Supersymmetric lattices: theory and applications

Simon Catterall

Syracuse University

Exact lattice SUSY: approaches

- Topological twisting:
  - Link constructions
  - Sugino formulation
  - Geometrical discretization
- Orbifolding

Talk about connections between last 2.
Simulations

General ideas

- Decompose fields under diagonal subgroup $SO_{\rm rot}(D) \times SO_R(D)$
- Exposes scalar nilpotent supercharge
- Action (often) Q-exact.
- Fermions represented as Kähler-Dirac field
- Lattice:
  - Subalgebra $Q^2 = 0$ intact on lattice
  - Doubling evaded
  - Link/orbifold/geometrical schemes: link fields. Novel gauge transformation properties.

$\mathcal{Q} = 4$ SYM in 2D

In twisted form (adjoint fields AH generators)

$$S = \frac{1}{g^2}\mathcal{Q}\int \mathrm{Tr}\,\left(\chi_{\mu\nu}\mathcal{F}_{\mu\nu} + \eta[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu] - \frac{1}{2}\eta d\right)$$

$$\begin{aligned}
\mathcal{Q}\,\mathcal{A}_\mu &= \psi_\mu \\
\mathcal{Q}\,\psi_\mu &= 0 \\
\mathcal{Q}\,\overline{\mathcal{A}}_\mu &= 0 \\
\mathcal{Q}\,\chi_{\mu\nu} &= -\overline{\mathcal{F}}_{\mu\nu} \\
\mathcal{Q}\,\eta &= d \\
\mathcal{Q}\,d &= 0
\end{aligned}$$

Note: complexified gauge field $\mathcal{A}_\mu = A_\mu + iB_\mu$

Action

Q-variation, integrate d:

$$S = \frac{1}{g^2}\int \mathrm{Tr}\ \left(-\overline{\mathcal{F}}_{\mu\nu}\mathcal{F}_{\mu\nu} + \frac{1}{2}[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu]^2 - \chi_{\mu\nu}\mathcal{D}_{[\mu}\psi_{\nu]} - \eta\overline{\mathcal{D}}_\mu\psi_\mu\right)$$

Rewrite as

$$S = \frac{1}{g^2}\int \mathrm{Tr}\ \left(-F_{\mu\nu}^2 + 2B_\mu D_\nu D_\nu B_\mu - [B_\mu, B_\nu]^2 + L_F\right)$$

where

$$L_F = \left(\begin{array}{cc} \chi_{12} & \frac{\eta}{2} \end{array}\right)\left(\begin{array}{cc} -D_2 - iB_2 & D_1 + iB_1 \\ D_1 - iB_1 & D_2 - iB_2 \end{array}\right)\left(\begin{array}{c} \psi_1 \\ \psi_2 \end{array}\right)$$

Lattice ?

- $\mathcal{A}_\mu(x) \to \mathcal{U}_\mu(n)$. Complexified Wilson links.
- Natural fermion assignment – η on sites, ψ_μ links, χ_{12} diagonal links of cubic lattice.
- $U(N)$ gauge transformations:

$$
\begin{aligned}
\eta(\mathbf{x}) &\to G(\mathbf{x})\eta(\mathbf{x})G^\dagger(\mathbf{x}) \\
\psi_\mu(\mathbf{x}) &\to G(\mathbf{x})\psi_\mu(\mathbf{x})G^\dagger(\mathbf{x}+\mu) \\
\chi_{\mu\nu}(\mathbf{x}) &\to G(\mathbf{x}+\mu+\nu)\chi_{\mu\nu}(\mathbf{x})G^\dagger(\mathbf{x}) \\
\mathcal{U}_\mu(\mathbf{x}) &\to G(\mathbf{x})\mathcal{U}_\mu(\mathbf{x})G^\dagger(\mathbf{x}+\mu) \\
\overline{\mathcal{U}}_\mu(\mathbf{x}) &\to G(\mathbf{x}+\mu)\overline{\mathcal{U}}_\mu(\mathbf{x})G^\dagger(\mathbf{x})
\end{aligned}
$$

