Relation among Supersymmetric Lattice Gauge Theories

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Krakow, Poland

based on the works
arXiv:0704.2696
arXiv:0706.3007
arXiv:0708.4129
arXiv:0801.2936
arXiv:0805:4491
with P.H.Damgaard

Introduction

Supersymmetric Gauge Theory

- Supersymmetry seems a fundamental symmetry of space-time.

  (an extension of the translational symmetry $x \to (x, \theta)$)

- Supersymmetry seems to be necessary to unify the interactions.

- Exact results in quantum field theory.

  (Seiberg-Witten theory, Nekrasov's formula, Dijkgraaf-Vafa etc...)

- Gauge/Gravity duality (AdS/CFT Correspondence)

- Connection to superstring theory

➡ We want a way to analyze SUSY gauge theory non-perturbatively. ➡ lattice?

Difficulty

It seems impossible to construct a SUSY invariant theory on a lattice.

SUSY invariant action in continuum space-time

Suppose an action is written as

$$S = \int dx d\theta F(\Phi(x,\theta)) \qquad \Phi(x,\theta) \text{ ; superfield}$$

Essentially, a SUSY generator can be represented as

$$\delta\Phi = \epsilon Q \Phi \qquad Q = \partial_\theta + \theta\Gamma\partial_x$$

Variation of the action

$$\begin{aligned} \delta_\epsilon S &= \int dx d\theta F(\Phi + \epsilon Q\Phi) - F(\Phi) \\ &= \int dx d\theta \, \epsilon Q F(\Phi) \\ &= \int dx d\theta \, \epsilon \left(\partial_\theta + \theta\Gamma\partial_x\right) F(\Phi) = 0 \end{aligned}$$

Leibniz rule

continuum theory

differential operator

$$\partial_\mu$$

Leibniz rule

$$\partial_\mu \left(f(x)g(x)\right) = \left(\partial_\mu f(x)\right) g(x) + f(x) \left(\partial_\mu g(x)\right)$$

lattice theory

difference operator

$$\Delta_\mu f(\mathbf{n}) = f(\mathbf{n} + \hat{\mu}) - f(\mathbf{n})$$

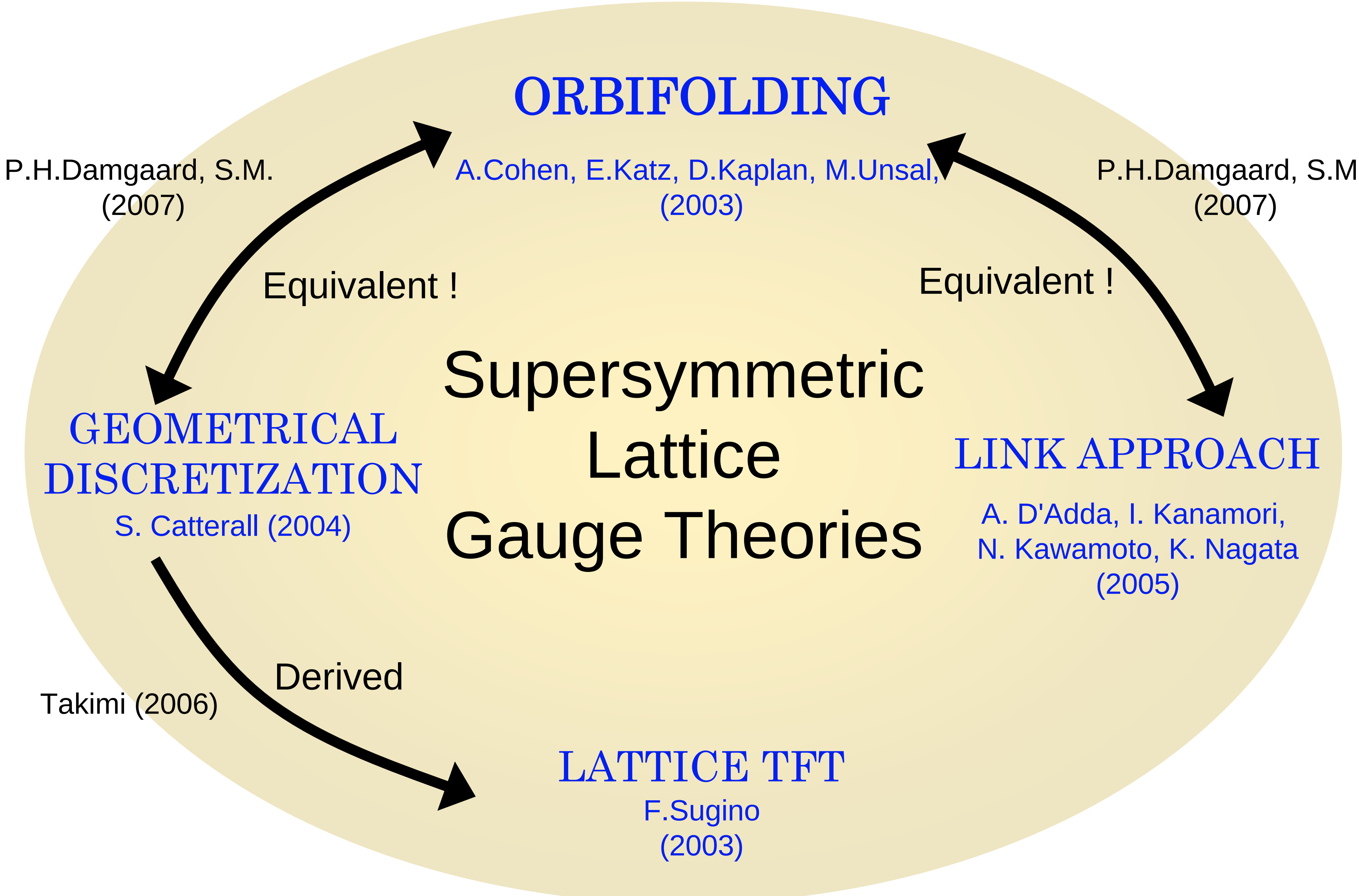
deformed Leibniz rule

$$\Delta_\mu \left(f(n)g(n)\right) = f(n+\hat{\mu})g(n+\hat{\mu}) - f(n)g(n) = \left(\Delta_\mu f(n)\right) g(n) + f(n+\hat{\mu}) \left(\Delta_\mu g(n)\right)$$

It seems impossible to keep all SUSY on a lattice.

Can we keep a part of SUSY on a lattice?

Yes!



