Relation among Supersymmetric Lattice Gauge Theories

So Matsuura
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based on the works
arXiv:0704.2696
arXiv:0706.3007
arXiv:0708.4129
arXiv:0801.2936
arXiv:0805.4491
with P.H.Damgaard
Introduction

Supersymmetric Gauge Theory

- Supersymmetry seems a fundamental symmetry of space-time.
  
  (an extension of the translational symmetry $x \rightarrow (x, \theta)$)

- Supersymmetry seems to be necessary to unify the interactions.

- Exact results in quantum field theory.
  
  (Seiberg-Witten theory, Nekrasov’s formula, Dijkgraaf-Vafa etc...)

- Gauge/Gravity duality (AdS/CFT Correspondence)

- Connection to superstring theory

We want a way to analyze SUSY gauge theory non-perturbatively.

lattice?
**Difficulty**

It seems impossible to construct a SUSY invariant theory on a lattice.

SUSY invariant action in continuum space-time

Suppose an action is written as

$$S = \int dx d\theta F(\Phi(x, \theta))$$

$\Phi(x, \theta)$; superfield

Essentially, a SUSY generator can be represented as

$$\delta \Phi = \epsilon Q \Phi \quad Q = \partial_\theta + \theta \Gamma \partial_x$$

Variation of the action

$$\delta_\epsilon S = \int dx d\theta F(\Phi + \epsilon Q \Phi) - F(\Phi)$$

$$= \int dx d\theta \epsilon Q F(\Phi)$$

$$= \int dx d\theta \epsilon (\partial_\theta + \theta \Gamma \partial_x) F(\Phi) = 0$$

Leibniz rule
It seems impossible to keep all SUSY on a lattice. Can we keep a part of SUSY on a lattice? Yes!
Supersymmetric Lattice Gauge Theories

ORBIFOLDING

Equivalent!
Equivalent!

GEOMETRICAL DISCRETIZATION
S. Catterall (2004)
Derived

LINK APPROACH

LATTICE TFT
F.Sugino (2003)
Contents

1. Introduction

2. Review of Orbifold Lattice Theory

3. Equivalence between Geometrical Discretization and Orbifolding

4. SUSY in Link Approach from Orbifolding Point of View

5. A Comment and Future Works

A. Orbifold Lattice Gauge Theories with Matter
§ 2 Review of Orbifold Lattice Theory

Basic Idea ~ matrix as a collection of lattice fields ~

**MATRIX**

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**(1,2)x(1,2)**

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**(2,2)x(3,2)**

**LATTICE**

1. Starting with a matrix theory (mother theory)
2. Project out "non-local" elements properly
3. We interpret the projected matrix theory as a lattice theory.

Strategy

with keeping SUSY

27/11/2008

NBIA
Construction of 2D $N=(2,2)$ SYM on lattice

Mother theory

dimensional reduction of $4D \, N=1$ SYM theory with a gauge group $\mathbf{SU}(N)$. \\

$$S_m = \frac{1}{g^2} \text{Tr} \left( -\frac{1}{4} [v_\alpha, v_\beta]^2 + i \bar{\psi} \sigma_\alpha [v_\alpha, \psi] \right) \quad \alpha, \beta = 0, \cdots, 3$$ \\

\begin{align*}
\mathbf{v}_\alpha : & \text{ four hermitian matrices (gauge boson)} \\
\psi, \, \bar{\psi} : & \text{ 2 component spinors (gaugino)}
\end{align*}

Symmetries

maximal $U(1)$ subgroup

1) global symmetry $SO(4) \times U(1)_R \supset U(1)_1 \times U(1)_2 \times U(1)_R$

2) gauge symmetry $v_\alpha \rightarrow g v_\alpha g^{-1}, \quad g \in U(N_c N^2)$
Equivalent expression in which the U(1) symmetries are manifest:

\[ S_m = \frac{1}{g^2} \text{Tr} \left( \frac{1}{4} |z_m, z_n|^2 + \frac{1}{8} [z_m, \bar{z}_m]^2 + \psi_m[\bar{z}_m, \eta] - \chi_{mn}[z_m, \psi_n] \right) \]

\[
\begin{align*}
  z_1 &\equiv v_1 + iv_2, \\
  z_2 &\equiv v_0 + iv_3,
\end{align*}
\]

\[
\psi = \begin{pmatrix} \chi_{12} \\ \eta \end{pmatrix}, \quad \bar{\psi} = (\psi_1, \psi_2)
\]

**U(1) charges**

<table>
<thead>
<tr>
<th></th>
<th>(z_1)</th>
<th>(z_2)</th>
<th>(\eta)</th>
<th>(\chi_{12})</th>
<th>(\psi_1)</th>
<th>(\psi_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(q_1)</td>
<td>1</td>
<td>0</td>
<td>1/2</td>
<td>-1/2</td>
<td>1/2</td>
<td>-1/2</td>
</tr>
<tr>
<td>(q_2)</td>
<td>0</td>
<td>1</td>
<td>1/2</td>
<td>-1/2</td>
<td>-1/2</td>
<td>1/2</td>
</tr>
<tr>
<td>(q_3)</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td>1/2</td>
<td>-1/2</td>
<td>-1/2</td>
</tr>
</tbody>
</table>

We can take any linear combination.

