2008/11/28 NBIA workshop “Lattice Supersymmetry and Beyond”

Ginsparg-Wilson formulation of 2D $\mathcal{N} = (2, 2)$ SQCD with exact supersymmetry

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This talk is mainly based on

- **[1] F. S., Nucl. Phys. B 808 (2009) 292** [`arXiv:0807.2683 [hep-lat]`].
- **[2] Y. Kikukawa and F. S.,** `arXiv:0811.0916 [hep-lat]`.

1 Introduction

$\diamondsuit$ **Supersymmetric gauge theories ($\supset$ matrix models) are promising approaches to the physics beyond the standard model ($\supset$ string theories).**
$\Rightarrow$ Their nonperturbative formulations (e.g. lattice fomulation) are desired.

$\diamondsuit$ **Notorious difficulty for realization of SUSY on lattice**

[D'Adda-Kawamoto et al, Bergner-Bruckmann-Pawlowski]

$\diamondsuit$ **A part of supercharges can be preserved on the lattice: (We focus on it.)**

- **2D Wess-Zumino model** [Sakai-Sakamoto, Kikukawa-Nakayama, Catterall]
- **pure SYM models** [Kaplan et al, Ishii et al] $\leftarrow$ **orbifolding**,
  [F.S., Catterall] $\leftarrow$ **TFT approach**
- **SYM + matter fields** [Endre-Kaplan, Matsuura] $\leftarrow$ **orbifolding**,
  [1,2] This Talk $\leftarrow$ **TFT approach**

Here, we construct lattice models for
2D $\mathcal{N} = (2,2)$ SQCD
(SYM + n_+ fundamental and n_- anti-fundamental matter multiplets)
with $G = \mathrm{U}(N)$ or $\mathrm{SU}(N)$
2D regular lattice (with the spacing a)
compact gauge fields U_μ
general matter superpotentials and general twisted mass terms,
keeping one of the supercharges Q.

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[1]: The Wilson terms are introduced in order to supress bosonic and fermionic doublers in the matter sector ($\leftarrow$ consistent with Q SUSY).
$\Rightarrow$ The lattice action is defined only when $n_+ = n_-$ and $\widetilde{m}_{+I} = \widetilde{m}_{-I}$.
Nevertheless, since the anti-holomorphic twisted masses $\widetilde{m}^*_{\pm I}$ can be chosen freely, we can analyze the case $n_+ \neq n_-$ by making some multiplets decoupled.

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Nevertheless, since the anti-holomorphic twisted masses $\widetilde{m}^*_{\pm I}$ can be chosen freely, we can analyze the case $n_+ \neq n_-$ by making some multiplets decoupled.

[2]: The overlap Dirac operators, which satisfy the Ginsparg-Wilson relation, are introduced to realize the chiral flavor symmetry on the lattice.
$\Rightarrow$ The lattice action can be defined for general $n_\pm$ and general $\widetilde{m}_{+I}, \widetilde{m}_{-I'}$.

Superpotentials are exactly holomorphic or anti-holomorphic on the lattice.
$\Rightarrow$ Nonrenormalization theorem is expected to hold.

The first example of lattice gauge models introduced the overlap operators with exactly preserving some of supersymmetry
(c.f. `[Kikukawa-Nakayama]` **for 2D WZ models)**

◇ Plan of Talk

§ 1: Introduction

§ 2: Continuum 2D $\mathcal{N} = (2, 2)$ SQCD

§ 3: Latticization of SYM part

§ 4: Lattice Formulation of [1]

§ 5: Lattice Formulation of [2]

§ 6: Lattice Formulation of Gauged Linear Sigma Models

§ 7: Summary and Discussion

Appendix A: Gauged Linear Sigma Models $\Rightarrow$ Grassmannian

Appendix B: Summary of Workshop and Outlook (maybe my personal)

2 Continuum 2D $\mathcal{N}=(2,2)$ SQCD

The continuum lagrangian is obtained by dimensional reduction from 4D $\mathcal{N}=1$ SQCD:

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\rm SYM} + \mathcal{L}_{\rm mat} + \mathcal{L}_{\rm pot} + \mathcal{L}_{{\rm FI},\vartheta}, \\
\mathcal{L}_{\rm SYM} &= \frac{1}{8g^2}{\rm tr}\left(W^\alpha W_\alpha|_{\theta\theta} + \bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}\Big|_{\bar\theta\bar\theta}\right), \\
\mathcal{L}_{\rm mat} &= \left[\sum_{I=1}^{n_+} \Phi^\dagger_{+I} e^{V-\widetilde{V}_{+I}}\Phi_{+I} + \sum_{I=1}^{n_-}\Phi_{-I}e^{-V+\widetilde{V}_{-I}}\Phi^\dagger_{-I}\right]\Bigg|_{\theta\theta\bar\theta\bar\theta} \\
\mathcal{L}_{\rm pot} &= W(\Phi_+,\Phi_-)|_{\theta\theta} + \overline{W}(\Phi^\dagger_+,\Phi^\dagger_-)\Big|_{\bar\theta\bar\theta} \\
\mathcal{L}_{{\rm FI},\vartheta} &= {\rm tr}\left(-\kappa D + \frac{\vartheta}{2\pi}F_{01}\right),
\end{aligned}$$

where $\widetilde{V}_{\pm I} \equiv 2\theta_R\bar\theta_L\widetilde{m}_{\pm I} + 2\theta_L\bar\theta_R\widetilde{m}^*_{\pm I}$: twisted masses.

$V=(A_\mu,\phi,\bar\phi;\lambda;D)$: Dim. Red. 4D $\mathcal{N}=1$ vector superfield
$\Phi_{+I}=(\phi_{+I};\psi_{+IR},\psi_{+IL};F_{+I})$: Dim. Red. of 4D $\mathcal{N}=1$ chiral superfield (fundamental repre., Flavors: $I=1,\cdots,n_+$)
$\Phi_{-I}=(\phi_{-I};\psi_{-IR},\psi_{-IL};F_{-I})$: Dim. Red. of 4D $\mathcal{N}=1$ chiral superfield (anti-fundamental repre., Flavors: $I=1,\cdots,n_-$)

Note

Two kinds of fermion mass terms can be introduced.

- **Complex mass terms** $(\subset \boldsymbol{W}, \bar{\boldsymbol{W}})$:
  $m_I \left(\psi_{-IL}\psi_{+IR} - \psi_{-IR}\psi_{+IL}\right) + m_I^* \left(\bar{\psi}_{+IR}\bar{\psi}_{-IL} - \bar{\psi}_{+IL}\bar{\psi}_{-IR}\right)$
- **Twisted mass terms** $(\not\subset \boldsymbol{W}, \bar{\boldsymbol{W}})$:
  $\widetilde{m}_{+I}\bar{\psi}_{+IL}\psi_{+IR} + \widetilde{m}^*_{+I}\bar{\psi}_{+IR}\psi_{+IL} + \widetilde{m}_{-I}\psi_{-IR}\bar{\psi}_{-IL} + \widetilde{m}^*_{-I}\psi_{-IL}\bar{\psi}_{-IR}$

$\diamondsuit$ **Flavor symmetry of $\mathcal{L}_{\rm mat}$:**

$$\begin{array}{rl} \mathrm{U}(n_+) \times \mathrm{U}(n_-) & \text{for } \widetilde{m}_{\pm 1} = \cdots = \widetilde{m}_{\pm n_\pm}, \widetilde{m}^*_{\pm 1} = \cdots = \widetilde{m}^*_{\pm n_\pm} \\ \updownarrow & \\ \mathrm{U}(1)^{n_+} \times \mathrm{U}(1)^{n_-} & \text{for general } \widetilde{m}_{\pm I}, \widetilde{m}^*_{\pm I} \end{array}$$

3 Latticization of SYM Part

4D $\mathcal{N}=1$ SYM $\Rightarrow$ (dim. red.) $\Rightarrow$ 2D $\mathcal{N}=(2,2)$ SYM

$A_\mu \quad (\mu=0,1)$ — $A_\mu \Rightarrow U_\mu(x)$ (link variables on the lattice)

A_2, A_3 — $\phi(x), \bar{\phi}(x)$ (site variables)

Fermions : 4-component Majorana spinor

$\Psi(x) = \left(\psi_0(x), \psi_1(x), \chi(x), \frac{1}{2}\eta(x)\right)^T$ (site variables)

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Q-SUSY on the lattice $\quad (Q \equiv -\frac{1}{\sqrt{2}}(Q_L + \bar{Q}_R))$

For admissible gauge fields $(||1 - U_{01}(x)|| < \epsilon)$

$$
\begin{aligned}
&QU_\mu(x) = i\psi_\mu(x)U_\mu(x)\\
&Q\psi_\mu(x) = i\psi_\mu(x)\psi_\mu(x) + ia\nabla_\mu\phi(x)\\
&Q\phi(x) = 0\\
&Q\bar{\phi}(x) = \eta(x), \quad Q\eta(x) = [\phi(x), \bar{\phi}(x)]\\
&Q\chi(x) = iD(x) + \frac{i}{2}\widehat{\Phi}(x), \quad QD(x) = -\frac{1}{2}Q\widehat{\Phi}(x) - i[\phi(x), \chi(x)],
\end{aligned}
$$

where $a\nabla_\mu\phi(x) \equiv U_\mu(x)\phi(x+\hat{\mu})U_\mu(x)^{-1} - \phi(x)$

(covariant difference for adjoint fields) ,

$\widehat{\Phi}(x) = \frac{-i(U_{01}(x) - U_{10}(x))}{1 - \frac{1}{\epsilon^2}||1 - U_{01}(x)||^2} \sim 2F_{01}$

$\Rightarrow Q^2 =$ (infinitesimal gauge tr. with the parameter $\phi(x)$)

Lattice Action: Q-exact form $\Rightarrow$ Exact Q-SUSY

For admissible gauge fields ($||1 - U_{01}(x)|| < \epsilon$ for $\forall x$),

$$\begin{aligned} S_{\rm SYM}^{\rm (lat)} &= Q \frac{1}{g_0^2} \sum_x \mathrm{tr} \left[\chi(x) \left\{ -\frac{i}{2} \widehat{\Phi}(x) + iD(x) \right\} + \frac{1}{4} \eta(x) [\phi(x), \bar{\phi}(x)] - i \sum_\mu \psi_\mu(x) a \nabla_\mu \bar{\phi}(x) \right] \\ &= \frac{1}{g_0^2} \sum_x \mathrm{tr} \left[\frac{1}{4} \widehat{\Phi}(x)^2 + a^2 \sum_\mu \nabla_\mu \phi(x) \nabla_\mu \bar{\phi}(x) + i\chi(x) Q \widehat{\Phi}(x) + i \sum_\mu \psi_\mu(x) a \nabla_\mu \eta(x) \right. \\ &\qquad + \frac{1}{4} [\phi(x), \bar{\phi}(x)]^2 - \chi(x) [\phi(x), \chi(x)] - \frac{1}{4} \eta(x) [\phi(x), \eta(x)] \\ &\qquad \left. - \sum_\mu \psi_\mu(x) \psi_\mu(x) \left(\bar{\phi}(x) + U_\mu(x) \bar{\phi}(x + \hat{\mu}) U_\mu(x)^{-1} \right) - D(x)^2 \right], \end{aligned}$$

For the other cases, $S_{\rm SYM}^{\rm (lat)} = +\infty$. (i.e. The Boltzmann weight is zero.)

Note

Without the admissibility and the denominator of $\widehat{\Phi}$, the configurations

$$U_{01}(x) = \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix} \qquad \text{(up to gauge tr.)} \tag{3.1}$$

for $\forall x$ give the vacua of the action.

To get the target theory, we should consider excitations around the single vacuum $U_{01}(x) = 1$.

The admissibility and the denominator of $\widehat{\Phi}$ smoothly remove the degenerated vacua $U_{01}(x)^2 = 1, U_{01}(x) \neq 1$ with preserving the Q-SUSY.
(Take the traceless part of the numerator of $\widehat{\Phi}$ for $G = \mathrm{SU}(N)$ case)

c.f. $f(t) = \begin{cases} e^{-c/t} & t \geq 0 \\ 0 & t < 0 \end{cases}$ **with $c > 0$ is smooth and infinitely differentiable w.r.t. $t \in \mathrm{R}$**

The Q-SUSY forbids the mass term $\phi\bar{\phi}$ appearing as radiative corrections in the lattice perturbation.
$\Rightarrow$ The continuum theory is expected to be constructed without any fine-tuning.
(Computer simulations will give the nonperturbative check `[Kanamori-Suzuki]`**.**
$\Rightarrow$ Care of the flat directions!)

FI and ϑ terms:

For $G = \mathrm{U}(N)$, the FI and topological ϑ-terms can be introduced to the action as

$$S^{\mathrm{LAT}}_{\mathrm{FI},\,\vartheta} = Q\kappa \sum_x \mathrm{tr}\,(-i\chi(x)) - \frac{\vartheta - 2\pi i\kappa}{2\pi} \sum_x \mathrm{tr}\ \ln U_{01}(x),$$

where the second term is Q-invariant by its topological nature

$(\delta \sum_x \mathrm{tr}\ \ln U_{01}(x) = 0)$.

