Current Status of Link Approach for Twisted Lattice SUSY

Noboru Kawamoto

Hokkaido University, Sapporo, Japan

In collaboration with
S.Arianos, A.D'adda, A.Feo,
I.Kanamori, K.Nagata, J.Saito

Plan of talk

1. Introduction
2. Summary of Link approach
3. Claimed “inconsistency”
4. Essence of difficulties
5. A new proposal

Success of Lattice Regularization

Lattice QCD

Exact gauge symmetry
Broken Lorentz invariance

2-dim. quantum gravity

Random lattice
(dynamical triangulation, matrix model)

Lattice gauge theory
+
Gravity
on random lattice

Regularization of
fermion on a lattice

Lattice SUSY

What is the guiding principle ?

SUSY algebra on a Lattice

$$\{Q, \overline{Q}\} = i\partial \rightarrow i\Delta$$

Difficulties

1） No Poincare invariance on lattice

No small fermionic parameter

2） $\partial \rightarrow \Delta$ (difference operator)

No Leibniz rule

3) Chiral fermion on lattice → Doubling of fermions

Unbalance of degrees of freedom between
#boson $\neq$ #fermion

A Clue to a possible solution to 3)

Kato, Miyake, N.K.,
Tsukioka, Uchida

Continuum

$$S = \int d^2x \phi \epsilon^{\mu\nu} \partial_\mu \omega_\nu$$

$$\delta\phi = 0, \quad \delta\omega_\mu = \partial_\mu v$$

(Two dimensional Abelian BF)

$$S = \int d^2x [\epsilon^{\mu\nu} \phi \partial_\mu \omega_\nu + \underline{b\partial^\mu \omega_\mu - i\bar{c}\partial^\mu \partial_\mu c} - i\lambda \underline{\rho}]$$

$$= \int d^2x s\tilde{s} \tfrac{1}{2} \epsilon^{\mu\nu} s_\mu s_\nu (-i\bar{c}c)$$

Auxiliary field
Off-shell invariance

$$\underline{s^2} = \{s, \tilde{s}\} = \tilde{s}^2 = \{s_\mu, s_\nu\} \underline{=0},$$

$$\{s, s_\mu\} = -i\partial_\mu, \{\tilde{s}, s_\mu\} = i\epsilon_{\mu\nu}\partial^\nu$$

Nilpotency of
BRS charge s

ϕ^A	$s\phi^A$	$s_\mu \phi^A$	$\tilde{s}\phi^A$
ϕ	$i\rho$	$-\epsilon_{\mu\nu}\partial^\nu \bar{c}$	0
ω_ν	$\partial_\nu c$	$-i\epsilon_{\mu\nu}\lambda$	$-\epsilon_{\nu\rho}\partial^\rho c$
c	0	$-i\omega_\mu$	0
$\bar{c}$	$-ib$	0	$-i\phi$
b	0	$\partial_\mu \bar{c}$	$-i\rho$
λ	$\epsilon^{\mu\nu}\partial_\mu\omega_\nu$	0	$-\partial_\mu \omega^\mu$
ρ	0	$-\partial_\mu \phi - \epsilon_{\mu\nu}\partial^\nu b$	0

N=D=2 Twisted SUSY

Kato,N.K.&Uchida

$$\{Q_{\alpha i}, Q_{\beta j}\} = 2\delta_{ij}\gamma^{\mu}{}_{\alpha\beta}P_{\mu}$$ **N=D=2 SUSY**

$$Q_{\alpha i} = \left(\mathbf{1}s + \gamma^{\mu}s_{\mu} + \gamma^{5}\tilde{s}\right)_{\alpha i}$$ **Dirac-Kaehler Twist**

$$(\Psi)_{\alpha i} = (\chi + \chi_{\mu}\gamma^{\mu} + \chi_{\mu\nu}\gamma^{[\mu\nu]} + \cdots)_{\alpha i}$$ **Dirac-Kaehler fermion**

$$\boxed{\{s, s_{\mu}\} = -i\partial_{\mu}, \quad \{\tilde{s}, s_{\mu}\} = i\epsilon_{\mu\nu}\partial^{\nu}}$$

$$s^2 = \{s, \tilde{s}\} = \tilde{s}^2 = \{s_{\mu}, s_{\nu}\} = 0,$$ **N=D=2 Twisted SUSY**

Continuum $\longrightarrow$ Lattice: $\partial_{\mu} \to \Delta_{\pm\mu}$

$$\boxed{\{Q, Q_{\mu}\} = i\Delta_{\pm\mu} \qquad \{\tilde{Q}, Q_{\mu}\} = -i\epsilon_{\mu\nu}\Delta_{\pm\nu}}$$

on a Lattice

$$\left(\Delta_{\pm\mu}\Phi\right)(x) = \pm(\Phi(x \pm n_{\mu}) - \Phi(x))$$

3)

N=D=2 SUSY

$$\{Q_{\alpha i}, Q_{\beta j}\} = 2\delta_{ij}\gamma^{\mu}{}_{\alpha\beta}P_{\mu}$$

Dirac-Kaehler Twist

$$Q_{\alpha i} = \left(1Q + \gamma^{\mu}Q_{\mu} + \gamma^{5}\tilde{Q}\right)_{\alpha i}$$

$$\{Q, Q_{\mu}\} = i\Delta_{\pm\mu} \qquad \{\tilde{Q}, Q_{\mu}\} = -i\epsilon_{\mu\nu}\Delta_{\pm\nu}$$

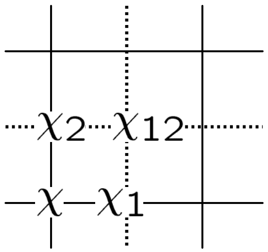
$$(\Psi)_{\alpha i} = (\chi + \chi_{\mu}\gamma^{\mu} + \chi_{\mu\nu}\gamma^{[\mu\nu]} + \cdots)_{\alpha i}$$

Dirac-Kaehler fermion

i : flavour ? → $\boxed{\text{Extended SUSY}}$

suffix

$$\Phi = \chi + \chi_{\mu}dx^{\mu} + \chi_{\mu\nu}dx^{\mu} \wedge dx^{\nu} + \cdots$$



y ↔ 2x

$2^{d/2}$ super charges in d-dim.

2-dim. N=2
3-dim. N=4
4-dim. N=4

Dirac-Kaehler twisting

#boson = #fermion

Lattice Fermions

$\int \overline{\psi}\ \not{\partial}\psi$ **Free Dirac**

$\eta_\mu(x) = \pm 1$

$\sum_{x,\widehat{\mu}} \eta_\mu(x)\overline{\chi}(x)\{\chi(x+\widehat{\mu}) - \chi(x-\widehat{\mu})\}$ $\longleftarrow$ $\sum_{x,\widehat{\mu}} \overline{\psi}(x)\gamma_\mu\{\psi(x+\widehat{\mu}) - \psi(x-\widehat{\mu})\}$

Staggered

$\psi(x) = \gamma_1^{x_1}\gamma_2^{x_2}\cdots\chi(x)$

Naïve

(Kluberg-Stern et.al. &Gliozzi)

(N.K. & J.Smit)

$$\sum_{y,\widehat{\mu}} \overline{\psi}_{i\alpha}(y)(\gamma_\mu)_{\alpha\beta}\{\psi(y+2\widehat{\mu}) - \psi(y-2\widehat{\mu})\}_{\beta i} + Tr(\overline{\psi}\gamma_5(\psi(y+\widehat{\mu}) - 2\psi(y) + \psi(y-\widehat{\mu}))\gamma_\mu\gamma_5)$$

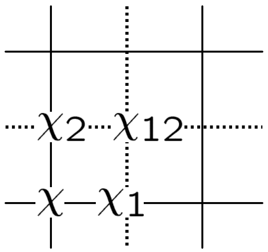
Kogut-Susskind

(N.K. & I.Kanamori)

$dx^\mu\chi(x) = \chi(x+\widehat{\mu})dx^\mu$

$\sum_x \overline{\Phi}(x)(d + \delta)\Phi(x)$ **Dirac-Kaehler**

Ivanenko&Landau '28



y $\leftrightarrow$ 2x

$$\Phi = \chi + \chi_\mu dx^\mu + \chi_{\mu\nu}dx^\mu \wedge dx^\nu + \cdots$$

$$(\Psi)_{\alpha i} = (\chi + \chi_\mu\gamma^\mu + \chi_{\mu\nu}\gamma^{[\mu\nu]} + \cdots)_{\alpha i}$$

i : flavour ?