\[
\begin{align*}
  r_1 &= q_1 - q_3, \\
  r_2 &= q_2 - q_3
\end{align*}
\]

where

\[
\begin{array}{c}
  r_1 = 1 \ 0 \ 0 \ -1 \ 1 \ 0 \\
  r_2 = 0 \ 1 \ 0 \ -1 \ 0 \ 1
\end{array}
\]
Orbifold projection

Combining the $U(1)$ and gauge symmetry, we consider a $\mathbb{Z}_N^2$ transformation generated by

$$\gamma_a : \Phi \rightarrow \omega^{r_a} \Omega_a \Phi \Omega_a^{-1}, \quad (a = 1, 2)$$

where $\omega = e^{2\pi i/N}$ and

$$\begin{align*}
\Omega_1 & \equiv 1_{N_c} \otimes U \otimes 1_N, \\
\Omega_2 & \equiv 1_{N_c} \otimes 1_N \otimes U,
\end{align*}$$

$c.f.$

$$(U \Phi U^{-1})_{ij} = \omega^{i-j} \Phi_{ij}$$

We keep only components that are invariant under this transformation.
<table>
<thead>
<tr>
<th>((r_1, r_2))</th>
<th>(z_1)</th>
<th>(z_2)</th>
<th>(\eta)</th>
<th>(\chi_{12})</th>
<th>(\psi_1)</th>
<th>(\psi_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1, 0))</td>
<td>((0, 1))</td>
<td>((0, 0))</td>
<td>((-1, -1))</td>
<td>((1, 0))</td>
<td>((0, 1))</td>
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</tbody>
</table>

\[
Z_1 \sim \sum_{k \in \mathbb{Z}_N^2} z_1(k) \otimes E_{k_1, k_1 + 1} \otimes E_{k_2, k_2}
\]

\[
Z_2 \sim \sum_{k \in \mathbb{Z}_N^2} z_2(k) \otimes E_{k_1, k_1} \otimes E_{k_2, k_2 + 1}
\]

\[
\eta \sim \sum_{k \in \mathbb{Z}_N^2} \eta(k) \otimes E_{k_1, k_1} \otimes E_{k_2, k_2}
\]

\[
\chi_{12} \sim \sum_{k \in \mathbb{Z}_N^2} \chi_{12}(k) \otimes E_{k_1 + 1, k_1} \otimes E_{k_2 + 1, k_2}
\]

\[
\psi_1 \sim \sum_{k \in \mathbb{Z}_N^2} \psi_1(k) \otimes E_{k_1, k_1 + 1} \otimes E_{k_2, k_2}
\]

\[
\psi_2 \sim \sum_{k \in \mathbb{Z}_N^2} \psi_2(k) \otimes E_{k_1, k_1} \otimes E_{k_2, k_2 + 1}
\]
Orbifolded action \( S_{\text{orb}} = \frac{1}{g^2} \text{Tr} \sum_{k} \left( \frac{1}{4} z_m(k)z_n(k + e_m) - z_n(k)z_m(k + e_n) \right)^2 \)

\[ + \frac{1}{8} \left( z_m(k)z_m(k) - z_m(k - e_m)z_m(k - e_m) \right)^2 \]

\[ + \psi_m(k) \left( \bar{z}_m(k) \eta(k) - \eta(k + e_m)\bar{z}_m(k) \right) \]

\[ - \frac{1}{2} \chi_{mn}(k) \left( z_m(k)\psi_n(k + e_n) - \psi_n(k)z_m(k + e_n) - (m \leftrightarrow n) \right) \]

Kinetic terms

We introduce kinetic terms and a lattice spacing by shifting

\[ z_m(k) \rightarrow \frac{1}{a} + z_m(k), \quad \bar{z}_m(k) \rightarrow \frac{1}{a} + \bar{z}_m(k), \quad a \in \mathbb{R}_+ \]

\[ \text{easy to see kinetic terms} \]

or equivalently, \( z_m(k) \) are regarded as link variables:

\[ z_m(k) \rightarrow \frac{1}{a} e^{az_m(k)} \equiv U_m(k), \quad \bar{z}_m(k) \rightarrow \frac{1}{a} e^{a\bar{z}_m(k)} \equiv \bar{U}_m(k) \]
Finally, we get the action:

\[
S_{\text{lat}}^{d=2} = \frac{1}{g^2} \text{Tr} \sum_k \left( \frac{1}{4} \left| \nabla_m^+ z_n(k) - \nabla_n^+ z_m(k) + z_m(k) z_n(k+e_m) - z_n(k) z_m(k+e_n) \right|^2 \\
+ \frac{1}{8} \left( \nabla_m^+ \tilde{z}_m(k) + z_m(k+e_m) \tilde{z}_m(k+e_m) - \tilde{z}_m(k) \tilde{z}_m(k) \right)^2 \\
+ \psi_m(k) \left( \nabla_m^+ \eta(k) - \tilde{z}_m(k) \eta(k) + \eta(k+e_m) \tilde{z}_m(k) \right) \\
+ \frac{1}{2} \chi_{mn}(k) \left( \nabla_m^+ \psi_n(k) + z_m(k) \psi_n(k+e_m) - \psi_n(k) z_m(k+e_n) - (m \leftrightarrow n) \right) \right)
\]

where

\[
\nabla_m^+ \phi(k) = \frac{1}{a} (\phi(k+e_m) - \phi(k)).
\]