In order for the logarithm of the plaquette fields to be well-defined, it is sufficient to choose ϵ as

$$0 < \epsilon < \frac{1}{\sqrt{N}} \qquad \text{for } G = \mathrm{U}(N) \text{ with } \vartheta\text{-term.}$$

4 Lattice Formulation of [1]

◇ Forward (backward) covariant differences $D_\mu (D_\mu^*)$:

$$\begin{aligned}
aD_\mu \Phi_{+I}(x) &\equiv U_\mu(x)\Phi_{+I}(x+\hat{\mu}) - \Phi_{+I}(x) \\
aD_\mu^* \Phi_{+I}(x) &\equiv \Phi_{+I}(x) - U_\mu(x-\hat{\mu})^{-1}\Phi_{+I}(x-\hat{\mu}) \\
aD_\mu \Phi_{-I}(x) &\equiv \Phi_{-I}(x+\hat{\mu})U_\mu(x)^{-1} - \Phi_{-I}(x) \\
aD_\mu^* \Phi_{-I}(x) &\equiv \Phi_{-I}(x) - \Phi_{-I}(x-\hat{\mu})U_\mu(x-\hat{\mu}) \\
&\vdots
\end{aligned}$$

and

$$D_\mu^S \equiv \frac{1}{2}\left(D_\mu + D_\mu^*\right), \qquad D_\mu^A \equiv \frac{1}{2}\left(D_\mu - D_\mu^*\right), \qquad D^A \equiv \sum_\mu D_\mu^A.$$

Q-SUSY on the lattice [Consider the case $n_+ = n_- \equiv n$]

$$Q\phi_{+I}(x) = -\psi_{+IL}(x), \quad Q\psi_{+IL}(x) = -(\phi(x) - \widetilde{m}_{+I})\phi_{+I}(x),$$
$$Q\psi_{+IR}(x) = a\left(D_0^S + iD_1^S\right)\phi_{+I}(x) + F_{+I}(x) - raD^A\phi_{-I}(x)^\dagger, \quad \leftarrow \text{Wilson term}$$
$$QF_{+I}(x) = (\phi(x) - \widetilde{m}_{+I})\psi_{+IR}(x) + a\left(D_0^S + iD_1^S\right)\psi_{+IL}(x) - raD^A\bar{\psi}_{-IR}(x)$$
$$-a\left(Q(D_0^S + iD_1^S)\right)\phi_{+I}(x) + ra\left(QD^A\right)\phi_{-I}(x)^\dagger,$$

$$Q\phi_{-I}(x) = -\psi_{-IL}(x), \quad Q\psi_{-IL}(x) = \phi_{-I}(x)(\phi(x) - \widetilde{m}_{-I}),$$
$$Q\psi_{-IR}(x) = a\left(D_0^S + iD_1^S\right)\phi_{-I}(x) + F_{-I}(x) - raD^A\phi_{+I}(x)^\dagger,$$
$$QF_{-I}(x) = -\psi_{-IR}(x)(\phi(x) - \widetilde{m}_{-I}) + a\left(D_0^S + iD_1^S\right)\psi_{-IL}(x) - raD^A\bar{\psi}_{+IR}(x)$$
$$-a\left(Q(D_0^S + iD_1^S)\right)\phi_{-I}(x) + ra\left(QD^A\right)\phi_{+I}(x)^\dagger,$$
$$\vdots.$$

$\Rightarrow$The nilpotency of Q holds for variables besides $F_{\pm I}$.
However, we have, for example,

$$Q^2F_{+I}(x) = (\phi(x) - \widetilde{m}_{+I})F_{+I}(x) + (\widetilde{m}_{+I} - \widetilde{m}_{-I})raD^A\phi_{-I}(x)^\dagger.$$

$\Rightarrow$ When $\widetilde{m}_{+I} = \widetilde{m}_{-I} (\equiv \widetilde{m}_I)$, Q is nilpotent for all variables, i.e.

$$\begin{aligned} Q^2 \;=\;& (\text{infinitesimal gauge tr. with the parameter } \phi(x)) \\ & +(\text{infinitesimal flavor rotation with the parameter } \widetilde{m}_I). \end{aligned}$$

$$\delta\Phi_{\pm I} = \mp\widetilde{m}_I\Phi_{\pm I}, \quad \delta\Phi^\dagger_{\pm I} = \pm\widetilde{m}_I\Phi^\dagger_{\pm I}$$

Lattice Action: Q-exact form

$$\begin{aligned}
S_{\text{mat}}^{(\text{lat})} &= S_{\text{mat},+}^{(\text{lat})} + S_{\text{mat},-}^{(\text{lat})} \\
S_{\text{mat},+}^{(\text{lat})} &= Q\sum_{x}\sum_{I=1}^{n}\left[\frac{1}{2}\bar{\psi}_{+IL}(x)\left\{a\left(D_0^S+iD_1^S\right)\phi_{+I}(x)-F_{+I}(x)-raD^A\phi_{-I}(x)^\dagger\right\}\right. \\
&\qquad\qquad +\frac{1}{2}\left\{a\left(D_0^S-iD_1^S\right)\phi_{+I}(x)^\dagger-F_{+I}(x)^\dagger-raD^A\phi_{-I}(x)\right\}\psi_{+IR}(x) \\
&\qquad +\frac{1}{2}\bar{\psi}_{+IR}(x)(\bar{\phi}(x)-\widetilde{m}_{+I}^*)\phi_{+I}(x)-\frac{1}{2}\phi_{+I}(x)^\dagger(\bar{\phi}(x)-\widetilde{m}_{+I}^*)\psi_{+IL}(x) \\
&\qquad \left. + i\phi_{+I}(x)^\dagger\chi(x)\phi_{+I}(x)\right], \\
S_{\text{mat},-}^{(\text{lat})} &= Q\sum_{x}\sum_{I=1}^{n}\left[\frac{1}{2}\left\{a\left(D_0^S+iD_1^S\right)\phi_{-I}(x)-F_{-I}(x)-raD^A\phi_{+I}(x)^\dagger\right\}\bar{\psi}_{-IL}(x)\right. \\
&\qquad\qquad +\frac{1}{2}\psi_{-IR}(x)\left\{a\left(D_0^S-iD_1^S\right)\phi_{-I}(x)^\dagger-F_{-I}(x)^\dagger-raD^A\phi_{+I}(x)\right\} \\
&\qquad +\frac{1}{2}\psi_{-IL}(x)(\bar{\phi}(x)-\widetilde{m}_{-I}^*)\phi_{-I}(x)^\dagger-\frac{1}{2}\phi_{-I}(x)(\bar{\phi}(x)-\widetilde{m}_{-I}^*)\bar{\psi}_{-IR}(x) \\
&\qquad \left. - i\phi_{-I}(x)\chi(x)\phi_{-I}(x)^\dagger\right],
\end{aligned}$$

Superpotential terms: (i: gauge group index)

$$S_{\rm pot}^{\rm (lat)} = Q\sum_x \sum_I \sum_{i=1}^{N} \left[-\frac{\partial W}{\partial \phi_{+Ii}(x)} \psi_{+IRi}(x) - \frac{\partial W}{\partial \phi_{-Ii}(x)} \bar{\psi}_{-IRi}(x) \right.$$
$$\left. -\bar{\psi}_{+ILi}(x) \frac{\partial \bar{W}}{\partial \phi^*_{+Ii}(x)} - \psi_{-ILi}(x) \frac{\partial \bar{W}}{\partial \phi^*_{-Ii}(x)} \right]$$

Note

Due to the Wilson term,

- the flavor symmetry of $S_{\rm mat}^{\rm LAT}$ is down to $\mathrm{U}(1)^n$ (diagonal subgroup of $\mathrm{U}(1)^n \times \mathrm{U}(1)^n$).
- the superpotential terms are not exactly holomorphic or anti-holomorphic on the lattice.

$\Rightarrow$ The lattice action is Q-SUSY invariant when $\widetilde{m}_{+I} = \widetilde{m}_{-I} (\equiv \widetilde{m}_I)$.
(We can still choose $\widetilde{m}^*_{+I}, \widetilde{m}^*_{-I}$ freely!)

4.1 U(1)$_A$ Anomaly

◇ **U(1)$_A$-symmetry with the charges:**

$$\begin{aligned} +2 &: \phi \\ +1 &: \psi_\mu, \quad \psi_{\pm IL}, \quad \bar{\psi}_{\pm IR} \\ -1 &: \chi, \quad \eta, \quad \psi_{\pm IR}, \quad \bar{\psi}_{\pm IL} \\ -2 &: \bar{\phi}, \\ 0 &: \text{the others} \end{aligned}$$

is realized in the lattice action when all the twisted masses are zero. In particular, the Wilson terms are consistent with the U(1)$_A$-symmetry. Since U(1)$_A$ transforms the left-handed fermions and the right-handed fermions differently, it can be anomalous at the quantum level.

Note

- **The gaugino fields (ψ_μ, χ, η) belong to the adjoint representation and do not contribute to the anomaly.**
- **U(1)$_A$ is not anomalous when $n_+ = n_-$.**
  $\Rightarrow$ consistent with the present lattice formulation

◇ **U(1)$_A$-WT identity:**

$$\partial^*_\mu \left\langle j_\mu^{U(1)_A}(x) \right\rangle = \left\langle \sum_{I=1}^{n} \left(\mathcal{M}_{+I}(x) + \mathcal{M}_{-I}(x)\right) \right\rangle,$$

with ∂^*_μ: backward difference operators,

$$\begin{aligned}
\mathcal{M}_{+I}(x) &= 2\widetilde{m}_I \left(\phi_{+I}(x)^\dagger \bar{\phi}(x) \phi_{+I}(x) + \bar{\psi}_{+IL}(x)\psi_{+IR}(x)\right) \\
&\quad -2\widetilde{m}^*_{+I} \left(\phi_{+I}(x)^\dagger \phi(x) \phi_{+I}(x) + \bar{\psi}_{+IR}(x)\psi_{+IL}(x)\right) \\
\mathcal{M}_{-I}(x) &= 2\widetilde{m}_I \left(\phi_{-I}(x) \bar{\phi}(x) \phi_{-I}(x)^\dagger + \psi_{-IR}(x)\bar{\psi}_{-IL}(x)\right) \\
&\quad -2\widetilde{m}^*_{-I} \left(\phi_{-I}(x) \phi(x) \phi_{-I}(x)^\dagger + \psi_{-IL}(x)\bar{\psi}_{-IR}(x)\right).
\end{aligned}$$

◇ U(1)$_A$-WT identity:

$$\partial_\mu^* \left\langle j_\mu^{U(1)_A}(x) \right\rangle = \left\langle \sum_{I=1}^{n} \left(\mathcal{M}_{+I}(x) + \mathcal{M}_{-I}(x)\right) \right\rangle,$$

with ∂_μ^*: backward difference operators,

$$\begin{aligned}
\mathcal{M}_{+I}(x) &= 2\widetilde{m}_I \left(\phi_{+I}(x)^\dagger \bar{\phi}(x) \phi_{+I}(x) + \bar{\psi}_{+IL}(x)\psi_{+IR}(x)\right) \\
&\quad -2\widetilde{m}_{+I}^* \left(\phi_{+I}(x)^\dagger \phi(x) \phi_{+I}(x) + \bar{\psi}_{+IR}(x)\psi_{+IL}(x)\right) \\
\mathcal{M}_{-I}(x) &= 2\widetilde{m}_I \left(\phi_{-I}(x) \bar{\phi}(x) \phi_{-I}(x)^\dagger + \psi_{-IR}(x)\bar{\psi}_{-IL}(x)\right) \\
&\quad -2\widetilde{m}_{-I}^* \left(\phi_{-I}(x) \phi(x) \phi_{-I}(x)^\dagger + \psi_{-IL}(x)\bar{\psi}_{-IR}(x)\right).
\end{aligned}$$

We can investigate the general case of $n_+ \neq n_-$, if the fields $\Phi_{+I}, \bar{\Phi}_{+I}$ $(I = n_+ + 1, \cdots, n)$ and $\Phi_{-I'}, \bar{\Phi}_{-I'}$ $(I' = n_- + 1, \cdots, n)$ are decoupled by sending $\widetilde{m}_{+I}^* \to \infty$ $(I = n_+ + 1, \cdots, n)$, $\widetilde{m}_{-I'}^* \to \infty$ $(I' = n_- + 1, \cdots, n)$.

Regarding U(1)$_A$-anomaly, we can check that such decoupling is achieved in the lattice perturbation.

The anomalous WT-identity for n_+ fundamentals and n_- anti-fundamentals is correctly obtained:

$$\partial_\mu^* \left\langle j_\mu^{U(1)_A}(x) \right\rangle = -\frac{1}{\pi}(n_+ - n_-)\mathrm{tr}\, F_{01}(x) + \left\langle \sum_{I=1}^{n_+} \mathcal{M}_{+I}(x) + \sum_{I=1}^{n_-} \mathcal{M}_{-I}(x) \right\rangle.$$

(The SYM fields are assumed to be smooth.)
The anomaly term comes from one-loop diagrams of $\boldsymbol{\mathcal{M}_{+I}\,(I > n_+)}$ and $\boldsymbol{\mathcal{M}_{-I'}\,(I' > n_-)}$.

Note

- **The decoupling is not completely trivial, because the holomorphic parts $\widetilde{m}_I$ are kept finite.**
- **The Q-supersymmetry plays an important role to achieve the decoupling. (tr ϕ terms, seeming to be left finite, cancel between the bosonic and fermionic sectors.)**

5 Lattice Formulation of [2]

◇ Here, we introduce the overlap Dirac operator to construct the lattice action for general $n_\pm$ and general twisted masses.

