Supersymmetric lattices: theory and applications

Simon Catterall

Syracuse University
Exact lattice SUSY: approaches

- Topological twisting:
  - Link constructions
  - Sugino formulation
  - Geometrical discretization
- Orbifolding

Talk about connections between last 2.

Simulations
General ideas

- Decompose fields under diagonal subgroup $SO_{\text{rot}}(D) \times SO_R(D)$

- Exposes scalar nilpotent supercharge

- Action (often) $Q$-exact.

- Fermions represented as Kähler-Dirac field

- Lattice:
  - Subalgebra $Q^2 = 0$ intact on lattice
  - Doubling evaded
\[ Q = 4 \text{ SYM in 2D} \]

In twisted form (adjoint fields AH generators)

\[ S = \frac{1}{g^2} Q \int \text{Tr} \left( \chi_{\mu\nu} F_{\mu\nu} + \eta [\overline{D}_\mu, D_\mu] - \frac{1}{2} \eta d \right) \]

\[ \begin{align*}
Q A_\mu &= \psi_\mu \\
Q \psi_\mu &= 0 \\
Q \overline{A}_\mu &= 0 \\
Q \chi_{\mu\nu} &= -\overline{F}_{\mu\nu} \\
Q \eta &= d \\
Q d &= 0 
\end{align*} \]

Note: complexified gauge field \( A_\mu = A_\mu + iB_\mu \)
Action

$Q$-variation, integrate $d$:

$$S = \frac{1}{g^2} \int \text{Tr} \left( -\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu} + \frac{1}{2}[\overline{D}_\mu, D_\mu]^2 - \chi_{\mu\nu}D_{[\mu} \psi_{\nu]} - \eta \overline{D}_\mu \psi_\mu \right)$$

Rewrite as

$$S = \frac{1}{g^2} \int \text{Tr} \left( -F^2_{\mu\nu} + 2B_\mu D_\nu D_\nu B_\mu - [B_\mu, B_\nu]^2 + L_F \right)$$

where

$$L_F = \begin{pmatrix} \chi_{12} & \eta/2 \\ \end{pmatrix} \begin{pmatrix} -D_2 - iB_2 & D_1 + iB_1 \\ D_1 - iB_1 & D_2 - iB_2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$
Lattice?

- \( A_\mu(x) \rightarrow U_\mu(n) \). Complexified Wilson links.

- Natural fermion assignment – \( \eta \) on sites, \( \psi_\mu \) links, \( \chi_{12} \) diagonal links of cubic lattice.

- \( U(N) \) gauge transformations:

  \[
  \eta(x) \rightarrow G(x)\eta(x)G^\dagger(x)
  \]

  \[
  \psi_\mu(x) \rightarrow G(x)\psi_\mu(x)G^\dagger(x + \mu)
  \]

  \[
  \chi_{\mu\nu}(x) \rightarrow G(x + \mu + \nu)\chi_{\mu\nu}(x)G^\dagger(x)
  \]

  \[
  U_\mu(x) \rightarrow G(x)U_\mu(x)G^\dagger(x + \mu)
  \]

  \[
  \overline{U}_\mu(x) \rightarrow G(x + \mu)\overline{U}_\mu(x)G^\dagger(x)
  \]

- Note: choose orientation of \( \chi \) so that \( \text{Tr}F_{\mu\nu}\chi_{\mu\nu} \) G.I
Derivatives

\[ D^{(+)}_{\mu} f_{\nu}(x) = U_{\mu}(x) f_{\nu}(x + \mu) - f_{\nu}(x) U_{\mu}(x + \nu) \]

\[ \overline{D}^{(-)}_{\mu} f_{\mu}(x) = f_{\mu}(x) U_{\mu}(x) - U_{\mu}(x - \mu) f_{\mu}(x - \mu) \]

In naive continuum limit \( U_{\mu}(x) = 1 + A_{\mu}(x) + \ldots \) reduce to adjoint covariant derivatives.

Note: act like \( d, d^\dagger \) – eg \( D^{(+)} \) takes lattice p-form to (p+1)-form as evidenced by gauge transformation
Lattice supersymmetry

\begin{align*}
Q U_\mu &= \psi_\mu \\
Q \psi_\mu &= 0 \\
Q \overline{U}_\mu &= 0 \\
Q \chi_{\mu\nu} &= \mathcal{F}_{\mu\nu}^L \\
Q \eta &= d \\
Q d &= 0
\end{align*}

Lattice field strength as

\[ \mathcal{F}_{\mu\nu}^L = D_\mu^{(+)} U_\nu(x) = U_\mu(x) U_\nu(x + \mu) - U_\nu(x) U_\mu(x + \nu) \]
Lattice Action

\[ S = \kappa \sum_x \text{Tr} \left( F^{L \dagger}_\mu \nabla^L_\mu + \frac{1}{2} \left( D_\mu (-) U_\mu \right)^2 - \chi_{\mu \nu} D^{(+)}_{[\mu} \psi_{\nu]} - \eta \overline{D}^{(-)}_\mu \psi_\mu \right) \]

with \( \kappa = \frac{V}{g_{\text{phys}}^2 A_{\text{phys}}} \)