2D $N=(2,2)$ SYM theory with the gauge group $U(N_c)$. 
Preserved Supersymmetry

Original matrix theory

\[ S_m = \frac{1}{g^2} \text{Tr} \left( -\frac{1}{4}[\nu_\alpha, \nu_\beta]^2 + i\bar{\psi}\sigma_\alpha[\nu_\alpha, \psi] \right) \]

SUSY

\[ \delta \nu_\alpha = -i\bar{\psi}\sigma_\alpha \xi + i\xi\bar{\sigma}_\alpha \psi, \]

\[ \delta \psi = -i\nu_{\alpha\beta}\sigma_{\alpha\beta}\xi, \]

\[ \delta \bar{\psi} = i\nu_{\alpha\beta}\bar{\xi}\bar{\sigma}_{\alpha\beta}, \]

with

\[ \xi = \begin{pmatrix} \hat{\kappa}_{12} \\ \hat{\kappa} \end{pmatrix}, \quad \bar{\xi} = (\hat{\kappa}_1, \hat{\kappa}_2) \]

Recall

\[ \psi = \begin{pmatrix} \chi_{12} \\ \eta \end{pmatrix}, \quad \bar{\psi} = (\psi_1, \psi_2) \]
The variation of the action is zero when the SUSY parameters are c-numbers:

$$\kappa, \kappa_{12}, \kappa_1, \kappa_2 \propto \frac{1}{N_c N^2}$$

Projection of the supersymmetry

The supersymmetry parameters have definite U(1) charges:

$$\begin{pmatrix} \kappa & \kappa_{12} & \kappa_1 & \kappa_2 \\ (r_1, r_2) & 0 & -e_1 - e_2 & e_1 & e_2 \end{pmatrix}$$

They are projected out by orbifolding.

The only preserved supersymmetry is the one corresponding to $\kappa$.

$$\begin{align*}
\delta z_m &= 2i\tilde{\kappa}_m \psi_m + 2i\tilde{\kappa}_m \eta, \\
\delta \bar{z}_m &= -2i\tilde{\kappa}_m \psi_m - 2i\tilde{\kappa}_m \chi_m, \\
\delta \eta &= \frac{i}{2} \tilde{\kappa} \left[ z_m, \bar{z}_m \right] + \frac{i}{2} \tilde{\kappa}_{mn} \left[ z_m, z_n \right], \\
\delta \chi_{12} &= -i\tilde{\kappa} \left[ \bar{z}_1, \bar{z}_2 \right] - \frac{i}{2} \tilde{\kappa}_{12} \left[ z_m, \bar{z}_m \right], \\
\delta \psi_m &= i\tilde{\kappa}_n \left( \left[ z_m, \bar{z}_n \right] - \frac{1}{2} \delta_{mn} \left[ z_l, \bar{z}_l \right] \right),
\end{align*}$$

$$\delta \Phi \equiv 2i\tilde{\kappa} Q \Phi$$

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Q-invariant expression of the lattice action

\[ S_{\text{lat}}^{d=2} = \frac{1}{g^2} \text{Tr} \sum_k \left( \frac{1}{4} \left| \nabla_m^+ z_n(k) - \nabla_n^+ z_m(k) + z_m(k) z_n(k + e_m) - z_n(k) z_m(k + e_n) \right|^2 \\
+ \frac{1}{8} \left( \nabla_m^+ (z_m(k) + \bar{z}_m(k)) + z_m(k + e_m) \bar{z}_m(k + e_m) - \bar{z}_m(k) z_m(k) \right)^2 \\
+ \psi_m(k) \left( \nabla_m^+ \eta(k) - \bar{z}_m(k) \eta(k) + \eta(k + e_m) \bar{z}_m(k) \right) \\
+ \frac{1}{2} \chi_{mn}(k) \left( \nabla_m^+ \psi_n(k) + z_m(k) \psi_n(k + e_m) - \psi_n(k) z_m(k + e_m) - (m \leftrightarrow n) \right) \right) \]

\[ = \frac{1}{g^2} \text{Tr} \sum_k Q \left( \eta(k) \left( \nabla_m^-(z_m(k) + \bar{z}_m(k)) \\
+ z_m(k) \bar{z}_m(k) - \bar{z}_m(k - e_m) z_m(k - e_m) \right) \\
+ \chi_{mn}(k) \left( \nabla_n^+ \bar{z}_m(k) - \nabla_m^+ \bar{z}_n(k) \\
+ \bar{z}_m(k + e_n) \bar{z}_n(k) - \bar{z}_n(k + e_m) \bar{z}_m(k) \right) \right) \]

Q-invariance is manifest since Q is nilpotent: Q^2 = 0.
List of constructed lattice theories by orbifolding

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<thead>
<tr>
<th>Mother Theory (Matrix Theory)</th>
<th>Lattice Gauge Theory</th>
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<tbody>
<tr>
<td>4D N=1 SYM (4 SUSY)</td>
<td>2D N=(2,2) SYM</td>
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<tr>
<td>6D N=1 SYM (8 SUSY)</td>
<td>2D N=(4,4) SYM</td>
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<td>3D SYM with 8 SUSY</td>
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<tr>
<td>10D N=1 SYM (IIB matrix model) (16 SUSY)</td>
<td>BFSS matrix theory (1D)</td>
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<td>2D N=(8,8) SYM</td>
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<tr>
<td></td>
<td>3D SYM with 16 SUSY</td>
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<td>4D N=4 SYM</td>
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§ 3 Relation between Geometrical Discretization and Orbifolding