5.1 Doublet Notation

We start from the continuum theory with n_+ fundamentals and n_- anti-fundamentals. Adding some matter multiplets to prepare the same number of the fundamentals and anti-fundamentals ($n_0 \equiv \max(n_+, n_-)$), we combine them as doublets:

$$
\begin{aligned}
&\Phi_I \equiv \begin{pmatrix} \phi_{+I} \\ \phi^\dagger_{-I} \end{pmatrix}, \qquad \Phi^\dagger_I \equiv \left(\phi^\dagger_{+I}, \phi_{-I}\right), \\
&\Psi_{uI} \equiv \begin{pmatrix} \psi_{+IL} \\ \bar\psi_{-IR} \end{pmatrix}, \qquad \Psi_{dI} \equiv \begin{pmatrix} \bar\psi_{-IL} \\ \psi_{+IR} \end{pmatrix}, \\
&\Psi^\dagger_{uI} \equiv \left(\bar\psi_{+IL}, \psi_{-IR}\right), \qquad \Psi^\dagger_{dI} \equiv \left(\psi_{-IL}, \bar\psi_{+IR}\right), \\
&F_I \equiv \begin{pmatrix} F_{+I} \\ F^\dagger_{-I} \end{pmatrix}, \qquad F^\dagger_I \equiv \left(F^\dagger_{+I}, F_{-I}\right) \qquad (I = 1, \cdots, n_0).
\end{aligned}
$$

The upper and down components of each doublet have the same gauge transformation property.

We define the γ-matrices in terms of the Pauli matrices as

$$\gamma_0 \equiv \boldsymbol{\sigma}_1, \qquad \gamma_1 \equiv \boldsymbol{\sigma}_2, \qquad \gamma_3 \equiv -i\gamma_0\gamma_1 = \boldsymbol{\sigma}_3,$$

and use the notation

$$\bar{\Psi}_{uI} \equiv \Psi^{\dagger}_{uI}\gamma_0, \qquad \bar{\Psi}_{dI} \equiv \Psi^{\dagger}_{dI}\gamma_0.$$

The fundamental or anti-fundamental degrees of freedom are extracted by acting the chiral projectors $P_{\pm} = \frac{1}{2}(1 \pm \gamma_3)$ to the doublets.

Then, the Euclidean actions for the matters $S^{(E)}_{\mathrm{mat},\pm}$ are rewritten as

$$\begin{aligned}
S^{(E)}_{\mathrm{mat},+} = \int \mathrm{d}^2x \sum_{I=1}^{n_+} \Big[& -\Phi_I^\dagger P_+ \mathcal{D}_\mu \mathcal{D}_\mu P_+ \Phi_I + \frac{1}{2}\Phi_I^\dagger P_+ \{\phi - \widetilde{m}_{+I}, \bar{\phi} - \widetilde{m}^*_{+I}\} P_+ \Phi_I \\
& - F_I^\dagger P_+ F_I - \Phi_I^\dagger P_+ D P_+ \Phi_I + \bar{\Psi}_{uI} P_- \not{\mathcal{D}} P_+ \Psi_{uI} - \bar{\Psi}_{dI} P_+ \not{\mathcal{D}}^\dagger P_- \Psi_{dI} \\
& + \bar{\Psi}_{uI} P_- \left(\phi - \widetilde{m}_{+I}\right) P_- \Psi_{dI} + \bar{\Psi}_{dI} P_+ \left(\bar{\phi} - \widetilde{m}^*_{+I}\right) P_+ \psi_{uI} \\
& - i \bar{\Psi}_{uI} P_- \gamma_\mu \psi_\mu P_+ \Phi_I - i \Phi_I^\dagger P_+ \gamma_\mu \psi_\mu P_- \Psi_{dI} \\
& - \bar{\Psi}_{dI} P_+ \left(\frac{1}{2}\eta + i\chi\right) P_+ \Phi_I - \Phi_I^\dagger P_+ \left(\frac{1}{2}\eta - i\chi\right) P_+ \Psi_{uI} \Big],
\end{aligned}$$

$$\begin{aligned}
S^{(E)}_{\mathrm{mat},-} = \int \mathrm{d}^2x \sum_{I'=1}^{n_-} \Big[& -\Phi_{I'}^\dagger P_- \mathcal{D}_\mu \mathcal{D}_\mu P_- \Phi_{I'} + \frac{1}{2}\Phi_{I'}^\dagger P_- \{\phi - \widetilde{m}_{-I'}, \bar{\phi} - \widetilde{m}^*_{-I'}\} P_- \Phi_{I'} \\
& - F_{I'}^\dagger P_- F_{I'} + \Phi_{I'}^\dagger P_- D P_- \Phi_{I'} + \bar{\Psi}_{uI'} P_+ \not{\mathcal{D}} P_- \Psi_{uI'} + \bar{\Psi}_{dI'} P_- \not{\mathcal{D}}^\dagger P_+ \Psi_{dI'} \\
& + \bar{\Psi}_{uI'} P_+ \left(\phi - \widetilde{m}_{-I'}\right) P_+ \Psi_{dI'} + \bar{\Psi}_{dI'} P_- \left(\bar{\phi} - \widetilde{m}^*_{-I'}\right) P_- \Psi_{uI'} \\
& - i \bar{\Psi}_{uI'} P_+ \gamma_\mu \psi_\mu P_- \Phi_{I'} - i \Phi_{I'}^\dagger P_- \gamma_\mu \psi_\mu P_+ \Psi_{dI'} \\
& - \bar{\Psi}_{dI'} P_- \left(\frac{1}{2}\eta - i\chi\right) P_- \Phi_{I'} - \Phi_{I'}^\dagger P_- \left(\frac{1}{2}\eta + i\chi\right) P_- \Psi_{uI'} \Big].
\end{aligned}$$

Q SUSY

$$
\begin{aligned}
&Q\Phi_I = -\Psi_{uI}, \qquad Q\Psi_{uI} = -\left(\phi - \widetilde{m}_{+I}P_+ - \widetilde{m}_{-I}P_-\right)\Phi_I,\\
&Q\Psi_{dI} = \not{\mathcal{D}}\Phi_I + \gamma_0 F_I,\\
&Q(\gamma_0 F_I) = \left(\phi - \widetilde{m}_{+I}P_- - \widetilde{m}_{-I}P_+\right)\Psi_{dI} + \not{\mathcal{D}}\Psi_{uI} - i\gamma_\mu\psi_\mu\Phi_I,\\
&Q\Phi_I^\dagger = -\bar{\Psi}_{dI}, \qquad Q\bar{\Psi}_{dI} = \Phi_I^\dagger\left(\phi - \widetilde{m}_{+I}P_+ - \widetilde{m}_{-I}P_-\right),\\
&Q\bar{\Psi}_{uI} = \Phi_I^\dagger\not{\mathcal{D}}^\dagger + F_I^\dagger\gamma_0,\\
&Q(F_I^\dagger\gamma_0) = -\bar{\Psi}_{uI}\left(\phi - \widetilde{m}_{+I}P_- - \widetilde{m}_{-I}P_+\right) + \bar{\Psi}_{dI}\not{\mathcal{D}}^\dagger + i\Phi_I^\dagger\gamma_\mu\psi_\mu,
\end{aligned}
\tag{5.1}
$$

is nilpotent in the sense of

$$
\begin{aligned}
Q^2 \;=\;& (\text{infinitesimal gauge transformation with the parameter } \phi)\\
&+(\text{infinitesimal flavor rotations (5.2)})
\end{aligned}
$$

with

$$
\begin{aligned}
\delta\Phi_I &= -\left(\widetilde{m}_{+I}P_+ + \widetilde{m}_{-I}P_-\right)\Phi_I, & \delta\Phi_I^\dagger &= \Phi_I^\dagger\left(\widetilde{m}_{+I}P_+ + \widetilde{m}_{-I}P_-\right),\\
\delta\Psi_{uI} &= -\left(\widetilde{m}_{+I}P_+ + \widetilde{m}_{-I}P_-\right)\Psi_{uI}, & \delta\bar{\Psi}_{uI} &= \bar{\Psi}_{uI}\left(\widetilde{m}_{+I}P_- + \widetilde{m}_{-I}P_+\right),\\
\delta\Psi_{dI} &= -\left(\widetilde{m}_{+I}P_- + \widetilde{m}_{-I}P_+\right)\Psi_{dI}, & \delta\bar{\Psi}_{dI} &= \bar{\Psi}_{dI}\left(\widetilde{m}_{+I}P_+ + \widetilde{m}_{-I}P_-\right),\\
\delta F_I &= -\left(\widetilde{m}_{+I}P_+ + \widetilde{m}_{-I}P_-\right)F_I, & \delta F_I^\dagger &= F_I^\dagger\left(\widetilde{m}_{+I}P_+ + \widetilde{m}_{-I}P_-\right).
\end{aligned}
\tag{5.2}
$$

Note

(5.1) for each I splits into four irreducible parts consisting of

$$\{P_+\Phi_I, P_+\Psi_{uI}, P_-\Psi_{dI}, P_+F_I\}, \quad \{\Phi_I^\dagger P_+, \bar{\Psi}_{dI}P_+, \bar{\Psi}_{uI}P_-, F_I^\dagger P_+\},$$
$$\{P_-\Phi_I, P_-\Psi_{uI}, P_+\Psi_{dI}, P_-F_I\}, \quad \{\Phi_I^\dagger P_-, \bar{\Psi}_{dI}P_-, \bar{\Psi}_{uI}P_+, F_I^\dagger P_-\}.$$

$\Rightarrow$ Chiral decomposition OK.

$\diamondsuit$ The latticization in the previous section corresponds to

$$\not{D} \longrightarrow D_W \equiv \textstyle\sum_{\mu=0}^{1} \gamma_\mu D_\mu^S - rD^A.$$

$\Rightarrow$ Due to the Wilson terms, the chiral decomposition is not possible on the lattice.

The previous lattice action is rewritten in the doublet notation as

$$\begin{aligned} S_{\rm mat}^{(\rm lat)} \;=\; Q\sum_x \sum_{I=1}^{n} \frac{1}{2}\Big[& \bar{\Psi}_{uI}(x)\left(aD_W\Phi_I(x) - \gamma_0 F_I(x)\right) \\ & + \left(\Phi_I(x)^\dagger aD_W^\dagger - F_I(x)^\dagger\gamma_0\right)\Psi_{dI}(x) \\ & -\Phi_I(x)^\dagger\left(\bar{\phi}(x) - \widetilde{m}^*_{+I}P_+ - \widetilde{m}^*_{-I}P_-\right)\Psi_{uI}(x) \\ & +\bar{\Psi}_{dI}(x)\left(\bar{\phi}(x) - \widetilde{m}^*_{+I}P_+ - \widetilde{m}^*_{-I}P_-\right)\Phi_I(x) \\ & + 2i\Phi_I(x)^\dagger\gamma_3\chi(x)\Phi_I(x)\Big]. \end{aligned} \tag{5.3}$$

In order to resolve the difficulty, we introduce the overlap Dirac operator.

5.2 The Overlap Dirac Operator

The overlap Dirac operator $\widehat{D}$ satisfies the Ginsparg-Wilson relation

$$\gamma_3 \widehat{D} + \widehat{D} \gamma_3 = a \widehat{D} \gamma_3 \widehat{D}.$$

$\widehat{D}$ has been explicitly given by [Neuberger]

$$\widehat{D} \equiv \frac{1}{a}\left(1 - X \frac{1}{\sqrt{X^\dagger X}}\right), \qquad X = 1 - aD_W.$$

(In order for $\widehat{D}$ to express the propagation of physical modes with doublers decoupled, we have to take $r > \frac{1}{2}$ [Kikukawa-Yamada, Suzuki].
In what follows, r is fixed to $r = 1$.)

Note
From the requirement $||X^\dagger X|| > 0$, the admissibility condition with $0 < \epsilon < \frac{1}{5}$ is imposed [Hernandez-Jansen-Lüscher].