(2) Difference Operator $\longrightarrow$ No Leibniz rule ?

$$\left(\Delta_{+\mu}\Phi\right)(x) = \Phi(x+n_\mu) - \Phi(x)$$

$$x-n_\mu \xleftarrow{\Delta_{-\mu}} x \xrightarrow{\Delta_{+\mu}} x+n_\mu$$

$$\left(\Delta_{+\mu}\Phi_1\Phi_2\right)(x) = \Phi_1(x+n_\mu)\Phi_2(x+n_\mu) - \Phi_1(x)\Phi_2(x)$$

$$= (\Phi_1(x+n_\mu) - \Phi_1(x))\Phi_2(x) + \Phi_1(x+n_\mu)(\Phi_2(x+n_\mu) - \Phi_2(x))$$

$$= (\Delta_{+\mu}\Phi_1)(x)\Phi_2(x) + \Phi_1(x+n_\mu)(\Delta_{+\mu}\Phi_2)(x)$$

$$= (\Delta_{+\mu}\Phi_1)(x)\Phi_2(x+n_\mu) + \Phi_1(x)(\Delta_{+\mu}\Phi_2)(x)$$

Modified Leibniz rule

left-right shift symmetric

To prove exact translational invariance of lattice action we need **modified Leibniz rule**

$$\sum_x \Delta_+ L(\phi(x), \nabla_\mu \phi(x))\Delta_- = \sum_x L(\phi(x+n_\mu), \nabla_\mu \phi(x+n_\mu))$$
$$= \sum_x L(\phi(x), \nabla_\mu \phi(x))$$

Translational shift → modified Leibniz

$$\delta_{+\mu}\Phi_i = \Delta_{+\mu}\Phi_i = \Phi_i(x+n_\mu) - \Phi_i(x) \equiv \Delta_+ \Phi_i \Delta_- - \Phi_i$$

$$\Delta_{+\mu}(\Phi_1(x)\Phi_2(x)) = \Delta_+ \Phi_1(x)\Phi_2(x)\Delta_- - \Phi_1(x)\Phi_2(x)$$
$$= \Phi_1(x+n_\mu)\Phi_2(x+n_\mu) - \Phi_1(x)\Phi_2(x)$$

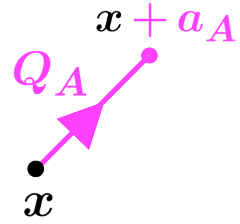
Matrix representation:

$$= (\Phi_1 + \delta_\mu \Phi_1)(\Phi_2 + \delta_\mu \Phi_2) - \Phi_1\Phi_2$$
$$= (\delta_{+\mu}\Phi_1)\Phi_2 + \Delta_+ \Phi_1 \Delta_- \delta_{+\mu}\Phi_2 = \delta_{+\mu}\Phi_1 \Delta_+ \Phi_2 \Delta_- + \Phi_1 \delta_{+\mu}\Phi_2$$

left-right shift symmetric

$$\left(\Delta_{+\mu}\Phi\right)(x) = \Delta_{+\mu}\Phi(x) - \Phi(x+n_\mu)\Delta_{+\mu}$$

We need a modified Leibniz rule for Q_A too !



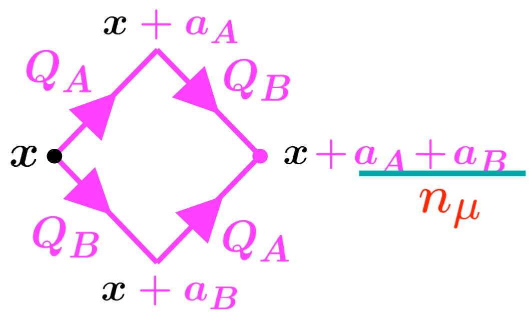
$$s_A\Phi(x,\theta) = Q_A\Phi(x,\theta) - \Phi(x+a_A,\theta)Q_A$$

$$\{Q, Q_\mu\} = i\Delta_{+\mu} \qquad \{\tilde{Q}, Q_\mu\} = -i\epsilon_{\mu\nu}\Delta_{-\nu}$$

$$a + a_\mu = +n_\mu \qquad \tilde{a} + a_\mu = -|\epsilon_{\mu\nu}|n_\nu$$

Compatibility of Shifts

$$\boxed{\begin{aligned}(\{Q_A, Q_B\}\Phi)(x) &= \{Q_A, Q_B\}\Phi(x) \\ &\quad -\Phi(x+a_A+a_B)\{Q_A, Q_B\}\end{aligned}}$$

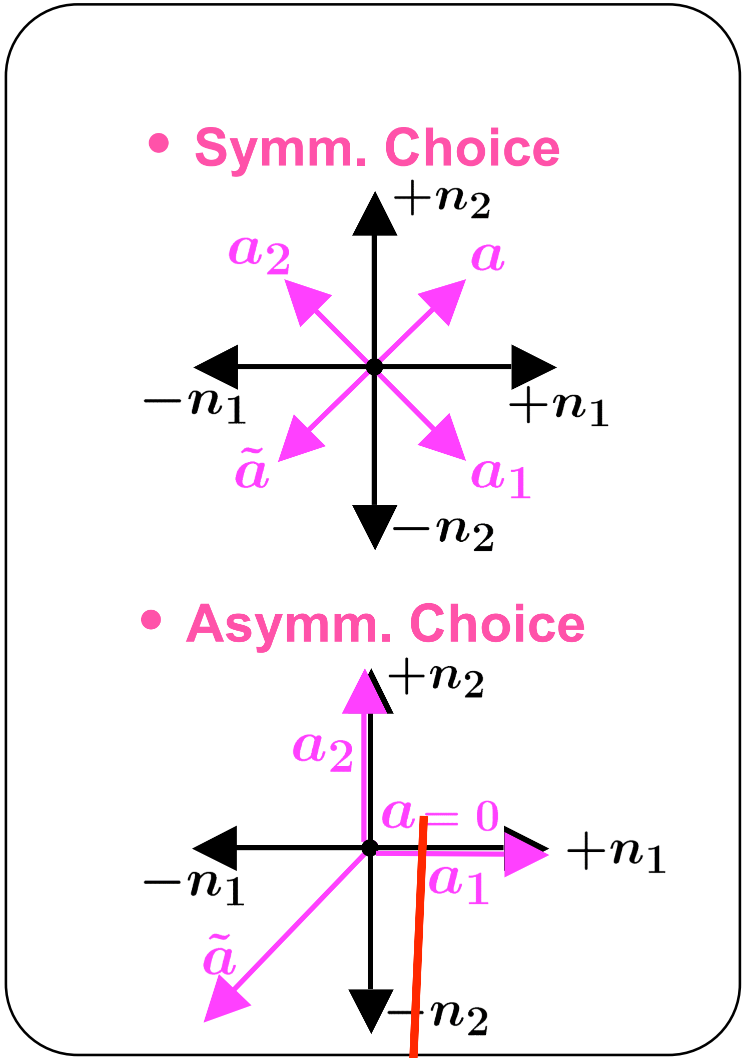


Cond. for Twisted N=D=2

$$a + a_\mu = +n_\mu$$
$$\tilde{a} + a_\mu = -|\epsilon_{\mu\nu}|n_\nu$$

Solutions

$$a = (arbitrary)$$
$$a_\mu = +n_\mu - a$$
$$\tilde{a} = -n_1 - n_2 + a$$
$$a + a_1 + a_2 + \tilde{a} = 0$$