Catterall’s discretization rules (review) S.Catterall (2004)

Starting with a BRST invariant continuum theory with conditions:

1) kinetic terms are written by complex differential derivatives:

\[ \mathcal{D}_\mu = \partial_\mu + i(A_\mu + iB_\mu) = \partial_\mu + A_\mu \]
\[ \bar{\mathcal{D}}_\mu = \partial_\mu - i(A_\mu - iB_\mu) = \partial_\mu + \bar{A}_\mu \]

2) all the fields (including fermions) are in p-forms:

\[ S = S[\mathcal{D}_\mu, \bar{\mathcal{D}}_\mu, f_{\mu_1 \cdots \mu_p}] \]
Prescription to construct lattice action by Catterall

1) Complex covariant derivatives are mapped to link variables:

\[ \mathcal{D}_\mu \rightarrow \mathcal{U}_\mu(k) \]

\[ \bar{\mathcal{D}}_\mu \rightarrow \bar{\mathcal{U}}_\mu(k) \]

2) p-form field is mapped to a variable on a p-cell:

\[ f_{\mu_1 \ldots \mu_p}(x) \rightarrow f_{\mu_1 \ldots \mu_p}(k) \]

3) Curl-like differential is mapped to a forward covariant difference:

\[ \mathcal{D}_\mu f_\nu(x) \rightarrow \mathcal{U}_\mu(k) f_\nu(k + \hat{\mu}) - f_\nu(k) \mathcal{U}_\mu(k + \hat{\nu}) \]

4) Divergent-like differential is mapped to a backward covariant difference:

\[ \bar{\mathcal{D}}_\mu f_\mu(x) \rightarrow f_\mu(k) \bar{\mathcal{U}}_\mu(k) - \bar{\mathcal{U}}_\mu(k - \hat{\mu}) f_\mu(k - \hat{\mu}) \]
Claim

This prescription is automatically reproduced by orbifolding.


Additional condition

1) all the fields (including fermions) are in p-forms

\[ S = S[D_{\mu}, \overline{D}_{\mu}, f_{\mu_1 \cdots \mu_p}] \]

2) kinetic terms are written by complex differential derivatives

\[ D_{\mu} = \partial_{\mu} + i(A_{\mu} + iB_{\mu}) = \partial_{\mu} + A_{\mu} \]
\[ \overline{D}_{\mu} = \partial_{\mu} - i(A_{\mu} - iB_{\mu}) = \partial_{\mu} + \overline{A}_{\mu} \]

3) the theory has \( U(1)^d \) symmetries with charge assignment:

\[ D_{\mu} : \hat{\mu} = (0, \cdots, 1, \cdots, 0) \]
\[ \overline{D}_{\mu} : -\hat{\mu} = (0, \cdots, -1, \cdots, 0) \]
\[ f_{\mu_1 \cdots \mu_p} : \pm(\hat{\mu}_1 + \cdots + \hat{\mu}_p) \]
1) dimensionally reduce the continuum theory to 0-dim

We get the action of a **mother theory** (matrix theory)

\[ S = S[\mathcal{A}_\mu, \bar{\mathcal{A}}_\mu, f_{\mu_1 \ldots \mu_p}^\pm] \]

2) Using the **U(1) charge**, we carry out the **orbifold projection**:

\[ \mathcal{A}_\mu = \sum \mathcal{A}_\mu(k) \otimes E_{k,k+\hat{\mu}} \]

\[ \bar{\mathcal{A}}_\mu = \sum \bar{\mathcal{A}}_\mu(k) \otimes E_{k+\hat{\mu},k} \]

\[ f_{\mu_1 \ldots \mu_p}^+ = \sum f_{\mu_1 \ldots \mu_p}(k) \otimes E_{k,k+\hat{\mu}_1 + \ldots + \hat{\mu}_p} \]

\[ f_{\mu_1 \ldots \mu_p}^- = \sum f_{\mu_1 \ldots \mu_p}(k) \otimes E_{k+\hat{\mu}_1 + \ldots + \hat{\mu}_p,k} \]

---

**NOTE**

This is more than the prescription since we can decide the direction of the cell-variables automatically from the assignment of the U(1) charge.
Since the continuum theory is supposed to be Lorentz and U(1) invariant, possible derivative terms are in the form:

**curl-like differential**

\[
\mathcal{D}_\nu f_{\mu_1 \ldots \mu_p}^{\pm} \sim [A_\nu, f_{\mu_1 \ldots \mu_p}^{\pm}]
\]

\[
\mathcal{D}_\nu f_{\mu_1 \ldots \mu_p}^{\pm} \sim [\bar{A}_\nu, f_{\mu_1 \ldots \mu_p}^{\pm}]
\]

orbifolding & deconstruction

\[
\mathcal{D}_\nu f_{\mu_1 \ldots \mu_p}^{\pm} (x) \rightarrow \mathcal{U}_\nu (k) f_{\mu_1 \ldots \mu_p}^{\pm} (k + e_\nu) - f_{\mu_1 \ldots \mu_p}^{\pm} (k) \mathcal{U}_\nu (k + \mu),
\]

\[
\mathcal{D}_\nu f_{\mu_1 \ldots \mu_p}^{-} (x) \rightarrow \mathcal{U}_\nu (k + \mu) f_{\mu_1 \ldots \mu_p}^{-} (k + e_\nu) - f_{\mu_1 \ldots \mu_p}^{-} (k) \mathcal{U}_\nu (k),
\]