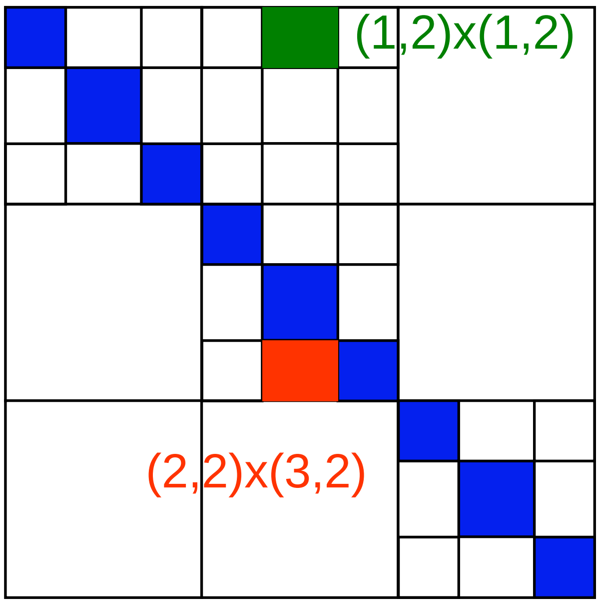
ORBIFOLDING
A.Cohen, E.Katz, D.Kaplan, M.Unsal,
(2003)
P.H.Damgaard, S.M.
(2007)
Equivalent !
P.H.Damgaard, S.M.
(2007)
Equivalent !
Supersymmetric
Lattice
Gauge Theories
GEOMETRICAL
DISCRETIZATION
S. Catterall (2004)
LINK APPROACH
A. D'Adda, I. Kanamori,
N. Kawamoto, K. Nagata
(2005)
Takimi (2006)
Derived
LATTICE TFT
F.Sugino
(2003)

Contents

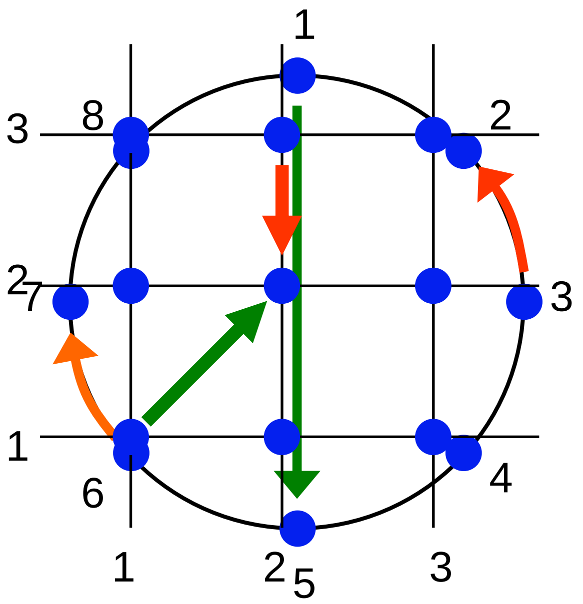
§ 2 Review of Orbifold Lattice Theory

Basic Idea ~ matrix as a collection of lattice fields ~

MATRIX



LATTICE



Strategy

with keeping SUSY

1. Starting with a matrix theory (mother theory)
2. Project out "non-local" elements properly
3. We interpret the projected matrix theory as a lattice theory.

Construction of 2D N=(2,2) SYM on lattice

Mother theory

dimensional reduction of 4D *N*=1 SYM theory with a gauge group .

$$S_{\rm m} = \frac{1}{g^2}{\rm Tr}\left(-\frac{1}{4}[v_\alpha, v_\beta]^2 + i\bar{\psi}\bar{\sigma}_\alpha[v_\alpha, \psi]\right) \qquad \alpha, \beta = 0, \cdots, 3$$

v_α : four hermitian matrices (gauge boson)

$\psi,\ \bar{\psi}$: 2 component spinors (gaugino)

Symmetries

maximal U(1) subgroup

1) global symmetry $SO(4) \times U(1)_{\rm R} \supset U(1)_1 \times U(1)_2 \times U(1)_R$

2) gauge symmetry $v_\alpha \to g v_\alpha g^{-1}, \quad g \in U(N_c N^2)$

Equivalent expression in which the U(1) symmetries are manifest:

$$S_{\mathrm{m}} = \frac{1}{g^2}\mathrm{Tr}\left(\frac{1}{4}|[z_m, z_n]|^2 + \frac{1}{8}[z_m, \bar{z}_m]^2 + \psi_m[\bar{z}_m, \eta] - \chi_{mn}[z_m, \psi_n]\right)$$

$$\boxed{\begin{aligned} z_1 &\equiv v_1 + iv_2, \\ z_2 &\equiv v_0 + iv_3, \end{aligned} \qquad \psi = \begin{pmatrix} \chi_{12} \\ \eta \end{pmatrix} \qquad \bar{\psi} = (\psi_1, \psi_2)}$$

U(1) charges

We can take any linear combination.

	z_1	z_2	η	χ_{12}	ψ_1	ψ_2
q_1	1	0	1/2	-1/2	1/2	-1/2
q_2	0	1	1/2	-1/2	-1/2	1/2
q_3	0	0	1/2	1/2	-1/2	-1/2
r_1	1	0	0	-1	1	0
r_2	0	1	0	-1	0	1

where

$$r_1 = q_1 - q_3, \qquad r_2 = q_2 - q_3$$

Orbifold projection

Combining the U(1) and gauge symmetry,
we consider a Z_N^2 transformation generated by

$$\gamma_a^N = 1$$

$$\gamma_a : \Phi \to \omega^{r_a} \Omega_a \Phi \Omega_a^{-1}, \quad (a = 1, 2)$$

where $\omega = e^{2\pi i/N}$ and

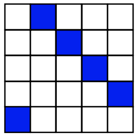
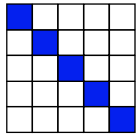
$$\begin{cases} \Omega_1 & \equiv 1_{N_c} \otimes U \otimes 1_N, \\ \Omega_2 & \equiv 1_{N_c} \otimes 1_N \otimes U, \end{cases} \qquad U \equiv \begin{pmatrix} \omega^1 & & \\ & \ddots & \\ & & \omega^N \end{pmatrix}$$: clock matrix

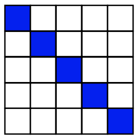
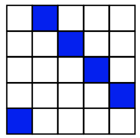
c.f.)

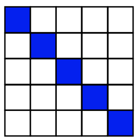
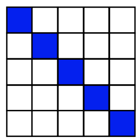
$$(U \Phi U^{-1})_{ij} = \omega^{i-j} \Phi_{ij}$$

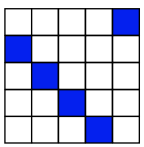
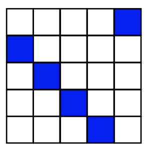
We keep only components
that are invariant under this transformation.

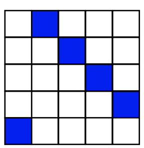
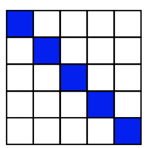
	z_1	z_2	η	χ_{12}	ψ_1	ψ_2
(r_1, r_2)	$(1,0)$	$(0,1)$	$(0,0)$	$(-1,-1)$	$(1,0)$	$(0,1)$

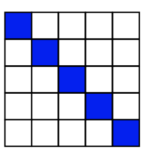
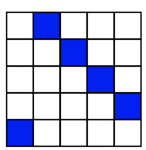
$Z_1 \sim$  $\otimes$  $= \sum_{\mathbf{k}\in\mathbb{Z}_N^2} z_1(\mathbf{k}) \otimes E_{k_1,k_1+1} \otimes E_{k_2,k_2}$

$Z_2 \sim$  $\otimes$  $= \sum z_2(\mathbf{k}) \otimes E_{k_1,k_1} \otimes E_{k_2,k_2+1}$

$\eta \sim$  $\otimes$  $= \sum \eta(\mathbf{k}) \otimes E_{k_1,k_1} \otimes E_{k_2,k_2}$

$\chi_{12} \sim$  $\otimes$  $= \sum \chi_{12}(\mathbf{k}) \otimes E_{k_1+1,k_1} \otimes E_{k_2+1,k_2}$

$\psi_1 \sim$  $\otimes$  $= \sum \psi_1(\mathbf{k}) \otimes E_{k_1,k_1+1} \otimes E_{k_2,k_2}$