Twisted N=D=2 Lattice SUSY Algebra

$$\{Q, Q_\mu\} = +i\Delta_{+\mu}$$
$$\{\tilde{Q}, Q_\mu\} = -i\epsilon_{\mu\nu}\Delta_{-\nu}$$

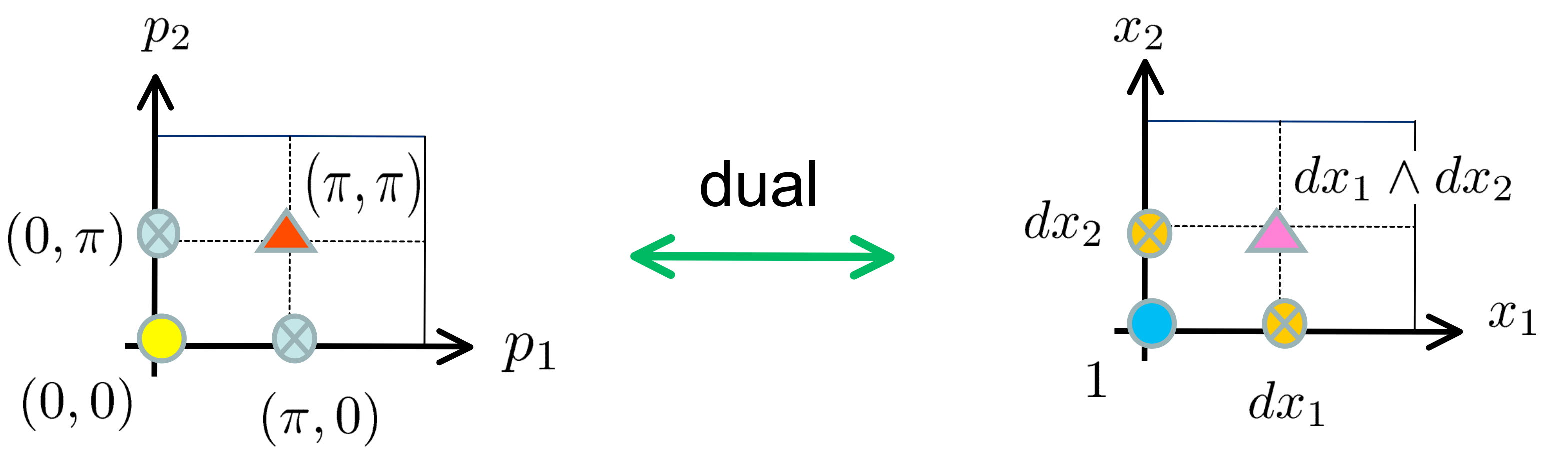
Equivalent to orbifold construction: $Q^2 = 0$
by Kaplan et.al.

3) Massless fermion propagator on a lattice

$$\overline{\psi}(x)\gamma_\mu\{\psi(x+\widehat{\mu})-\psi(x-\widehat{\mu})\}$$

$$\frac{1}{\gamma^\mu sin p_\mu} \qquad p_\mu = 0, \pi \qquad \text{(poles)}$$

doubling of fermions



Dirac-Kaehler fermion

Lattice SUSY

$$\{Q_\alpha, \overline{Q}_\beta\} = 2(\gamma^\mu)_{\alpha\beta} P\mu$$

Difficulties

1) No Poincare invariance

2) No Leibniz rule

3) (Chiral) fermion

$\downarrow$

species doublers

Our Solution

1) Discrete SUSY algebra

2) (A) non-commutativity for
the difference operator

(B) modified Leibniz rule

3) Dirac-Kaehler twist
SUSY algebra

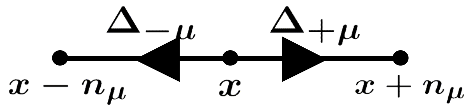
Extended SUSY d.o.f.

BRST charge = scalar twisted SUSY

$$Q^2 = 0$$

1), 2) Difference Operator → Shifted Leibniz rule

$$\left(\Delta_{+\mu}\Phi\right)(x) = \Phi(x+n_\mu) - \Phi(x)$$



$$\begin{aligned}
\left(\Delta_{+\mu}\Phi_1\Phi_2\right)(x) &= \Phi_1(x+n_\mu)\Phi_2(x+n_\mu) - \Phi_1(x)\Phi_2(x) \\
&= (\Phi_1(x+n_\mu) - \Phi_1(x))\Phi_2(x) + \Phi_1(x+n_\mu)(\Phi_2(x+n_\mu) - \Phi_2(x)) \\
&= (\Delta_{+\mu}\Phi_1)(x)\Phi_2(x) + \Phi_1(x+n_\mu)(\Delta_{+\mu}\Phi_2)(x)
\end{aligned}$$

(A) Non-commutative $\vec{\Delta}_{+\mu}$

$$\left(\vec{\Delta}_{+\mu}\Phi\right)(x) = \vec{\Delta}_{+\mu}\Phi(x) - \Phi(x+n_\mu)\vec{\Delta}_{+\mu}$$

(B) Shifted Leibniz rule $\Delta_{+\mu}$

$$\begin{aligned}
\left(\Delta_{+\mu}\Phi\right)_{x+n_\mu,x} &= \left(\Delta_{+\mu}\right)_{x+n_\mu,x}\Phi_x - \Phi_{x+n_\mu}\left(\Delta_{+\mu}\right)_{x+n_\mu,x} \\
&= [\Delta_{+\mu}, \Phi]_{x+n_\mu,x}
\end{aligned}$$

$$(\Delta_{+\mu})_{x+n_\mu,x} = -\delta_{x+n_\mu,x} \qquad \Phi_x = \Phi_{x,x} = \Phi_x\delta_{x,x}$$

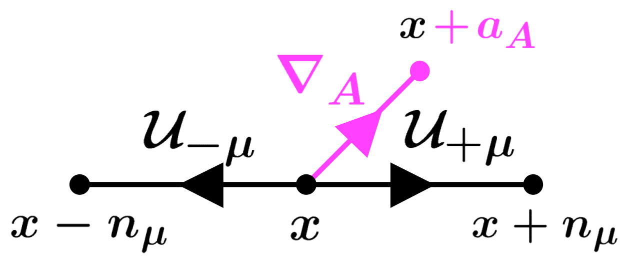
N=D=2 Twisted Super Yang-Mills

Introduce Bosonic & Fermionic Link variables

$$
\boxed{
\begin{aligned}
(\Delta_{\pm\mu})_{x\pm n_\mu,x} &\rightarrow \mp(\mathcal{U}_{\pm\mu})_{x\pm n_\mu,x} \\
(Q_A)_{x+a_A,x} &\rightarrow (\nabla_A)_{x+a_A,x}
\end{aligned}
}
$$