\[
\mathcal{D}_\nu f_{\mu_1 \ldots \mu_p}^{+} (x) \rightarrow f_{\mu_1 \ldots \mu_p}^{+} (k + e_\nu) \mathcal{U}_\nu (k + \mu) - \mathcal{U}_\nu (k) f_{\mu_1 \ldots \mu_p}^{+} (k),
\]

\[
\mathcal{D}_\nu f_{\mu_1 \ldots \mu_p}^{-} (x) \rightarrow f_{\mu_1 \ldots \mu_p}^{-} (k + e_\nu) \mathcal{U}_\nu (k) - \mathcal{U}_\nu (k + \mu) f_{\mu_1 \ldots \mu_p}^{-} (k),
\]

**Covariant Forward Difference**
We obtain 3) and 4) in the prescription

Catterall’s scheme to construct a lattice theory is a short-cut rule of orbifolding.
§ 4 SUSY in Link Approach from Orbifolding

Brief review of Link Approach

Continuum (topologically twisted) N=(2,2) SUSY algebra

\[ \{Q, Q_\mu\} = i \partial_\mu, \quad \{Q_{12}, Q_\mu\} = -i \epsilon_{\mu\nu} \partial_\nu \]

on lattice

\[ \{Q, Q_\mu\} = i \Delta^+_\mu \rightarrow z_\mu(k), \quad \{Q_{12}, Q_\mu\} = -i \epsilon_{\mu\nu} \Delta^-_\nu \rightarrow \epsilon_{\mu\nu} z_\nu(k) \]

with a modified Leibniz rule:

\[ Q_A(F(k)G(k)) = (Q_A F(k)) G(k) + (-1)^{|F|} F(k - a_A) (Q_A G(k)) , \]
\{Q, Q_1, Q_2, Q_{12}\} \text{ are supposed to live on links,}
(\text{k, k + a), (k, k + a}_1), (k, k + a_2), (k, k + a_{12})
\text{respectively and } a_A \text{ satisfy}
\begin{align*}
a + a_m &= e_m, \quad a_{12} + a_m = -|e_{mn}|e_n, \quad a + a_1 + a_2 + a_{12} = 0.
\end{align*}

A typical example
They showed that the algebra is satisfied by

$$\delta \Phi(k) = 2i\kappa (Q \Phi)(k) - 2i\kappa_{12} (Q_{12} \Phi)(k) + 2i\kappa_m (Q_m \Phi)(k)$$

with

$$\delta z_m(k) = 2i\kappa \psi_m(k) + 2i\kappa_m \eta(k),$$

$$\delta \bar{z}_m(k) = -2i\kappa_{mn} \psi_n(k - e_n) - 2i\kappa \chi_{mn}(k),$$

$$\delta \eta(k) = \frac{i}{2} \kappa \left( z_m(k) \bar{z}_m(k) - \bar{z}_m(k - e_m) z_m(k - e_m) \right)$$

$$+ i\kappa_{12} \left( z_1(k - e_1 - e_2) z_2(k - e_2) - z_2(k - e_1 - e_2) z_1(k - e_1) \right),$$

$$\delta \chi_{12}(k) = -i\kappa \left( \bar{z}_1(k + e_2) \bar{z}_2(k) - \bar{z}_2(k + e_1) \bar{z}_1(k) \right)$$

$$- \frac{i}{2} \kappa_{12} \left( z_m(k) \bar{z}_m(k) - \bar{z}_m(k - e_m) z_m(k - e_m) \right),$$

$$\delta \psi_m(k) = i\kappa_n \left( z_m(k + e_n) \bar{z}_n(k + e_m) - \bar{z}_n(k) z_m(k) \right)$$

$$- \frac{1}{2} \delta_{mn} \left( z_l(k) \bar{z}_l(k) - \bar{z}_l(k - e_l) z_l(k - e_l) \right).$$
and the lattice action,

\[ S = \frac{1}{g^2} \text{Tr} \sum_k \left( \frac{1}{4} |z_\mu(k)z_\nu(k + e_\mu) - z_\nu(k)z_\mu(k + e_\nu)|^2 \right. \\
+ \frac{1}{8} \left( z_\mu(k)\bar{z}_\mu(k) - \bar{z}_\mu(k - e_\mu)z_\mu(k - e_\mu) \right)^2 \\
+ \eta(k) \left( \bar{z}_\mu(k + a - e_\mu)\psi_\mu(k + a - e_\mu) - \psi_\mu(k + a)\bar{z}_\mu(k + a) \right) \\
- \frac{1}{2} \chi_{\mu\nu}(k) \left( z_\mu(k)\psi_\nu(k + e_\mu) - \psi_\nu(k)z_\mu(k + a_\nu) \right. \\
\left. \left. - z_\nu(k)\psi_\mu(k + e_\nu) + \psi_\mu(k)z_\nu(k + a_\mu) \right) \right), \]

satisfy

\[ Q_A S = 0 \]

almost the same with the orbifold action

The continuum limit is 2D N=(2,2) SYM theory.
Recall the U(1) charges of the fields in the mother theory with 4 SUSY