$\psi_2 \sim$  $\otimes$  $= \sum \psi_2(\mathbf{k}) \otimes E_{k_1,k_1} \otimes E_{k_2,k_2+1}$

Orbifolded action

$$\mathbf{e}_1 = (1,0), \quad \mathbf{e}_2 = (0,1), \quad \mathbf{k} = (k_1, k_2) \in \mathbb{Z}_N^2$$

$$\begin{aligned} S_{\rm orb} =& \frac{1}{g^2} \mathrm{Tr} \sum_{\mathbf{k}} \Bigg(\frac{1}{4} \Big| z_m(\mathbf{k}) z_n(\mathbf{k}+\mathbf{e}_m) - z_n(\mathbf{k}) z_m(\mathbf{k}+\mathbf{e}_n) \Big|^2 \\ &+ \frac{1}{8} \Big(z_m(\mathbf{k}) \bar{z}_m(\mathbf{k}) - \bar{z}_m(\mathbf{k}-\mathbf{e}_m) z_m(\mathbf{k}-\mathbf{e}_m) \Big)^2 \\ &+ \psi_m(\mathbf{k}) \Big(\bar{z}_m(\mathbf{k}) \eta(\mathbf{k}) - \eta(\mathbf{k}+\mathbf{e}_m) \bar{z}_m(\mathbf{k}) \Big) \\ &- \frac{1}{2} \chi_{mn}(\mathbf{k}) \Big(z_m(\mathbf{k}) \psi_n(\mathbf{k}+\mathbf{e}_n) - \psi_n(\mathbf{k}) z_m(\mathbf{k}+\mathbf{e}_n) - (m \leftrightarrow n) \Big) \Bigg) \end{aligned}$$

Kinetic terms

We introduce kinetic terms and a lattice spacing by shifting

$$z_m(\mathbf{k}) \to \frac{1}{a} + z_m(\mathbf{k}), \quad \bar{z}_m(\mathbf{k}) \to \frac{1}{a} + \bar{z}_m(\mathbf{k}), \quad a \in \mathbb{R}_+$$

$\Rightarrow$ easy to see kinetic terms

or equivalently, $z_m(\mathbf{k})$ are regarded as link variables:

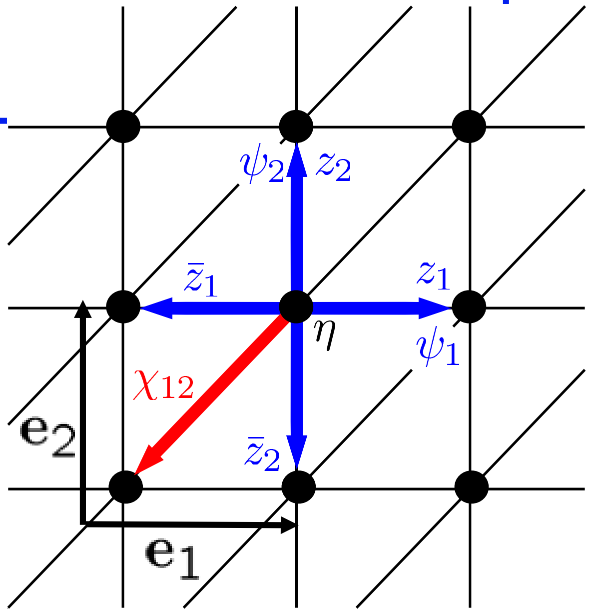
$$z_m(\mathbf{k}) \to \frac{1}{a} e^{a z_m(\mathbf{k})} \equiv U_m(\mathbf{k}), \;\; \bar{z}_m(\mathbf{k}) \to \frac{1}{a} e^{a \bar{z}_m(\mathbf{k})} \equiv \bar{U}_m(\mathbf{k})$$

Finally, we get the action:

$$
\begin{aligned}
S^{d=2}_{\mathrm{lat}} = \frac{1}{g^2}\mathrm{Tr}\sum_{\mathbf{k}}\Bigg(& \frac{1}{4}\Big|\nabla^+_m z_n(\mathbf{k}) - \nabla^+_n z_m(\mathbf{k}) + z_m(\mathbf{k})z_n(\mathbf{k}+\mathbf{e}_m) - z_n(\mathbf{k})z_m(\mathbf{k}+\mathbf{e}_n)\Big|^2 \\
& + \frac{1}{8}\Big(\nabla^+_m\Big(z_m(\mathbf{k}) + \bar{z}_m(\mathbf{k})\Big) + z_m(\mathbf{k}+\mathbf{e}_m)\bar{z}_m(\mathbf{k}+\mathbf{e}_m) - \bar{z}_m(\mathbf{k})z_m(\mathbf{k})\Big)^2 \\
& + \psi_m(\mathbf{k})\Big(\nabla^+_m\eta(\mathbf{k}) - \bar{z}_m(\mathbf{k})\eta(\mathbf{k}) + \eta(\mathbf{k}+\mathbf{e}_m)\bar{z}_m(\mathbf{k})\Big) \\
& + \frac{1}{2}\chi_{mn}(\mathbf{k})\Big(\nabla^+_m\psi_n(\mathbf{k}) + z_m(\mathbf{k})\psi_n(\mathbf{k}+\mathbf{e}_m) - \psi_n(\mathbf{k})z_m(\mathbf{k}+\mathbf{e}_n) - (m\leftrightarrow n)\Big)\Bigg)
\end{aligned}
$$

where

$$
\nabla^+_m\phi(\mathbf{k}) = \frac{1}{a}\left(\phi(\mathbf{k}+\mathbf{e}_m) - \phi(\mathbf{k})\right).
$$



continuum theory

2D *N*=(2,2) SYM theory with the gauge group $U(N_c)$.

Preserved Supersymmetry

Original matrix theory

$$S_{\mathrm{m}} = \frac{1}{g^2}\mathrm{Tr}\left(-\frac{1}{4}[v_\alpha, v_\beta]^2 + i\bar{\psi}\bar{\sigma}_\alpha[v_\alpha, \psi]\right)$$

SUSY

$$\begin{aligned} \delta v_\alpha &= -i\bar{\psi}\bar{\sigma}_\alpha\xi + i\bar{\xi}\bar{\sigma}_\alpha\psi, \\ \delta\psi &= -iv_{\alpha\beta}\sigma_{\alpha\beta}\xi, \\ \delta\bar{\psi} &= iv_{\alpha\beta}\bar{\xi}\bar{\sigma}_{\alpha\beta}, \end{aligned}$$

with

$$\xi = \begin{pmatrix} \widehat{\kappa}_{12} \\ \widehat{\kappa} \end{pmatrix}, \ \ \bar{\xi} = (\widehat{\kappa}_1, \widehat{\kappa}_2)$$

Recall

$$\psi = \begin{pmatrix} \chi_{12} \\ \eta \end{pmatrix} \ \ \bar{\psi} = (\psi_1, \psi_2)$$

The variation of the action is zero when the SUSY parameters are c-numbes;

$$\kappa, \kappa_{12}, \kappa_1, \kappa_2 \propto 1_{N_c N^2}$$

Projection of the supersymmetry

The supersymmetry parameters have definite U(1) charges:

	κ	κ_{12}	κ_1	κ_2
(r_1, r_2)	0	$-\mathbf{e}_1 - \mathbf{e}_2$	$\mathbf{e}_1$	$\mathbf{e}_2$

They are projected out by orbifolding.