Gauge trans.

$$
\begin{aligned}
(\mathcal{U}_{\pm\mu})' &= G_{x\pm n_\mu}\,(\mathcal{U}_{\pm\mu})\,G_x^{-1} \\
(\nabla_A)' &= G_{x+a_A}\,(\nabla_A)\,G_x^{-1}
\end{aligned}
$$



- $(\mathcal{U}_{\pm\mu})_{x\pm n_\mu,x} = (e^{\pm i(A_\mu \pm i\phi^{(\mu)})})_{x\pm n_\mu,x},$

  $\phi^{(\mu)}\ (\mu = 1,2)$: Scalar fields in SYM multiplet

- $\mathcal{U}_{+\mu}\mathcal{U}_{-\mu} \neq 1$

N=D=2 SUSY Algebra

(continuum)

$$\boxed{\{Q_{\alpha i}, Q_{\beta j}\} = 2i\delta_{ij}(\gamma^\mu)_{\alpha\beta}\partial_\mu}$$

$$[J, Q_{\alpha i}] = \frac{i}{2}(\gamma^5)_\alpha{}^\beta Q_{\beta i}$$

$$[R, Q_{\alpha i}] = \frac{i}{2}(\gamma^5)_i{}^j Q_{\alpha j}$$

J : $SO(2)$ Lorentz generator

R : $SO(2)$ Internal rotation

$$[J, \partial_\mu] = i\epsilon_{\mu\nu}\partial_\nu \quad [R, \partial_\mu] = 0$$

$$J' \equiv J + R$$

Dirac-KaehlerTwist:

(Ghost > matter fermion)

J' : Twisted Lorentz generator

Lattice Formulation of N=D=2 Twisted SYM

c.f. N=D=2 Twisted SYM multiplet in continuum spacetime

Gauge field : A_μ Fermions : $(\rho, \lambda_\mu, \tilde{\rho})$

Scalar field : $\phi^{(\mu)}$ $(\mu = 1, 2)$

	J	R	$J' = J + R$
A_μ	vector $[J, A_\mu] = i\epsilon_{\mu\nu}A_\nu$	singlet $[R, A_\mu] = 0$	vector $[J', A_\mu] = i\epsilon_{\mu\nu}A_\nu$
$\phi^{(\mu)}$	singlet $[J, \phi^{(\mu)}] = 0$	vector $[R, \phi^{(\mu)}] = i\epsilon_{\mu\nu}\phi^{(\nu)}$	vector $[J', \phi^{(\mu)}] = i\epsilon_{\mu\nu}\phi^{(\nu)}$

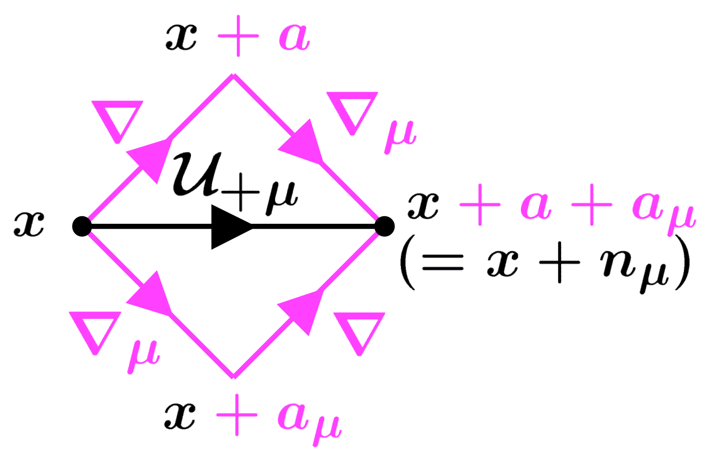
⇩

- $A_\mu \pm i\phi^{(\mu)}$: Covariant expression w.r.t. J'

N=D=2 Twisted Lattice SUSY Algebra for SYM

$$
\begin{aligned}
\{\nabla, \nabla_\mu\}_{x+a+a_\mu,x} &= +i(\mathcal{U}_{+\mu})_{x+n_\mu,x} \\
\{\tilde{\nabla}, \nabla_\mu\}_{x+\tilde{a}+a_\mu,x} &= +i\epsilon_{\mu\nu}\ (\mathcal{U}_{-\nu})_{x-n_\nu,x} \\
\{others\} &= 0
\end{aligned}
$$

$$
\begin{aligned}
&\{\nabla, \nabla_\mu\}_{x+a+a_\mu,x} \\
&\equiv (\nabla)_{x+a+a_\mu,x+a_\mu}(\nabla_\mu)_{x+a_\mu,x} \\
&\quad +(\nabla_\mu)_{x+a+a_\mu,x+a}(\nabla)_{x+a,x}
\end{aligned}
$$



“Shifted” Anti-commutator

$$
\left[\ \because\ a+a_\mu = +n_\mu \\ \vdots \ \right]
$$

Jacobi Identities

$$[\nabla_\mu\{\nabla_\nu, \nabla\}]_{x+a_\mu+n_\nu,x} + (cyclic) = 0,$$

$$[\nabla_\mu, \mathcal{U}_{+\nu}]_{x+a_\mu+n_\nu,x} + [\nabla_\nu, \mathcal{U}_{+\mu}]_{x+a_\nu+n_\mu,x} \;=\; 0,$$

$$\vdots$$

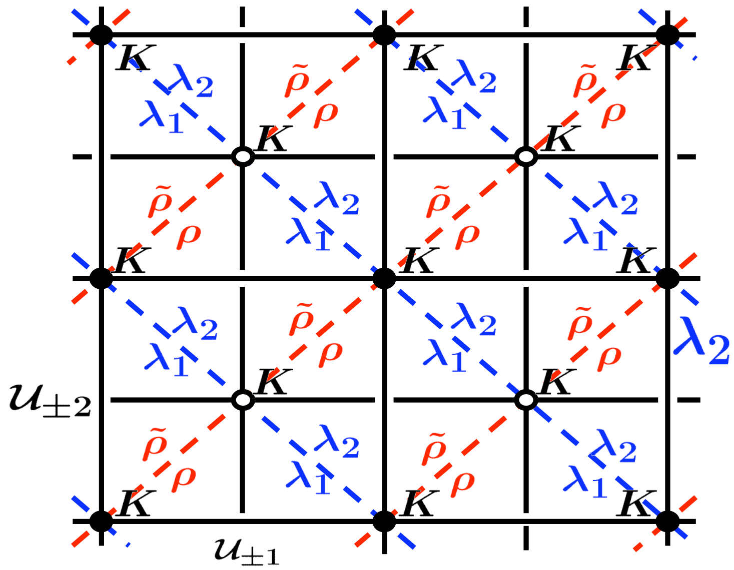
Define fermionic link components

$$[\nabla_\mu, \mathcal{U}_{+\nu}]_{x+a_\mu+n_\nu,x} \equiv -\epsilon_{\mu\nu}(\tilde{\rho})_{x-\tilde{a},x} \;,$$

