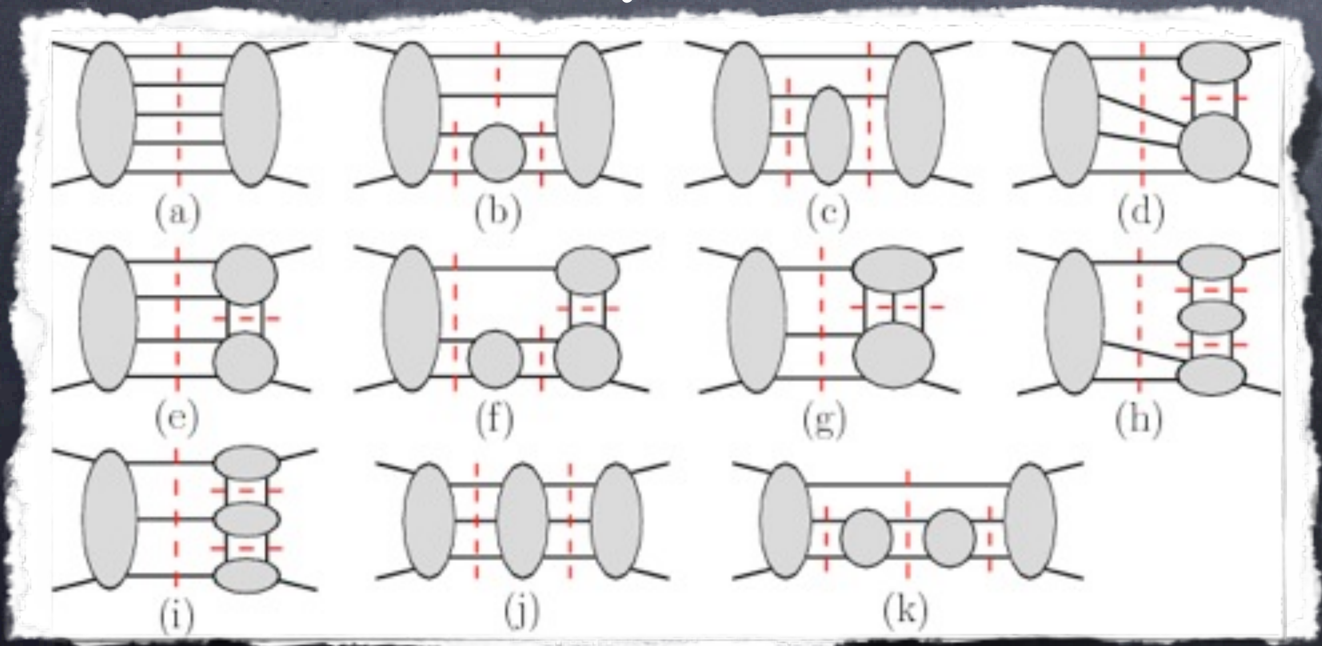
Bern, Dixon and Kosower



Correct?

all cuts:

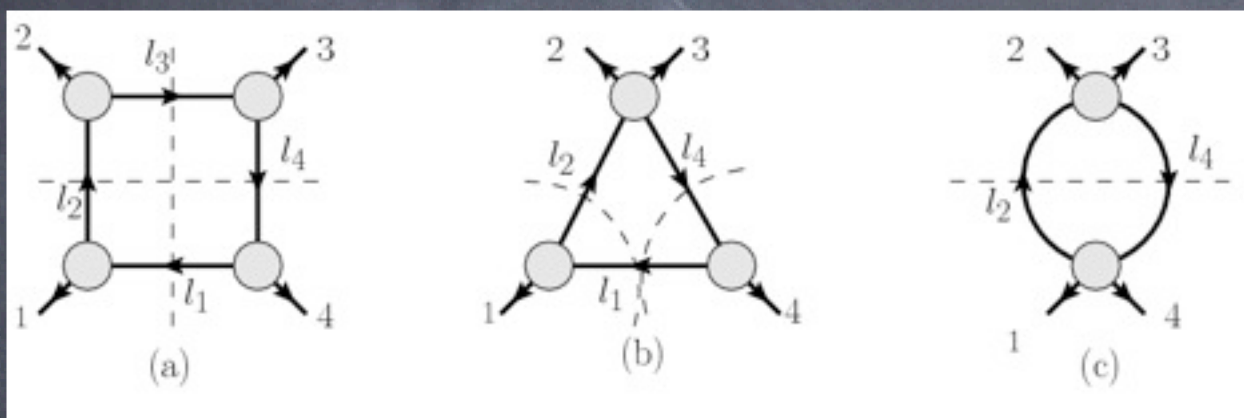
- Leaves no topologies untouched for Feynman rule contributions to be hiding in.
- **spanning set:** any set sufficient to guarantee satisfaction of all cuts given the theory

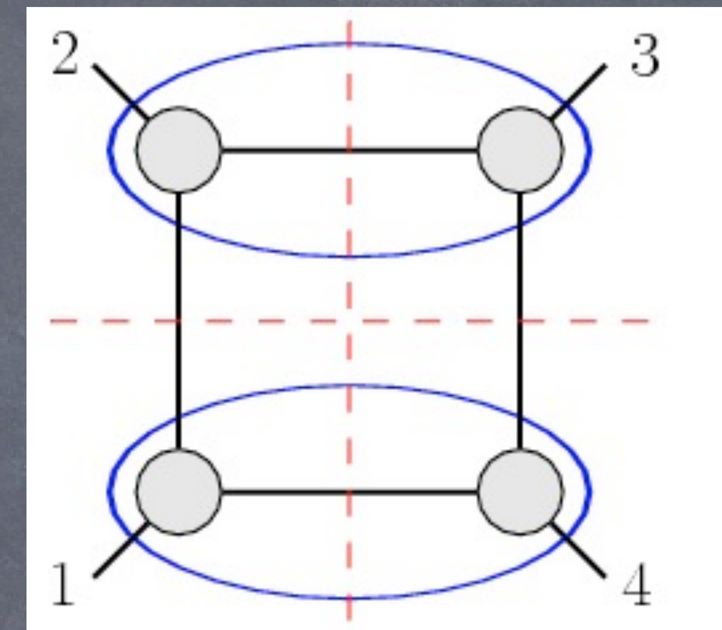


Bern, JJMC, Dixon, Johansson, Roiban (2010)

Correct?

- **D-dimensional:**





Venerable: N=1 in 10D

(as tree multiplicity increases expressions can be unwieldy)

New Shiny: N=2 in 6D

Cheung, O'Connell; Dennen, Huang, Siegel; Boels; Bern, JJMC, Dennen, Huang, Ita

Super New Shiny: N=1 in 10D

Caron-Hout, O'Connell;

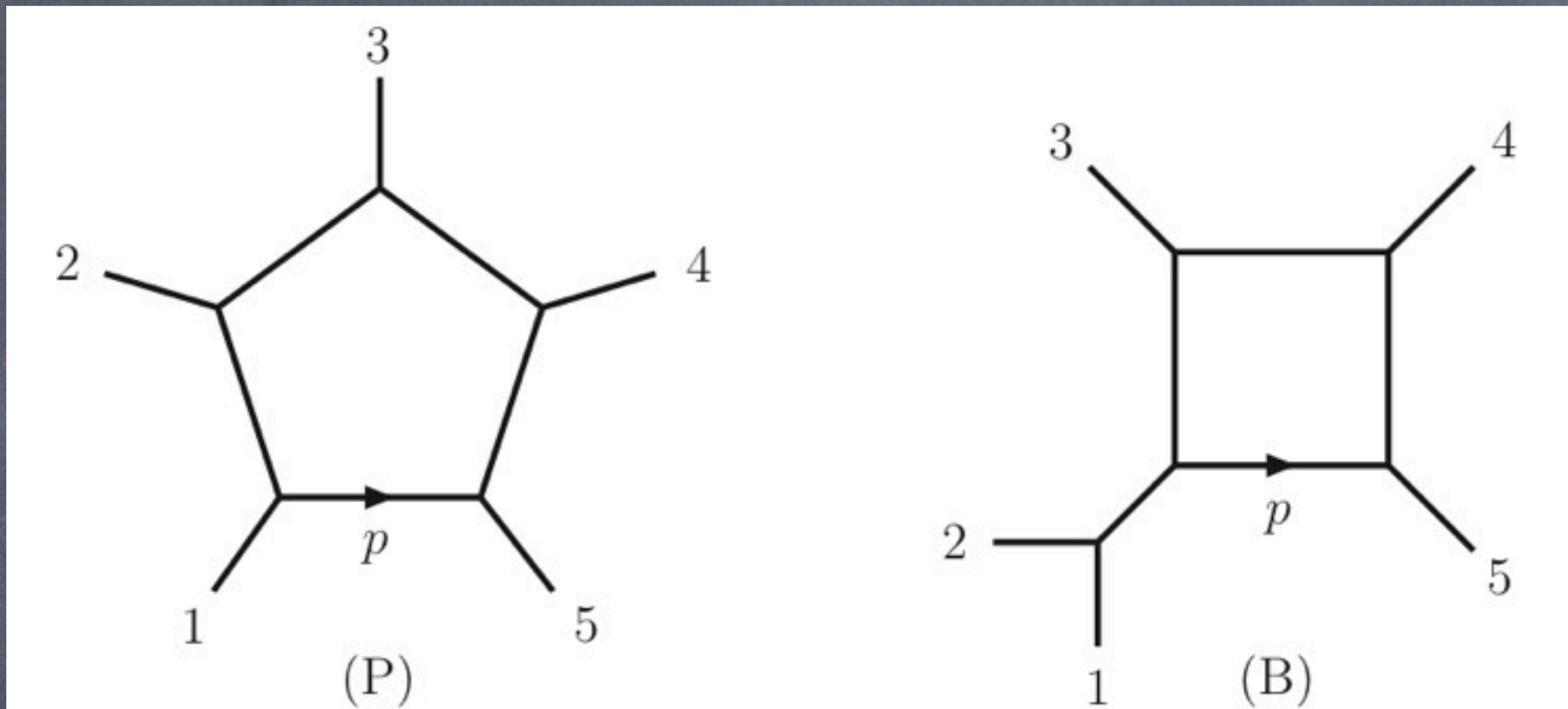
Solved D-dim. cuts special to maximal susy: Iterated 2-particle, Box, Pentacuts

Bern, JJMC, Dixon, Johansson, Roiban; Broedel, JJMC

Ok -- we've seen it work through three-loops -- anywhere else?