- Note: choose orientation of χ so that $\mathrm{Tr} F_{\mu\nu}\chi_{\mu\nu}$ G.I

Derivatives

$$
\begin{aligned}
\mathcal{D}^{(+)}_\mu f_\nu(\mathbf{x}) &= \mathcal{U}_\mu(\mathbf{x}) f_\nu(\mathbf{x}+\mu) - f_\nu(\mathbf{x})\mathcal{U}_\mu(\mathbf{x}+\nu) \\
\overline{\mathcal{D}}^{(-)}_\mu f_\mu(\mathbf{x}) &= f_\mu(\mathbf{x})\overline{\mathcal{U}}_\mu(\mathbf{x}) - \overline{\mathcal{U}}_\mu(\mathbf{x}-\mu) f_\mu(\mathbf{x}-\mu)
\end{aligned}
$$

In naive continuum limit $U_\mu(x) = 1 + A_\mu(x) + \ldots$ reduce to adjoint covariant derivatives.
Note: act like $d, d^\dagger$ – eg $\mathcal{D}^{(+)}$ takes lattice p-form to (p+1)-form as evidenced by gauge transformation

Lattice supersymmetry

$$
\begin{aligned}
\mathcal{Q}\,\mathcal{U}_\mu &= \psi_\mu \\
\mathcal{Q}\,\psi_\mu &= 0 \\
\mathcal{Q}\,\overline{\mathcal{U}}_\mu &= 0 \\
\mathcal{Q}\,\chi_{\mu\nu} &= \mathcal{F}_{\mu\nu}^{L\dagger} \\
\mathcal{Q}\,\eta &= d \\
\mathcal{Q}\,d &= 0
\end{aligned}
$$

Lattice field strength as

$$
\mathcal{F}_{\mu\nu}^{L} = \mathcal{D}_\mu^{(+)}\mathcal{U}_\nu(\mathbf{x}) = \mathcal{U}_\mu(\mathbf{x})\mathcal{U}_\nu(\mathbf{x}+\mu) - \mathcal{U}_\nu(\mathbf{x})\mathcal{U}_\mu(\mathbf{x}+\nu)
$$

Lattice Action

$$S = \kappa \sum_{\mathbf{x}} \mathrm{Tr} \left(\mathcal{F}_{\mu\nu}^{L\dagger} \mathcal{F}_{\mu\nu}^{L} + \frac{1}{2} \left(\overline{\mathcal{D}}_{\mu}^{(-)} \mathcal{U}_{\mu} \right)^2 - \chi_{\mu\nu} \mathcal{D}_{[\mu}^{(+)} \psi_{\nu]} - \eta \overline{\mathcal{D}}_{\mu}^{(-)} \psi_{\mu} \right)$$

with $\kappa = \frac{V}{g_{\mathrm{phys}}^2 A_{\mathrm{phys}}}$
Formally similar to continuum expression.
Contains (complexified) Wilson plaquette term+...
Identical to $\mathcal{Q} = 4$ orbifold action
Fermion EOM discrete Kähler-Dirac equation – no doubles (staggered quarks)

Summary

- Twisted picture gives a nice reinterpretation of orbifold lattices.
- New content: coupling constant indep, continuum limit, why no doubling
- Quick way to derive orbifold theories (Matsuura, Damgaard)
- Metric independence – possibility of lattice theories ...

$\mathcal{Q} = 16$ SYM in 4D

- Twist: diagonal subgroup of $SO_{\rm Lorenz}(4) \times SO_{\rm R}(4)$
- Again after twisting regard fermions as 4×4 matrix.
- Expand on basis of products of gamma matrices – 16 twisted fermions as expected for $\mathcal{N} = 4$ SYM.
- But notice: to represent 10 bosons of $\mathcal{N} = 4$ theory with complex connections is most natural in five dimensions.
- Fermion counting requires multiplet $(\eta, \psi_a, \chi_{ab})$ where $a, b = 1 \ldots 5$
- Action contains same $\mathcal{Q}$-exact term as for $\mathcal{Q} = 4$ plus new $\mathcal{Q}$-closed piece.