$\diamondsuit$ **For the kinetic part of the action (5.3) with D_W replaced by $\widehat{D}$:**

$$\bar{\Psi}_{uI}(x)a\widehat{D}\Phi_I(x) + \Phi_I(x)^\dagger a\widehat{D}^\dagger\Psi_{dI}(x),$$

there are two possibilities of the chiral decomposition:

$$\begin{aligned}
&\bar{\Psi}_{uI}(x)P_\pm a\widehat{D}\Phi_I(x) + \Phi_I(x)^\dagger a\widehat{D}^\dagger P_\pm\Psi_{dI}(x) \;\Rightarrow\; \text{Formulation I},\\
&\bar{\Psi}_{uI}(x)a\widehat{D}P_\pm\Phi_I(x) + \Phi_I(x)^\dagger P_\pm a\widehat{D}^\dagger\Psi_{dI}(x) \;\Rightarrow\; \text{Formulation II}.
\end{aligned}$$

Formulation I

$$\widehat{P}_\pm \equiv \frac{1\pm\widehat{\gamma}_3}{2}, \qquad \widehat{\gamma}_3 \equiv \gamma_3(1-a\widehat{D})$$

are projection operators ($\widehat{P}_\pm^2 = \widehat{P}_\pm$), which we use in Formulation I, because

$$P_\pm\widehat{D} = \widehat{D}\widehat{P}_\mp, \qquad \widehat{D}^\dagger P_\pm = \widehat{P}_\mp\widehat{D}^\dagger, \qquad \widehat{P}_\pm^\dagger = \widehat{P}_\pm.$$

Formulation II

$$\bar{P}_\pm \equiv \frac{1\pm\bar{\gamma}_3}{2}, \qquad \bar{\gamma}_3 \equiv (1-a\widehat{D})\gamma_3$$

are projection operators ($\bar{P}_\pm^2 = \bar{P}_\pm$), which we use in Formulation II, because

$$\bar{P}_\pm\widehat{D} = \widehat{D}P_\mp, \qquad \widehat{D}^\dagger\bar{P}_\pm = P_\mp\widehat{D}^\dagger, \qquad \bar{P}_\pm^\dagger = \bar{P}_\pm.$$

We pick

$$\widehat{P}_+\Phi_I, \qquad \widehat{P}_+\Psi_{uI}, \qquad P_-\Psi_{dI}, \qquad P_+F_I \quad \text{as chiral fields,} \tag{5.4}$$

$$\Phi_I^\dagger\widehat{P}_+, \qquad \bar{\Psi}_{dI}\widehat{P}_+, \qquad \bar{\Psi}_{uI}P_-, \qquad F_I^\dagger P_+ \quad \text{as anti-chiral fields,} \tag{5.5}$$

for fundamental matters ($I = 1, \cdots, n_+$), and

$$\Phi_{I'}^\dagger\widehat{P}_-, \qquad \bar{\Psi}_{dI'}\widehat{P}_-, \qquad \bar{\Psi}_{uI'}P_+, \qquad F_{I'}^\dagger P_- \quad \text{as chiral fields,} \tag{5.6}$$

$$\widehat{P}_-\Phi_{I'}, \qquad \widehat{P}_-\Psi_{uI'}, \qquad P_+\Psi_{dI'}, \qquad P_-F_{I'} \quad \text{as anti-chiral fields,} \tag{5.7}$$

for anti-fundamental matters ($I' = 1, \cdots, n_-$).

If we use a naive transformation in the previous section, it leads to

$$\begin{aligned} Q(\widehat{P}_+\Phi_I(x)) &= \widehat{P}_+(Q\Phi_I(x)) + (Q\widehat{P}_+)\Phi_I(x) \\ &= -\widehat{P}_+\Psi_{uI}(x) + (Q\widehat{P}_+)\widehat{P}_+\Phi_I(x) + (Q\widehat{P}_+)\widehat{P}_-\Phi_I(x). \end{aligned}$$

Note that $Q\widehat{P}_\pm$ generally do not vanish since $\widehat{P}_\pm$ involve the link variables. Due to the last term in the r.h.s., the transformation does not close among the chiral variables (5.4).

Instead, we regard (5.4), (5.5), (5.6), (5.7) as fundamental contents of the theory, and let us define their transformation by starting with

$$
\begin{aligned}
Q(\widehat{P}_+\Phi_I(x)) &= -\widehat{P}_+\Psi_{uI}(x) + (Q\widehat{P}_+)\widehat{P}_+\Phi_I(x), \\
Q(\Phi_I^\dagger\widehat{P}_+(x)) &= -\bar{\Psi}_{dI}\widehat{P}_+(x) + \Phi_I^\dagger\widehat{P}_+(Q\widehat{P}_+)(x), \\
Q(\widehat{P}_-\Phi_{I'}(x)) &= -\widehat{P}_-\Psi_{uI'}(x) + (Q\widehat{P}_-)\widehat{P}_-\Phi_{I'}(x), \\
Q(\Phi_{I'}^\dagger\widehat{P}_-(x)) &= -\bar{\Psi}_{dI'}\widehat{P}_-(x) + \Phi_{I'}^\dagger\widehat{P}_-(Q\widehat{P}_-)(x).
\end{aligned}
$$

It turns out that the Q supersymmetry transformation can be consistently determined as a closed form among the (anti-)chiral variables, satisfying the nilpotency.

Concretely, we have

$$
\begin{aligned}
Q(\widehat{P}_+\Phi_I(x)) &= -\widehat{P}_+\Psi_{uI}(x) + (Q\widehat{P}_+)\widehat{P}_+\Phi_I(x), \\
Q(\widehat{P}_+\Psi_{uI}(x)) &= -(\widehat{P}_+\phi - \widetilde{m}_{+I})\widehat{P}_+\Phi_I(x) + (Q\widehat{P}_+)\widehat{P}_+\Psi_{uI}(x) - (Q\widehat{P}_+)^2\widehat{P}_+\Phi_I(x), \\
Q(P_-\Psi_{dI}(x)) &= a\widehat{D}\widehat{P}_+\Phi_I(x) + \gamma_0 P_+ F_I(x), \\
Q(\gamma_0 P_+ F_I(x)) &= (\phi(x) - \widetilde{m}_{+I})P_-\Psi_{dI}(x) + a\widehat{D}\widehat{P}_+\Psi_{uI}(x) - P_- Q(a\widehat{D})\widehat{P}_+\Phi_I(x)
\end{aligned}
$$

$$
\begin{aligned}
Q(\Phi_I^\dagger\widehat{P}_+(x)) &= -\bar{\Psi}_{dI}\widehat{P}_+(x) + \Phi_I^\dagger\widehat{P}_+(Q\widehat{P}_+)(x), \\
Q(\bar{\Psi}_{dI}\widehat{P}_+(x)) &= \Phi_I^\dagger\widehat{P}_+(\phi\widehat{P}_+ - \widetilde{m}_{+I})(x) - \bar{\Psi}_{dI}\widehat{P}_+(Q\widehat{P}_+)(x) + \Phi_I^\dagger\widehat{P}_+(Q\widehat{P}_+)^2(x), \\
Q(\bar{\Psi}_{uI}(x)P_-) &= \Phi_I^\dagger\widehat{P}_+(x)a\widehat{D}^\dagger + F_I(x)^\dagger P_+\gamma_0, \\
Q(F_I(x)^\dagger P_+\gamma_0) &= -\bar{\Psi}_{uI}(x)P_-(\phi(x) - \widetilde{m}_{+I}) + \bar{\Psi}_{dI}\widehat{P}_+(x)a\widehat{D}^\dagger - \Phi_I^\dagger\widehat{P}_+(x)Q(a\widehat{D}^\dagger)P_-,
\end{aligned}
$$

$$
\begin{aligned}
Q(\widehat{P}_-\Phi_{I'}(x)) &= -\widehat{P}_-\Psi_{uI'}(x) + (Q\widehat{P}_-)\widehat{P}_-\Phi_{I'}(x), \\
Q(\widehat{P}_-\Psi_{uI'}(x)) &= -(\widehat{P}_-\phi - \widetilde{m}_{-I'})\widehat{P}_-\Phi_{I'}(x) + (Q\widehat{P}_-)\widehat{P}_-\Psi_{uI'}(x) - (Q\widehat{P}_-)^2\widehat{P}_-\Phi_{I'}(x), \\
Q(P_+\Psi_{dI'}(x)) &= a\widehat{D}\widehat{P}_-\Phi_{I'}(x) + \gamma_0 P_- F_{I'}(x), \\
Q(\gamma_0 P_- F_{I'}(x)) &= (\phi(x) - \widetilde{m}_{-I'})P_+\Psi_{dI'}(x) + a\widehat{D}\widehat{P}_-\Psi_{uI'}(x) - P_+ Q(a\widehat{D})\widehat{P}_-\Phi_{I'}(x),
\end{aligned}
$$

$$
Q(\Phi_{I'}^\dagger\widehat{P}_-(x)) = -\bar{\Psi}_{dI'}\widehat{P}_-(x) + \Phi_{I'}^\dagger\widehat{P}_-(Q\widehat{P}_-)(x),
$$

$$
\begin{aligned}
Q(\bar{\Psi}_{dI'}\widehat{P}_-(x)) &= \Phi^\dagger_{I'}\widehat{P}_-(\phi\widehat{P}_- - \widetilde{m}_{-I'})(x) - \bar{\Psi}_{dI'}\widehat{P}_-(Q\widehat{P}_-)(x) + \Phi^\dagger_{I'}\widehat{P}_-(Q\widehat{P}_-)^2(x), \\
Q(\bar{\Psi}_{uI'}(x)P_+) &= \Phi^\dagger_{I'}\widehat{P}_-(x)a\widehat{D}^\dagger + F_{I'}(x)^\dagger P_-\gamma_0, \\
Q(F_{I'}(x)^\dagger P_-\gamma_0) &= -\bar{\Psi}_{uI'}(x)P_+(\phi(x) - \widetilde{m}_{-I'}) + \bar{\Psi}_{dI'}\widehat{P}_-(x)a\widehat{D}^\dagger - \Phi^\dagger_{I'}\widehat{P}_-(x)Q(a\widehat{D}^\dagger)P_+.
\end{aligned}
$$

The nilpotency holds as

$$
\begin{aligned}
Q^2 = {} & \text{(infinitesimal gauge transformation with the parameter } \phi(x)) \\
& +\text{(infinitesimal flavor rotations (5.9) and (5.10))}
\end{aligned}
\tag{5.8}
$$

with

$$
\begin{aligned}
\delta(\widehat{P}_+\Phi_I) &= -\widetilde{m}_{+I}\widehat{P}_+\Phi_I, & \delta(\Phi^\dagger_I\widehat{P}_+) &= \widetilde{m}_{+I}\Phi^\dagger_I\widehat{P}_+, \\
\delta(\widehat{P}_+\Psi_{uI}) &= -\widetilde{m}_{+I}\widehat{P}_+\Psi_{uI}, & \delta(\bar{\Psi}_{uI}P_-) &= \widetilde{m}_{+I}\bar{\Psi}_{uI}P_-, \\
\delta(P_-\Psi_{dI}) &= -\widetilde{m}_{+I}P_-\Psi_{dI}, & \delta(\bar{\Psi}_{dI}\widehat{P}_+) &= \widetilde{m}_{+I}\bar{\Psi}_{dI}\widehat{P}_+, \\
\delta(P_+F_I) &= -\widetilde{m}_{+I}P_+F_I, & \delta(F^\dagger_I P_+) &= \widetilde{m}_{+I}F^\dagger_I P_+,
\end{aligned}
\tag{5.9}
$$

$$
\begin{aligned}
\delta(\Phi^\dagger_{I'}\widehat{P}_-) &= \widetilde{m}_{-I'}\Phi^\dagger_{I'}\widehat{P}_-, & \delta(\widehat{P}_-\Phi_{I'}) &= -\widetilde{m}_{-I'}\widehat{P}_-\Phi_{I'}, \\
\delta(\bar{\Psi}_{uI'}P_+) &= \widetilde{m}_{-I'}\bar{\Psi}_{uI'}P_+, & \delta(\widehat{P}_-\Psi_{uI'}) &= -\widetilde{m}_{-I'}\widehat{P}_-\Psi_{uI'}, \\
\delta(\bar{\Psi}_{dI'}\widehat{P}_-) &= \widetilde{m}_{-I'}\bar{\Psi}_{dI'}\widehat{P}_-, & \delta(P_+\Psi_{dI'}) &= -\widetilde{m}_{-I'}P_+\Psi_{dI'}, \\
\delta(F^\dagger_{I'}P_-) &= \widetilde{m}_{-I'}F^\dagger_{I'}P_-, & \delta(P_-F_{I'}) &= -\widetilde{m}_{-I'}P_-F_{I'}.
\end{aligned}
\tag{5.10}
$$

We used the identity

$$\boldsymbol{\widehat{P}_{\pm}(Q\widehat{P}_{\pm})\widehat{P}_{\pm} = 0}, \tag{5.11}$$

which is derived from the $\boldsymbol{Q}$ transformation of $\boldsymbol{\widehat{P}_{\pm}^2 = \widehat{P}_{\pm}}$.
Differently from the situation in the previous section, we here have no requirement to $\boldsymbol{n_{\pm}}$ nor to the twisted masses for the $\boldsymbol{Q}$ supersymmetry being closed and nilpotent.