JJMC, Johansson (2011)

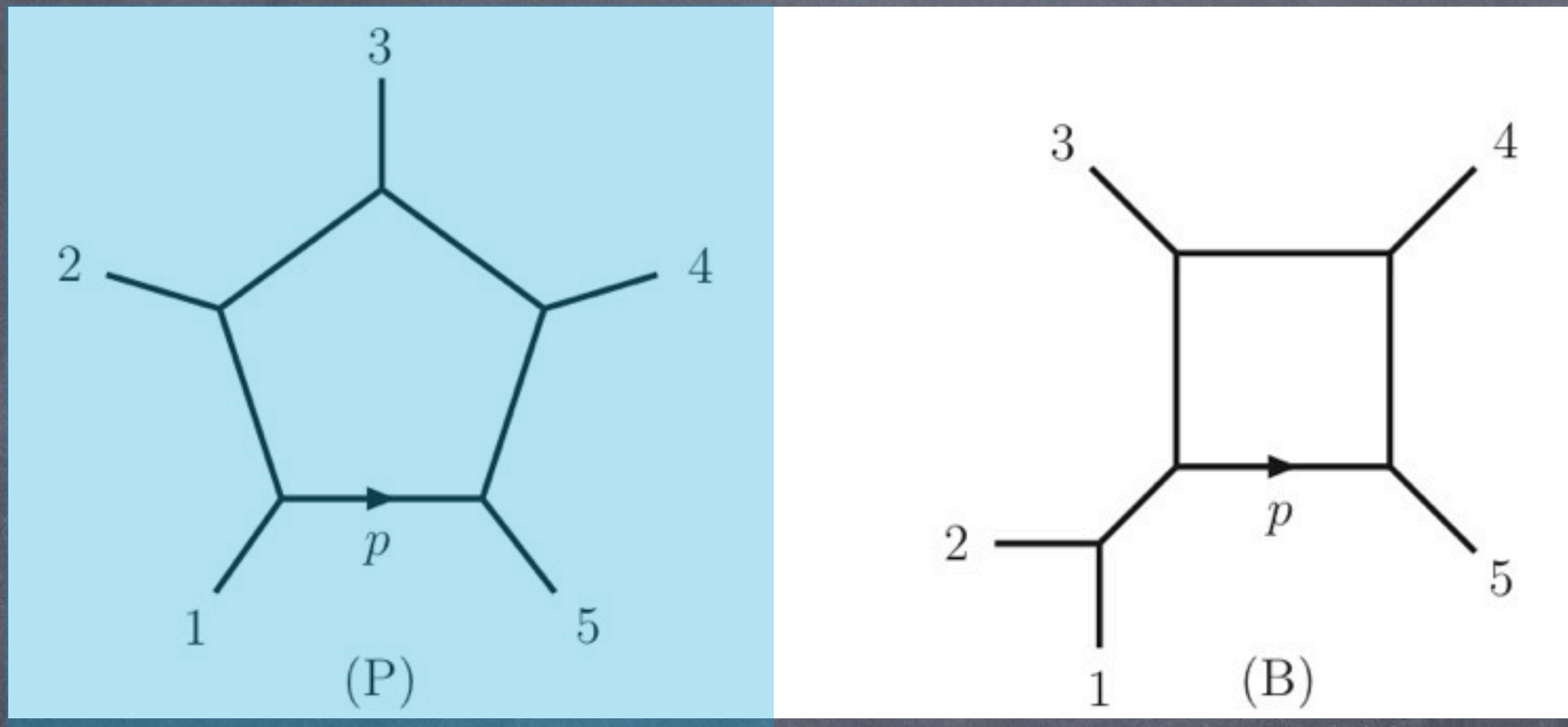
Five point 1-loop N=4 SYM & N=8 SUGRA



Venerable form satisfies duality (no freedom)

Bern, Dixon, Dunbar, Kosower;
Cachazo

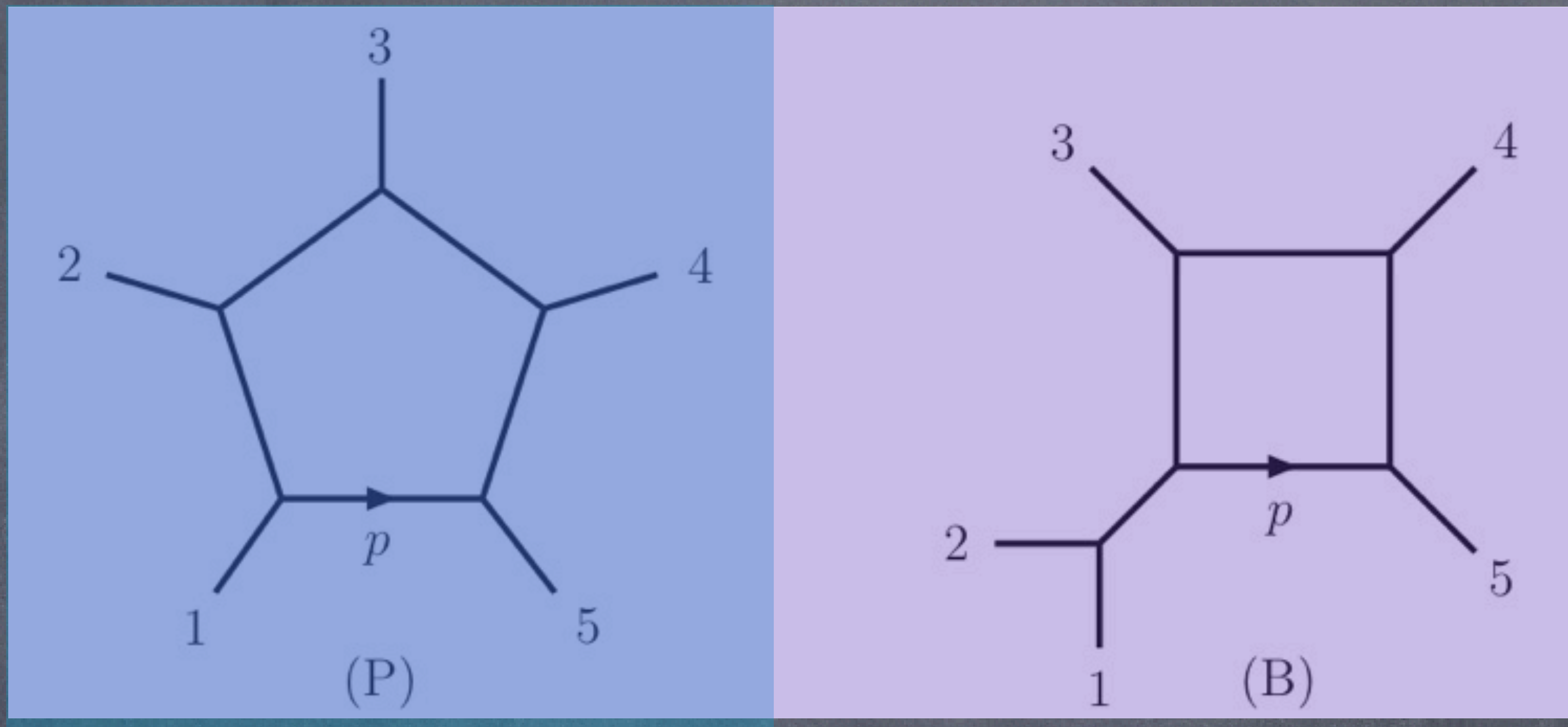
Five point 1-loop N=4 SYM & N=8 SUGRA



Venerable form satisfies duality (no freedom)

Bern, Dixon, Dunbar, Kosower;
Cachazo

Five point 1-loop N=4 SYM & N=8 SUGRA

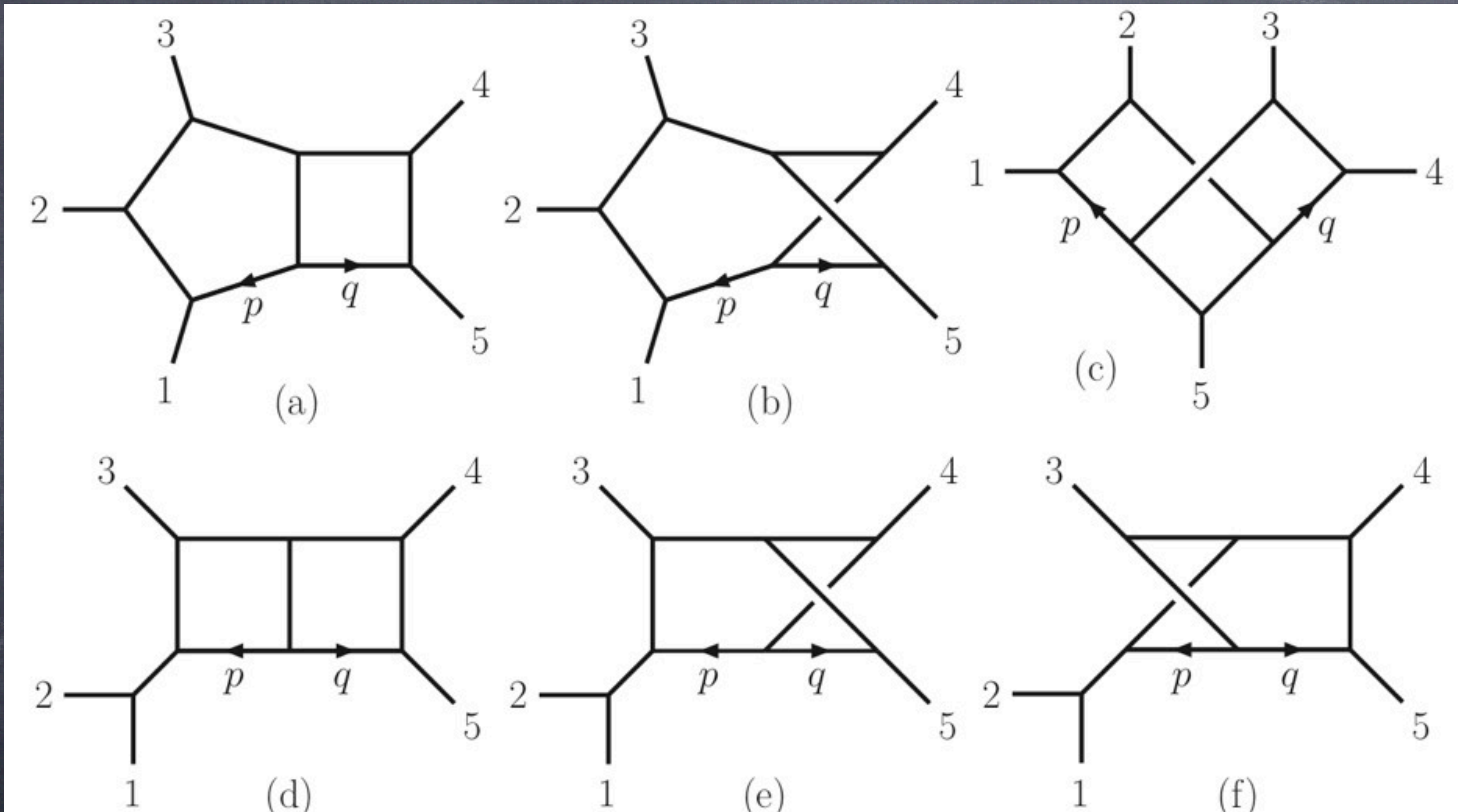


Venerable form satisfies duality (no freedom)

Bern, Dixon, Dunbar, Kosower;
Cachazo

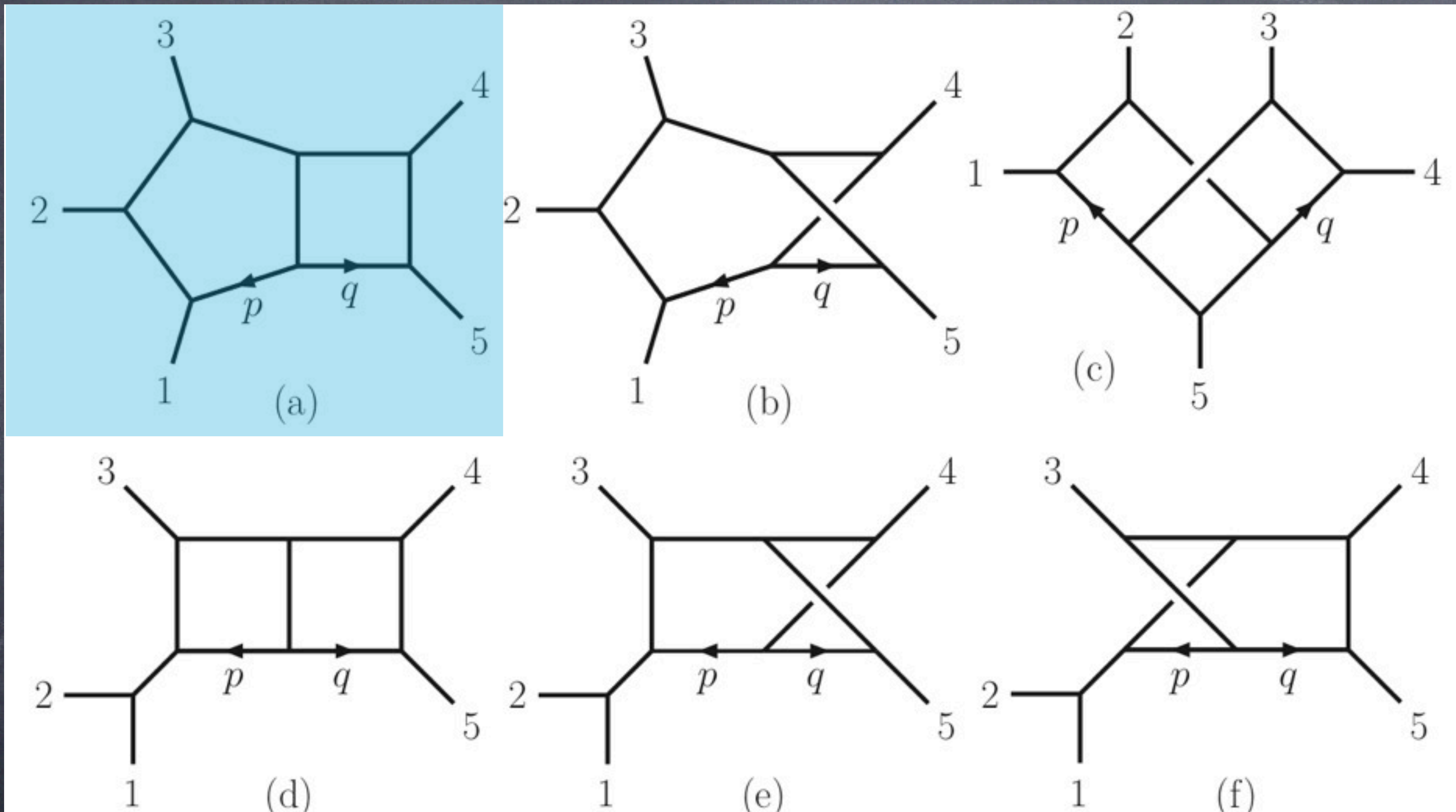
JJMC, Johansson (2011)