The only preserved supersymmetry is the one corresponding to κ.

$$\begin{aligned} \delta z_m &= 2i\hat{\kappa}\psi_m + 2i\hat{\kappa}_m \eta, \\ \delta \bar{z}_m &= -2i\hat{\kappa}_{mn}\psi_n - 2i\hat{\kappa}_n \chi_{mn}, \\ \delta\eta &= \frac{i}{2}\hat{\kappa}[z_m, \bar{z}_m] + \frac{i}{2}\hat{\kappa}_{mn}[z_m, z_n], \\ \delta\chi_{12} &= -i\hat{\kappa}[\bar{z}_1, \bar{z}_2] - \frac{i}{2}\hat{\kappa}_{12}[z_m, \bar{z}_m], \\ \delta\psi_m &= i\hat{\kappa}_n \left([z_m, \bar{z}_n] - \frac{1}{2}\delta_{mn}[z_l, \bar{z}_l] \right), \end{aligned}$$

$$\Rightarrow \quad \delta\Phi \equiv 2i\hat{\kappa}Q\Phi$$

Q-invariant expression of the lattice action

$$
\begin{aligned}
S_{\mathrm{lat}}^{d=2} = & \frac{1}{g^2}\mathrm{Tr}\sum_{\mathbf{k}}\Bigg(\frac{1}{4}\Big|\nabla_m^+ z_n(\mathbf{k}) - \nabla_n^+ z_m(\mathbf{k}) + z_m(\mathbf{k})z_n(\mathbf{k}+\mathbf{e}_m) - z_n(\mathbf{k})z_m(\mathbf{k}+\mathbf{e}_n)\Big|^2 \\
& + \frac{1}{8}\Big(\nabla_m^+\Big(z_m(\mathbf{k}) + \bar{z}_m(\mathbf{k})\Big) + z_m(\mathbf{k}+\mathbf{e}_m)\bar{z}_m(\mathbf{k}+\mathbf{e}_m) - \bar{z}_m(\mathbf{k})z_m(\mathbf{k})\Big)^2 \\
& + \psi_m(\mathbf{k})\Big(\nabla_m^+\eta(\mathbf{k}) - \bar{z}_m(\mathbf{k})\eta(\mathbf{k}) + \eta(\mathbf{k}+\mathbf{e}_m)\bar{z}_m(\mathbf{k})\Big) \\
& + \frac{1}{2}\chi_{mn}(\mathbf{k})\Big(\nabla_m^+\psi_n(\mathbf{k}) + z_m(\mathbf{k})\psi_n(\mathbf{k}+\mathbf{e}_m) - \psi_n(\mathbf{k})z_m(\mathbf{k}+\mathbf{e}_n) - (m\leftrightarrow n)\Big)\Bigg) \\
= & \frac{1}{g^2}\mathrm{Tr}\sum_{\mathbf{k}} Q\Big(\eta(\mathbf{k})\Big(\nabla_m^-\left(z_m(\mathbf{k}) + \bar{z}_m(\mathbf{k})\right) \\
& \qquad + z_m(\mathbf{k})\bar{z}_m(\mathbf{k}) - \bar{z}_m(\mathbf{k}-\mathbf{e}_m)z_m(\mathbf{k}-\mathbf{e}_m)\Big) \\
& \quad + \chi_{mn}(\mathbf{k})\Big(\nabla_n^+\bar{z}_m(\mathbf{k}) - \nabla_m^+\bar{z}_n(\mathbf{k}) \\
& \qquad + \bar{z}_m(\mathbf{k}+\mathbf{e}_n)\bar{z}_n(\mathbf{k}) - \bar{z}_n(\mathbf{k}+\mathbf{e}_m)\bar{z}_m(\mathbf{k})\Big)\Big)
\end{aligned}
$$

Q-invariance is manifest since Q is nilpotent: Q^2=0.

List of constructed lattice theories by orbifolding

mother theory (matrix theory)		lattice gauge theory
4D N=1 SYM (4 SUSY)	→	➢ 2D N=(2,2) SYM
6D N=1 SYM (8 SUSY)	→	➢ 2D N=(4,4) SYM ➢ 3D SYM with 8 SUSY
10D N=1 SYM (IIB matrix model) (16 SUSY)	→	➢ BFSS matrix theory (1D) ➢ 2D N=(8,8) SYM ➢ 3D SYM with 16 SUSY ➢ 4D N=4 SYM

§ 3 Relation between Geometrical Discretization and Orbifolding

P.H.Damgaard and S.M. (2007)
P.H.Damgaard and S.M. (2008)

Catterall's discretization rules (review) S.Catterall (2004)

Starting with a BRST invariant **continuum** theory with conditions:

1) kinetic terms are written by complex differential derivatives:

$$\mathcal{D}_\mu = \partial_\mu + i(A_\mu + iB_\mu) = \partial_\mu + \mathcal{A}_\mu$$
$$\bar{\mathcal{D}}_\mu = \partial_\mu - i(A_\mu - iB_\mu) = \partial_\mu + \bar{\mathcal{A}}_\mu$$

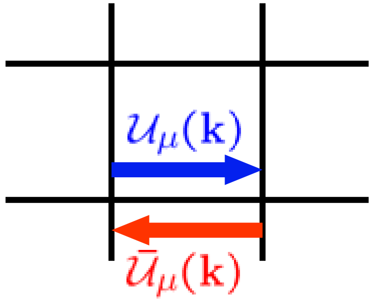
2) all the fields (including fermions) are in p-forms:

$$S = S[\mathcal{D}_\mu,\, \bar{\mathcal{D}}_\mu,\, f_{\mu_1 \cdots \mu_p}]$$

Prescription to construct lattice action by Catterall

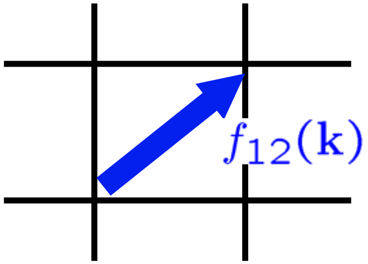
1) Complex covariant derivatives are mapped to link variables:

$$\mathcal{D}_\mu \to \mathcal{U}_\mu(\mathbf{k})$$
$$\bar{\mathcal{D}}_\mu \to \bar{\mathcal{U}}_\mu(\mathbf{k})$$



2) p-form field is mapped to a variable on a p-cell:

$$f_{\mu_1\cdots\mu_p}(x) \to f_{\mu_1\cdots\mu_p}(\mathbf{k})$$



3) Curl-like differential is mapped to a forward covariant difference:

$$\mathcal{D}_\mu f_\nu(x) \to \mathcal{U}_\mu(\mathbf{k}) f_\nu(\mathbf{k}+\widehat{\mu}) - f_\nu(\mathbf{k})\mathcal{U}_\mu(\mathbf{k}+\widehat{\nu})$$

4) Divergent-like differential is mapped to a backward covariant difference:

$$\bar{\mathcal{D}}_\mu f_\mu(x) \to f_\mu(\mathbf{k})\bar{\mathcal{U}}_\mu(\mathbf{k}) - \bar{\mathcal{U}}_\mu(\mathbf{k}-\widehat{\mu}) f_\mu(\mathbf{k}-\widehat{\mu})$$

Claim

This prescription is automatically reproduced by orbifolding.