Formally similar to continuum expression.
Contains (complexified) Wilson plaquette term + ...
Identical to \( Q = 4 \) orbifold action
Fermion EOM discrete Kähler-Dirac equation – no doubles
(staggered quarks)
Summary

- Twisted picture gives a nice reinterpretation of orbifold lattices.
- New content: coupling constant indep, continuum limit, why no doubling
- Quick way to derive orbifold theories (Matsuura, Damgaard)
- Metric independence – possibility of lattice theories ...
$Q = 16$ SYM in 4D

- Twist: diagonal subgroup of $SO_{\text{Lorenz}}(4) \times SO_{\text{R}}(4)$
- Again after twisting regard fermions as $4 \times 4$ matrix.
- Expand on basis of products of gamma matrices – 16 twisted fermions as expected for $\mathcal{N} = 4$ SYM.
- But notice: to represent 10 bosons of $\mathcal{N} = 4$ theory with complex connections is most natural in five dimensions.
- Fermion counting requires multiplet $(\eta, \psi_a, \chi_{ab})$ where $a, b = 1 \ldots 5$
- Action contains same $Q$-exact term as for $Q = 4$ plus new $Q$-closed piece.
Naturally formulated as 5D theory.

After dimensional reduction to 4D – $A_5$ plus imag parts of $A_\mu, \mu = 1 \ldots 4$ yield 6 scalars of $\mathcal{N} = 4$

Fermions: $\chi_{ab} \rightarrow \chi_{\mu\nu} \oplus \overline{\psi}_\mu, \psi_a \rightarrow \psi_\mu \oplus \eta$

$S = Q\Lambda - \frac{1}{8} \int \epsilon_{abcde} \chi_{de} \overline{D}_c \chi_{ab}$

Twisted action reduces to Marcus topological twist of $\mathcal{N} = 4$ (GL-twist). Equivalent to usual theory in flat space.

Identical to $Q = 16$ orbifold action
Marcus twist

Reduces to:

\[ S = \int \text{Tr} \left( -\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu} + \frac{1}{2} \left[ \overline{\mathcal{D}}_\mu, \mathcal{D}_\mu \right]^2 + \frac{1}{2} \left[ \phi, \phi \right]^2 + (\mathcal{D}_\mu \phi)^\dagger (\mathcal{D}_\mu \phi) 
- \chi_{\mu\nu} \mathcal{D}_{[\mu} \psi_{\nu]} - \overline{\psi}_\mu \mathcal{D}_\mu \overline{\eta} - \overline{\psi}_\mu \left[ \phi, \psi_\mu \right] 
- \eta \overline{\mathcal{D}}_\mu \psi_\mu - \eta \left[ \phi, \overline{\eta} \right] - \chi^*_{\mu\nu} \overline{\mathcal{D}}_\mu \overline{\psi}_\nu - \chi_{\mu\nu}^* \left[ \phi, \chi_{\mu\nu} \right] \right) \]
Transition to lattice

- Introduce cubic lattice with unit vectors
  \[ \mu_a^i = \delta_a^i, \ a = 1 \ldots 4. \] Additional vector
  \[ \mu_5 = (-1, -1, -1, -1). \]

- Notice: \[ \sum_a \mu_a = 0. \] Needed for G.I.

- Assign fields to links in cubic lattice (plus diagonals). Eg \[ \chi_{ab}(x) \] lives on link from \( (x + \mu_a + \mu_b) \rightarrow x. \) (link vectors correspond to r-charges of orbifold)

- Derivatives similar to \( Q = 4. \) eg
  \[ D^{(+)}_a f(x) = \mathcal{U}_a(x) f(x + a) - f(x) \mathcal{U}_a(x) \]

- Remarkably \( Q \)-closed term still supersymmetric since
  \[ \epsilon_{abcde} D^{(+)}_a F^{L}_{bc} = 0 ! \]
Simulations

- Bosonic action local, real, positive semidefinite.
- Integrate out fermions – Pfaffian.
- Monte Carlo simulation possible using lattice QCD algorithms – RHMC to handle Pfaffian.
- Periodic and antiperiodic (thermal) bcs.
- Code allows for dimensional reduction – eg $Q = 4, 16$

SYMQM
Vacuum stability - trace mode

Correspondance to continuum requires
\[ U_\mu = 1 + aA_\mu + O(a^2). \]
For \( U(N) \) this is not true
\[ \frac{1}{N} \text{Tr} \, U_\mu^\dagger(x)U_\mu(x) > \sim 0.5 \]
det\((U_\mu^\dagger(x)U_\mu(x)) \to 0! \]
Vacuum instability \(-\det(U_\mu^\dagger U_\mu) \sim e^{B_\mu^0} \) implies \( B_\mu^0 \to -\infty \)
Truncation