Details

- Naturally formulated as 5D theory.
- After dimensional reduction to 4D – $\mathcal{A}_5$ plus imag parts of $\mathcal{A}_\mu, \mu = 1 \ldots 4$ yield 6 scalars of $\mathcal{N} = 4$
- Fermions: $\chi_{ab} \rightarrow \chi_{\mu\nu} \oplus \overline{\psi}_\mu$, $\psi_a \rightarrow \psi_\mu \oplus \overline{\eta}$
- $S = \mathcal{Q}\Lambda - \frac{1}{8} \int \epsilon_{abcde} \chi_{de} \overline{\mathcal{D}}_c \chi_{ab}$
- Twisted action reduces to Marcus topological twist of $\mathcal{N} = 4$ (GL-twist). Equivalent to usual theory in flat space.
- Identical to $Q = 16$ orbifold action

Marcus twist

Reduces to:

$$
\begin{aligned}
S &= \int \mathrm{Tr}\,\Big(-\overline{\mathcal{F}}_{\mu\nu}\mathcal{F}_{\mu\nu} + \frac{1}{2}\left[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu\right]^2 + \frac{1}{2}\left[\overline{\phi}, \phi\right]^2 + (\mathcal{D}_\mu\phi)^\dagger(\mathcal{D}_\mu\phi) \\
&- \chi_{\mu\nu}\mathcal{D}_{[\mu}\psi_{\nu]} - \overline{\psi}_\mu\mathcal{D}_\mu\overline{\eta} - \overline{\psi}_\mu\left[\phi, \psi_\mu\right] \\
&- \eta\overline{\mathcal{D}}_\mu\psi_\mu - \eta\left[\overline{\phi}, \overline{\eta}\right] - \chi^*_{\mu\nu}\overline{\mathcal{D}}_\mu\overline{\psi}_\nu - \chi^*_{\mu\nu}\left[\overline{\phi}, \chi_{\mu\nu}\right]\Big)
\end{aligned}
$$

Transition to lattice

- Introduce cubic lattice with unit vectors $\mu_a^i = \delta_a^i, a = 1 \ldots 4$. Additional vector $\mu_5 = (-1, -1, -1, -1)$.
- Notice: $\sum_a \mu_a = 0$. Needed for G.I.
- Assign fields to links in cubic lattice (plus diagonals). Eg $\chi_{ab}(\mathbf{x})$ lives on link from $(\mathbf{x} + \mu_a + \mu_b) \to \mathbf{x}$. (link vectors correspond to r-charges of orbifold)
- Derivatives similar to $\mathcal{Q} = 4$. eg $D_a^{(+)} f(\mathbf{x}) = \mathcal{U}_a(\mathbf{x}) f(\mathbf{x} + \mathbf{a}) - f(\mathbf{x}) \mathcal{U}_a(\mathbf{x})$
- Remarkably $\mathcal{Q}$-closed term still supersymmetric since $\epsilon_{abcde} D_a^{(+)} F_{bc}^L = 0$!

Simulations

- Bosonic action local, real, positive semidefinite.
- Integrate out fermions – Pfaffian.
- Monte Carlo simulation possible using lattice QCD algorithms – RHMC to handle Pfaffian.
- Periodic and antiperiodic (thermal) bcs.
- Code allows for dimensional reduction – eg $\mathcal{Q} = 4, 16$ SYMQM