$$\vdots$$

Auxiliary Field

$$K = \frac{1}{2}\{\nabla_\mu, \lambda_\mu\}$$



Fermionic Link Fields

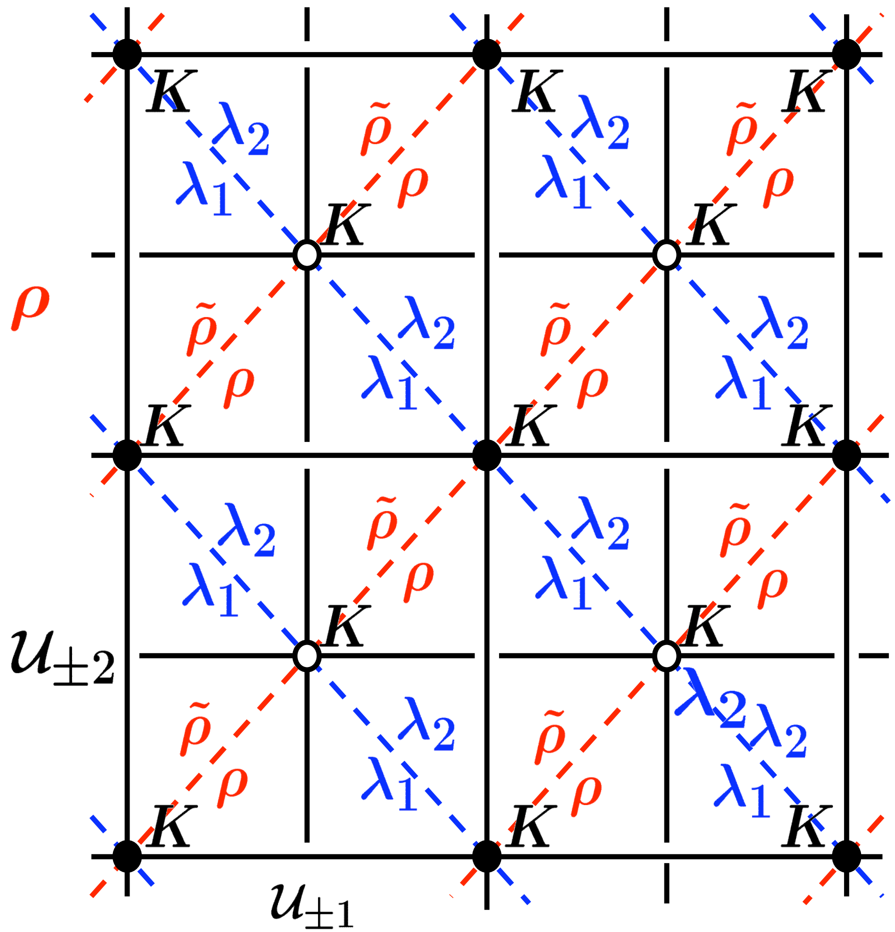
$$[\nabla_\mu, \mathcal{U}_{+\nu}] \equiv -\epsilon_{\mu\nu}\tilde{\rho}$$

$$[\nabla_\mu, \mathcal{U}_{-\nu}] \equiv -\delta_{\mu\nu}\rho$$

$$\epsilon_{\mu\nu}[\nabla, \mathcal{U}_{-\nu}] \equiv -\epsilon_{\mu\nu}\lambda_\nu$$

Auxiliary Field

$$K = \frac{1}{2}\{\nabla_\mu, \lambda_\mu\}$$



Twisted N=2 Lattice SUSY Transformation

$$s_A(\varphi) \equiv [\nabla_A, \varphi\}_{x+a_A+a_\varphi,x}$$

Shifts of Fields

∇	$\tilde{\nabla}$	∇_μ	$\mathcal{U}_{\pm\mu}$	ρ	$\tilde{\rho}$	λ_μ	K
a	$\tilde{a}$	a_μ	$\pm n_\mu$	$-a$	$-\tilde{a}$	$-a_\mu$	0

	s	$\tilde{s}$	s_μ
$\mathcal{U}_{+\nu}$	0	$+\epsilon_{\nu\rho}\lambda_\rho$	$-\epsilon_{\mu\nu}\tilde{\rho}$
$\mathcal{U}_{-\nu}$	$-\lambda_\nu$	0	$-\delta_{\mu\nu}\rho$
λ_ν	0	0	$-i[\mathcal{U}_{+\mu}, \mathcal{U}_{-\nu}] + \delta_{\mu\nu}(K + \frac{i}{2}[\mathcal{U}_{+\rho}, \mathcal{U}_{-\rho}])$
ρ	$-\frac{i}{2}[\mathcal{U}_{+\rho}, \mathcal{U}_{-\rho}] - K$	$+\frac{i}{2}\epsilon_{\rho\sigma}[\mathcal{U}_{-\rho}, \mathcal{U}_{-\sigma}]$	0
$\tilde{\rho}$	$-\frac{i}{2}\epsilon_{\rho\sigma}[\mathcal{U}_{+\rho}, \mathcal{U}_{+\sigma}]$	$+\frac{i}{2}[\mathcal{U}_{+\rho}, \mathcal{U}_{-\rho}] - K$	0
K	$+\frac{i}{2}[\mathcal{U}_{+\rho}, \lambda_\rho]$	$-\frac{i}{2}\epsilon_{\rho\sigma}[\mathcal{U}_{-\rho}, \lambda_\sigma]$	$-\frac{i}{2}[\mathcal{U}_{+\mu}, \rho] - \frac{i}{2}\epsilon_{\mu\nu}[\mathcal{U}_{-\nu}, \tilde{\rho}]$

Twisted SUSY Algebra closes off-shell

$$\begin{aligned}
\{s, s_\mu\}(\varphi)_{x+a_\varphi,x} &= +i[\mathcal{U}_{+\mu}, \varphi]_{x+a_\varphi+n_\mu,x} \\
\{\tilde{s}, s_\mu\}(\varphi)_{x+a_\varphi,x} &= +i\epsilon_{\mu\nu}[\mathcal{U}_{-\nu}, \varphi]_{x+a_\varphi-n_\nu,x} \\
s^2(\varphi)_{x+a_\varphi,x} &= \tilde{s}^2(\varphi)_{x+a_\varphi,x} = 0 \\
\{s, \tilde{s}\}(\varphi)_{x+a_\varphi,x} &= \{s_\mu, s_\nu\}(\varphi)_{x+a_\varphi,x} = 0
\end{aligned}$$

Twisted N=2 Super Yang-Mills Action

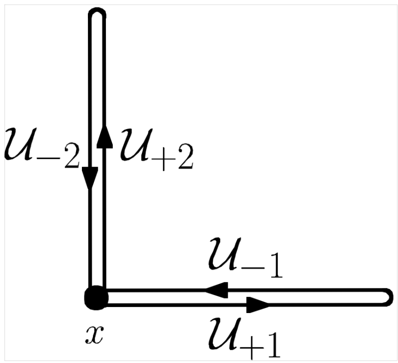
Action has twisted SUSY exact form. → **Off-shell SUSY invariance for all twisted super charges.**