Cannot cure with mass \( m^2 \sum \text{Tr} (\mathcal{U}_\mu^\dagger \mathcal{U}_\mu - I)^2 \)

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \langle \mathcal{U}<em>\mu^\dagger \mathcal{U}</em>\mu \rangle )</th>
</tr>
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<tr>
<td>0.01</td>
<td>0.45(2)</td>
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<tr>
<td>0.1</td>
<td>0.57(6)</td>
</tr>
<tr>
<td>0.5</td>
<td>0.38(2)</td>
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\[ S_B(e^{-\delta B_\mu^0 \mathcal{U}}) \sim e^{-4\delta B_\mu^0} S(\mathcal{U}_\mu) \] any \( \{\mathcal{U}_\mu\} \)

Exponential effective potential for \( B_\mu^0 \).

Fix ? - truncate to \( SU(N) - \delta S \sim \frac{1}{N^2} O(a) \)

Also removes exact 0 mode in fermion op.
Vacuum stability - flat directions

What about moduli space \([B_\mu, B_\nu] = 0\) for \(SU(N)\)?

\(Q = 4\)

\(Q = 16\)

\(D = 0.\) \(SU(2)\). Periodic bcs. Eigenvalues of \(U_\mu U_\mu - 1\)
Scalars localized close to origin. Power law tails.

\(p(Q = 4) \sim 3, p(Q = 16) \sim 15\) (Staudacher et al.)
Supersymmetric Ward identity

$Q$-exactness ensures that $\frac{\partial \ln Z_{pbc}}{\partial \kappa} = 0$

Ensures: $\langle \kappa S_B \rangle = \frac{1}{2} V (N^2 - 1)(n_{\text{bosons}} - 1)$

Example: $D = 0 \ SU(2)$

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>$\kappa S_B$</th>
<th>exact</th>
</tr>
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<tbody>
<tr>
<td>1.0</td>
<td>4.40(2)</td>
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</tr>
<tr>
<td>10.0</td>
<td>4.47(2)</td>
<td>4.5</td>
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<tr>
<td>100.0</td>
<td>4.49(1)</td>
<td>4.5</td>
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<thead>
<tr>
<th>$\kappa$</th>
<th>$\kappa S_B$</th>
<th>exact</th>
</tr>
</thead>
<tbody>
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<td>1.0</td>
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<td>13.5</td>
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<tr>
<td>10.0</td>
<td>13.52(2)</td>
<td>13.5</td>
</tr>
<tr>
<td>100.0</td>
<td>13.48(2)</td>
<td>13.5</td>
</tr>
</tbody>
</table>
Pfaffian phase

Simulation uses $|Pf(U)|$. Measure phase $\alpha(U)$.

$$< O > = \frac{< O e^{\alpha} >_{\text{phase quenched}}}{< e^{\alpha} >_{\text{phase quenched}}}$$

$SU(2) \, D = 2: \, 4^2.$

<table>
<thead>
<tr>
<th>$Q$</th>
<th>$S_B^q$</th>
<th>$S_B$</th>
<th>$S_B^e$</th>
<th>$\cos \alpha$</th>
</tr>
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<tbody>
<tr>
<td>4</td>
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<td>65(5)</td>
<td>72.0</td>
<td>-0.016(6)</td>
</tr>
<tr>
<td>16</td>
<td>214.7(4)</td>
<td>214.6(3)</td>
<td>216.0</td>
<td>0.999994(3)</td>
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</table>

$< e^{i\alpha(U_{\mu})} >_{\text{phase quenched pbc}} = W = 0 \text{ for } Q = 4$?

SUSY breaking (Tong et. al)?
Fermion eigenvalue distribution

\[ SU(2) \ D = 2: \ 2^2 \]

Non-zero density for \( Q = 4 \) close to origin – linked to log divergence of \( \langle \delta \lambda^2 \rangle \)?
Potential Goldstino?
\[ \mathcal{N} = 4 \text{ SYM in four dimensions} \]

Initial results encouraging: 6000 trajs on \( SU(2) 2^4 \) lattice (1000 hrs)
\[ \frac{S_B}{S_B^{\text{exact}}} = 0.98 < \cos(\alpha) > = 0.98(1) \]

Parallel code under development.
Renormalization

Gauge invariance strongly constrains counterterms;

- no fermion bilinears allowed! No scalar masses allowed by $Q$.
- Few additional constraints from broken supersymmetries eg $<\delta Q_\mu O_{G.I\ loop}> = 0$
- Numerical evidence in $D = 2$ supports no fine tuning.
Future

- Nonperturbative exploration $\mathcal{N} = 4$ YM. Tests of AdSCFT.
- But – what residual fine tuning needed to get full SUSY as $a \rightarrow 0$?
- Dimensional reductions – duality between strings with Dp-branes and $(p + 1)$-SYM?
- Add fermions in fundamental .. (Matsuura, Sugino in $D = 2$ recently).