The matter-part action is given as the Q-exact form:

$$
\begin{aligned}
S^{\mathrm{LAT}}_{\mathrm{mat},+\widetilde{m}} \;=\; Q\sum_x\sum_{I=1}^{n_+}\frac{1}{2}\Big[&\bar{\Psi}_{uI}(x)P_-\left(a\widehat{D}\widehat{P}_+\Phi_I(x)-\gamma_0P_+F_I(x)\right)\\
&+\left(\Phi_I^\dagger\widehat{P}_+(x)\,a\widehat{D}^\dagger-F_I(x)^\dagger P_+\gamma_0\right)P_-\Psi_{dI}(x)\\
&-\Phi_I^\dagger\widehat{P}_+(x)\,\left(\bar{\phi}(x)-\widetilde{m}^*_{+I}\right)\widehat{P}_+\Psi_{uI}(x)\\
&+\bar{\Psi}_{dI}\widehat{P}_+(x)\,\left(\bar{\phi}(x)-\widetilde{m}^*_{+I}\right)\widehat{P}_+\Phi_I(x)\\
&+2i\Phi_I^\dagger\widehat{P}_+(x)\,\chi(x)\widehat{P}_+\Phi_I(x)\Big],
\end{aligned}
\tag{5.12}
$$

$$
\begin{aligned}
S^{\mathrm{LAT}}_{\mathrm{mat},-\widetilde{m}} \;=\; Q\sum_x\sum_{I'=1}^{n_-}\frac{1}{2}\Big[&\bar{\Psi}_{uI'}(x)P_+\left(a\widehat{D}\widehat{P}_-\Phi_{I'}(x)-\gamma_0P_-F_{I'}(x)\right)\\
&+\left(\Phi_{I'}^\dagger\widehat{P}_-(x)\,a\widehat{D}^\dagger-F_{I'}(x)^\dagger P_-\gamma_0\right)P_+\Psi_{dI'}(x)\\
&-\Phi_{I'}^\dagger\widehat{P}_-(x)\,\left(\bar{\phi}(x)-\widetilde{m}_{-I'}\right)\widehat{P}_-\Psi_{uI'}(x)\\
&+\bar{\Psi}_{dI'}\widehat{P}_-(x)\,\left(\bar{\phi}(x)-\widetilde{m}^*_{-I'}\right)\widehat{P}_-\Phi_{I'}(x)\\
&-2i\Phi_{I'}^\dagger\widehat{P}_-(x)\,\chi(x)\widehat{P}_-\Phi_{I'}(x)\Big].
\end{aligned}
\tag{5.13}
$$

After the Q operation,

$$\begin{aligned}
S^{\mathrm{LAT}}_{\mathrm{mat},+\widetilde{m}} = \sum_x \sum_{I=1}^{n_+} \Big[& a^2 \Phi_I^\dagger \widehat{P}_+(x)\, \widehat{D}^\dagger \widehat{D} \widehat{P}_+ \Phi_I(x) - \left(F_I(x)^\dagger P_+ \right) \left(P_+ F_I(x) \right) \\
& + \bar{\Psi}_{uI}(x) P_-\, a\widehat{D} \widehat{P}_+ \Psi_{uI}(x) - \bar{\Psi}_{dI} \widehat{P}_+(x)\, a\widehat{D}^\dagger P_- \Psi_{dI}(x) \\
& + \frac{1}{2} \Phi_I^\dagger \widehat{P}_+(x) \left\{ \phi \widehat{P}_+ - \widetilde{m}_{+I}, \bar{\phi} \widehat{P}_+ - \widetilde{m}^*_{+I} \right\} \widehat{P}_+ \Phi_I(x) \\
& - \Phi_I^\dagger \widehat{P}_+(x) \left(D(x) + \frac{1}{2} \widehat{\Phi}(x) \right) \widehat{P}_+ \Phi_I(x) \\
& + \bar{\Psi}_{uI}(x) P_- \left(\phi(x) - \widetilde{m}_{+I} \right) P_- \Psi_{dI}(x) + \bar{\Psi}_{dI} \widehat{P}_+(x) \left(\bar{\phi}(x) - \widetilde{m}^*_{+I} \right) \widehat{P}_+ \Psi_{uI}(x) \\
& - \bar{\Psi}_{uI}(x) P_-\, Q(a\widehat{D}) \widehat{P}_+ \Phi_I(x) + \Phi_I^\dagger \widehat{P}_+(x)\, Q(a\widehat{D}^\dagger) P_- \Psi_{dI}(x) \\
& - \bar{\Psi}_{dI} \widehat{P}_+(x) \left(\frac{1}{2} \eta(x) + i\chi(x) \right) \widehat{P}_+ \Phi_I(x) \\
& - \Phi_I^\dagger \widehat{P}_+(x) \left(\frac{1}{2} \eta(x) - i\chi(x) \right) \widehat{P}_+ \Psi_{uI}(x) \\
& - \frac{1}{2} \Phi_I^\dagger \widehat{P}_+(x) \left\{ (Q\widehat{P}_+), \bar{\phi} \right\} \widehat{P}_+ \Psi_{uI}(x) - \frac{1}{2} \bar{\Psi}_{dI} \widehat{P}_+(x) \left\{ (Q\widehat{P}_+), \bar{\phi} \right\} \widehat{P}_+ \Phi_I(x) \\
& + \frac{1}{2} \Phi_I^\dagger \widehat{P}_+(x) \left\{ (Q\widehat{P}_+)^2, \bar{\phi} \right\} \widehat{P}_+ \Phi_I(x) + i \Phi_I^\dagger \widehat{P}_+(x) \left[(Q\widehat{P}_+), \chi \right] \widehat{P}_+ \Phi_I(x) \Big],
\end{aligned} \tag{5.14}$$

$$
\begin{aligned}
S^{\mathrm{LAT}}_{\mathrm{mat},-\widetilde{m}} = \sum_x \sum_{I'=1}^{n_-} \Big[& a^2 \Phi^\dagger_{I'} \widehat{P}_-(x)\, \widehat{D}^\dagger \widehat{D} \widehat{P}_- \Phi_{I'}(x) - \left(F_{I'}(x)^\dagger P_-\right) (P_- F_{I'}(x)) \\
& + \bar{\Psi}_{uI'}(x) P_+\, a\widehat{D} \widehat{P}_- \Psi_{uI'}(x) - \bar{\Psi}_{dI'} \widehat{P}_-(x)\, a\widehat{D}^\dagger P_+ \Psi_{dI'}(x) \\
& + \frac{1}{2} \Phi^\dagger_{I'} \widehat{P}_-(x) \left\{ \phi \widehat{P}_- - \widetilde{m}_{-I'}, \bar{\phi} \widehat{P}_- - \widetilde{m}^*_{-I'} \right\} \widehat{P}_- \Phi_{I'}(x) \\
& + \Phi^\dagger_{I'} \widehat{P}_-(x) \left(D(x) + \frac{1}{2} \widehat{\Phi}(x) \right) \widehat{P}_- \Phi_{I'}(x) \\
& + \bar{\Psi}_{uI'}(x) P_+ \left(\phi(x) - \widetilde{m}_{-I'}\right) P_+ \Psi_{dI'}(x) + \bar{\Psi}_{dI'} \widehat{P}_-(x) \left(\bar{\phi}(x) - \widetilde{m}^*_{-I'} \right) \widehat{P}_- \Psi_{uI'}(x) \\
& - \bar{\Psi}_{uI'}(x) P_+\, Q(a\widehat{D}) \widehat{P}_- \Phi_{I'}(x) + \Phi^\dagger_{I'} \widehat{P}_-(x)\, Q(a\widehat{D}^\dagger) P_+ \Psi_{dI'}(x) \\
& - \bar{\Psi}_{dI'} \widehat{P}_-(x) \left(\frac{1}{2} \eta(x) - i\chi(x) \right) \widehat{P}_- \Phi_{I'}(x) \\
& - \Phi^\dagger_{I'} \widehat{P}_-(x) \left(\frac{1}{2} \eta(x) + i\chi(x) \right) \widehat{P}_- \Psi_{uI'}(x) \\
& - \frac{1}{2} \Phi^\dagger_{I'} \widehat{P}_-(x) \left\{ (Q\widehat{P}_-), \bar{\phi} \right\} \widehat{P}_- \Psi_{uI'}(x) - \frac{1}{2} \bar{\Psi}_{dI'} \widehat{P}_-(x) \left\{ (Q\widehat{P}_-), \bar{\phi} \right\} \widehat{P}_- \Phi_{I'}(x) \\
& + \frac{1}{2} \Phi^\dagger_{I'} \widehat{P}_-(x) \left\{ (Q\widehat{P}_-)^2, \bar{\phi} \right\} \widehat{P}_- \Phi_{I'}(x) - i \Phi^\dagger_{I'} \widehat{P}_-(x) \left[(Q\widehat{P}_-), \chi \right] \widehat{P}_- \Phi_{I'}(x) \Big].
\end{aligned}
\tag{5.15}
$$

The last four terms both in (5.14) and (5.15) are lattice artifacts having no counterparts in the continuum theory.

5.4 Formulation II

We pick

$$P_+\Phi_I, \qquad P_+\Psi_{uI}, \qquad \bar{P}_-\Psi_{dI}, \qquad \bar{P}_-\gamma_0 F_I \quad \text{as chiral fields,} \tag{5.16}$$

$$\Phi_I^\dagger P_+, \qquad \bar{\Psi}_{dI}P_+, \qquad \bar{\Psi}_{uI}\bar{P}_-, \qquad F_I^\dagger\gamma_0\bar{P}_- \quad \text{as anti-chiral fields,} \tag{5.17}$$

for fundamental matters ($I = 1, \cdots, n_+$), and

$$\Phi_{I'}^\dagger P_-, \qquad \bar{\Psi}_{dI'}P_-, \qquad \bar{\Psi}_{uI'}\bar{P}_+, \qquad F_{I'}^\dagger\gamma_0\bar{P}_+ \quad \text{as chiral fields,} \tag{5.18}$$

$$P_-\Phi_{I'}, \qquad P_-\Psi_{uI'}, \qquad \bar{P}_+\Psi_{dI'}, \qquad \bar{P}_+\gamma_0 F_{I'} \quad \text{as anti-chiral fields,} \tag{5.19}$$

for anti-fundamental matters ($I' = 1, \cdots, n_-$).

Q SUSY transformation:

$$\begin{aligned}
Q(P_+\Phi_I(x)) &= -P_+\Psi_{uI}(x), \\
Q(P_+\Psi_{uI}(x)) &= -(\phi(x) - \widetilde{m}_{+I})P_+\Phi_I(x), \\
Q(\bar{P}_-\Psi_{dI}(x)) &= a\widehat{D}P_+\Phi_I(x) + \bar{P}_-\gamma_0 F_I(x) + (Q\bar{P}_-)\bar{P}_-\Psi_{dI}(x), \\
Q(\bar{P}_-\gamma_0 F_I(x)) &= (\bar{P}_-\phi - \widetilde{m}_{+I})\bar{P}_-\Psi_{dI}(x) + a\widehat{D}P_+\Psi_{uI}(x) - \bar{P}_-Q(a\widehat{D})P_+\Phi_I(x) \\
&\quad +(Q\bar{P}_-)\bar{P}_-\gamma_0 F_I(x) + (Q\bar{P}_-)^2\bar{P}_-\Psi_{dI}(x)
\end{aligned}$$

$$
\begin{aligned}
Q(\Phi_I(x)^\dagger P_+) &= -\bar\Psi_{dI}(x)P_+, \\
Q(\bar\Psi_{dI}(x)P_+) &= \Phi_I(x)^\dagger P_+(\phi(x)-\widetilde{m}_{+I}), \\
Q(\bar\Psi_{uI}\bar P_-(x)) &= \Phi_I(x)^\dagger P_+ a\widehat{D}^\dagger + F_I^\dagger\gamma_0\bar P_-(x) - \bar\Psi_{uI}\bar P_-(Q\bar P_-)(x), \\
Q(F_I^\dagger\gamma_0\bar P_-(x)) &= -\bar\Psi_{uI}\bar P_-(\phi\bar P_- - \widetilde{m}_{+I})(x) + \bar\Psi_{dI}(x)P_+ a\widehat{D}^\dagger - \Phi_I(x)^\dagger P_+ Q(a\widehat{D}^\dagger)\bar P_- \\
&\quad + F_I^\dagger\gamma_0\bar P_-(Q\bar P_-)(x) - \bar\Psi_{uI}\bar P_-(Q\bar P_-)^2(x),
\end{aligned}
$$

$$
\begin{aligned}
Q(P_-\Phi_{I'}(x)) &= -P_-\Psi_{uI'}(x), \\
Q(P_-\Psi_{uI'}(x)) &= -(\phi(x)-\widetilde{m}_{-I'})P_-\Phi_{I'}(x), \\
Q(\bar P_+\Psi_{dI'}(x)) &= a\widehat{D}P_-\Phi_{I'}(x) + \bar P_+\gamma_0 F_{I'}(x) + (Q\bar P_+)\bar P_+\Psi_{dI'}(x), \\
Q(\bar P_+\gamma_0 F_{I'}(x)) &= (\bar P_+\phi - \widetilde{m}_{-I'})\bar P_+\Psi_{dI'}(x) + a\widehat{D}P_-\Psi_{uI'}(x) - \bar P_+ Q(a\widehat{D})P_-\Phi_{I'}(x) \\
&\quad + (Q\bar P_+)\bar P_+\gamma_0 F_{I'}(x) + (Q\bar P_+)^2\bar P_+\Psi_{dI'}(x),
\end{aligned}
$$

$$
\begin{aligned}
Q(\Phi_{I'}(x)^\dagger P_-) &= -\bar\Psi_{dI'}(x)P_-, \\
Q(\bar\Psi_{dI'}(x)P_-) &= \Phi_{I'}(x)^\dagger P_-(\phi(x)-\widetilde{m}_{-I'}), \\
Q(\bar\Psi_{uI'}\bar P_+(x)) &= \Phi_{I'}(x)^\dagger P_- a\widehat{D}^\dagger + F_{I'}^\dagger\gamma_0\bar P_+(x) - \bar\Psi_{uI'}\bar P_+(Q\bar P_+)(x), \\
Q(F_{I'}^\dagger\gamma_0\bar P_+(x)) &= -\bar\Psi_{uI'}\bar P_+(\phi\bar P_+ - \widetilde{m}_{-I'})(x) + \bar\Psi_{dI'}(x)P_- a\widehat{D}^\dagger - \Phi_{I'}(x)^\dagger P_- Q(a\widehat{D}^\dagger)\bar P_+ \\
&\quad + F_{I'}^\dagger\gamma_0\bar P_+(Q\bar P_+)(x) - \bar\Psi_{uI'}\bar P_+(Q\bar P_+)^2(x),
\end{aligned}
$$