Five point 2-loop N=4 SYM & N=8 SUGRA



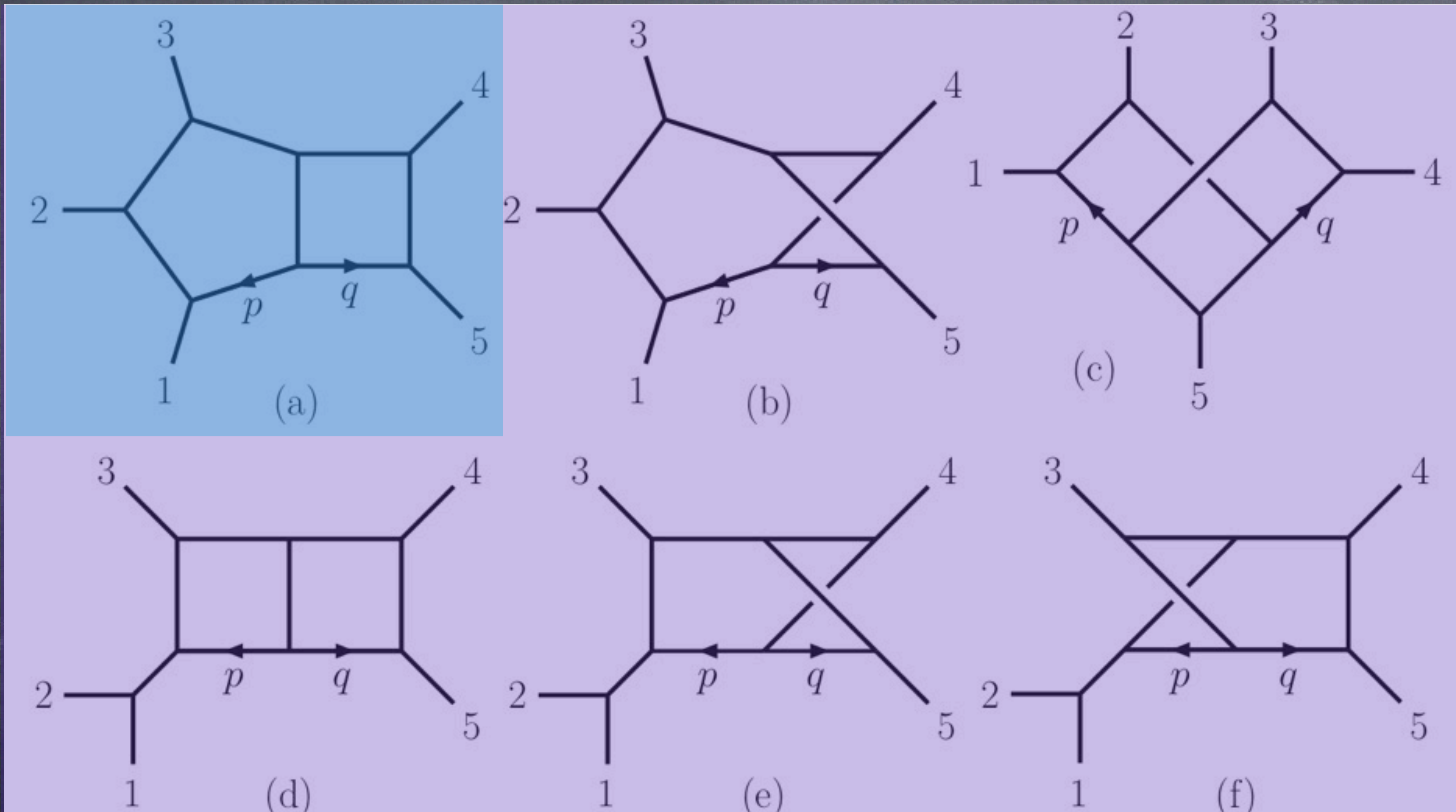
JJMC, Johansson (2011)

Five point 2-loop N=4 SYM & N=8 SUGRA



JJMC, Johansson (2011)

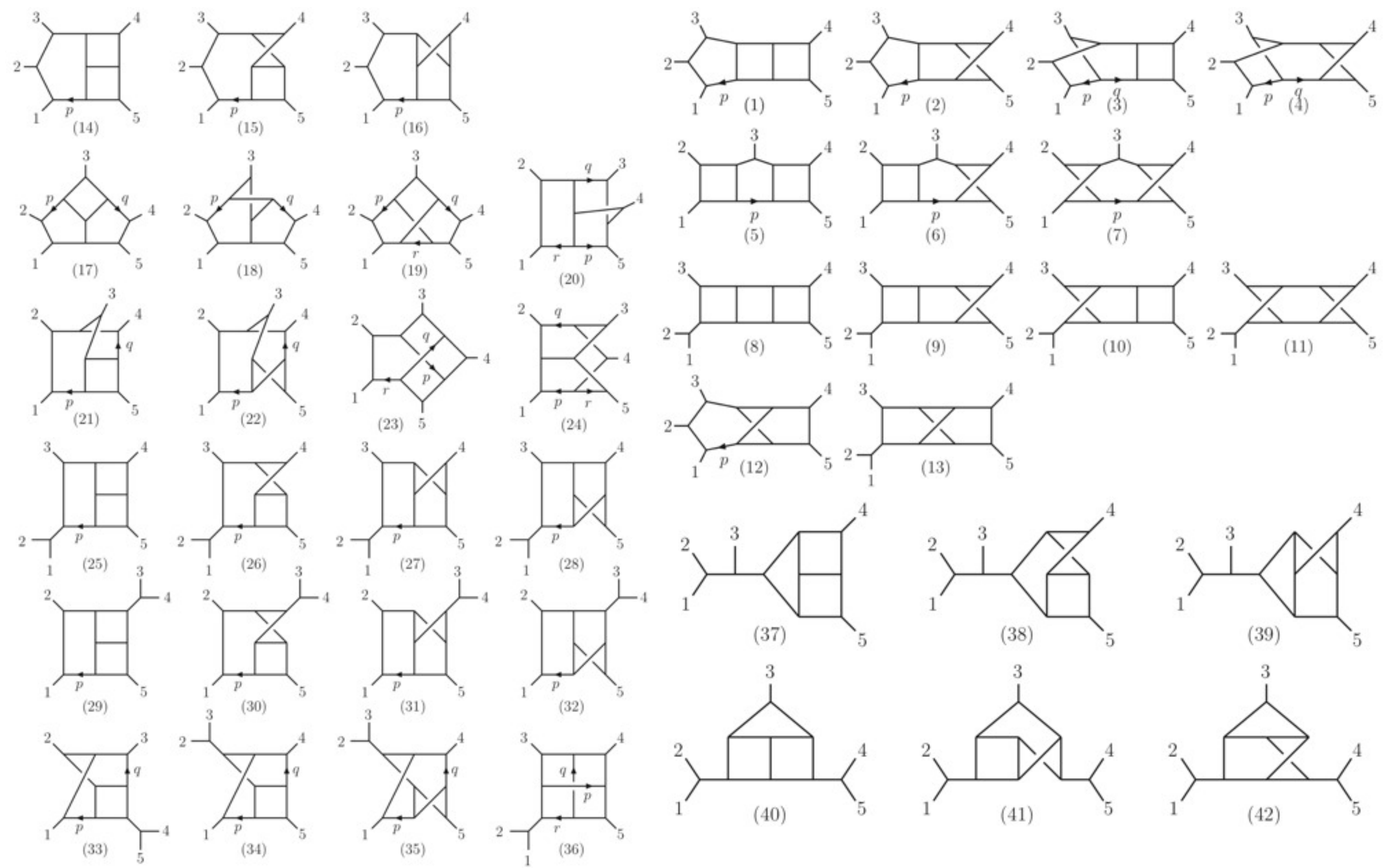
Five point 2-loop N=4 SYM & N=8 SUGRA



well -- that's it for published N=4, but here's a preview of results to come...

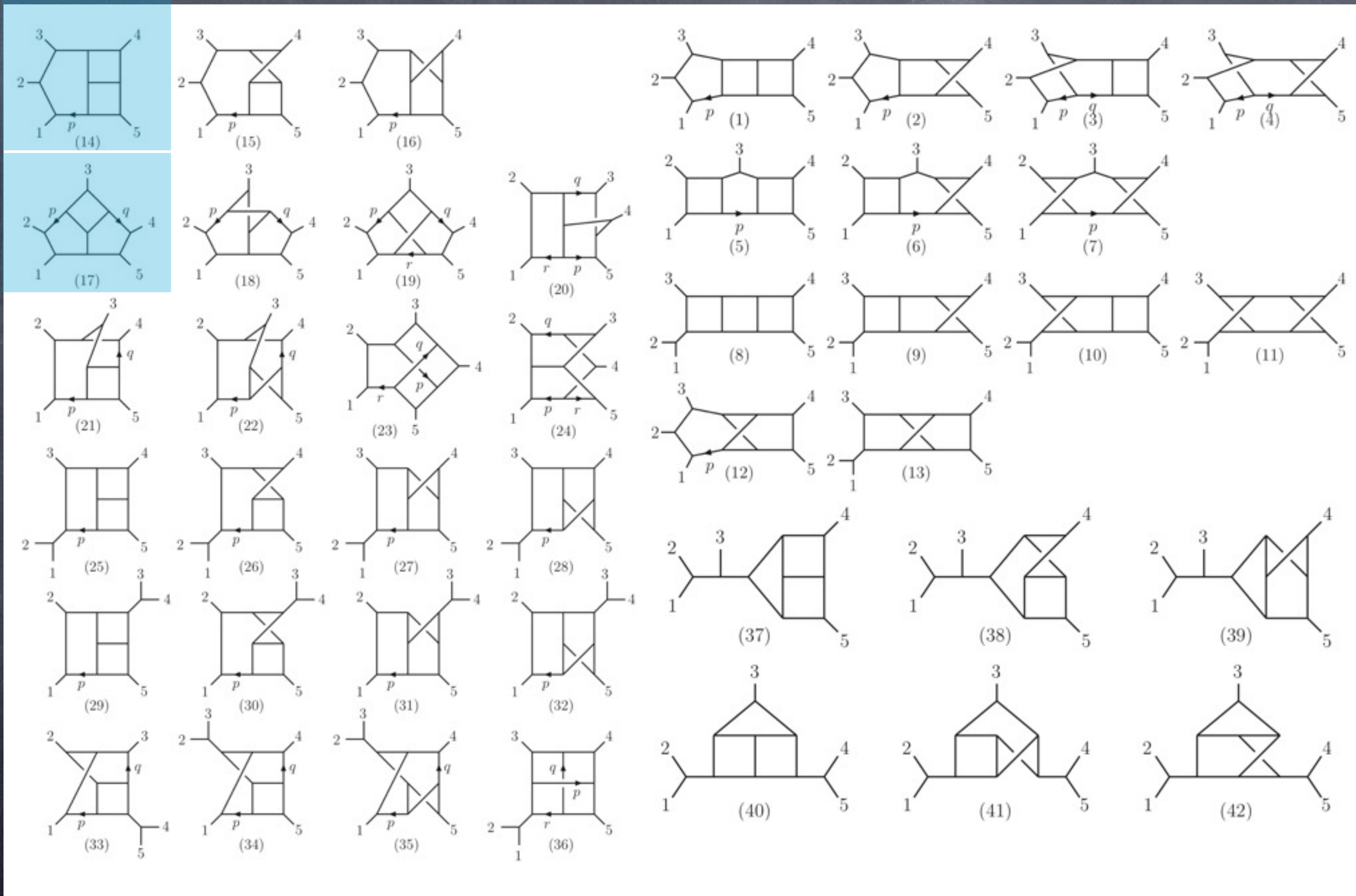
Five point 3-loop N=4 SYM & N=8 SUGRA

JJMC, Johansson (to appear)