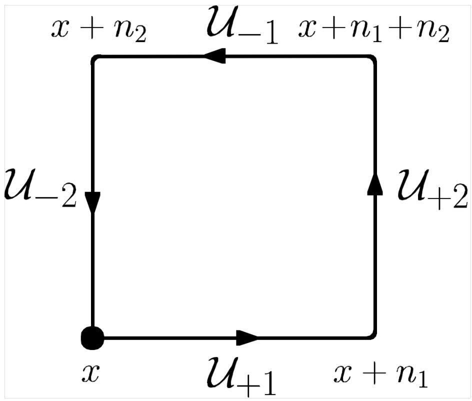
$$
\begin{aligned}
S &\equiv \frac{1}{2}\sum_x \mathrm{Tr}\ s\tilde{s}s_1 s_2\ \mathcal{U}_{+\mu}\mathcal{U}_{-\mu} \\
&= S_B + S_F \\
S_B &= \sum_x \mathrm{Tr}\Big[\frac{1}{4}[\mathcal{U}_{+\mu},\mathcal{U}_{-\mu}]_{x,x}[\mathcal{U}_{+\nu},\mathcal{U}_{-\nu}]_{x,x} + K^2_{x,x} \\
&\quad -\frac{1}{4}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}[\mathcal{U}_{+\mu},\mathcal{U}_{+\nu}]_{x,x-n_\mu-n_\nu}[\mathcal{U}_{-\rho},\mathcal{U}_{-\sigma}]_{x-n_\rho-n_\sigma,x}\Big] \\
S_F &= \sum_x \mathrm{Tr}\Big[-i[\mathcal{U}_{+\mu},\lambda_\mu]_{x,x-a}(\rho)_{x-a,x} \\
&\quad - i(\tilde{\rho})_{x,x+\tilde{a}}\epsilon_{\mu\nu}[\mathcal{U}_{-\mu},\lambda_\nu]_{x+\tilde{a},x}\Big]
\end{aligned}
$$

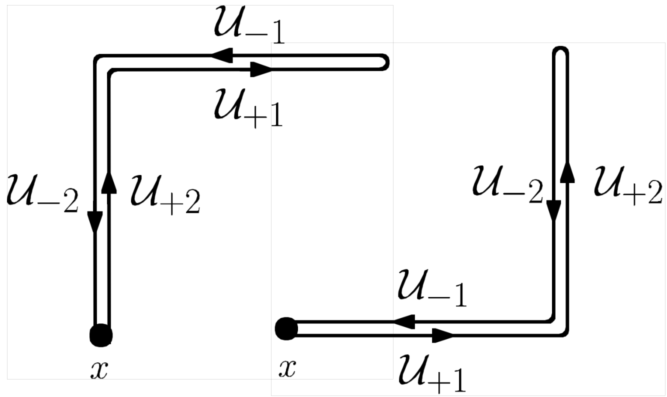
Bosonic part of the Action

$$S_B = \sum_x \mathrm{Tr}\Bigg[\frac{1}{4}[\mathcal{U}_{+\mu},\mathcal{U}_{-\mu}]_{x,x}[\mathcal{U}_{+\nu},\mathcal{U}_{-\nu}]_{x,x} + K^2_{x,x}$$

$$-\frac{1}{4}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}[\mathcal{U}_{+\mu},\mathcal{U}_{+\nu}]_{x,x-n_\mu-n_\nu}[\mathcal{U}_{-\rho},\mathcal{U}_{-\sigma}]_{x-n_\rho-n_\sigma,x}\Bigg]$$



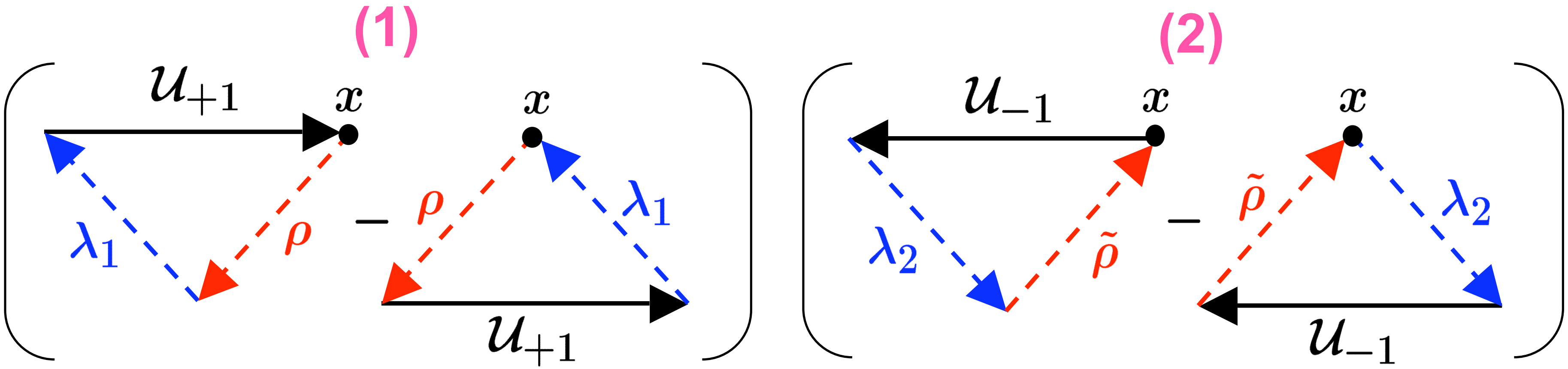




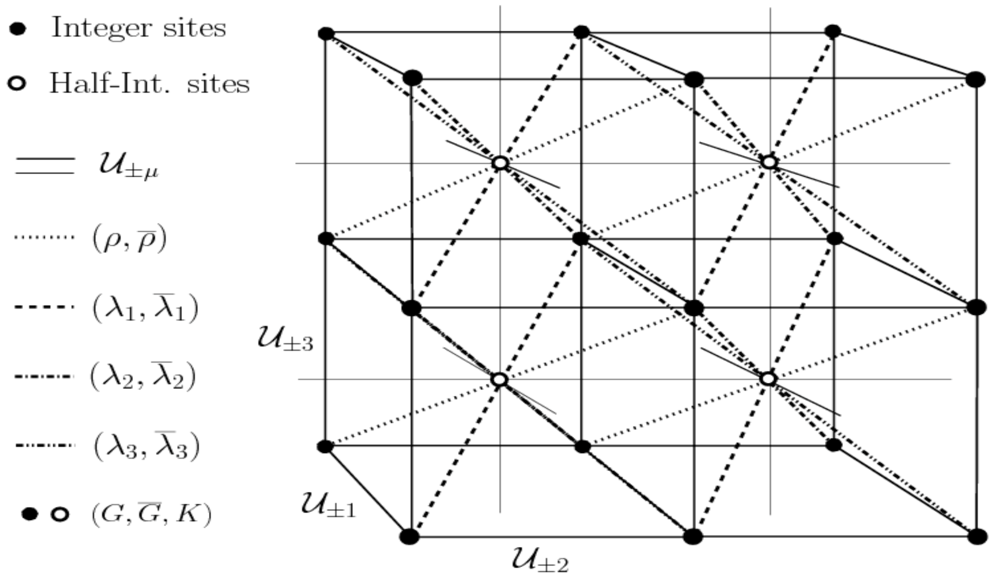
Fermionic part of the Action

$$S_F = \sum_x \mathrm{Tr}\Big[-i[\mathcal{U}_{+\mu}, \lambda_\mu]_{x,x-a}(\rho)_{x-a,x} \quad \cdots (1)$$

$$- i(\tilde{\rho})_{x,x+\tilde{a}}\epsilon_{\mu\nu}[\mathcal{U}_{-\mu}, \lambda_\nu]_{x+\tilde{a},x}\Big] \quad \cdots (2)$$



Higer dimensional extension is possible:



3-dim. N=4 super Yang-Mills

Origin of "inconsistency"

(non-gauge case)

Bruckmann
Kok

Presentation by N=D=1 model with superfield

Superfield: $\Phi = \hat{\phi} + \theta\hat{\psi}$ $\quad \theta^2 = \left(\frac{\partial}{\partial\theta}\right)^2 = 0$ $\quad \{\frac{\partial}{\partial\theta}, \theta\} = 1$