P.H.Damgaard and S.M. (2008)

Additional condition

1) all the fields (including fermions) are in p-forms

$$S = S[\mathcal{D}_\mu,\ \bar{\mathcal{D}}_\mu,\ f_{\mu_1 \cdots \mu_p}]$$

2) kinetic terms are written by complex differential derivatives

$$\mathcal{D}_\mu = \partial_\mu + i(A_\mu + iB_\mu) = \partial_\mu + \mathcal{A}_\mu$$
$$\bar{\mathcal{D}}_\mu = \partial_\mu - i(A_\mu - iB_\mu) = \partial_\mu + \bar{\mathcal{A}}_\mu$$

3) the theory has U(1)^d symmetries with charge assignment:

$$\mathcal{D}_\mu : \ \ \widehat{\mu} = (0, \cdots, 1, \cdots, 0)$$
$$\bar{\mathcal{D}}_\mu : \ -\widehat{\mu} = (0, \cdots, -1, \cdots, 0)$$
$$f^{\pm}_{\mu_1 \cdots \mu_p} : \pm(\widehat{\mu}_1 + \cdots + \widehat{\mu}_p)$$

1) dimensionally reduce the continuum theory to 0-dim

We get the action of a mother theory (matrix theory)

$$S = S[\mathcal{A}_\mu,\ \bar{\mathcal{A}}_\mu,\ f^{\pm}_{\mu_1\cdots\mu_p}]$$

2) Using the U(1) charge, we carry out the orbifold projection:

$$\begin{aligned}
\mathcal{A}_\mu &= \sum \mathcal{A}_\mu(\mathbf{k}) \otimes E_{\mathbf{k},\mathbf{k}+\widehat{\mu}} \\
\bar{\mathcal{A}}_\mu &= \sum \bar{\mathcal{A}}_\mu(\mathbf{k}) \otimes E_{\mathbf{k}+\widehat{\mu},\mathbf{k}} \\
f^{+}_{\mu_1\cdots\mu_p} &= \sum f^{+}_{\mu_1\cdots\mu_p}(\mathbf{k}) \otimes E_{\mathbf{k},\mathbf{k}+\widehat{\mu}_1+\cdots+\widehat{\mu}_p} \\
f^{-}_{\mu_1\cdots\mu_p} &= \sum f^{-}_{\mu_1\cdots\mu_p}(\mathbf{k}) \otimes E_{\mathbf{k}+\widehat{\mu}_1+\cdots+\widehat{\mu}_p,\mathbf{k}}
\end{aligned}$$

$\Rightarrow$ We obtain 1) and 2) in the prescription

NOTE

This is more than the prescription since we can decide the direction of the cell-variables automatically from the assignment of the U(1) charge.

Since the continuum theory is supposed to be Lorentz and U(1) invariant, possible derivative terms are in the form:

curl-like differential

$$\mathcal{D}_\nu f^{\pm}_{\mu_1\cdots\mu_p} \sim [\mathcal{A}_\nu, f^{\pm}_{\mu_1\cdots\mu_p}]$$
$$\bar{\mathcal{D}}_\nu f^{\pm}_{\mu_1\cdots\mu_p} \sim [\bar{\mathcal{A}}_\nu, f^{\pm}_{\mu_1\cdots\mu_p}]$$

$$\Downarrow \quad \text{orbifolding \& deconstruction}$$

$$\begin{aligned}
\mathcal{D}_\nu f^{+}_{\mu_1\cdots\mu_p}(x) &\rightarrow \mathcal{U}_\nu(\mathbf{k}) f^{+}_{\mu_1\cdots\mu_p}(\mathbf{k}+\mathrm{e}_\nu) - f^{+}_{\mu_1\cdots\mu_p}(\mathbf{k})\mathcal{U}_\nu(\mathbf{k}+\boldsymbol{\mu}),\\
\mathcal{D}_\nu f^{-}_{\mu_1\cdots\mu_p}(x) &\rightarrow \mathcal{U}_\nu(\mathbf{k}+\boldsymbol{\mu}) f^{-}_{\mu_1\cdots\mu_p}(\mathbf{k}+\mathrm{e}_\nu) - f^{-}_{\mu_1\cdots\mu_p}(\mathbf{k})\mathcal{U}_\nu(\mathbf{k}),\\
\bar{\mathcal{D}}_\nu f^{+}_{\mu_1\cdots\mu_p}(x) &\rightarrow f^{+}_{\mu_1\cdots\mu_p}(\mathbf{k}+\mathrm{e}_\nu)\bar{\mathcal{U}}_\nu(\mathbf{k}+\boldsymbol{\mu}) - \bar{\mathcal{U}}_\nu(\mathbf{k}) f^{+}_{\mu_1\cdots\mu_p}(\mathbf{k}),\\
\bar{\mathcal{D}}_\nu f^{-}_{\mu_1\cdots\mu_p}(x) &\rightarrow f^{-}_{\mu_1\cdots\mu_p}(\mathbf{k}+\mathrm{e}_\nu)\bar{\mathcal{U}}_\nu(\mathbf{k}) - \bar{\mathcal{U}}_\nu(\mathbf{k}+\boldsymbol{\mu}) f^{-}_{\mu_1\cdots\mu_p}(\mathbf{k}),
\end{aligned}$$

Covariant Forward Difference

divergent-like differential

$$\mathcal{D}_{\mu} f^{-}_{\mu\nu_2\cdots\nu_p} \sim [\mathcal{A}_{\mu}, f^{-}_{\mu\nu_2\cdots\nu_p}]$$

$$\bar{\mathcal{D}}_{\mu} f^{+}_{\mu\nu_2\cdots\nu_p} \sim [\bar{\mathcal{A}}_{\mu}, f^{+}_{\mu\nu_2\cdots\nu_p}]$$

orbifolding & deconstruction

$$\mathcal{D}_{\mu_i} f^{-}_{\mu_1\cdots\mu_p}(x) \to \mathcal{U}_{\mu_i}(\mathbf{k}+\boldsymbol{\mu}-\widehat{\mu}_i) f^{-}_{\mu_1\cdots\mu_p}(\mathbf{k}) - f^{-}_{\mu_1\cdots\mu_p}(\mathbf{k}-\widehat{\mu}_i)\mathcal{U}_{\mu_i}(\mathbf{k}-\mathbf{e}_{\mu_i}),$$

$$\bar{\mathcal{D}}_{\mu_i} f^{+}_{\mu_1\cdots\mu_p}(x) \to f^{+}_{\mu_1\cdots\mu_p}(\mathbf{k})\bar{\mathcal{U}}_{\mu_i}(\mathbf{k}+\boldsymbol{\mu}-\widehat{\mu}_i) - \bar{\mathcal{U}}_{\mu_i}(\mathbf{k}-\widehat{\mu}_i) f^{+}_{\mu_1\cdots\mu_p}(\mathbf{k}-\widehat{\mu}_i),$$

$$(\boldsymbol{\mu} \equiv \widehat{\mu}_1 + \cdots + \widehat{\mu}_p)$$

Covariant Backward Difference

$\Rightarrow$ We obtain 3) and 4) in the prescription

Catterall's scheme to construct a lattice theory is a short-cut rule of orbifolding.