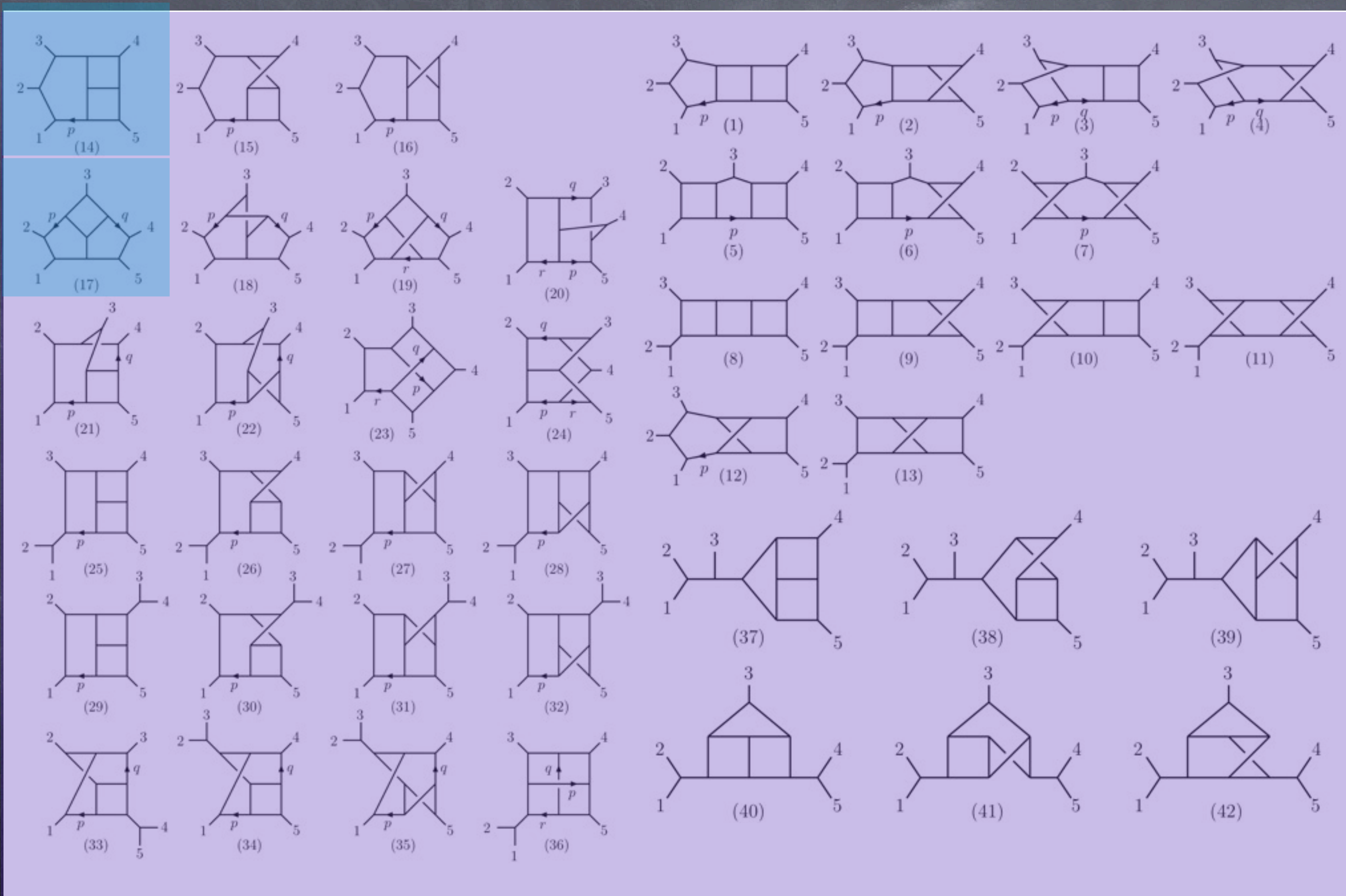
Five point 3-loop N=4 SYM & N=8 SUGRA

JJMC, Johansson (to appear)



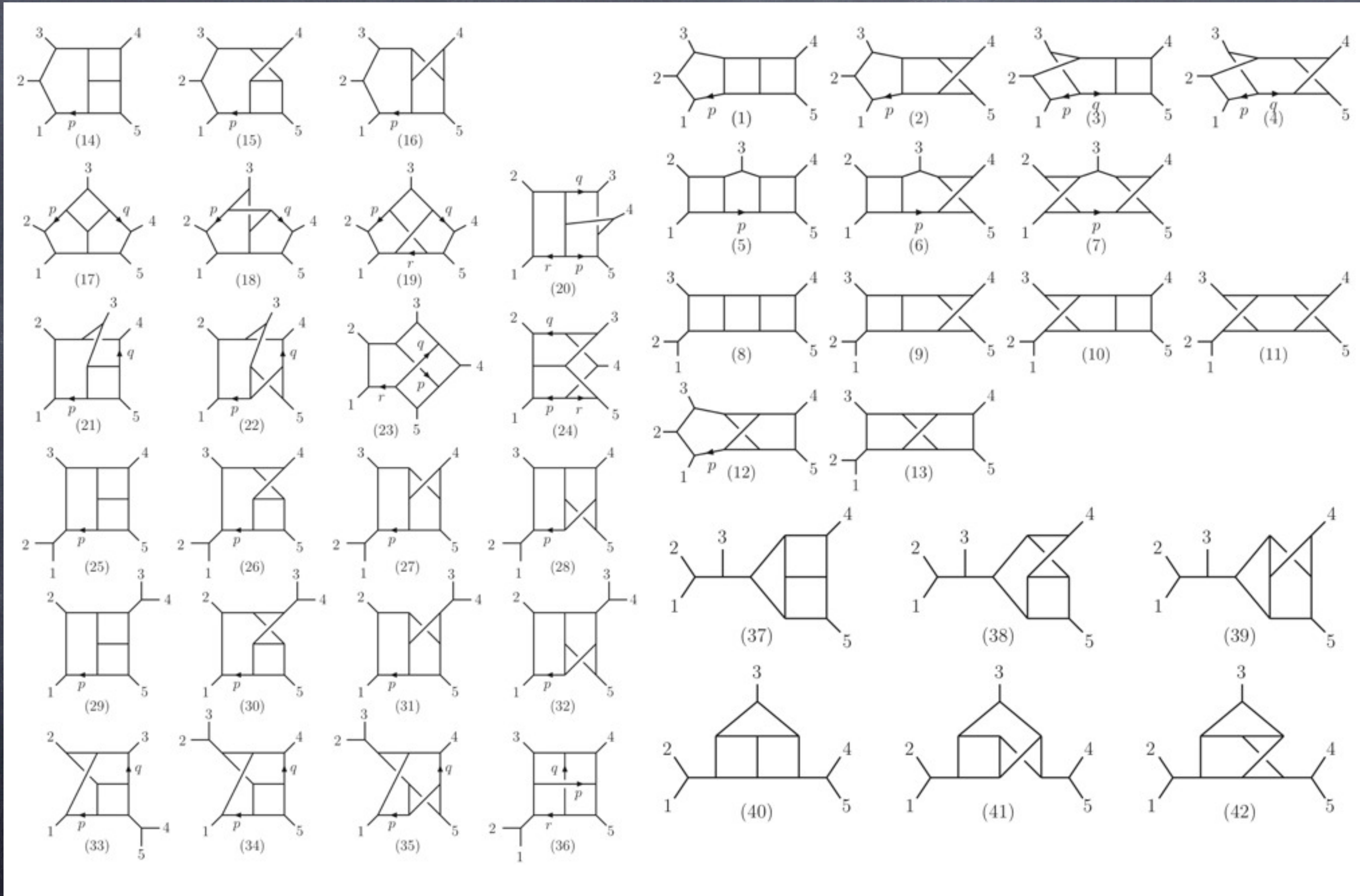
Five point 3-loop N=4 SYM & N=8 SUGRA

JJMC, Johansson (to appear)



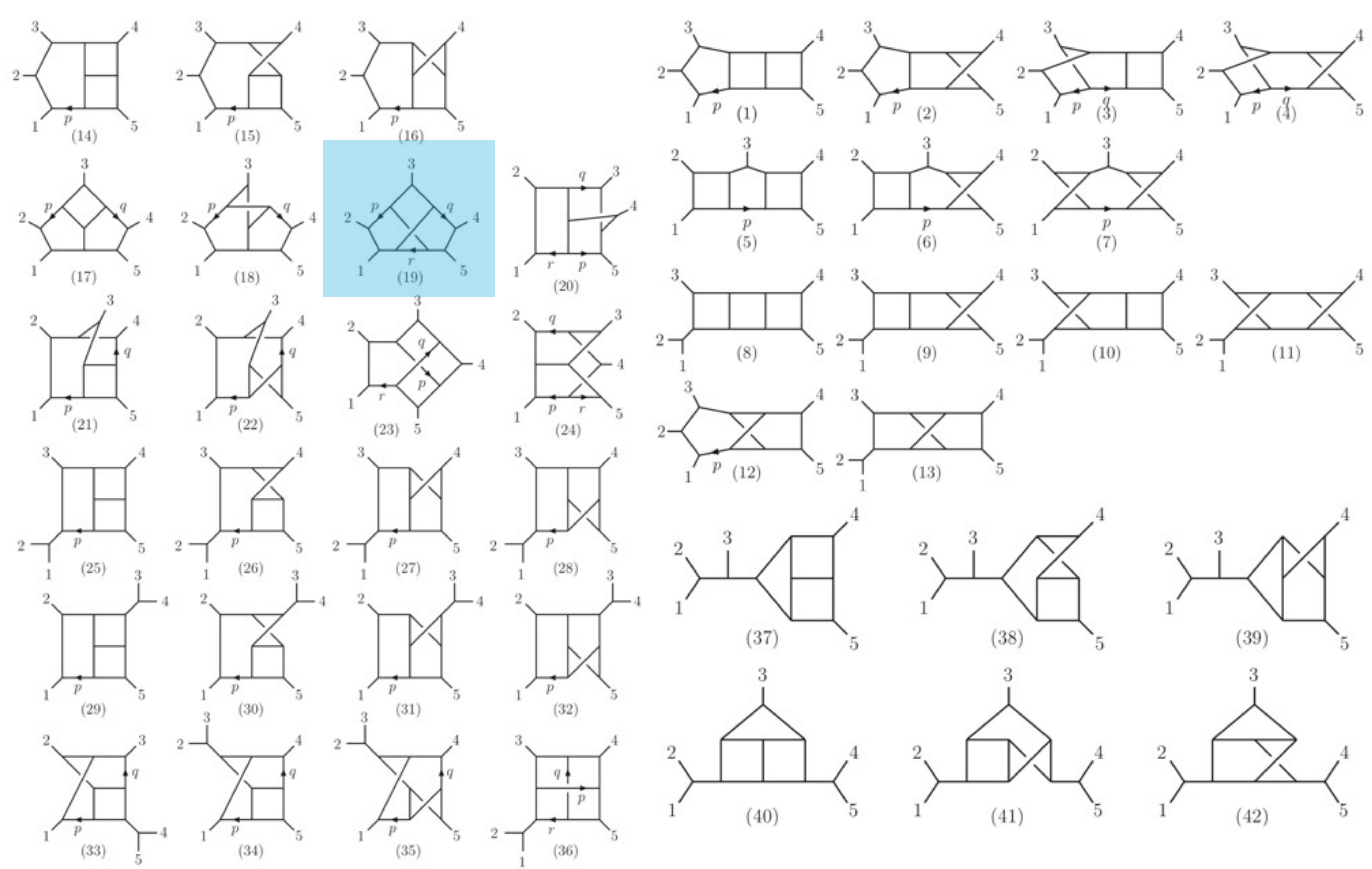
Five point 3-loop N=4 SYM & N=8 SUGRA

JJMC, Johansson (to appear)