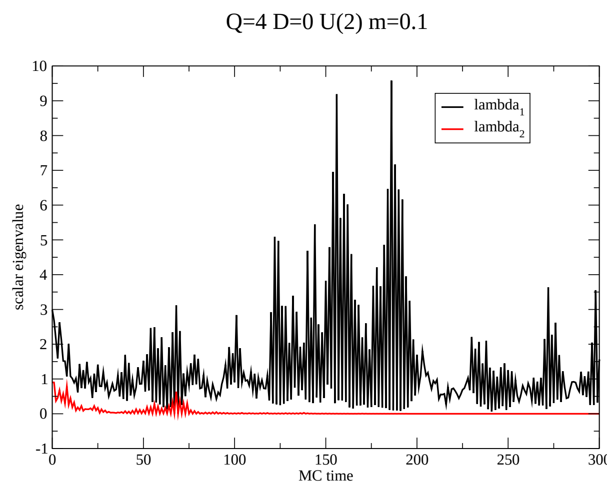
Vacuum stability - trace mode

Correspondance to continuum requires $\mathcal{U}_\mu = 1 + aA_\mu + O(a^2)$.

For $U(N)$ this is not true $< \frac{1}{N}\mathrm{Tr}\,\mathcal{U}_\mu^\dagger(x)\mathcal{U}_\mu(x) > \sim 0.5$

$\det(\mathcal{U}_\mu^\dagger(x)\mathcal{U}_\mu(x)) \longrightarrow 0$!

Vacuum instability – $\det(\mathcal{U}_\mu^\dagger\mathcal{U}_\mu) \sim e^{B_\mu^0}$ implies $B_\mu^0 \to -\infty$



Truncation

Cannot cure with mass $m^2 \sum \mathrm{Tr}\, (\mathcal{U}_\mu^\dagger \mathcal{U}_\mu - I)^2$

m	$< \mathcal{U}_\mu^\dagger \mathcal{U}_\mu >$
0.01	0.45(2)
0.1	0.57(6)
0.5	0.38(2)

$S_B(e^{-\delta B_\mu^0}\mathcal{U}) \sim e^{-4\delta B_\mu^0} S(\mathcal{U}_\mu)$ any $\{\mathcal{U}_\mu\}$

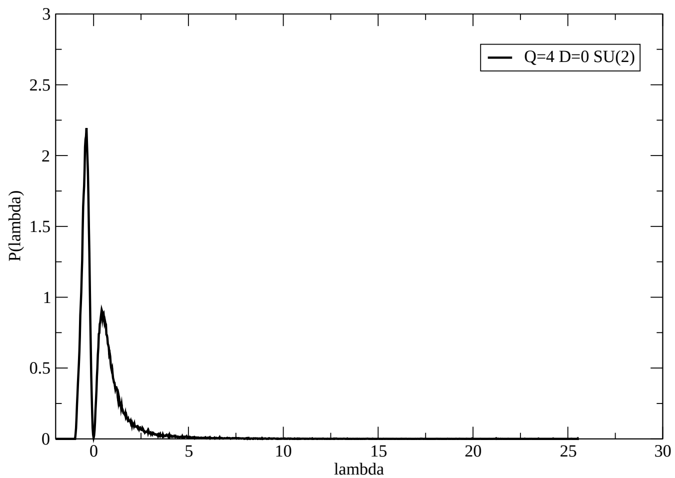
Exponential effective potential for B_μ^0.

Fix ? - truncate to $SU(N)$ – $\delta S \sim \frac{1}{N^2} O(a)$

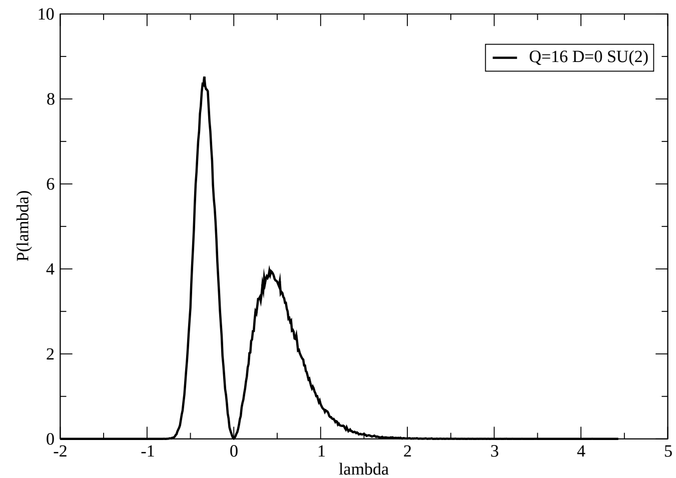
Also removes exact 0 mode in fermion op.