is nilpotent in the sense of

$$
\begin{aligned}
Q^2 &= (\text{infinitesimal gauge transformation with the parameter } \phi(x)) \\
&\quad +(\text{infinitesimal flavor rotations (5.21) and (5.22)})
\end{aligned} \tag{5.20}
$$

with

$$
\begin{aligned}
\delta(P_+\Phi_I) &= -\widetilde{m}_{+I}P_+\Phi_I, \quad & \delta(\Phi_I^\dagger P_+) &= \widetilde{m}_{+I}\Phi_I^\dagger P_+, \\
\delta(P_+\Psi_{uI}) &= -\widetilde{m}_{+I}P_+\Psi_{uI}, \quad & \delta(\bar{\Psi}_{uI}\bar{P}_-) &= \widetilde{m}_{+I}\bar{\Psi}_{uI}\bar{P}_-, \\
\delta(\bar{P}_-\Psi_{dI}) &= -\widetilde{m}_{+I}\bar{P}_-\Psi_{dI}, \quad & \delta(\bar{\Psi}_{dI}P_+) &= \widetilde{m}_{+I}\bar{\Psi}_{dI}P_+, \\
\delta(\bar{P}_-\gamma_0 F_I) &= -\widetilde{m}_{+I}\bar{P}_-\gamma_0 F_I, \quad & \delta(F_I^\dagger\gamma_0\bar{P}_-) &= \widetilde{m}_{+I}F_I^\dagger\gamma_0\bar{P}_-,
\end{aligned} \tag{5.21}
$$

$$
\begin{aligned}
\delta(\Phi_{I'}^\dagger P_-) &= \widetilde{m}_{-I'}\Phi_{I'}^\dagger P_-, \quad & \delta(P_-\Phi_{I'}) &= -\widetilde{m}_{-I'}P_-\Phi_{I'}, \\
\delta(\bar{\Psi}_{uI'}\bar{P}_+) &= \widetilde{m}_{-I'}\bar{\Psi}_{uI'}\bar{P}_+, \quad & \delta(P_-\Psi_{uI'}) &= -\widetilde{m}_{-I'}P_-\Psi_{uI'}, \\
\delta(\bar{\Psi}_{dI'}P_-) &= \widetilde{m}_{-I'}\bar{\Psi}_{dI'}P_-, \quad & \delta(\bar{P}_+\Psi_{dI'}) &= -\widetilde{m}_{-I'}\bar{P}_+\Psi_{dI'}, \\
\delta(F_{I'}^\dagger\gamma_0\bar{P}_+) &= \widetilde{m}_{-I'}F_{I'}^\dagger\gamma_0\bar{P}_+, \quad & \delta(\bar{P}_+\gamma_0 F_{I'}) &= -\widetilde{m}_{-I'}\bar{P}_+\gamma_0 F_{I'}.
\end{aligned} \tag{5.22}
$$

Similarly to (5.11), we have

$$
\bar{P}_\pm(Q\bar{P}_\pm)\bar{P}_\pm = 0. \tag{5.23}
$$

The matter-part action :

$$S^{\rm LAT}_{{\rm mat},\widetilde{m}} = S^{\rm LAT}_{{\rm mat},+\widetilde{m}} + S^{\rm LAT}_{{\rm mat},-\widetilde{m}},$$

$$\begin{aligned} S^{\rm LAT}_{{\rm mat},+\widetilde{m}} = Q\sum_x \sum_{I=1}^{n_+} \frac{1}{2}\Big[& \bar{\Psi}_{uI}\bar{P}_-(x)\left(a\widehat{D}P_+\Phi_I(x) - \bar{P}_-\gamma_0 F_I(x)\right) \\ & + \left(\Phi_I(x)^\dagger P_+\, a\widehat{D}^\dagger - F_I^\dagger\gamma_0\bar{P}_-(x)\right)\bar{P}_-\Psi_{dI}(x) \\ & -\Phi_I(x)^\dagger P_+\,\left(\bar{\phi}(x) - \widetilde{m}^*_{+I}\right)P_+\Psi_{uI}(x) \\ & +\bar{\Psi}_{dI}(x)P_+\,\left(\bar{\phi}(x) - \widetilde{m}^*_{+I}\right)P_+\Phi_I(x) \\ & +2i\Phi_I(x)^\dagger P_+\,\chi(x)P_+\Phi_I(x)\Big], \end{aligned} \tag{5.24}$$

$$\begin{aligned} S^{\rm LAT}_{{\rm mat},-\widetilde{m}} = Q\sum_x \sum_{I'=1}^{n_-} \frac{1}{2}\Big[& \bar{\Psi}_{uI'}\bar{P}_+(x)\left(a\widehat{D}P_-\Phi_{I'}(x) - \bar{P}_+\gamma_0 F_{I'}(x)\right) \\ & + \left(\Phi_{I'}(x)^\dagger P_-\, a\widehat{D}^\dagger - F_{I'}^\dagger\gamma_0\bar{P}_+(x)\right)\bar{P}_+\Psi_{dI'}(x) \\ & -\Phi_{I'}(x)^\dagger P_-\,\left(\bar{\phi}(x) - \widetilde{m}^*_{-I'}\right)P_-\Psi_{uI'}(x) \\ & +\bar{\Psi}_{dI'}(x)P_-\,\left(\bar{\phi}(x) - \widetilde{m}^*_{-I'}\right)P_-\Phi_{I'}(x) \\ & -2i\Phi_{I'}(x)^\dagger P_-\,\chi(x)P_-\Phi_{I'}(x)\Big]. \end{aligned} \tag{5.25}$$

The last three terms both in (5.24) and (5.25) yield interactions without the projectors depending on $\widehat{D}$.

After the Q operation, we have

$$
\begin{aligned}
S^{\text{LAT}}_{\text{mat},+\widetilde{m}} = \sum_x \sum_{I=1}^{n_+} \Big[& a^2 \Phi_I(x)^\dagger P_+ \widehat{D}^\dagger \widehat{D} P_+ \Phi_I(x) - \left(F_I^\dagger \gamma_0 \bar{P}_-(x)\right)\left(\bar{P}_- \gamma_0 F_I(x)\right) \\
& + \bar{\Psi}_{uI} \bar{P}_-(x)\, a\widehat{D} P_+ \Psi_{uI}(x) - \bar{\Psi}_{dI}(x) P_+\, a\widehat{D}^\dagger \bar{P}_- \Psi_{dI}(x) \\
& + \frac{1}{2} \Phi_I(x)^\dagger P_+ \left\{\phi(x) - \widetilde{m}_{+I}, \bar{\phi}(x) - \widetilde{m}^*_{+I}\right\} P_+ \Phi_I(x) \\
& - \Phi_I(x)^\dagger P_+ \left(D(x) + \frac{1}{2}\widehat{\Phi}(x)\right) P_+ \Phi_I(x) \\
& + \bar{\Psi}_{uI} \bar{P}_-(x) \left(\phi(x) - \widetilde{m}_{+I}\right) \bar{P}_- \Psi_{dI}(x) + \bar{\Psi}_{dI}(x) P_+ \left(\bar{\phi}(x) - \widetilde{m}^*_{+I}\right) P_+ \Psi_{uI}(x) \\
& - \bar{\Psi}_{uI} \bar{P}_-(x)\, Q(a\widehat{D}) P_+ \Phi_I(x) + \Phi_I(x)^\dagger P_+\, Q(a\widehat{D}^\dagger) \bar{P}_- \Psi_{dI}(x) \\
& - \bar{\Psi}_{dI}(x) P_+ \left(\frac{1}{2}\eta(x) + i\chi(x)\right) P_+ \Phi_I(x) \\
& - \Phi_I(x)^\dagger P_+ \left(\frac{1}{2}\eta(x) - i\chi(x)\right) P_+ \Psi_{uI}(x) \\
& + \bar{\Psi}_{uI} \bar{P}_-(x) \left(Q\bar{P}_-\right)^2 \bar{P}_- \Psi_{dI}(x) \Big],
\end{aligned}
\tag{5.26}
$$

$$
\begin{aligned}
S^{\mathrm{LAT}}_{\mathrm{mat},-\widetilde{m}} = \sum_x \sum_{I'=1}^{n_-} \Big[& a^2 \Phi_{I'}(x)^\dagger P_- \widehat{D}^\dagger \widehat{D} P_- \Phi_{I'}(x) - \left(F^\dagger_{I'} \gamma_0 \bar{P}_+(x)\right) \left(\bar{P}_+ \gamma_0 F_{I'}(x)\right) \\
& + \bar{\Psi}_{uI'} \bar{P}_+(x)\, a\widehat{D} P_- \Psi_{uI'}(x) - \bar{\Psi}_{dI'}(x) P_-\, a\widehat{D}^\dagger \bar{P}_+ \Psi_{dI'}(x) \\
& + \frac{1}{2} \Phi_{I'}(x)^\dagger P_- \left\{ \phi(x) - \widetilde{m}_{-I'}, \bar{\phi}(x) - \widetilde{m}^*_{-I'} \right\} P_- \Phi_{I'}(x) \\
& + \Phi_{I'}(x)^\dagger P_- \left(D(x) + \frac{1}{2} \widehat{\Phi}(x) \right) P_- \Phi_{I'}(x) \\
& + \bar{\Psi}_{uI'} \bar{P}_+(x) \left(\phi(x) - \widetilde{m}_{-I'}\right) \bar{P}_+ \Psi_{dI'}(x) + \bar{\Psi}_{dI'}(x) P_- \left(\bar{\phi}(x) - \widetilde{m}^*_{-I'}\right) P_- \Psi_{uI'}(x) \\
& - \bar{\Psi}_{uI'} \bar{P}_+(x)\, Q(a\widehat{D}) P_- \Phi_{I'}(x) + \Phi_{I'}(x)^\dagger P_-\, Q(a\widehat{D}^\dagger) \bar{P}_+ \Psi_{dI'}(x) \\
& - \bar{\Psi}_{dI'}(x) P_- \left(\frac{1}{2} \eta(x) - i\chi(x) \right) P_- \Phi_{I'}(x) \\
& - \Phi_{I'}(x)^\dagger P_- \left(\frac{1}{2} \eta(x) + i\chi(x) \right) P_- \Psi_{uI'}(x) \\
& + \bar{\Psi}_{uI'} \bar{P}_+(x) \left(Q\bar{P}_+\right)^2 \bar{P}_+ \Psi_{dI'}(x) \Big],
\end{aligned}
\tag{5.27}
$$

where the last terms both in (5.26) and (5.27) are lattice artifacts.

Since Formulation II seems to give a simpler expression than Formulation I, we will mainly develop Formulation II in what follows.

Superpotentials

We can latticize the superpotential terms as

$$S_{\rm pot}^{\rm LAT} = Q\sum_x \sum_{i=1}^{N}\sum_{I=1}^{n_+}\left[-\frac{\partial W}{\partial(P_+\Phi_I(x))_i}\left(\gamma_0\bar{P}_-\Psi_{dI}(x)\right)_i - \left(\bar{\Psi}_{uI}\bar{P}_-(x)\gamma_0\right)_i\frac{\partial\bar{W}}{\partial(\Phi_I(x)^\dagger P_+)_i}\right]$$
$$+Q\sum_x \sum_{i=1}^{N}\sum_{I'=1}^{n_-}\left[-\frac{\partial\bar{W}}{\partial(P_-\Phi_{I'}(x))_i}\left(\gamma_0\bar{P}_+\Psi_{dI'}(x)\right)_i - \left(\bar{\Psi}_{uI'}\bar{P}_+(x)\gamma_0\right)_i\frac{\partial W}{\partial(\Phi_{I'}(x)^\dagger P_-)_i}\right]$$

with

$$W = W(P_+\Phi_I, \Phi_{I'}^\dagger P_-), \qquad \bar{W} = \bar{W}(\Phi_I^\dagger P_+, P_-\Phi_{I'}).$$

$(\cdots)_i$ represent independent color degrees of freedom of the projected doublet by $P_\pm$ or $\bar{P}_\pm$.

Note

$S_{\rm pot}^{\rm LAT}$ exactly realizes holomorphic or anti-holomorphic structure on the lattice, i.e.

- terms containing W depend only on the chiral variables (5.16) and (5.18),
- terms containing $\bar{W}$ depend only on the anti-chiral variables (5.17) and (5.19),

besides the SYM variables which come in via $\bar{P}_\pm$ or $Q\bar{P}_\pm$.

(Recall that the holomorphy is not exact in the previous section due to the Wilson terms.
$\leftarrow$ Q transformation does not respect the chiral decomposition there.)

Similarly to the continuum case, the holomorphy tempts us to expect that the superpotential terms receive no radiative correction on lattice perturbative computations concerning the matter sector.