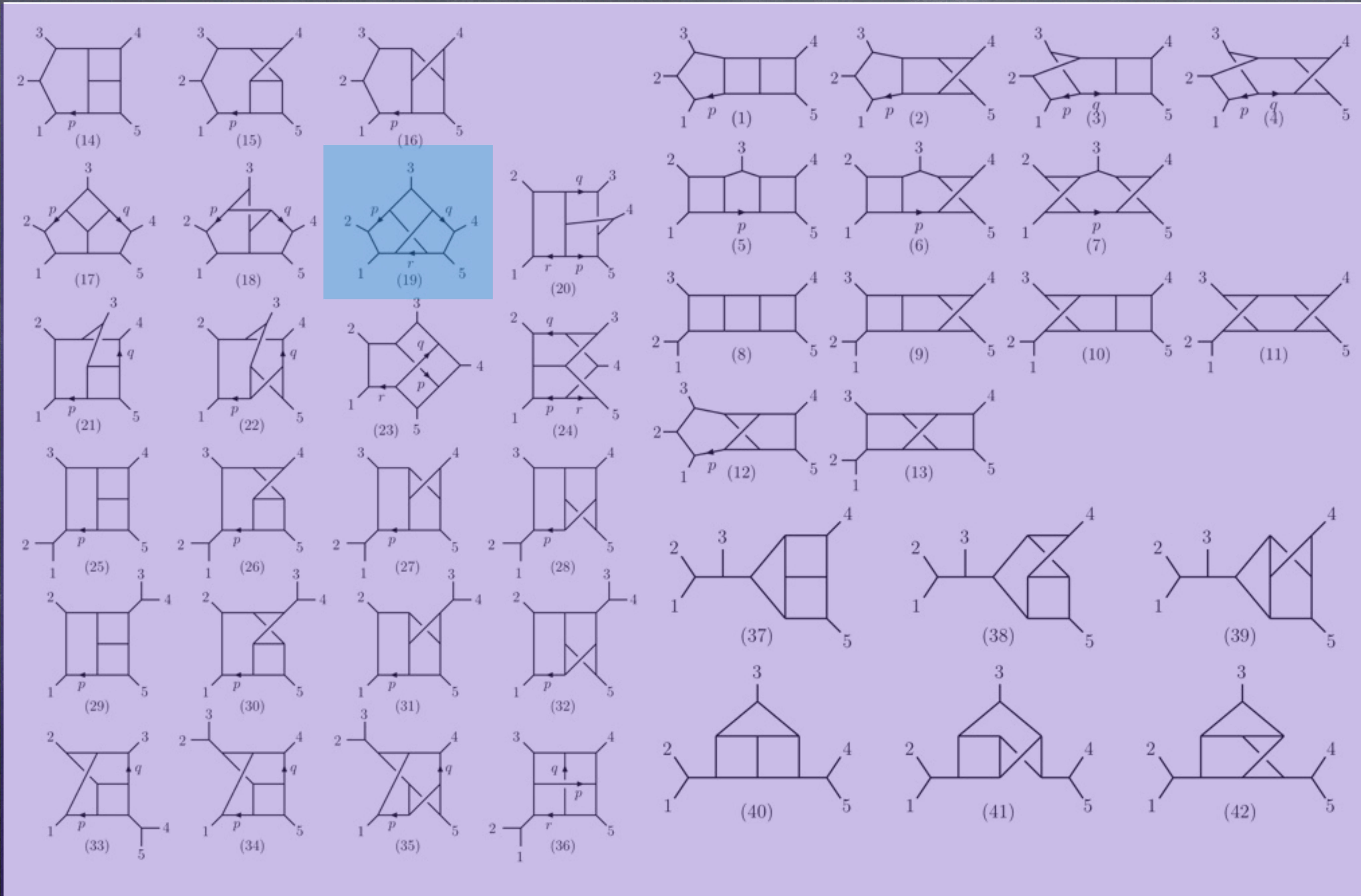
Five point 3-loop N=4 SYM & N=8 SUGRA

JJMC, Johansson (to appear)

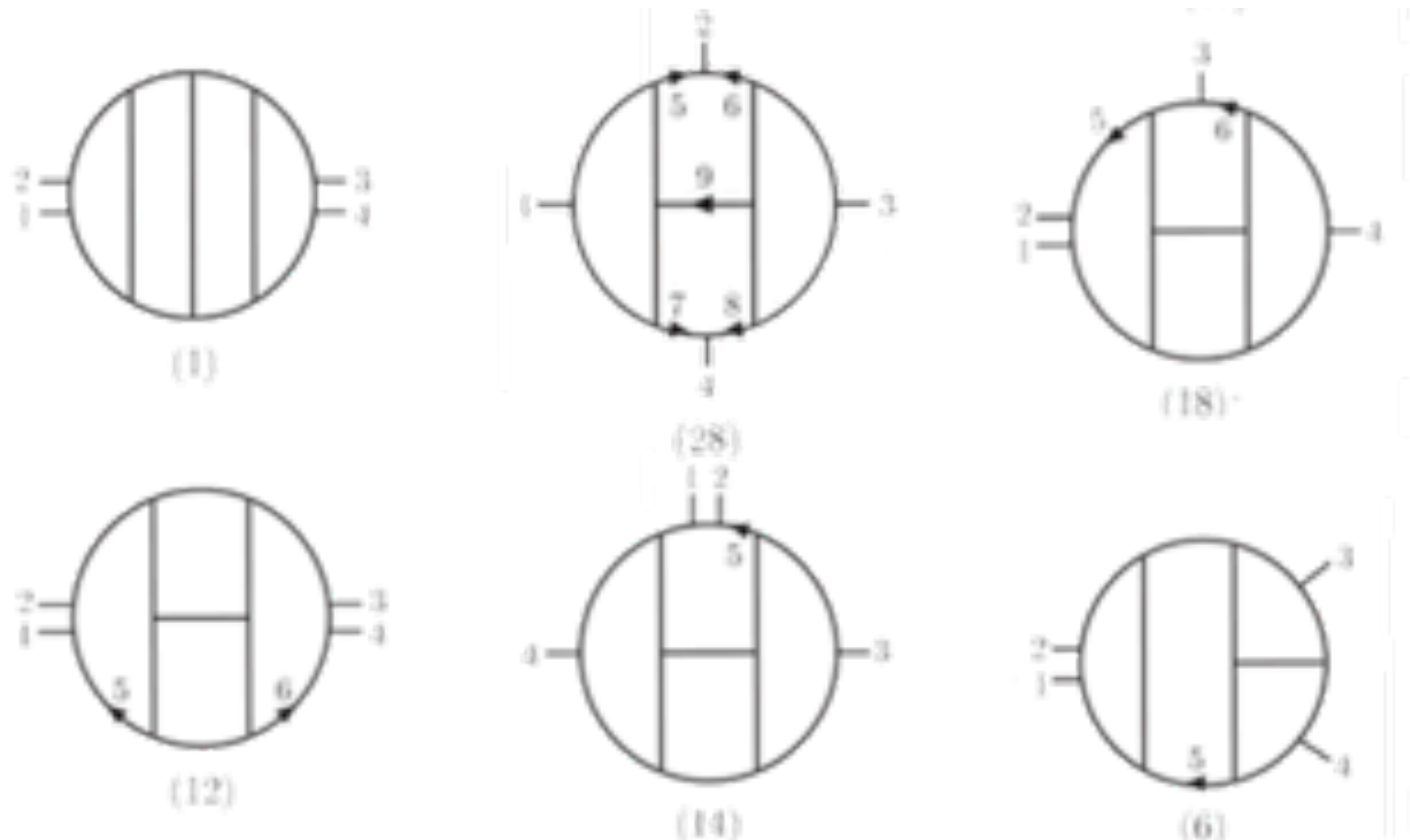


Five point 3-loop N=4 SYM & N=8 SUGRA

JJMC, Johansson (to appear)



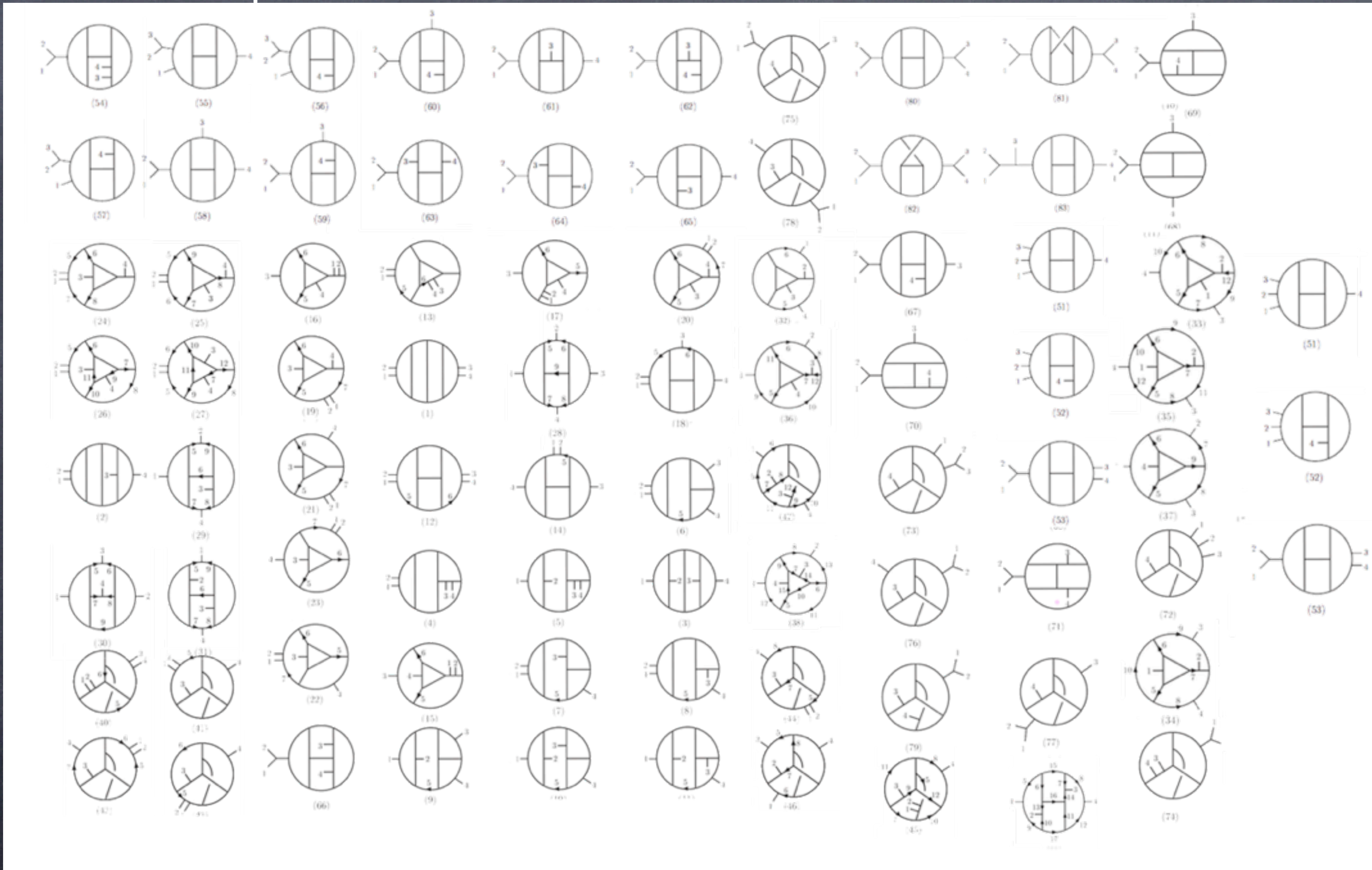
Four loop planar (extracted cusp anom. dim)