Vacuum stability - flat directions

What about moduli space $[B_\mu, B_\nu] = 0$ for $SU(N)$?



$\mathcal{Q} = 4$



$\mathcal{Q} = 16$

$D = 0$. $SU(2)$. Periodic bcs. Eigenvalues of $\mathcal{U}_\mu^\dagger \mathcal{U}_\mu - 1$
Scalars localized close to origin. Power law tails.
$p(\mathcal{Q} = 4) \sim 3$, $p(\mathcal{Q} = 16) \sim 15$ (Staudacher et al.)

Supersymmetric Ward identity

$\mathcal{Q}$-exactness ensures that $\frac{\partial \ln Z_{\rm pbc}}{\partial \kappa} = 0$

Ensures: $< \kappa S_B >= \frac{1}{2} V(N^2-1)(n_{\rm bosons}-1)$

Example: $D=0\ SU(2)$

κ	κS_B	exact
1.0	4.40(2)	4.5
10.0	4.47(2)	4.5
100.0	4.49(1)	4.5

κ	κS_B	exact
1.0	13.67(4)	13.5
10.0	13.52(2)	13.5
100.0	13.48(2)	13.5

Pfaffian phase

Simulation uses $|Pf(\mathcal{U})|$. Measure phase $\alpha(\mathcal{U})$.

$$< O >= \frac{< Oe^{\alpha} >_{\text{phase quenched}}}{< e^{\alpha} >_{\text{phase quenched}}}$$

$SU(2)$ $D = 2$: 4^2.

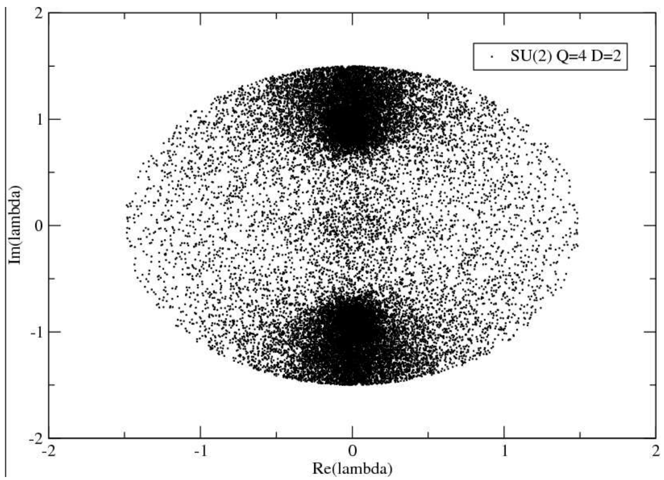
$\mathcal{Q}$	S_B^q	S_B	S_B^e	$\cos\alpha$
4	70.61(4)	65(5)	72.0	-0.016(6)
16	214.7(4)	214.6(3)	216.0	0.999994(3)

$< e^{i\alpha(\mathcal{U}_\mu)} >_{\text{phase quenched pbc}} = W = 0$ for $\mathcal{Q} = 4$?

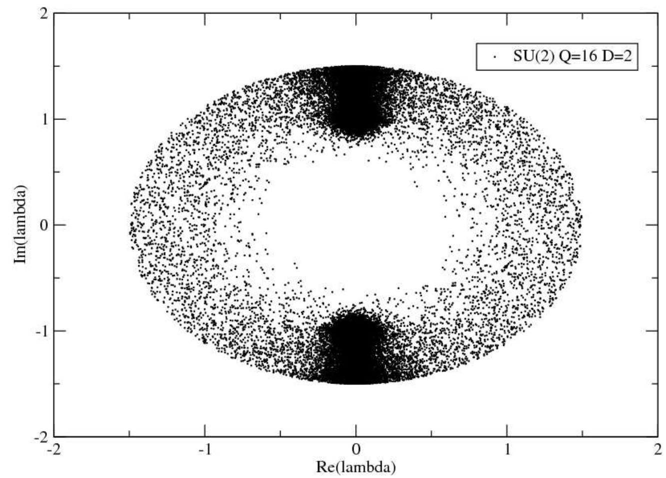
SUSY breaking (Tong et. al) ?