◇ Path-integral measure for the SYM part

$$
\begin{aligned}
(\mathrm{d}\mu_{\mathrm{2DSYM}}) \;\equiv\; & \prod_x \left[\prod_{\mu=0}^{1} \mathrm{d}U_\mu(x) \right] \\
& \times \prod_A \mathrm{d}\psi_0^A(x)\, \mathrm{d}\psi_1^A(x)\, \mathrm{d}\chi^A(x)\, \mathrm{d}\eta^A(x)\, \mathrm{d}\phi^A(x)\, \mathrm{d}\bar{\phi}^A(x)\, \mathrm{d}D^A(x),
\end{aligned}
$$

where $\mathrm{d}U_\mu(x)$ is the Haar measure of the gauge group G,
the index A labels the generators of G.

◇ Path-integral measure for the matter part

$$
\begin{aligned}
(\mathrm{d}\mu_{\mathrm{mat}}) \;=\; & \left(\prod_{I=1}^{n_+} \mathrm{d}\mu_{\mathrm{mat},+I} \right) \left(\prod_{I'=1}^{n_-} \mathrm{d}\mu_{\mathrm{mat},-I'} \right) \\
\mathrm{d}\mu_{\mathrm{mat},+I} \;\equiv\; & \prod_x \prod_{i=1}^{N} \mathrm{d}(P_+\Phi_I(x))_i\, \mathrm{d}(\Phi_I(x)^\dagger P_+)_i\, \mathrm{d}(\bar{P}_-\gamma_0 F_I(x))_i\, \mathrm{d}(F_I^\dagger \gamma_0 \bar{P}_-(x))_i \\
& \times \mathrm{d}(P_+\Psi_{uI}(x))_i\, \mathrm{d}(\bar{\Psi}_{uI}\bar{P}_-(x))_i\, \mathrm{d}(\bar{P}_-\Psi_{dI}(x))_i\, \mathrm{d}(\bar{\Psi}_{dI}(x)P_+)_i, \\
\mathrm{d}\mu_{\mathrm{mat},-I'} \;\equiv\; & \prod_x \prod_{i=1}^{N} \mathrm{d}\left(P_-\Phi_{I'}(x)\right)_i\, \mathrm{d}(\Phi_{I'}(x)^\dagger P_-)_i\, \mathrm{d}(\bar{P}_+\gamma_0 F_{I'}(x))_i\, \mathrm{d}(F_{I'}^\dagger \gamma_0 \bar{P}_+(x))_i \\
& \times \mathrm{d}(P_-\Psi_{uI'}(x))_i\, \mathrm{d}(\bar{\Psi}_{uI'}\bar{P}_+(x))_i\, \mathrm{d}(\bar{P}_+\Psi_{dI'}(x))_i\, \mathrm{d}(\bar{\Psi}_{dI'}(x)P_-)_i.
\end{aligned}
$$

Let us see transformation properties of the matter-part measure.

Gauge Invariance

For $g(x) = \mathrm{e}^{i\omega(x)} \in G$ ($\omega(x)$: infinitesimal) transforms the fundamental matters as

$$\begin{aligned} P_+\Phi_I(x) &\longrightarrow g(x)P_+\Phi_I(x) = (1+i\omega(x)P_+)P_+\Phi_I(x), \\ \Phi_I(x)^\dagger P_+ &\longrightarrow \Phi_I(x)^\dagger P_+ g(x)^{-1} = \Phi_I(x)^\dagger P_+(1-iP_+\omega(x)), \\ \bar{P}_-\gamma_0 F_I(x) &\longrightarrow g(x)\bar{P}_-\gamma_0 F_I(x) = (1+i\omega(x)\bar{P}_-)\bar{P}_-\gamma_0 F_I(x), \\ F_I^\dagger\gamma_0\bar{P}_-(x) &\longrightarrow F_I^\dagger\gamma_0\bar{P}_-(x)g(x)^{-1} = F_I^\dagger\gamma_0\bar{P}_-(1-i\bar{P}_-\omega)(x), \end{aligned}$$

$$\begin{aligned} P_+\Psi_{uI}(x) &\longrightarrow g(x)P_+\Psi_{uI}(x) = (1+i\omega(x)P_+)P_+\Psi_{uI}(x), \\ \bar{\Psi}_{uI}\bar{P}_-(x) &\longrightarrow \bar{\Psi}_{uI}\bar{P}_-(x)g(x)^{-1} = \bar{\Psi}_{uI}\bar{P}_-(1-i\bar{P}_-\omega)(x), \\ \bar{P}_-\Psi_{dI}(x) &\longrightarrow g(x)\bar{P}_-\Psi_{dI}(x) = (1+i\omega(x)\bar{P}_-)\bar{P}_-\Psi_{dI}(x), \\ \bar{\Psi}_{dI}(x)P_+ &\longrightarrow \bar{\Psi}_{dI}(x)P_+g(x)^{-1} = \bar{\Psi}_{dI}(x)P_+(1-iP_+\omega(x)). \end{aligned}$$

For bosons, $\mathcal{O}(\omega)$ parts of the jacobian cancel with their conjugates.
For fermions, they cancel between $P_+\Psi_{uI}$ and $\bar{\Psi}_{dI}P_+$,
and between $\bar{\Psi}_{uI}\bar{P}_-$ and $\bar{P}_-\Psi_{dI}$.
$\Rightarrow$ Gauge invariance of $\mathrm{d}\mu_{\mathrm{mat},+I}$ (and of $\mathrm{d}\mu_{\mathrm{mat},-I'}$ from the similar argument).

Q-SUSY Invariance

Under the Q-SUSY transformation with the Grassmann number ε, the fundamental matter fields change as

$$
\begin{aligned}
P_+\Phi_I(x) &\to (1+i\varepsilon Q)P_+\Phi_I(x) = P_+\Phi_I(x) + \cdots,\\
\Phi_I(x)^\dagger P_+ &\to (1+i\varepsilon Q)\Phi_I(x)^\dagger P_+ = \Phi_I(x)^\dagger P_+ + \cdots,\\
\bar{P}_-\gamma_0 F_I(x) &\to (1+i\varepsilon Q)\bar{P}_-\gamma_0 F_I(x) = \left[1+i\varepsilon(Q\bar{P}_-)\bar{P}_-\right]\bar{P}_-\gamma_0 F_I(x) + \cdots,\\
F_I^\dagger\gamma_0\bar{P}_-(x) &\to (1+i\varepsilon Q)F_I^\dagger\gamma_0\bar{P}_-(x) = F_I^\dagger\gamma_0\bar{P}_-\left[1+i\varepsilon\bar{P}_-(Q\bar{P}_-)\right](x) + \cdots,
\end{aligned}
$$

$$
\begin{aligned}
P_+\Psi_{uI}(x) &\to (1+i\varepsilon Q)P_+\Psi_{uI}(x) = P_+\Psi_{uI}(x) + \cdots,\\
\bar{\Psi}_{uI}\bar{P}_-(x) &\to (1+i\varepsilon Q)\bar{\Psi}_{uI}\bar{P}_-(x) = \bar{\Psi}_{uI}\bar{P}_-\left[1+i\varepsilon\bar{P}_-(Q\bar{P}_-)\right](x) + \cdots,\\
\bar{P}_-\Psi_{dI}(x) &\to (1+i\varepsilon Q)\bar{P}_-\Psi_{dI}(x) = \left[1+i\varepsilon(Q\bar{P}_-)\bar{P}_-\right]\bar{P}_-\Psi_{dI}(x) + \cdots,\\
\bar{\Psi}_{dI}(x)P_+ &\to (1+i\varepsilon Q)\bar{\Psi}_{dI}(x)P_+ = \bar{\Psi}_{dI}(x)P_+ + \cdots,
\end{aligned}
$$

where "$\cdots$" correspond to off-diagonal elements of Jacobi matrices and are irrelevant for the calculation.

For example, the measure $\Pi_x \Pi_{i=1}^N \mathrm{d}(\bar{P}_-\gamma_0 F_I(x))_i$ contributes to the Jacobian factor by

$$
\mathrm{Det}\left[1+i\varepsilon(Q\bar{P}_-)\bar{P}_-\right] = 1 + i\varepsilon\,\mathrm{Tr}\left[(Q\bar{P}_-)\bar{P}_-\right] = 1 + i\varepsilon\,\mathrm{Tr}\left[\bar{P}_-(Q\bar{P}_-)\bar{P}_-\right] = 1.
$$

($\bar{P}_- = \bar{P}_-^2$ and (5.23) was used.)
Repeating the same kind of computation $\Rightarrow$ $\mathrm{d}\mu_{\mathrm{mat},+I}$ and $\mathrm{d}\mu_{\mathrm{mat},-I'}$ are Q-invariant.

$\mathrm{U}(1)_A$ Transformation

The $\mathrm{U}(1)_A$ transformation (the parameter α infinitesimal) changes the fundamental fields as

$$\begin{aligned}
P_+\Psi_{uI}(x) &\longrightarrow \mathrm{e}^{i\alpha}P_+\Psi_{uI}(x) = (1+i\alpha P_+)P_+\Psi_{uI}(x),\\
\bar{\Psi}_{uI}\bar{P}_-(x) &\longrightarrow \bar{\Psi}_{uI}\bar{P}_-(x)\mathrm{e}^{-i\alpha} = \bar{\Psi}_{uI}\bar{P}_-(1-i\alpha\bar{P}_-)(x),\\
\bar{P}_-\Psi_{dI}(x) &\longrightarrow \mathrm{e}^{-i\alpha}\bar{P}_-\Psi_{dI}(x) = (1-i\alpha\bar{P}_-)\bar{P}_-\Psi_{dI}(x),\\
\bar{\Psi}_{dI}(x)P_+ &\longrightarrow \bar{\Psi}_{dI}(x)P_+\mathrm{e}^{i\alpha} = \bar{\Psi}_{dI}(x)P_+(1+i\alpha P_+),
\end{aligned}$$

$$\begin{aligned}
P_-\Psi_{uI'}(x) &\longrightarrow \mathrm{e}^{i\alpha}P_-\Psi_{uI'}(x) = (1+i\alpha P_-)P_-\Psi_{uI'}(x),\\
\bar{\Psi}_{uI'}\bar{P}_+(x) &\longrightarrow \bar{\Psi}_{uI'}\bar{P}_+(x)\mathrm{e}^{-i\alpha} = \bar{\Psi}_{uI'}\bar{P}_+(1-i\alpha\bar{P}_+)(x),\\
\bar{P}_+\Psi_{dI'}(x) &\longrightarrow \mathrm{e}^{-i\alpha}\bar{P}_+\Psi_{dI'}(x) = (1-i\alpha\bar{P}_+)\bar{P}_+\Psi_{dI'}(x),\\
\bar{\Psi}_{dI'}(x)P_- &\longrightarrow \bar{\Psi}_{dI'}(x)P_-\mathrm{e}^{i\alpha} = \bar{\Psi}_{dI'}(x)P_-(1+i\alpha P_-).
\end{aligned}$$

$\Rightarrow$ The measures change as

$$\begin{aligned}
\mathrm{d}\mu_{\mathrm{mat},+I} &\longrightarrow \left[1-2i\alpha\mathrm{Tr}(P_+-\bar{P}_-)\right]\mathrm{d}\mu_{\mathrm{mat},+I} = \left[1+i\alpha\mathrm{Tr}(\gamma_3 a\widehat{D})\right]\mathrm{d}\mu_{\mathrm{mat},+I},\\
\mathrm{d}\mu_{\mathrm{mat},-I'} &\longrightarrow \left[1+2i\alpha\mathrm{Tr}(\bar{P}_+-P_-)\right]\mathrm{d}\mu_{\mathrm{mat},-I'} = \left[1-i\alpha\mathrm{Tr}(\gamma_3 a\widehat{D})\right]\mathrm{d}\mu_{\mathrm{mat},-I'}.
\end{aligned}$$

Thus,

$$
\begin{aligned}
(\mathrm{d}\mu_{\mathrm{mat}}) &\longrightarrow \left[1 + i\alpha\,(n_+ - n_-)\,\mathrm{Tr}(\gamma_3 a\widehat{D})\right](\mathrm{d}\mu_{\mathrm{mat}}) \\
&\simeq \left[1 + i\alpha\,\frac{n_+ - n_-}{\pi}\int \mathrm{d}^2x\,\mathrm{tr}\,F_{01}\right](\mathrm{d}\mu_{\mathrm{mat}}) \qquad (a \longrightarrow 0)
\end{aligned}
$$

for the gauge fields assumed to be smooth [Kikukawa-Yamada]**.**

$\Rightarrow$ It reproduces the $\mathrm{U}(1)_A$ anomaly in the previous section.