Bern, Czakon, Dixon, Kosower, Smirnov (2006)

Full four loop N=4 SYM & N=8 SUGRA

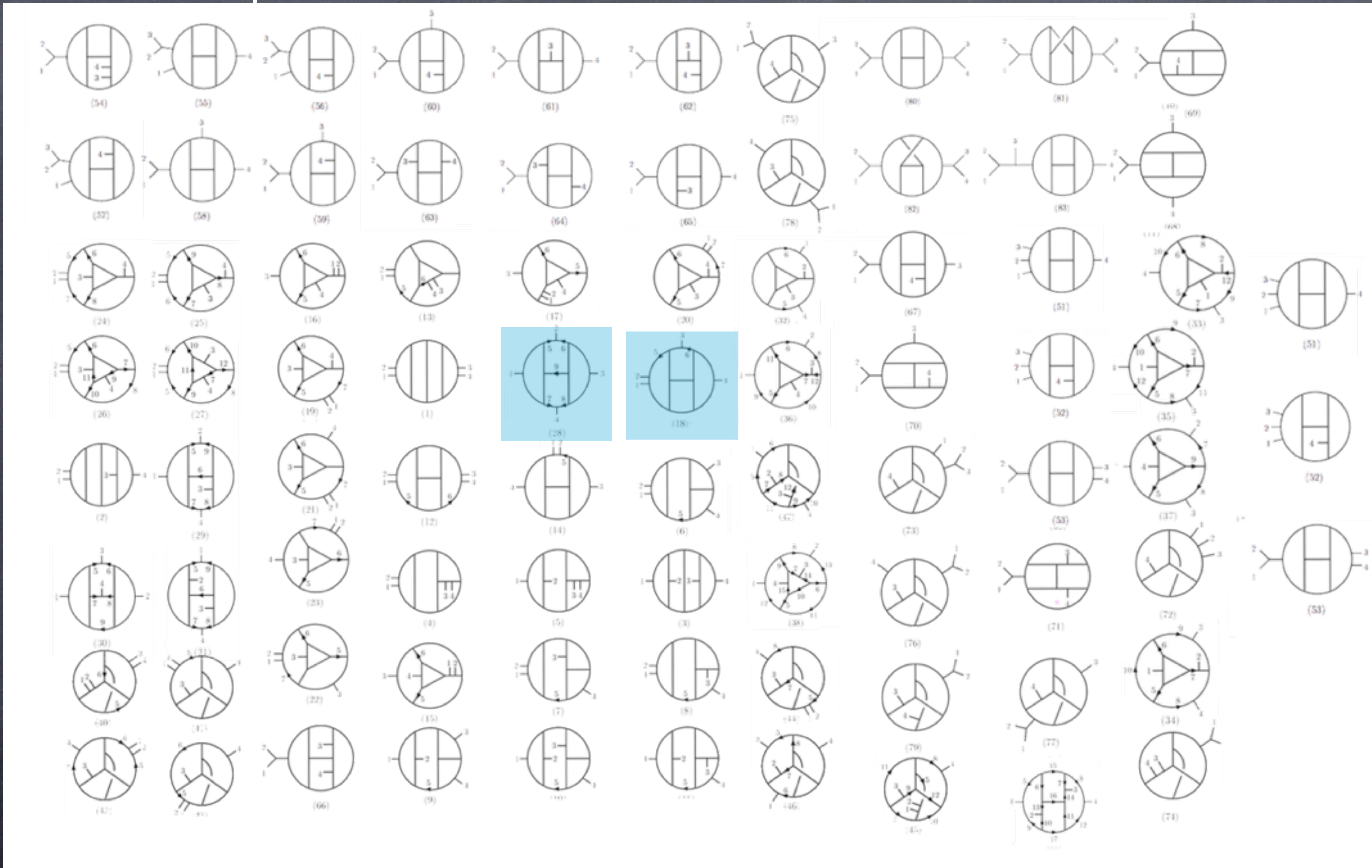
Bern, JJMC, Dixon, Johansson, Roiban



(to appear)

Full four loop N=4 SYM & N=8 SUGRA

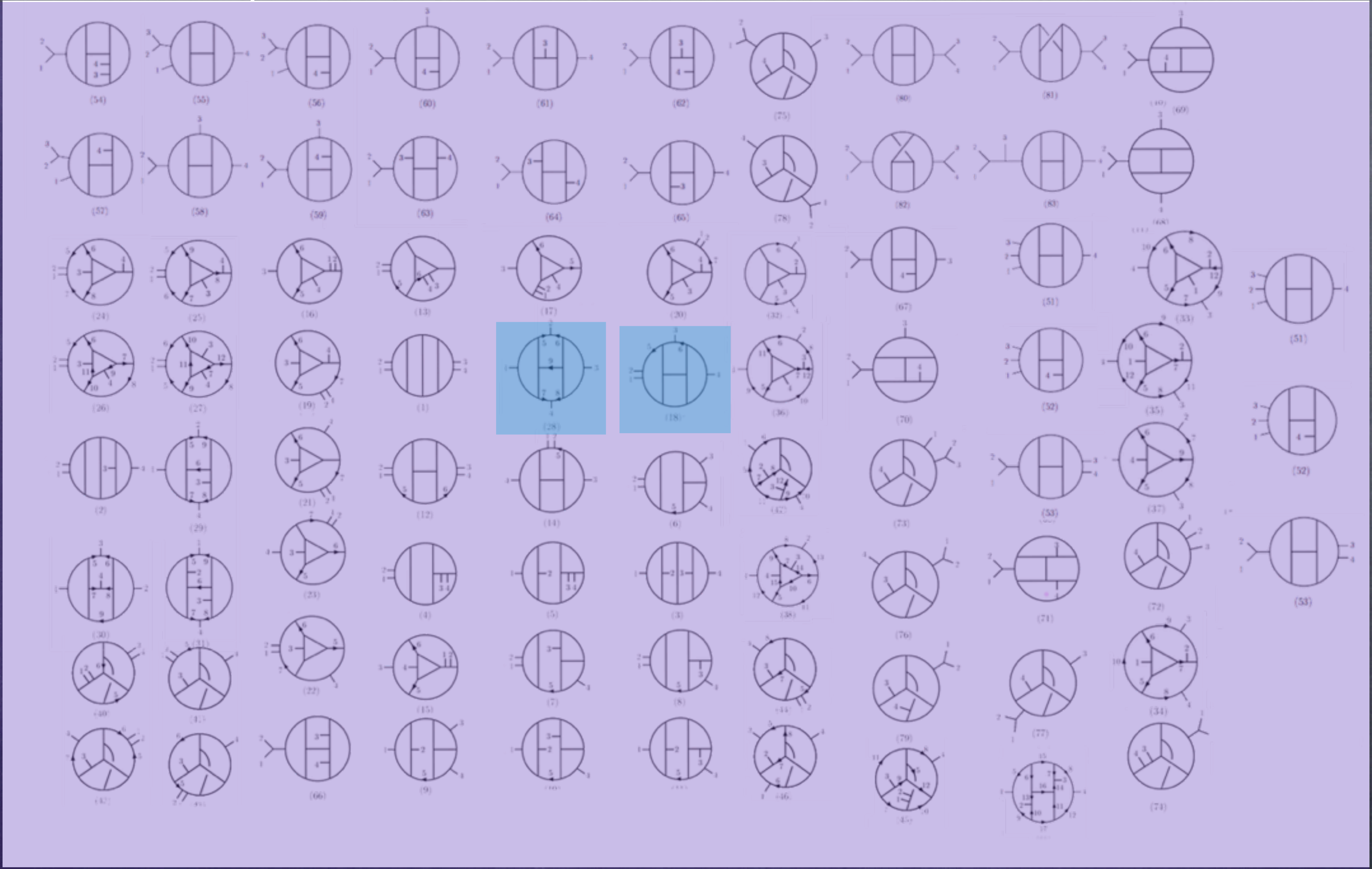
Bern, JJMC, Dixon, Johansson, Roiban



(to appear)

Full four loop N=4 SYM & N=8 SUGRA

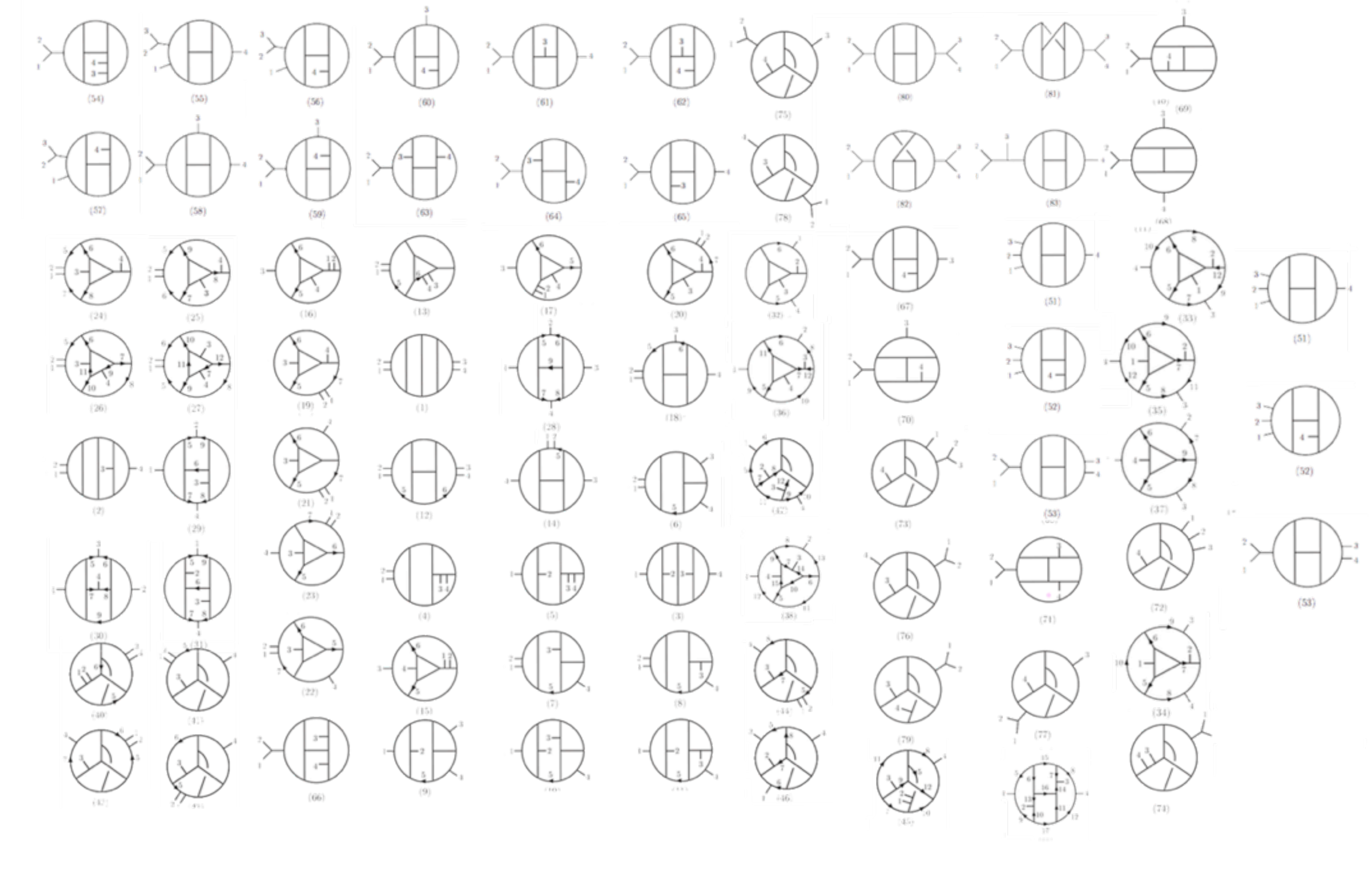
Bern, JJMC, Dixon, Johansson, Roiban



(to appear)