<table>
<thead>
<tr>
<th></th>
<th>( z_1 )</th>
<th>( z_2 )</th>
<th>( \eta )</th>
<th>( \chi_{12} )</th>
<th>( \psi_1 )</th>
<th>( \psi_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q_1 )</td>
<td>1</td>
<td>0</td>
<td>1/2</td>
<td>-1/2</td>
<td>1/2</td>
<td>-1/2</td>
</tr>
<tr>
<td>( q_2 )</td>
<td>0</td>
<td>1</td>
<td>1/2</td>
<td>-1/2</td>
<td>-1/2</td>
<td>1/2</td>
</tr>
<tr>
<td>( q_3 )</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td>1/2</td>
<td>-1/2</td>
<td>-1/2</td>
</tr>
</tbody>
</table>

We can carry out the orbifold projection using these U(1) charges.

\begin{align*}
    r_1 &= q_1 - m_1 q_3 \\
    r_2 &= q_2 - m_2 q_3
\end{align*}

\[ a + a_m = e_m, \]
\[ a_{12} + a_m = -|\epsilon_{mn}|e_n \]
\[ a + a_1 + a_2 + a_{12} = 0. \]

Coincide with the lattice action of Link Approach
Supersymmetry of this theory

The actions obtained by **Orbifolding** and **Link Approach** are identical.

- **Orbifolding**: Only one supercharge is preserved on a lattice.
- **Link Approach**: All the supersymmetry is preserved on a lattice in a modified sense.

\[
\text{Action of Lattice Theory} = \text{Action of Orbifolded Matrix Theory}
\]

from the viewpoint of matrix theory

- What is the “preserved” SUSY in the matrix theory sense?
- Are they really preserved?
Deformation of supersymmetry parameters

SUSY transformation of the mother theory (matrix theory):

\[ \delta z_m = 2i\hat{\kappa}_m \psi_m + 2i\hat{\kappa}_m \eta, \]
\[ \delta \bar{z}_m = -2i\hat{\kappa}_{mn} \psi_n - 2i\hat{\kappa}_n \chi_{mn}, \]
\[ \delta \eta = \frac{i}{2} \hat{\kappa}[z_m, \bar{z}_m] + \frac{i}{2} \hat{\kappa}_{mn}[z_m, z_n], \]
\[ \delta \chi_{12} = -i\hat{\kappa} [\bar{z}_1, \bar{z}_2] - \frac{i}{2} \hat{\kappa}_{12}[z_m, \bar{z}_m], \]
\[ \delta \psi_m = i\hat{\kappa}_n \left( [z_m, \bar{z}_n] - \frac{1}{2} \delta_{mn} [z_l, \bar{z}_l] \right), \]

RECALL

\[ \hat{\kappa}_A \] : anti-commuting c-numbers

with U(1) charges (a = 0)

<table>
<thead>
<tr>
<th>\hat{\kappa}</th>
<th>\hat{\kappa}_{12}</th>
<th>\hat{\kappa}_1</th>
<th>\hat{\kappa}_2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-e_1 - e_2</td>
<td>e_1</td>
<td>e_2</td>
</tr>
</tbody>
</table>

only SUSY corresponding to \( \hat{\kappa} \) survives.
In order that all $\widehat{\kappa}_A$ survive after orbifolding, we reinterpret them as matrices:

$$\widehat{\kappa}_A \equiv \kappa_A \otimes V_A$$

with

$$\widehat{\kappa} = \kappa \otimes \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$$

$$\widehat{\kappa}_{12} = \kappa_{12} \otimes \begin{pmatrix} 0 & 1 & \cdots \\ \vdots & \ddots & \vdots \\ 1 & 0 & \cdots \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 & \cdots \\ \vdots & \ddots & \vdots \\ 1 & 0 & \cdots \end{pmatrix}$$

$$\widehat{\kappa}_1 = \kappa_1 \otimes \begin{pmatrix} 0 & 1 \\ \vdots & \ddots \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$$

$$\widehat{\kappa}_2 = \kappa_2 \otimes \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ \vdots & \ddots \\ 0 & 1 \end{pmatrix}$$

Substituting to the SUSY transformation of matrices orbifolding the same SUSY transformation given in Link Approach.
A Comment
~ supersymmetric Wilson loop ~

Supersymmetric Wilson loop of N=4 SYM

\[ W(C) = \frac{1}{N} \text{Tr} \mathcal{P} \left( i \int (A_\mu \dot{x}^\mu(s) + i \Phi_i \dot{y}^i(s)) \, ds \right), \quad \mu, \nu = 0, \ldots, 3, \quad i, j = 1, \ldots, 6. \]

This is half-BPS when

\[ \dot{x}_\mu^2 - \dot{y}_i^2 = 0. \]

One specific choice

\[ x^0 = s, \quad y^1 = \pm s, \quad \text{others} = 0. \]

\[ W = \frac{1}{N} \text{Tr} \mathcal{P} \left( i \int (A_0 \pm i \Phi_1) \, ds \right) = \frac{1}{N} \text{Tr} \mathcal{P} \left( i \int Z_1(\tilde{Z}_1) ds \right) \]

The combination of \( A_\mu \) and \( \Phi_i \) is not unnatural in this sense.

Is the supersymmetric Wilson loop essence of the SUSY lattice formulation?
Future Problems

• Numerical simulations
  ✓ recovering of the supersymmetries in the continuum limit
  ✓ comparison with exact results
  ✓ non-perturbative estimation of non-BPS operators

• Connection to the superstring theory
  ✓ relation to IIB matrix theory?
  ✓ D-brane interpretation?
  ✓ AdS/CFT correspondence?