Combining the addmissibility conditions from the SYM part [F.S.] and from the matter part, we find

$G = \mathrm{U}(N)$ without ϑ-term :

$$0 < \epsilon < \frac{1}{5} \quad \text{for } N = 1, 2, \cdots, 100$$
$$0 < \epsilon < \frac{2}{\sqrt{N}} \quad \text{for } N \geq 101,$$

$G = \mathrm{U}(N)$ with ϑ-term :

$$0 < \epsilon < \frac{1}{5} \quad \text{for } N = 1, 2, \cdots, 25$$
$$0 < \epsilon < \frac{1}{\sqrt{N}} \quad \text{for } N \geq 26,$$

$G = \mathrm{SU}(N)$:

$$0 < \epsilon < \frac{1}{5} \quad \text{for } N = 2, 3, \cdots, 31$$
$$0 < \epsilon < 2\sin\left(\frac{\pi}{N}\right) \quad \text{for } N \geq 32.$$

6 Lattice Formulation of Gauged Linear Sigma Models

◇ Gauged linear sigma models we consider is
2D $\mathcal{N} = (2,2)$ SQCD ($G = \mathrm{U}(N)$) with n_+ fundamental matters
and ℓ_- matters in the $\det^{-q_{\mathtt{A}'}}$-representation. ($\mathtt{A}' = 1, \cdots, \ell_-$, $q_{\mathtt{A}'} \in \mathbf{Z}_{>0}$)

The $\det^{-q_{\mathtt{A}'}}$-matters are charged only under the overall $\mathrm{U}(1)$ of $G = \mathrm{U}(N)$
and gauge-transform as

$$\Xi_{-\mathtt{A}'}(x) \longrightarrow (\det g(x))^{-q_{\mathtt{A}'}}\, \Xi_{-\mathtt{A}'}(x) \qquad \text{for} \qquad g(x) \in G,$$

or

$$\delta \Xi_{-\mathtt{A}'}(x) = -i q_{\mathtt{A}'}\, (\mathrm{tr}\, \omega(x))\, \Xi_{-\mathtt{A}'}(x) \qquad \text{for} \qquad g(x) = 1 + i\omega(x)$$

with $\omega(x)$ infinitesimal.
⇒ Covariant derivatives $\mathcal{D}_\mu \Xi_{-\mathtt{A}'} = (\partial_\mu - i q_{\mathtt{A}'} (\mathrm{tr}\, A_\mu))\, \Xi_{-\mathtt{A}'}$.
⇒ Forward (Backward) covariant differences D_μ (D^*_μ):

$$\begin{aligned} aD_\mu\, \Xi_{-\mathtt{A}}(x) &= (\det U_\mu(x))^{q_{\mathtt{A}}}\, \Xi_{\mathtt{A}}(x+\hat{\mu}) - \Xi_{-\mathtt{A}}(x), \\ aD^*_\mu\, \Xi_{-\mathtt{A}}(x) &= \Xi_{-\mathtt{A}}(x) - (\det U_\mu(x-\hat{\mu}))^{-q_{\mathtt{A}}}\, \Xi_{-\mathtt{A}}(x-\hat{\mu}). \end{aligned}$$

Similarly to the (anti-)fundamental matters, 2D SQCD system with the
$\det^{-q_{\mathtt{A}'}}$-matters can be latticized preserving the chiral flavor symmetry.

$\Rightarrow$ Combining the n_+ fundamental matters and ℓ_- $\det^{-q_{\mathrm{A}'}}$-matters, the lattice formulation of gauged linear sigma models is possible.
(Thanks to the Ginsparg-Wilson formulation, it is possible that matters belonging to different representations are put in different chiral sectors.)

$\diamondsuit$ When $n_+ \geq N$, baryonic chiral superfields

$$B_{I_1\cdots I_N} \equiv \epsilon_{i_1\cdots i_N}\Phi_{+I_1 i_1}\cdots\Phi_{+I_N i_N}$$

are not trivial, and they gauge-transform as

$$B_{I_1\cdots I_N}(x) \longrightarrow (\det g(x))B_{I_1\cdots I_N}(x).$$

Let $\mathcal{G}_{\mathrm{A}'}(B)$ be a homogeneous polynomial of degree $q_{\mathrm{A}'}$ w.r.t. $B_{I_1\cdots I_N}$.
Then, the superpotential

$$\mathcal{W} = \sum_{\mathrm{A}'=1}^{\ell_-} \Xi_{-\mathrm{A}'}\,\mathcal{G}_{\mathrm{A}'}(B)$$

is gauge invariant.
Its lattice formulation is possible under the admissibility condition

$$0 < \epsilon < \frac{1}{8Nq} \qquad \text{with} \qquad q \equiv \max_{\mathrm{A}'=1,\cdots,\ell_-}(q_{\mathrm{A}'}).$$

Applications:

Gauged linear sigma models are discussed to flow in the infra-red limit to nonlinear sigma models with target spaces determined by the D-term and F-term conditions [Witten].

◇ Target spaces are in Grassmann manifolds ($\supset$ Calabi-Yau manifolds): $G(N, n_+) = \frac{\mathrm{U}(n_+)}{\mathrm{U}(N)\times\mathrm{U}(n_+-N)}$

$\Rightarrow$ Duality $G(N, n_+) \cong G(n_+ - N, n_+)$ suggests an analog of the Seiberg duality between the following gauged linear sigma models ($\ell_- = 1$):

- $G = \mathrm{U}(N)$, n_+ fundamental matters Φ_{+I}, one $\det^{-q}$-matter Ξ_- with $\mathcal{W} = \Xi_- \mathcal{G}(B)$ ($\mathcal{G}$: degree q)
- $G = \mathrm{U}(n_+ - N)$, n_+ fundamental matters Φ'_{+I}, one $\det^{-q}$-matter Ξ'_- with $\mathcal{W} = \Xi'_- \mathcal{G}'(B')$ ($\mathcal{G}'$: degree q)

where $\mathcal{G}(B) = \mathcal{G}'(B')$ with the replacement $B_{I_1\cdots I_N} = \epsilon_{I_1\cdots I_{n_+}} B'_{I_{N+1}\cdots I_{n_+}}$.
[Hori-Tong]

Now, this duality can be confirmed from the first principle by the lattice formulation!

7 Summary and Discussion

◇ We have presented a lattice formulation of 2D $\mathcal{N} = (2,2)$ SQCD (including gauged linear sigma models) with exactly preserving Q-SUSY.

- Gauge Group $G = \mathrm{U}(N)$ or $\mathrm{SU}(N)$, Compact link variables $U_\mu(x)$
- In order to resolve the matter doublers,
  - Use of D_W $\Rightarrow$ the lattice action is constructed in the case $n_+ = n_-$
  - Use of $\widehat{D}$ $\Rightarrow$ the lattice action is constructed for general $n_\pm$ (Exact chiral flavor symmetry on the lattice due to the Ginsparg-Wilson formulation)
- The Ginsparg-Wilson formulation makes possible to construct exactly holomorphic or anti-holomorphic superpotentials on the lattice.
  $\Rightarrow$ Nonrenormalization theorem on the lattice expected to hold.
- Use of $\widehat{D}$ yields another possibility of the FI and ϑ-term:
  $S^{\mathrm{LAT}}_{\mathrm{FI},\vartheta\,(\widehat{D})} \equiv Q\kappa \sum_x \mathrm{tr}\,(-i\chi(x)) - \frac{\vartheta - 2\pi i\kappa}{2\pi} i a^2 \sum_x \mathrm{tr}\,\widehat{F}_{01}(x)$
  with $\widehat{F}_{01}(x) \equiv \frac{\pi}{a}\,\mathrm{tr}_{\mathrm{spin}}\left(\gamma_3 \widehat{D}\right)(x,x)$ ($\mathrm{tr}_{\mathrm{spin}}$: trace over the Dirac indices).
  Note $\sum_x \mathrm{tr}\widehat{F}_{01}(x)$ is topological because $\delta\,\mathrm{Tr}\,(\gamma_3\widehat{D}) = 0$.

A Gauged Linear Sigma Models $\Rightarrow$ Grassmannian

$\diamondsuit$ Consider the case of all twisted masses zero and $\ell_- = 1$.

Superpotential: $\mathcal{W} = \Xi_- \mathcal{G}(B)$. ($\Xi_-$: $\det^{-q}$-repre., $\mathcal{G}$: degree q)

Bosonic potential is

$$
\begin{aligned}
U \;=\; & |\mathcal{G}(b)|^2 + |\xi_-|^2 \sum_{I=1}^{n_+} \sum_{i=1}^{N} \left| \sum_{I_1 < \cdots < I_N} \frac{\partial \mathcal{G}(b)}{\partial b_{I_1 \cdots I_N}} \frac{\partial b_{I_1 \cdots I_N}}{\partial \phi_{+Ii}} \right|^2 \\
& + \frac{g^2}{4} \operatorname{tr} \left\{ \left[\sum_{I=1}^{n_+} \phi_{+I} \phi_{+I}^\dagger - \left(q \xi_-^* \xi_- + \kappa \right) \mathbb{1}_N \right]^2 \right\} \\
& + \frac{1}{4g^2} \operatorname{tr} \left([\phi, \bar{\phi}]^2 \right) + \sum_{I=1}^{n_+} \frac{1}{2} \phi_{+I}^\dagger \left\{ \phi, \bar{\phi} \right\} \phi_{+I} + |q \operatorname{tr} \phi|^2 |\xi_-|^2 ,
\end{aligned}
$$

where $b_{I_1 \cdots I_N}$, ξ_-: the lowest components of the chiral superfields $B_{I_1 \cdots I_N}$, Ξ_-. The first and second lines come from the F-term and D-term conditions, respectively.

For the potential minimum $U = 0$,

The second term $\Rightarrow$ $\xi_- = 0$ (for generic $\mathcal{G}$),

The third term $\Rightarrow$ $\sum_{I=1}^{n_+} \phi_{+I} \phi_{+I}^\dagger = \kappa \mathbb{1}_N$

$\Rightarrow$ N vectors $v_1, \cdots, v_N \in \mathbf{C}^{n_+}$ ($(v_i)_I = \phi_{+Ii}$) are orthogonal and have $(\text{length})^2 = \kappa$. ($\kappa > 0$ assumed.)

$\Rightarrow$ $\{v_1, \cdots, v_N\}$ span the space of N-dim. planes in C^{n_+},

i.e. Grassmann manifold $G(N, n_+) = \frac{\mathrm{U}(n_+)}{\mathrm{U}(N)\times\mathrm{U}(n_+-N)}$.

$\Rightarrow$ Together with the first term, the F-term and D-term conditions yield

a hypersurface defined by $\mathcal{G}(b) = 0$ in $G(N, n_+)$.

A.1 Gauged Linear Sigma Models $\Rightarrow$ Calabi-Yau

$\diamondsuit$ On top of the above situation, consider the case $G = \mathrm{U}(1)$.

$b_I = \phi_{+I}\ (I = 1, \cdots, n_+)$

$$\begin{aligned} U \;=\; & |\mathcal{G}(\phi_+)|^2 + |\xi_-|^2 \sum_{I=1}^{n_+} \left|\frac{\partial \mathcal{G}(\phi_+)}{\partial \phi_{+I}}\right|^2 + \frac{g^2}{4}\left(\sum_{I=1}^{n_+} \phi_{+I}\phi^*_{+I} - q\xi^*_-\xi_- - \kappa\right)^2 \\ & + \sum_{I=1}^{n_+} |\phi|^2 |\phi_{+I}|^2 + |q\phi|^2 |\xi_-|^2, \end{aligned}$$

For $U = 0$,

the second and third terms $\Rightarrow$ $\xi_- = 0$, $\sum_{I=1}^{n_+} \phi_{+I}\phi^*_{+I} = \kappa$

$\Rightarrow$ Under the action of $G = \mathrm{U}(1)$, the above eq. represents CP^{n_+-1}.

$\Rightarrow$ The F-term and D-term conditions yield

a hypersurface defined by $\mathcal{G}(\phi_+) = 0$ (degree q) in CP^{n_+-1}.

$\Rightarrow$ When $q = n_+$, this becomes a Calabi-Yau manifold.

Also, then $\mathrm{U}(1)_A$ anomaly cancels and the coupling κ does not run.

B Summary of Workshop and Outlook (maybe my personal)

Summary

In this workshop "Lattice Supersymmetry and Beyond", many interesting ideas and results on lattice supersymmetry were presented.

- Kawamoto, D'Adda
  Ambitious attempt to realize full supersymmetry on lattice $\Rightarrow$ Finite supersymmetry transformation (η)

- Bruckmann
  Use blocking transformation to find lattice counterparts of continuum symmetries
  (Generalization of the derivation of GW relation)
  Apply to SUSY

- Endre
  Numerical study of 4D $\mathcal{N} = 1$ SYM using domain wall fermions
  overlap fermions?

- **Catterall**
  Numerical study of pfaffian phases of 2D $\mathcal{N} = (2,2)$ and 4D $\mathcal{N} = 4$ SYM
  $\Rightarrow$ SUSY breaking in 2D $\mathcal{N} = (2,2)$?
- **Suzuki**
  Numerical study of restoration of SUSY and some physics in 2D $\mathcal{N} = (2,2)$ lattice SYM with one exact supercharge
  PCSC relation
- **Nishimura**
  Nonlattice approach for SUSY matrix QM
  $\Rightarrow$ Confirmation and prediction for blackhole and string physics
  Appendix: SYM on R $\times$ S^2, R $\times$ S^3 from plane wave MM
- **Matsuura**
  Connection among SUSY lattice approches
  (Orbifolding, Geometrical, Link)
  Appendix A: Fundamental matters in 2D $\mathcal{N} = (2,2)$ orbifolding approach
- **F. S.**
  Ginsparg-Wilson formulation with exact SUSY on lattice for 2D $\mathcal{N} = (2,2)$ SQCD

Outlook

I felt really nice atmosphere in the workshop, NBIA and Denmark.
If this kind of next workshop is held, it will be pleasant.

In the workshop dinner last night, So Matsuura claimed that
Poland is a more beautiful place than Denmark by 100/70
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Thank you very much for the organizers

Poul H. Damgaard, Hidenori Fukaya, So Matsuura,

and thank you for all the speakers and all the participants!

See you again (hopfully in Poland)!