Full four loop N=4 SYM & N=8 SUGRA

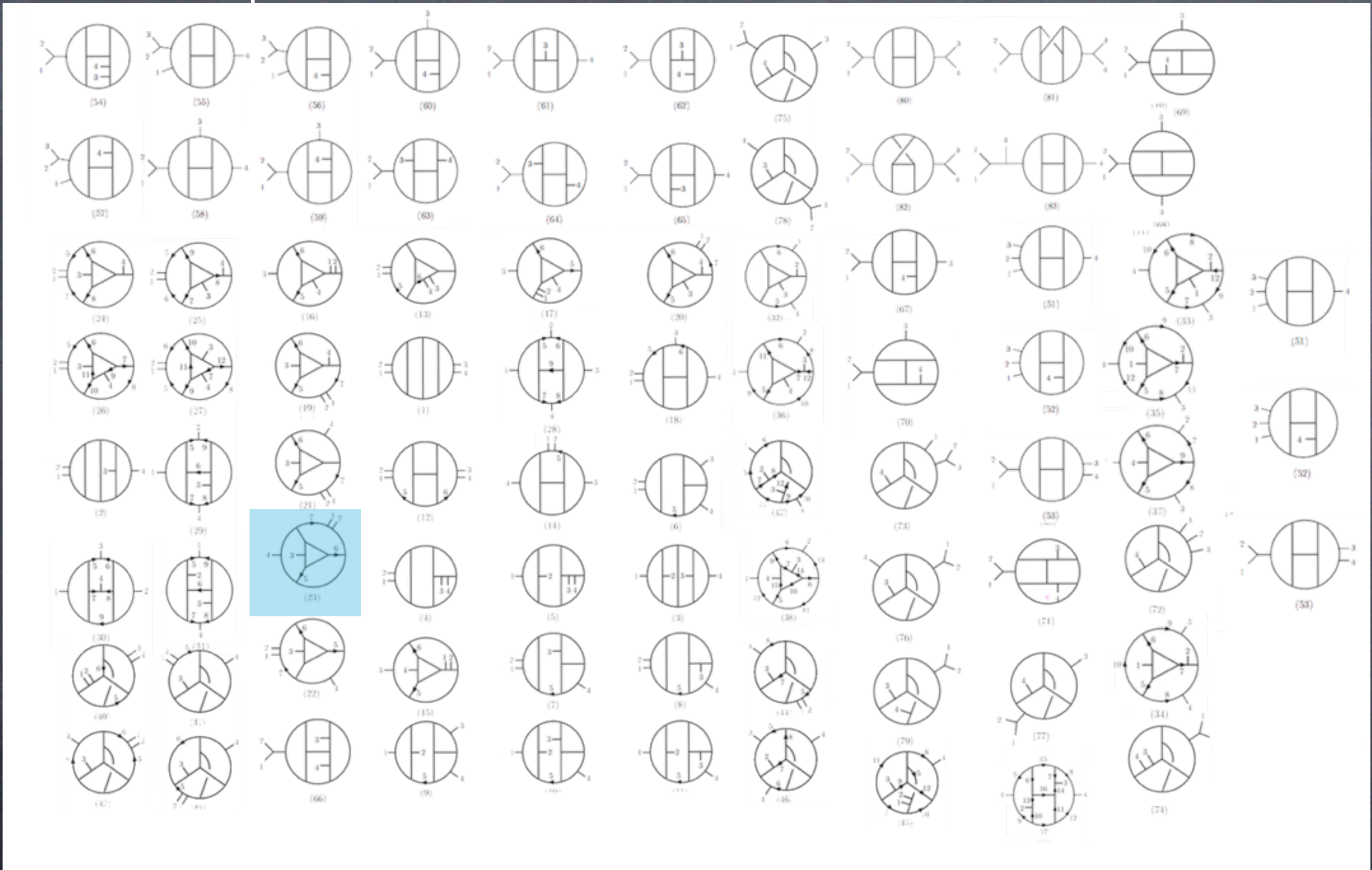
Bern, JJMC, Dixon, Johansson, Roiban



(to appear)

Full four loop N=4 SYM & N=8 SUGRA

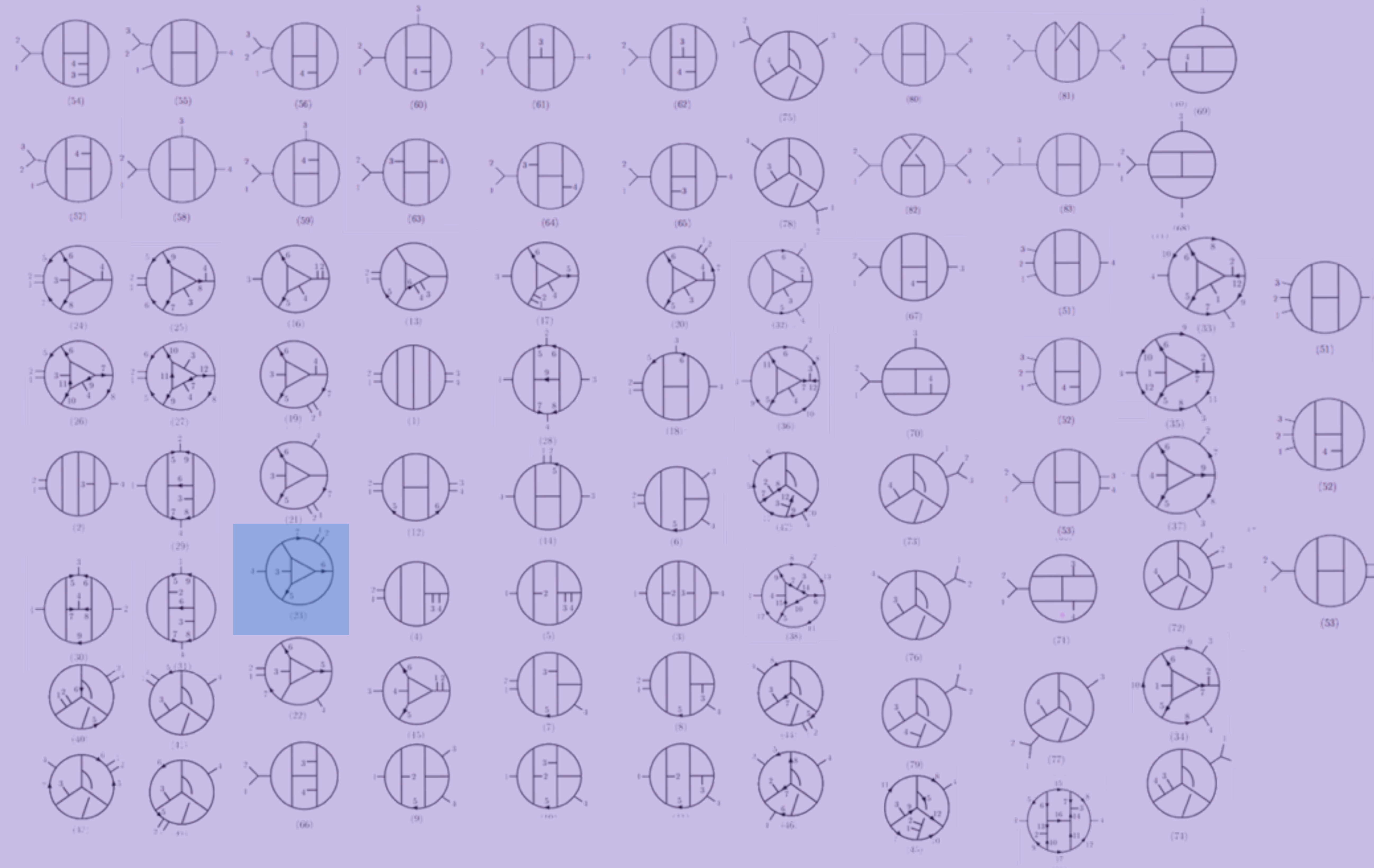
Bern, JJMC, Dixon, Johansson, Roiban



(to appear)

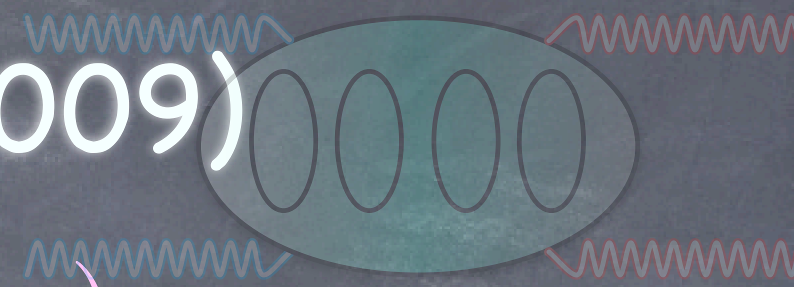
Full four loop N=4 SYM & N=8 SUGRA

Bern, JJMC, Dixon, Johansson, Roiban

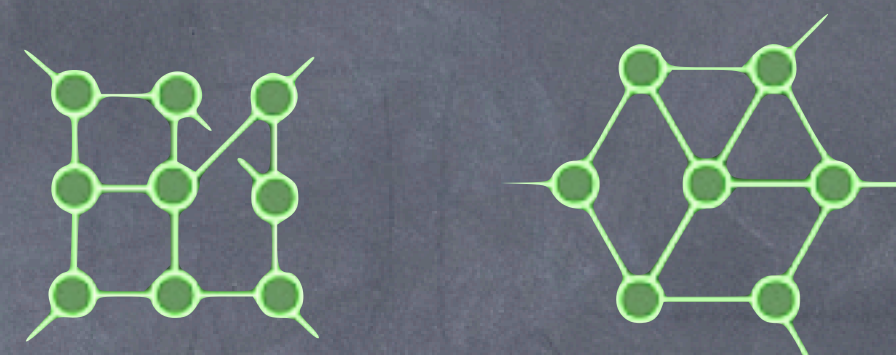


(to appear)

Contrast with BCDJR (2009)



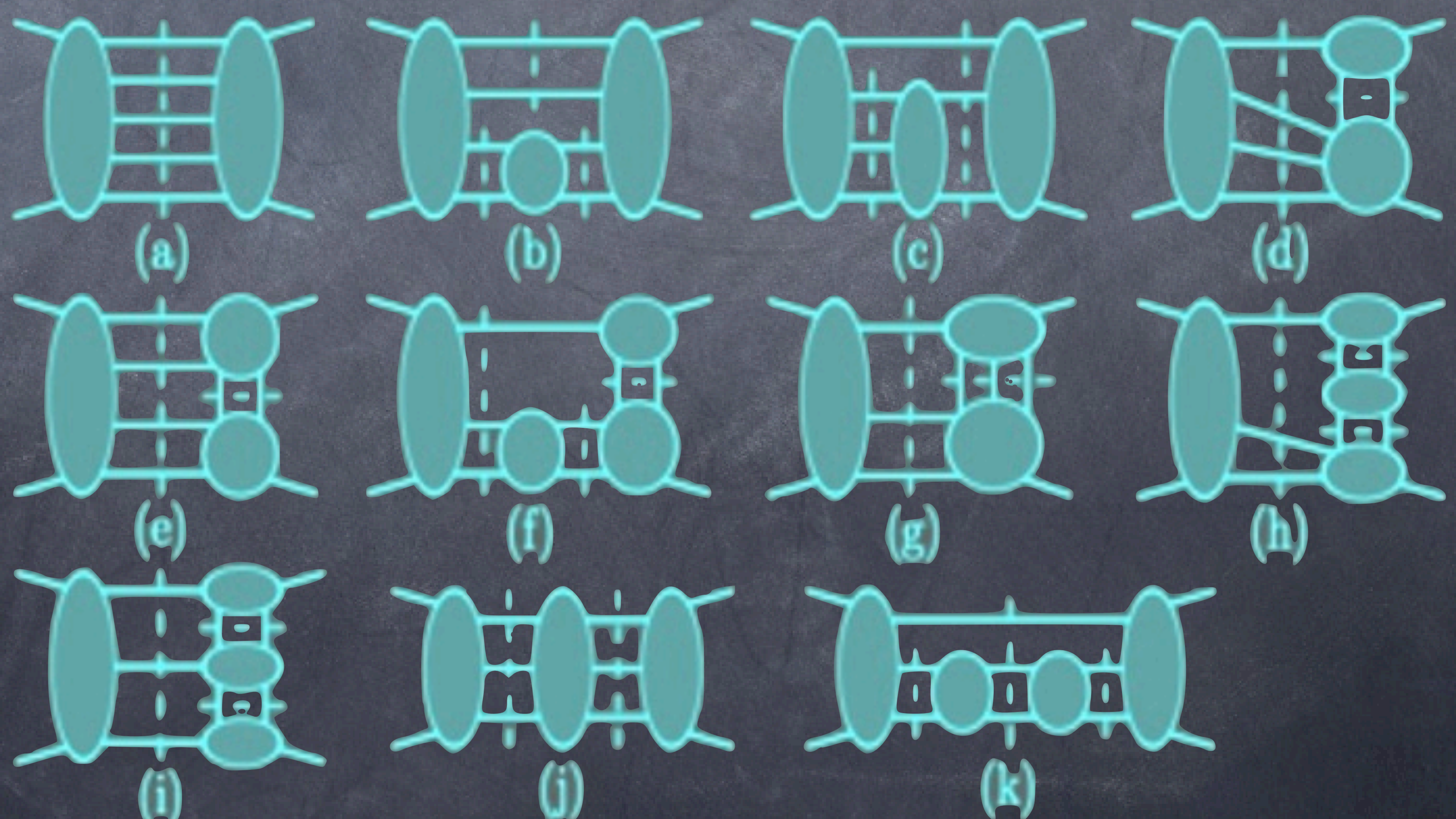
$$I_i = \int \left[\prod_{p=1}^{4} \frac{d^D l_{n_p}}{(2\pi)^D} \right] \frac{N_i(l_j, k_j)}{l_1 l_2 \ldots l_{13}}$$