§ 4 SUSY in Link Approach from Orbifolding

P.H.Damgaard and S.M. (2007)

Brief review of Link Apploach

A. D'Adda, I. Kanamori, N. Kawamoto, K. Nagata (2005)

Continuum (topologically twisted) N=(2,2) SUSY algebra

$$\{Q, Q_\mu\} = i\partial_\mu, \ \{Q_{12}, Q_\mu\} = -i\epsilon_{\mu\nu}\partial_\nu$$

on lattice

requirement

$$\{Q, Q_\mu\} = i\Delta_\mu^+ \to \bar{z}_\mu(\mathbf{k}), \ \{Q_{12}, Q_\mu\} = -i\epsilon_{\mu\nu}\Delta_\nu^- \to \epsilon_{\mu\nu}z_\nu(\mathbf{k})$$

with a modified Leibniz rule:

$$\begin{aligned} &Q_A(F(\mathbf{k})G(\mathbf{k})) \\ &= (Q_A F(\mathbf{k}))\,G(\mathbf{k}) + (-1)^{|F|}F(\mathbf{k}-\mathbf{a}_A)\,(Q_A G(\mathbf{k}))\,, \end{aligned}$$

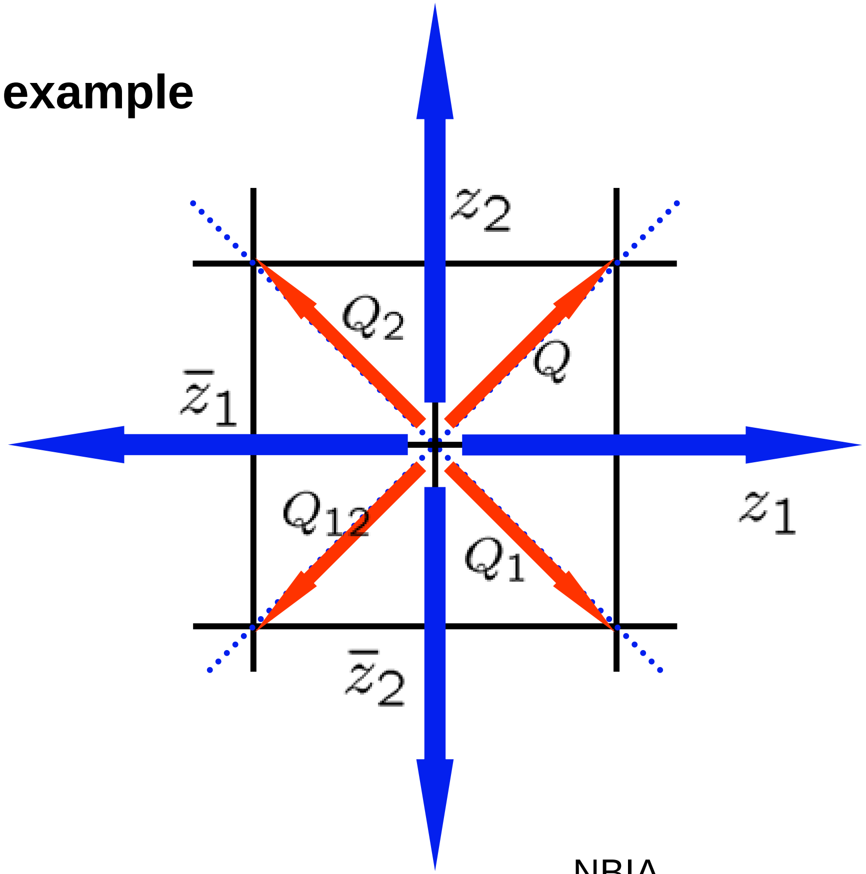
$\{Q, Q_1, Q_2, Q_{12}\}$ are supposed to live on links,

$$(\mathbf{k}, \mathbf{k}+\mathbf{a}), (\mathbf{k}, \mathbf{k}+\mathbf{a}_1), (\mathbf{k}, \mathbf{k}+\mathbf{a}_2), (\mathbf{k}, \mathbf{k}+\mathbf{a}_{12})$$

respectively and $\mathbf{a}_A$ satisfy

$$\mathbf{a}+\mathbf{a}_m = \mathbf{e}_m, \quad \mathbf{a}_{12}+\mathbf{a}_m = -|\epsilon_{mn}|\mathbf{e}_n, \quad \mathbf{a}+\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_{12} = 0.$$

A typical example



back

They showed that the algebra is satisfied by

$$\delta\Phi(\mathbf{k}) = 2i\kappa(Q\Phi)(\mathbf{k}) - 2i\kappa_{12}(Q_{12}\Phi)(\mathbf{k}) + 2i\kappa_m(Q_m\Phi)(\mathbf{k})$$

with

$$\begin{aligned}
\delta z_m(\mathbf{k}) &= 2i\kappa\psi_m(\mathbf{k}) + 2i\kappa_m\eta(\mathbf{k}),\\
\delta \bar{z}_m(\mathbf{k}) &= -2i\kappa_{mn}\psi_n(\mathbf{k}-\mathrm{e}_n) - 2i\kappa_n\chi_{mn}(\mathbf{k}),\\
\delta\eta(\mathbf{k}) &= \frac{i}{2}\kappa\Big(z_m(\mathbf{k})\bar{z}_m(\mathbf{k}) - \bar{z}_m(\mathbf{k}-\mathrm{e}_m)z_m(\mathbf{k}-\mathrm{e}_m)\Big)\\
&\quad + i\kappa_{12}\Big(z_1(\mathbf{k}-\mathrm{e}_1-\mathrm{e}_2)z_2(\mathbf{k}-\mathrm{e}_2) - z_2(\mathbf{k}-\mathrm{e}_1-\mathrm{e}_2)z_1(\mathbf{k}-\mathrm{e}_1)\Big),\\
\delta\chi_{12}(\mathbf{k}) &= -i\kappa\Big(\bar{z}_1(\mathbf{k}+\mathrm{e}_2)\bar{z}_2(\mathbf{k}) - \bar{z}_2(\mathbf{k}+\mathrm{e}_1)\bar{z}_1(\mathbf{k})\Big)\\
&\quad - \frac{i}{2}\kappa_{12}\Big(z_m(\mathbf{k})\bar{z}_m(\mathbf{k}) - \bar{z}_m(\mathbf{k}-\mathrm{e}_m)z_m(\mathbf{k}-\mathrm{e}_m)\Big),\\
\delta\psi_m(\mathbf{k}) &= i\kappa_n\Big(z_m(\mathbf{k}+\mathrm{e}_n)\bar{z}_n(\mathbf{k}+\mathrm{e}_m) - \bar{z}_n(\mathbf{k})z_m(\mathbf{k})\\
&\qquad - \frac{1}{2}\delta_{mn}\Big(z_l(\mathbf{k})\bar{z}_l(\mathbf{k}) - \bar{z}_l(\mathbf{k}-\mathrm{e}_l)z_l(\mathbf{k}-\mathrm{e}_l)\Big)\Big).
\end{aligned}$$