• Matter theories in detail
  ✓ Why did the procedure work?
  ✓ other theories with matter
  ✓ higher-dimensional theory
  ✓ connection to string theory?
§ A Orbifold Lattice Gauge Theory with Matter

We start with the dimensional reduced theory of 4D N=2 SYM:

\[ S_m = \frac{1}{g^2} \text{Tr} \left( \frac{1}{4} [v_\alpha, v_\beta]^2 + \bar{\psi} \Sigma_\alpha [v_\alpha, \psi] \right) \quad (\alpha, \beta = 0, \ldots, 5) \]

\[ \Gamma_\alpha = \begin{pmatrix} 0 & \Sigma_\alpha \\ \Sigma_\alpha & 0 \end{pmatrix} : 6D \text{ gamma matrices} \]

\[ = \text{Tr} \left( \frac{1}{4} |[z_a, z_b]|^2 + \frac{1}{2} [z_a, z_a] D - \frac{1}{2} D^2 \right) \quad (a, b, c = 1, 2, 3) \]

\[ + \psi_a [\bar{z}_a, \eta] + \xi_{ab} [z_a, \psi_b] + \frac{1}{2} \chi_{abc} [\bar{z}_a, \xi_{bc}] \]

\[ \begin{cases} 
  z_a \equiv v_{2a-2} + iv_{2a-1} \\
  \psi^T \equiv (\eta, \xi_{23}, \xi_{31}, \xi_{12}), \\
  \bar{\psi} \equiv (-\psi_1, \chi_{123}, \psi_3, -\psi_2). 
\end{cases} \]
\[
\begin{align*}
\Phi & \equiv z_3, \quad \bar{\Phi} \equiv \bar{z}_3, \\
\bar{\eta} & \equiv \psi_3, \quad \bar{\psi}_m \equiv \xi_{m3}, \quad \bar{\xi}_{12} \equiv \chi_{123}
\end{align*}
\]

\[m, n = 1, 2\]

\[
= \text{Tr} \left( \frac{1}{4} |z_m, z_n|^2 + \frac{1}{2} ([z_m, z_m] + [\Phi, \bar{\Phi}])D - \frac{1}{2} D^2 + \frac{1}{4} |Z_m, \Phi|^2 \\
+ \eta [\bar{z}_m, \psi_m] + \frac{1}{2} \xi_{mn} ([z_m, \psi_n] - [z_n, \psi_m]) \\
+ \bar{\eta} [z_m, \bar{\psi}_m] + \frac{1}{2} \bar{\xi}_{mn} ([\bar{z}_m, \bar{\psi}_n] - [\bar{z}_n, \bar{\psi}_m]) \\
+ \bar{\eta} [\bar{\Phi}, \eta] - \bar{\psi}_m [\Phi, \psi_m] + \frac{1}{2} \bar{\xi}_{mn} [\bar{\Phi}, \xi_{mn}] \right)
\]

usual orbifolding procedure

continuum limit

\[
\begin{align*}
\begin{array}{cc}
\text{orbifold lattice theory} & \text{of 2D N=(4,4) SYM} \\
\rightarrow & \\
\begin{array}{c}
\overline{z}_m(k) & \rightarrow \phi_m(x) + iA_m(x), \\
\Phi(k) & \rightarrow \phi_3(x) + i\phi_4(x)
\end{array}
\end{array}
\end{align*}
\]
**Idea**

1) Let us assume the size of the matrices to be $(N_c + N_f)N^2$.

2) The orbifold projection is carried out by

$$\gamma_a : \Phi \rightarrow \omega^{r_a} \Omega_a \Phi \Omega_a^{-1}, \quad (a = 1, 2)$$

with

$$\begin{align*}
\Omega_1 &\equiv \left(1_{N_c} \otimes U \otimes 1_N\right) \oplus \left(1_{N_f} \otimes U \otimes 1_N\right), \\
\Omega_2 &\equiv \left(1_{N_c} \otimes 1_N \otimes U\right) \oplus \left(1_{N_f} \otimes 1_N \otimes U\right),
\end{align*}$$

$$U \equiv \begin{pmatrix} \omega^1 & \cdots \\ & \ddots & \vdots \\ && \omega^N \end{pmatrix}$$

3) We further project out blocks using $Z_2$ transformation,

$$s : \Phi \rightarrow \pm P \Phi P \quad P = \begin{pmatrix} 1_{N_cN^2} & 0 \\ 0 & -1_{N_fN^2} \end{pmatrix}$$

"parity" of the field
\[
\begin{align*}
\begin{cases}
z_m &= \left( \frac{z_m(k)}{\bar{z}_m(k)} \right) \\
\psi_m &= \left( \frac{\psi_m(k)}{\bar{\psi}_m(k)} \right)
\end{cases}
\quad \begin{cases}
\bar{z}_m &= \left( \frac{\bar{z}_m(k)}{\tilde{z}_m(k)} \right) \\
\bar{\psi}_m &= \left( \frac{\bar{\psi}_m(k)}{\tilde{\psi}_m(k)} \right)
\end{cases}
\quad \begin{cases}
D &= \left( \frac{d(k)}{\tilde{d}(k)} \right) \\
\xi_{12} &= \left( \frac{\lambda_{12}(k)}{\tilde{\lambda}_{12}(k)} \right)
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
\Phi &= \left( \frac{\phi(k)}{\bar{\phi}(k)} \right) \\
\bar{\eta} &= \left( \frac{\bar{\psi}(k)}{\bar{\psi}(k)} \right)
\end{cases}
\quad \begin{cases}
\bar{\Phi} &= \left( \frac{\bar{\phi}(k)}{\bar{\phi}(k)} \right) \\
\bar{\psi}_m &= \left( \frac{\bar{\psi}_m(k)}{\bar{\psi}_m(k)} \right)
\end{cases}
\quad \begin{cases}
\bar{\xi}_{12} &= \left( \frac{\bar{\psi}_{12}(k)}{\bar{\psi}_{12}(k)} \right)
\end{cases}
\end{align*}
\]