Fermion eigenvalue distribution

$SU(2)$ $D = 2$: 2^2



$\mathcal{Q} = 4$

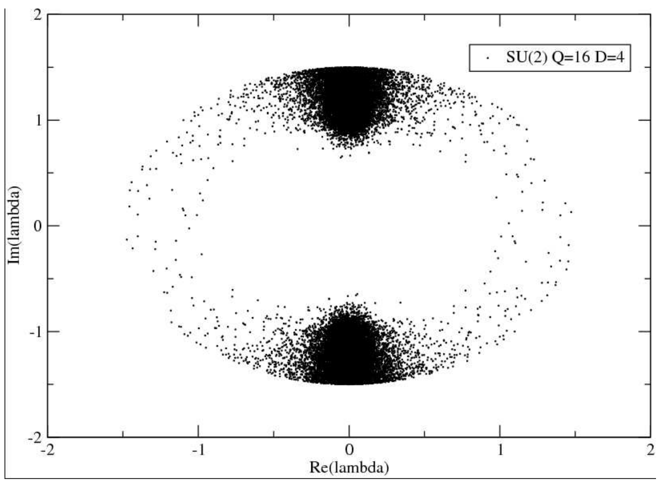


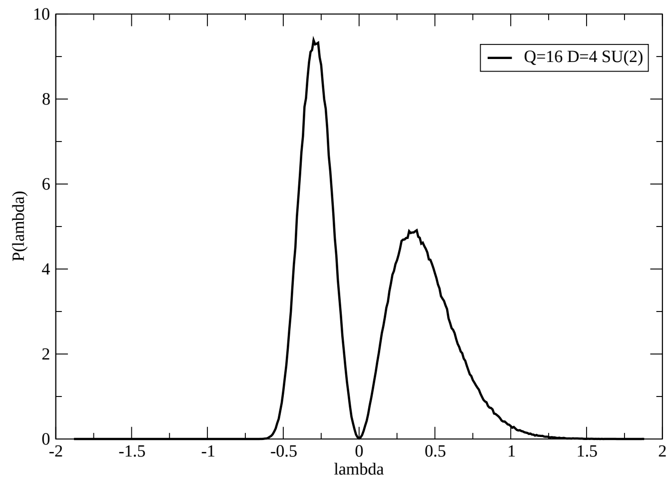
$\mathcal{Q} = 16$

Non-zero density for $\mathcal{Q} = 4$ close to origin – linked to log divergence of $< \delta\lambda^2 >$?
Potential Goldstino ?

$\mathcal{N} = 4$ SYM in four dimensions

Initial results encouraging: 6000 trajs on $SU(2)$ 2^4 lattice (1000 hrs)
$S_B/S_B^{\mathrm{exact}} = 0.98 < \cos(\alpha) >= 0.98(1)$





Parallel code under development.

Renormalization

Gauge invariance strongly constrains counterterms;

- no fermion bilinears allowed ! No scalar masses allowed by $\mathcal{Q}$.
- Few additional constraints from broken supersymmetries eg $<\delta_{Q_\mu} O_{\rm G.I\ loop}>=0$
- Numerical evidence in $D=2$ supports no fine tuning.

Future

- Nonperturbative exploration $\mathcal{N} = 4$ YM. Tests of AdSCFT.
- But – what residual fine tuning needed to get full SUSY as $a \to 0$?
- Dimensional reductions – duality between strings with Dp-branes and $(p+1)$-SYM ?
- Add fermions in fundamental .. (Matsuura, Sugino in $D = 2$ recently).