back

and the lattice action,

$$
\begin{aligned}
S =& \frac{1}{g^2}\mathrm{Tr}\sum_{\mathbf{k}}\Bigg(\frac{1}{4}\Big|z_\mu(\mathbf{k})z_\nu(\mathbf{k}+\mathbf{e}_\mu) - z_\nu(\mathbf{k})z_\mu(\mathbf{k}+\mathbf{e}_n)\Big|^2 \\
&+ \frac{1}{8}\Big(z_\mu(\mathbf{k})\bar{z}_\mu(\mathbf{k}) - \bar{z}_\mu(\mathbf{k}-\mathbf{e}_\mu)z_\mu(\mathbf{k}-\mathbf{e}_\mu)\Big)^2 \\
&+ \eta(\mathbf{k})\Big(\bar{z}_\mu(\mathbf{k}+\mathbf{a}-\mathbf{e}_\mu)\psi_\mu(\mathbf{k}+\mathbf{a}-\mathbf{e}_\mu) - \psi_\mu(\mathbf{k}+\mathbf{a})\bar{z}_\mu(\mathbf{k}+\mathbf{a})\Big) \\
&- \frac{1}{2}\chi_{\mu\nu}(\mathbf{k})\Big(z_\mu(\mathbf{k})\psi_n(\mathbf{k}+\mathbf{e}_\mu) - \psi_\nu(\mathbf{k})z_\mu(\mathbf{k}+\mathbf{a}_\nu) \\
&\qquad - z_\nu(\mathbf{k})\psi_\mu(\mathbf{k}+\mathbf{e}_\nu) + \psi_\mu(\mathbf{k})z_\nu(\mathbf{k}+\mathbf{a}_\mu)\Big)\Bigg),
\end{aligned}
$$

satisfy

$$Q_A S = 0$$

almost the same with the orbifold action

The continuum limit is 2D N=(2,2) SYM theory.

From Orbifolding to Link Apploach

Recall the U(1) charges of the fields in the mother theory with 4 SUSY

	z_1	z_2	η	χ_{12}	ψ_1	ψ_2
q_1	1	0	1/2	-1/2	1/2	-1/2
q_2	0	1	1/2	-1/2	-1/2	1/2
q_3	0	0	1/2	1/2	-1/2	-1/2
$\mathbf{r}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{a}$	$\mathbf{a}_{12}$	$\mathbf{a}_1$	$\mathbf{a}_2$

new combination $\begin{cases} r_1 = q_1 - m_1 q_3 \\ r_2 = q_2 - m_2 q_3 \end{cases}$

$$\boxed{\begin{aligned} &\mathbf{a} + \mathbf{a}_m = \mathbf{e}_m, \\ &\mathbf{a}_{12} + \mathbf{a}_m = -|\epsilon_{mn}|\mathbf{e}_n \\ &\mathbf{a} + \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_{12} = 0. \end{aligned}}$$

We can carry out the orbifold projection using these U(1) charges.

⬇

Coincide with the lattice action of Link Approach

Supersymmetry of this theory

The actions obtained by Orbifolding and Link Approach are identical.

but

?

$\Longleftrightarrow$

Orbifolding

Only one supercharge is preserved on a lattice

Link Approach

All the supersymmetry is preserved on a lattice in a modified sense.

Action of Lattice Theory = Action of Orbifolded Matrix Theory

$\Downarrow$ from the view point of matrix theory

- What is the “preserved” SUSY in the matrix theory sense?
- Are they really preserved?

Deformation of supersymmetry parameters

back

SUSY transformation of the mother theory (matrix theory):

$$
\begin{aligned}
\delta z_m &= 2i\widehat{\kappa}\psi_m + 2i\widehat{\kappa}_m\eta, \\
\delta \bar{z}_m &= -2i\widehat{\kappa}_{mn}\psi_n - 2i\widehat{\kappa}_n\chi_{mn}, \\
\delta\eta &= \frac{i}{2}\widehat{\kappa}[z_m, \bar{z}_m] + \frac{i}{2}\widehat{\kappa}_{mn}[z_m, z_n], \\
\delta\chi_{12} &= -i\widehat{\kappa}[\bar{z}_1, \bar{z}_2] - \frac{i}{2}\widehat{\kappa}_{12}[z_m, \bar{z}_m], \\
\delta\psi_m &= i\widehat{\kappa}_n\left([z_m, \bar{z}_n] - \frac{1}{2}\delta_{mn}[z_l, \bar{z}_l]\right),
\end{aligned}
$$

RECALL

$\widehat{\kappa}_A$: anti-commuting c-numbers

with U(1) charges ($\mathrm{a} = 0$)

$\widehat{\kappa}$	$\widehat{\kappa}_{12}$	$\widehat{\kappa}_1$	$\widehat{\kappa}_2$
0	$-\mathrm{e}_1 - \mathrm{e}_2$	e_1	e_2

orbifolding

only SUSY corresponding to $\widehat{\kappa}$ survives.

In order that all $\hat{\kappa}_A$ survive after orbifolding, we reinterpret them as matrices:

$$\hat{\kappa}_A \equiv \kappa_A \otimes V_A$$

with

$$\hat{\kappa} = \kappa \otimes \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \qquad \hat{\kappa}_{12} = \kappa_{12} \otimes \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$$

$$\hat{\kappa}_1 = \kappa_1 \otimes \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \qquad \hat{\kappa}_2 = \kappa_2 \otimes \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$

Substituting to
the SUSY transformation
of matrices

orbifolding

the same SUSY transformation given in Link Approach.

A Comment
~ supersymmetric Wilson loop ~

Supersymmetric Wilson loop of N=4 SYM

Drukker-Gross-Ooguri (1999)

$$W(C) = \frac{1}{N}\mathrm{Tr}\,\mathcal{P}\left(i\int\left(A_\mu \dot{x}^\mu(s) + i\Phi_i \dot{y}^i(s)\right)ds\right), \quad \begin{array}{l}\mu,\nu = 0,\cdots,3,\\ i,j = 1,\cdots,6.\end{array}$$

This is half-BPS when

$$\dot{x}_\mu^2 - \dot{y}_i^2 = 0.$$

One specific choice

$$x^0 = s, \quad y^1 = \pm s, \quad \text{others} = 0.$$

$$W = \frac{1}{N}\mathrm{Tr}\,\mathcal{P}\left(i\int(A_0 \pm i\Phi_1)\,ds\right) = \frac{1}{N}\mathrm{Tr}\,\mathcal{P}\left(i\int Z_1(\bar{Z}_1)ds\right)$$

The combination of A_μ and Φ_i is not unnatural in this sense.

Is the supersymmetric Wilson loop essence of the SUSY lattice formulation?

Future Problems

- Numerical simulations
  - ✓ recovering of the supersymmetries in the continuum limit
  - ✓ comparison with exact results
  - ✓ non-perturbative estimation of non-BPS operators
- Connection to the superstring theory
  - ✓ relation to IIB matrix theory?
  - ✓ D-brane interpretation?
  - ✓ AdS/CFT correspondence?
- Matter theories in detail
  - ✓ Why did the procedure work?
  - ✓ other theories with matter
  - ✓ higher-dimensional theory
  - ✓ connection to string theory?