"heaven" lattice

\[
\tilde{z}_m(k), \tilde{z}_m(k), \tilde{\lambda}(k), \tilde{\lambda}_m(k), \tilde{\lambda}_{12}(k)
\]

real lattice

\[
z_m(k), \bar{z}_m(k), \lambda(k), \lambda_m(k), \lambda_{12}(k)
\]
Strategy

1) Substitute the matrices into the action of mother theory

2) Shift \( \{z_m(k), \bar{z}_m(k)\} \) as well as \( \{\hat{z}_m(k), \bar{\hat{z}}_m(k)\} \) by \( 1/a \):

   \[
   z_m(k) \rightarrow 1/a + z_m(k) \quad \hat{z}_m(k) \rightarrow 1/a + \hat{z}_m(k) \\
   \bar{z}_m(k) \rightarrow 1/a + \bar{z}_m(k) \quad \bar{\hat{z}}_m(k) \rightarrow 1/a + \bar{\hat{z}}_m(k)
   \]

3) Fix the fields with “hat” to be zero with keeping Q-symmetry:

   \[
   \hat{z}_m(k) = \hat{\bar{z}}_m(k) = \hat{d}(k) = 0, \\
   \hat{\lambda}(k) = \hat{\lambda}_m(k) = \hat{\lambda}_{12}(k) = 0
   \]
\[ S_{\text{matter}} = \]
\[-\frac{1}{2} \bar{\phi}(k)[(\nabla_{m}^{-} + z_{m}(k))(\nabla_{m}^{+} - \bar{z}_{m}(k)) + (\nabla_{m}^{+} - \bar{z}_{m}(k - \hat{m}))(\nabla_{m}^{-} + z_{m}(k - \hat{m}))] \phi(k) \]
\[-\frac{1}{2} \bar{\phi}(k)[(\nabla_{m}^{-} + z_{m}(k))(\nabla_{m}^{+} - \bar{z}_{m}(k)) + (\nabla_{m}^{+} - \bar{z}_{m}(k - \hat{m}))(\nabla_{m}^{-} + z_{m}(k - \hat{m}))] \bar{\phi}(k) \]
\[+ \frac{1}{2} \text{Tr} \left( \phi(k) \bar{\phi}(k) - \bar{\phi}(k) \bar{\phi}(k) \right)^{2} \]
\[+ \bar{\psi}(k)(\nabla_{m}^{-} + z_{m}(k)) \psi_{m}(k + \hat{m}) \]
\[+ \bar{\psi}(k)(\nabla_{m}^{-} + z_{m}(k)) \bar{\psi}_{m}(k + \hat{m}) \]
\[ - \frac{1}{2} \bar{\psi}_{mn}(k + \hat{m} + \hat{n}) [(\nabla_{m}^{+} - \bar{z}_{m}(k + \hat{n})) \psi_{m}(k + \hat{n}) - (\nabla_{n}^{+} - \bar{z}_{n}(k + \hat{m})) \psi_{m}(k + \hat{m})] \]
\[ - \frac{1}{2} \left[ \bar{\psi}_{n}(k + \hat{m} + \hat{n}) (\nabla_{m}^{+} - \bar{z}_{m}(k)) \bar{\psi}_{mn}(k) - \bar{\psi}_{m}(k + \hat{n}) (\nabla_{n}^{+} - \bar{z}_{n}(k)) \bar{\psi}_{mn}(k)] \right] \]
\[+ \sqrt{2}i \left( \bar{\psi}(k) \lambda(k) \phi(k) - \bar{\phi}(k) \lambda(k) \bar{\psi}(k) - \bar{\psi}_{m}(k) \lambda_{m}(k) \lambda_{m}(k) \bar{\psi}(k + \hat{m}) + \bar{\phi}(k) \lambda_{m}(k) \psi_{m}(k + \hat{m}) \right) \]
\[+ \frac{1}{2} \bar{\psi}_{mn}(k + \hat{m} + \hat{n}) \lambda_{mn}(k) \phi(k) - \frac{1}{2} \bar{\psi}(k + \hat{m} + \hat{n}) \lambda_{mn}(k) \bar{\psi}_{mn}(k) \right) \]
properties

- The continuum limit gives the matter action of two-dimensional N=(2,2) Theory:

\[ \mathcal{L}_{4D} = \int d^2\theta d^2\bar{\theta} \left( \bar{\Phi} e^{2V} \Phi + \bar{\Phi} e^{-2V} \bar{\Phi} \right) \]

- Q-symmetry is preserved.

- There is no fermion/boson doubler.

\[ \left[ \text{kinetic term: } \phi \nabla_+ \nabla_- \phi \right] \]

- Chiral symmetry is explicitly broken.