Numerators determined from 2906 maximal and near maximal cuts

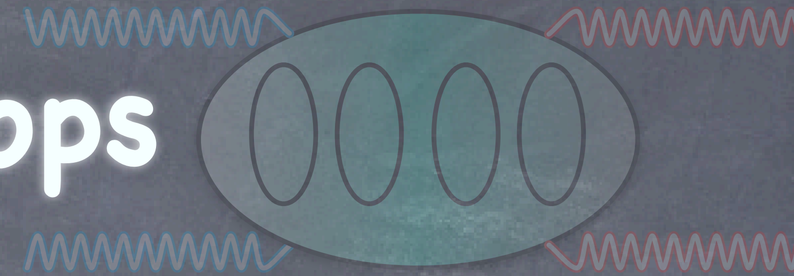
YM diags thru KLT used as truth.

Completeness of ansatz verified on 26 generalized cuts



(2009)

UV Divergence at Four Loops



$$I_i = \int \left[\prod_{p=1}^{4} \frac{d^D l_{n_p}}{(2\pi)^D} \right] \frac{N_i(l_j, k_j)}{l_1 l_2 \ldots l_{13}}$$

k external
l internal:
too many are
bad for UV

Leading numerators $N_i \sim O(k^4 l^8)$

would have D = 4.5 divergence

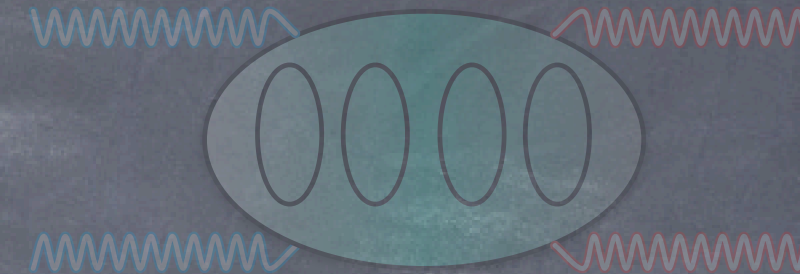
Represented by integrals which cancel in the full amplitude

Sub-leading divergence: $O(k^5 l^7)$

trivially vanishes under integration by Lorentz invariance

(2009)

UV Divergence at Four Loops



$N_i \sim O(k^6 l^6)$ corresponding to D = 5 div.

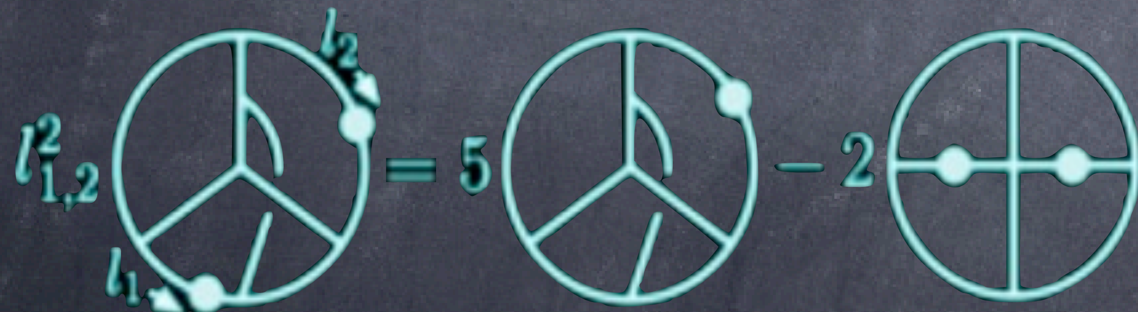
Expand the integrands about small external momenta:

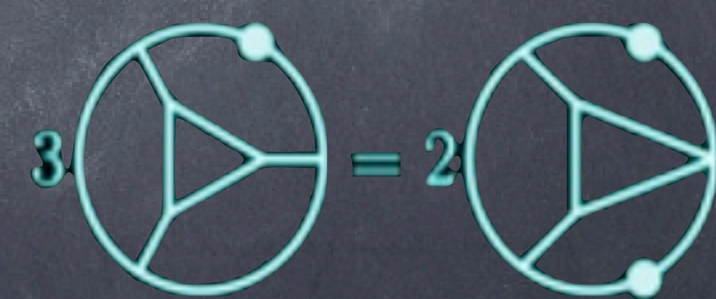
$$N_i^{(6)} + N_i^{(7)} \frac{K_n \cdot l_j}{l_j^2} + N_i^{(8)} \left(\frac{K_n^2}{l_j^2} + \frac{K_n \cdot l_j \; K_q \cdot l_p}{l_j^2 l_p^2} \right)$$

(K_i annotates sums over external momenta)

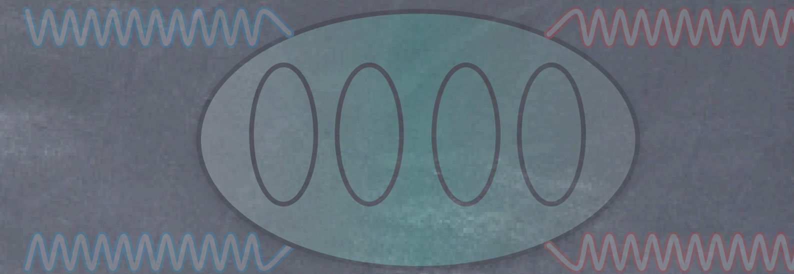
Marcus & Sagnotti UV extraction method

cancels after using D = 5 integral identities like:

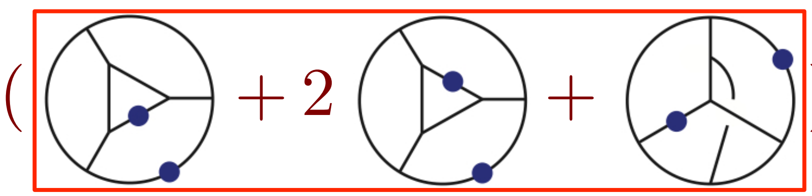




Understand divergence, but UV structure was obscured!



In the new manifest representation, as Radu told us, we have the power to identify remarkable structure between YM and Gravity

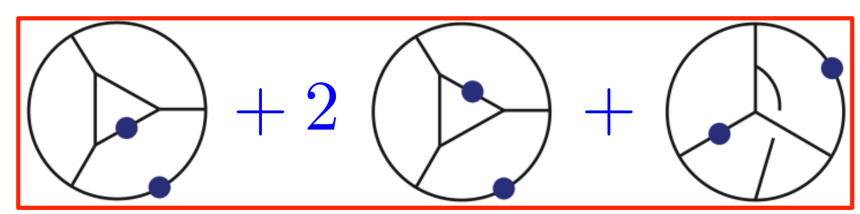
$$\mathcal{M}_4^{(4)}\Big|_{\text{pole}} = -\frac{23}{8}\left(\frac{\kappa}{2}\right)^{10} stu(s^2+t^2+u^2)^2\, M_4^{\text{tree}} \Big($$

$$\Big)$$

$$-256 + \frac{2025}{8}$$

12- and 13-propagator integrals

11-propagator integrals; same as in sYM

$$\mathcal{A}_4^{(4)}\Big|_{\text{pole}}^{SU(N_c)} = -6\, g^{10}\, \mathcal{K}\, N_c^2 \Big(N_c^2$$

$$+ 12\,\Big($$

$$\Big)\Big)$$
$$\times \Big(s\,(\text{Tr}_{1324} + \text{Tr}_{1423}) + t\,(\text{Tr}_{1243} + \text{Tr}_{1342}) + u\,(\text{Tr}_{1234} + \text{Tr}_{1432}) \Big)$$

(to appear)


DEN GODE,
DEN ONDE
OG DEN GRUSOMME
("THE GOOD, THE BAD AND THE UGLY")

Underlying Algebra?

- **Understanding in 4D in self-dual sector, translating into 4D MHV**

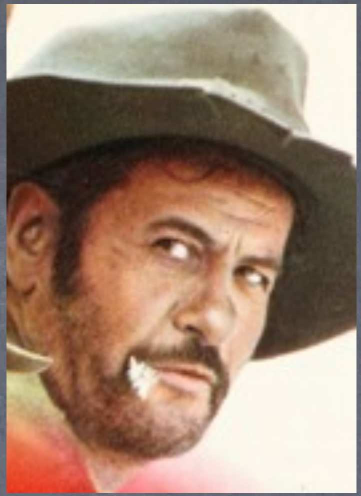
Monteiro, O'Connell

- **Inverting standard color decomposition, i.e. tracing over kinematics**

Bern, Dennen

$$\mathcal{A}_m^{\text{tree}} = g^{m-2} \sum_{\sigma} \tau_{(12\ldots m)} A_m^{\text{dual}}(1, 2, \ldots, m)$$

Solving the functional relations?



- These loop level calculations work beautifully!
  - but ... functional equation solving!

"Small problems at the multiloop level aren't small problems." -Z. Bern

Want to figure out new techniques of how to solve these guys.

- Tree-level imposition of symmetry provides many of the same challenges
  - Could be an interesting playground for techniques

Broedel, JJMC

String Theory & Tree-level Duality

- **Derivation of relations leading to (n-3)! amplitudes using monodromy of ST amps.** — Bjerrum-Bohr, Damgaard, Vanhove; Stieberger
    - **Duality first satisfied in 5-point ST using pure-spinor formalism** — Mafra
        - **Insights into nature of duality in Heterotic strings due to parallel treatment of color and kinematics** — Tye, Zhang
            - **n-point duality (local, asymmetric) satisfied in ST using pure-spinor formalism** — Mafra, Schlotterer, Stieberger

Field Theory & Tree Level Duality

- Proof of double-copy form of gravity assuming duality
- Existence of Lagrangian manifesting 6-point duality

  Bern, Dennen, Huang, Kiermaier

- Using (n-3)! relations via BCFW for field theoretic proofs of KLT relations, new forms etc.

  Bjerrum-Bohr, Damgaard, Feng, Sondegaard
  Feng, He, (R.) Huang, Jia

- Explicit (non-symmetric) duality-satisfying tree-level num. to all multiplicity.

  Kiermaier
  B-B,D,S, Vanhove

- Derivation of relations leading to (n-3)! amplitudes using BCFW

  Feng, (R.) Huang, Jia

- Relations with (some) non-SUSY matter

  Sondergaard

- Symmetric, amplitude encoded, duality satisfying tree-level representations from 4-6 points

  JB, JJMC

What's the endgame?

- We don't want to have to write an **ansatz.** Rather, a **direct** way to write down master.
- As an intermediate step, we'll be happy with greater control over more **fluidly flowing between representations (c.f. polytopes)**
- **Existence in higher-genus perturbative string theory?**
- **Connection to recent understanding from Higher-Spin work?**
- **What is non-perturbative implication/barrier to gravity as a double-copy?**

proofs, generalizations, etc... Lots to do!