Matrix representation $\theta = \begin{bmatrix} 0 & 0 \\ \Delta_- & 0 \end{bmatrix}, \quad \frac{\partial}{\partial\theta} = \begin{bmatrix} 0 & \Delta_+ \\ 0 & 0 \end{bmatrix}$

Super charge:
(one step translation) $Q = \frac{\partial}{\partial\theta} + \theta\Delta_+^2 \qquad Q^2 = \Delta_+^2$

Boson: $[\theta, \hat{\phi}] = [\frac{\partial}{\partial\theta}, \hat{\phi}] = 0 \ \rightarrow \hat{\phi} = \begin{bmatrix} \phi & 0 \\ 0 & \Delta_-\phi\Delta_+ \end{bmatrix}$

Fermion: $\{\theta, \hat{\psi}\} = \{\frac{\partial}{\partial\theta}, \hat{\psi}\} = 0 \ \rightarrow \hat{\psi} = \begin{bmatrix} \Delta_+\psi & 0 \\ 0 & \psi\Delta_+ \end{bmatrix}$

ϕ and ψ can be taken diagonal $N \times N$

$$\phi = \begin{bmatrix} \phi_1 & 0 & \cdots & \\ 0 & \phi_2 & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ & \cdots & 0 & \phi_N \end{bmatrix} \qquad \psi = \begin{bmatrix} \psi_1 & 0 & \cdots & \\ 0 & \psi_2 & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ & \cdots & 0 & \psi_N \end{bmatrix}$$

SUSY transformation

$$\delta_Q \hat{\phi} \equiv \Delta_-[Q, \Phi]|_0 = \Delta_-[\tfrac{\partial}{\partial\theta} + \theta\Delta_+^2, \hat{\phi} + \theta\hat{\psi}] = \Delta_-\hat{\psi}$$

Similarly,

$$\delta_Q(\Delta_-\hat{\psi}) = \tfrac{2}{N}\partial\hat{\phi},$$

diagonal: $\Delta_-\hat{\psi} = \begin{bmatrix} \psi & 0 \\ 0 & \Delta_-\psi\Delta_+ \end{bmatrix}$

Superfields product

$$\Phi_1\Phi_2 = (\hat{\phi}_1 + \theta\hat{\psi}_1)(\hat{\phi}_2 + \theta\hat{\psi}_2) = \hat{\phi}_1\hat{\phi}_2 + \theta(\hat{\phi}_1\hat{\psi}_2 + \hat{\psi}_1\hat{\phi}_2)$$

$$Q(\hat{\phi}_1\hat{\phi}_2) = \Delta_-(\hat{\phi}_1\hat{\psi}_2 + \hat{\psi}_1\hat{\phi}_2) = \Delta_-\hat{\psi}_1\hat{\phi}_2 + \Delta_-\hat{\phi}_1\Delta_+\Delta_-\hat{\psi}_2$$

similarly $= Q(\hat{\phi}_1)\hat{\phi}_2 + \Delta_-\hat{\phi}_1\Delta_+Q\hat{\phi}_2$

$\not\parallel$ "inconsistency"

$$Q(\hat{\phi}_2\hat{\phi}_1) = Q(\hat{\phi}_2)\hat{\phi}_1 + \Delta_-\hat{\phi}_2\Delta_+Q\hat{\phi}_1$$

while $\hat{\phi}_1\hat{\phi}_2 = \hat{\phi}_2\hat{\phi}_1$ since $\hat{\phi}_i = \begin{bmatrix} \phi_i & 0 \\ 0 & \Delta_-\phi_i\Delta_+ \end{bmatrix}$

$$\hat{\phi}_1\hat{\phi}_2 = \hat{\phi}_2\hat{\phi}_1$$

This equality is accidental because of the choice of diagonal $\hat{\phi}_i$. In general $\hat{\phi}_i$ may not be diagonal matrix. For example, if the scalar fields are defined not only on sites but also on links then, there will be one off-diagonal elements and thus $\hat{\phi}_1\hat{\phi}_2 \neq \hat{\phi}_2\hat{\phi}_1$.
In general superfields are not commutative on the lattice and this is the natural consequence of it.

$$\Phi_1\Phi_2 \neq \Phi_2\Phi_1 \rightarrow \{\delta_Q(\Phi_1\Phi_2)\}|_0 \neq \{\delta_Q(\Phi_2\Phi_1)\}|_0$$

In the products of component fields the product order should be arranged to inherit the original superfields order. If the order is properly kept:

$$\delta_Q(\Phi_1\Phi_2|_0) = \{\delta_Q(\Phi_1\Phi_2)\}|_0, \qquad \delta_Q(\Phi_2\Phi_1|_0) = \{\delta_Q(\Phi_2\Phi_1)\}|_0$$

Possible origin for the problem

SUSY transformation is asymmetric between left and right operation, while translation is left-right symmetric.

Translation: $\delta_+\phi_i \equiv \Delta_+^2\phi_i\Delta_-^2 - \phi_i$

$$\delta_+(\phi_1\phi_2) = \Delta_+^2\phi_1\phi_2\Delta_-^2 - \phi_1\phi_2 = (\phi_1 + \delta_+\phi_1)(\phi_2 + \delta_+\phi_2) - \phi_1\phi_2$$

$$= (\delta_+\phi_1)\phi_2 + \Delta_+^2\phi_1\Delta_-^2\delta_+\phi_2 = \delta_+\phi_1\Delta_+^2\phi_2\Delta_-^2 + \phi_1\delta_+\phi_2$$

SUSY transformation:

$$\begin{aligned}\delta_Q(\Phi_1\Phi_2) &= \Delta_-[Q,\Phi_1\Phi_2] = \Delta_-([Q,\Phi_1]\Phi_2 + \Phi_1[Q,\Phi_2]) \\ &= \delta_Q\Phi_1\Phi_2 + \Delta_-\Phi_1\Delta_+\Delta_-[Q,\Phi_2] = \delta_Q\Phi_1\Phi_2 + \Delta_-\Phi_1\Delta_+\delta_Q\Phi_2 \\ &\neq \delta_Q\Phi_1\Delta_-\Phi_2\Delta_+ + \Phi_1\delta_Q\Phi_2\end{aligned}$$

This generates ordering ambiguity !

Exact SUSY invariance ?

We need modified Leibniz rule for super charges too.
With the modified Leibniz rule we can construct twisted SUSY algebra representation on a lattice.
We can construct exact twisted SUSY algebra on a lattice. Then we can construct twisted **supercharge exact** action parallel to continuum formulation.

Example of action: $S = \sum_x L = \sum_x \frac{1}{2} s\tilde{s}\epsilon_{\mu\nu} s_\mu s_\nu (-i\bar{c}c)$

twisted SUSY algebra

$$\{Q, Q_\mu\} = i\Delta_{+\mu}, \qquad \{\tilde{Q}, Q_\mu\} = i\Delta_{-\mu}, \quad Q^2 = \tilde{Q}^2 = Q_\mu^2 = 0$$

SUSY transformation of the action

$$s_A S = \sum_x Q_A L = \sum_x \{Q_A^2 L' + \text{total derivatives}\} = 0$$

⟹ Exact SUSY invariance with the modified Leibniz rule, if we keep the proper order of the component fields.

Exact SUSY on a Lattice ??

Twisted lattice SUSY algebra is a modification of ordinary algebra with shifted (anti-)commutator on the lattice. Shifted commutator is needed to show exact translational invariance of lattice actions. This modification of algebra reminds us of the Modification of lattice chiral transformation for Ginzberg-Wilson relation.

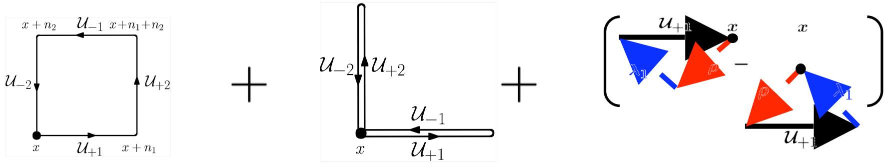
However, for super Yang-Mills another problem:

Link hole by SUSY operation

s_A operation generates a link hole to the action:

$$(s_A(\varphi) \equiv [\nabla_A, \varphi\}_{x+a_A+a_\varphi, x})$$

$$s_A(S) \equiv [\nabla_A, S]_{x+a_A, x}$$

$= \nabla_A($  $)$

$$= \ \{\text{link holes}\}$$

⟹ no gauge invariance ?

The gauge invariance is lost, since the edges of the links are gauge dependent.

(Bruckmann, Kok, Catterall)

But, there is, however, one free parameter $\mathbf{a}$ which corresponds to the shift of ∇ and can be taken to be 0. Thus the corresponding supercharge s doesn't create link holes and thus SUSY and gauge invariance are assured.

(Bruckmann, Kok, Catterall, Damgaard, Matsuura)

$\mathbf{a = 0}$ for asymmetric choice of the parameters corresponds to the orbifold construction by Kaplan et. al.

$\mathbf{a \neq 0}$ for symmetric choice of the parameters are not SUSY invariant !

A possible solution

We claim: if there is covariantly constant super parameter η_A which has opposite shift of ∇_A and commutes with all the super covariant derivatives:

$$\{\eta_A, \nabla_B\} = 0$$

$$\{\eta_A, \varphi\} = 0$$

$$\Longrightarrow \quad \{\eta_A \nabla_A, S\} = 0$$

lattice SUSY and gauge invariant !

η_A compensates the link holes.

We need to prove the existence of covariantly constant super parameter.

A trial

We need to find covariantly constant super parameter $\xi_A, \quad (\xi_A \nabla_A : \text{shiftless})$

$$\{\nabla_A, \xi_B\} = 0$$

$$(\nabla_A)_{x,x-a_A}(\xi_B)_{x-a_A,x-a_A+a_B} + (\xi_B)_{x,x+a_B}(\nabla_A)_{x+a_B,x+a_B-a_A} = 0$$

$$\nabla_A(x)\xi_B(x-a_A) = -\xi_B(x)\nabla_A(x+a_B)$$

We need to introduce non-commutative structure.
This can be realized by the following observation:

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & \\ 0 & 0 & 1 & \cdots & \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & & & 0 \end{bmatrix}}_{\mathbf{R}} \underbrace{\begin{bmatrix} \nabla(x) & 0 & \cdots & & \\ 0 & \nabla(x+a) & 0 & \cdots & \\ \vdots & 0 & \nabla(x+2a) & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \end{bmatrix}}_{\hat{\nabla}(x)} = \underbrace{\begin{bmatrix} \nabla(x+a) & 0 & \cdots & & \\ 0 & \nabla(x+2a) & 0 & \cdots & \\ \vdots & 0 & \nabla(x+3a) & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \end{bmatrix}}_{\hat{\nabla}(x+a)} \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & \\ 0 & 0 & 1 & \cdots & \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & & & 0 \end{bmatrix}}_{\mathbf{R}}$$

We introduce further matrix structure:

$$\nabla_A = \hat{\nabla}_A(x)\mathbf{R}_{a_A}, \qquad \xi_B = \hat{\xi}_B(x)\mathbf{R}_{a_B}$$

$$\begin{aligned}\{\nabla_A, \xi_B\} &= \hat{\nabla}_A(x)\mathbf{R}_{a_A}\hat{\xi}_B(x - a_A)\mathbf{R}_{a_B} + \hat{\xi}_B(x)\mathbf{R}_{a_B}\hat{\nabla}_A(x + a_B)\mathbf{R}_{a_A} \\ &= \hat{\nabla}_A(x)\mathbf{R}_{a_A}\hat{\xi}_B(x - a_A)\mathbf{R}_{a_B} - \hat{\nabla}_A(x)\mathbf{R}_{a_A}\hat{\xi}_B(x - a_A)\mathbf{R}_{a_B} \\ &= 0\end{aligned}$$

Gauge transformation

$$\nabla'_A(x) = \mathrm{G}(x)\hat{\nabla}_A(x)\mathrm{R}_{a_A}\mathrm{G}(x-a_A) = \mathrm{G}(x)\hat{\nabla}_A(x)\mathrm{G}(x)\mathrm{R}_{a_A}$$

$$\{\nabla, \nabla_\mu\} = iu_{+\mu}$$

$$\hat{\nabla}(x)\mathrm{R}_a\hat{\nabla}_\mu(x-a)\mathrm{R}_{a_\mu} + \hat{\nabla}_\mu(x)\mathrm{R}_{a_\mu}\hat{\nabla}(x-a_\mu)\mathrm{R}_a = i\mathrm{u}_\mu(x)\mathrm{R}_{n_\mu}$$

$$\hat{\nabla}(x)\hat{\nabla}_\mu(x)\mathrm{R}_a\mathrm{R}_{a_\mu} + \hat{\nabla}_\mu(x)\hat{\nabla}(x)\mathrm{R}_{a_\mu}\mathrm{R}_a = i\mathrm{u}_\mu(x)\mathrm{R}_{n_\mu}$$

$$\mathrm{R}_a\mathrm{R}_{a_\mu} = \mathrm{R}_{a_\mu}\mathrm{R}_a = \mathrm{R}_{n_\mu}$$

$$\hat{\nabla}(x)\hat{\nabla}_\mu(x) + \hat{\nabla}_\mu(x)\hat{\nabla}(x) = i\mathrm{u}_\mu(x)$$

After taking off the shift operators R_{a_A}, ∇_A loose a Link nature and space-time points shrink to a point.

Reduced model

No solution yet !

$\mathbf{a} = \mathbf{0}$ for asymmetric choice of the parameters
corresponds to the orbifold construction by Kaplan et. al.

$\mathbf{a} \neq \mathbf{0}$ for symmetric choice of the parameters
are not SUSY invariant !

Our Claim

There is one free parameter $\mathbf{a}$ in the formulation
and physical quantities may not depend on it.
The symmetric choice of $\mathbf{a} \neq \mathbf{0}$ takes into account
the dual lattice as well and geometrically most
symmetric and may lead to the continuum limit
quickly (Unsal).

A New Proposal for the first problem

This part will be presented later when it's ready.

Summary of the Proposal

Half translation = Lattice SUSY transformation

Lattice SUSY		Lattice QCD
exact half translation inv.	$\Longleftrightarrow$	exact gauge inv.
broken naïve statistics	$\Longleftrightarrow$	broken Lorentz inv.

Summary

- Discrete Twisted SUSY algebra is realized on the Lattice with modified Leibniz rule.
- Ordering ambiguity for the product of component fields is cleared up and the product order should inherits the original order of the superfields for the asymmetric definition of SUSY.
- We need covariantly constant super parameter for lattice super Yang-Mills. No proof yet !
- We propose that lattice SUSY transformation can be identified as half-translation with new super parameters.