§ A Orbifold Lattice Gauge Theory with Matter

S.M. 0805.4491

We start with the dimensional reduced theory of 4D N=2 SYM:

$$S_{\mathrm{m}} = \frac{1}{g^2}\mathrm{Tr}\left(-\frac{1}{4}[v_\alpha, v_\beta]^2 + \bar{\psi}\bar{\Sigma}_\alpha[v_\alpha, \psi]\right) \qquad (\alpha, \beta = 0, \cdots, 5)$$

$$\Gamma_\alpha = \begin{pmatrix} 0 & \Sigma_\alpha \\ \bar{\Sigma}_\alpha & 0 \end{pmatrix}$$: 6D gamma matrices

$$= \mathrm{Tr}\left(\frac{1}{4}|[z_a, z_b]|^2 + \frac{1}{2}[z_a, z_a]D - \frac{1}{2}D^2 \right. \qquad (a, b, c = 1, 2, 3)$$

$$\left. + \psi_a[\bar{z}_a, \eta] + \xi_{ab}[z_a, \psi_b] + \frac{1}{2}\chi_{abc}[\bar{z}_a, \xi_{bc}]\right)$$

$$\begin{cases} z_a \equiv v_{2a-2} + i v_{2a-1} \\ \psi^T \equiv (\eta, \xi_{23}, \xi_{31}, \xi_{12}), \\ \bar{\psi} \equiv (-\psi_1, \chi_{123}, \psi_3, -\psi_2). \end{cases}$$

$$\begin{bmatrix} \Phi \equiv z_3, & \bar{\Phi} \equiv \bar{z}_3, & \\ \bar{\eta} \equiv \psi_3, & \bar{\psi}_m \equiv \xi_{m3}, & \bar{\xi}_{12} \equiv \chi_{123} \end{bmatrix} \qquad (m, n = 1, 2)$$

$$= \mathrm{Tr}\left(\frac{1}{4}|[z_m, z_n]|^2 + \frac{1}{2}([z_m, z_m] + [\Phi, \bar{\Phi}])D - \frac{1}{2}D^2 \; + \frac{1}{4}|[Z_m, \Phi]|^2 \right.$$
$$+ \eta[\bar{z}_m, \psi_m] + \frac{1}{2}\xi_{mn}([z_m, \psi_n] - [z_n, \psi_m])$$
$$+ \bar{\eta}[z_m, \bar{\psi}_m] + \frac{1}{2}\bar{\xi}_{mn}([\bar{z}_m, \bar{\psi}_n] - [\bar{z}_n, \bar{\psi}_m])$$
$$\left. + \bar{\eta}[\bar{\Phi}, \eta] - \bar{\psi}_m[\Phi, \psi_m] + \frac{1}{2}\bar{\xi}_{mn}[\bar{\Phi}, \xi_{mn}] \right)$$

usual orbifolding procedure

orbifold lattice theory
of 2D N=(4,4) SYM

$\boxed{\text{continuum limit}}$

$$z_m(\mathbf{k}) \to \phi_m(x) + iA_m(x),$$
$$\Phi(\mathbf{k}) \to \phi_3(x) + i\phi_4(x)$$

Idea

1) Let us assume the size of the matrices to be $(N_c + N_f)N^2$.

$$\begin{array}{cc} & \begin{array}{cc} N_cN^2 & N_fN^2 \end{array} \\ \begin{array}{c} N_cN^2 \\ N_fN^2 \end{array} & \left(\begin{array}{ccc|ccc} \star & & & \star & & \\ & \star & & & \star & \\ & & \star & & & \star \\ \hline \star & & & \star & & \\ & \star & & & \star & \\ & & \star & & & \star \end{array}\right) \end{array}$$

parity even odd

2) The orbifold projection is carried out by

$$\gamma_a : \Phi \to \omega^{r_a} \Omega_a \Phi \Omega_a^{-1}, \ (a = 1, 2)$$

with

$$\begin{cases} \Omega_1 & \equiv \left(1_{N_c} \otimes U \otimes 1_N\right) \oplus \left(1_{N_f} \otimes U \otimes 1_N\right), \\ \Omega_2 & \equiv \left(1_{N_c} \otimes 1_N \otimes U\right) \oplus \left(1_{N_f} \otimes 1_N \otimes U\right), \end{cases} \qquad U \equiv \begin{pmatrix} \omega^1 & & \\ & \ddots & \\ & & \omega^N \end{pmatrix}$$

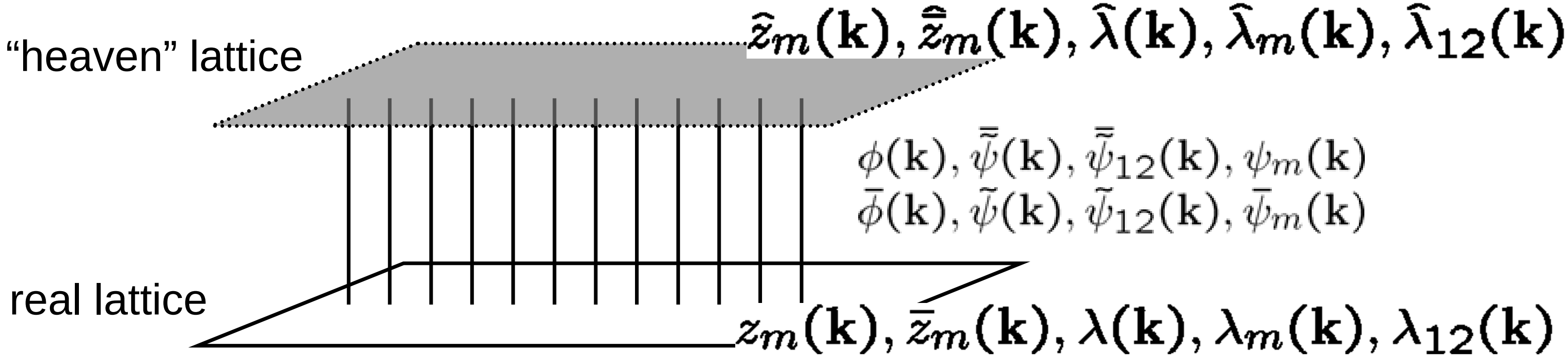
3) We further project out blocks using Z_2 transformation,

$$s : \Phi \to \pm P \Phi P \qquad P = \begin{pmatrix} \mathbf{1}_{N_cN^2} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1}_{N_fN^2} \end{pmatrix}$$

"parity" of the field

$$
\begin{cases}
z_m = \left(\begin{array}{c|c} z_m(\mathbf{k}) & \\ \hline & \hat{z}_m(\mathbf{k}) \end{array}\right) &
\bar{z}_m = \left(\begin{array}{c|c} \bar{z}_m(\mathbf{k}) & \\ \hline & \hat{\bar{z}}_m(\mathbf{k}) \end{array}\right) \qquad
D = \left(\begin{array}{c|c} d(\mathbf{k}) & \\ \hline & \hat{d}(\mathbf{k}) \end{array}\right) \\
\psi_m = \left(\begin{array}{c|c} \psi_m(\mathbf{k}) & \\ \hline & \hat{\psi}_m(\mathbf{k}) \end{array}\right) &
\eta = \left(\begin{array}{c|c} \lambda(\mathbf{k}) & \\ \hline & \hat{\lambda}(\mathbf{k}) \end{array}\right) \qquad
\xi_{12} = \left(\begin{array}{c|c} \lambda_{12}(\mathbf{k}) & \\ \hline & \hat{\lambda}_{12}(\mathbf{k}) \end{array}\right)
\end{cases}
$$

$$
\begin{cases}
\mathbf{\Phi} = \left(\begin{array}{c|c} & \phi(\mathbf{k}) \\ \hline \tilde{\phi}(\mathbf{k}) & \end{array}\right) &
\bar{\mathbf{\Phi}} = \left(\begin{array}{c|c} & \bar{\phi}(\mathbf{k}) \\ \hline \tilde{\bar{\phi}}(\mathbf{k}) & \end{array}\right) \\
\bar{\eta} = \left(\begin{array}{c|c} & \bar{\tilde{\psi}}(\mathbf{k}) \\ \hline \bar{\psi}(\mathbf{k}) & \end{array}\right) &
\bar{\psi}_m = \left(\begin{array}{c|c} & \psi_m(\mathbf{k}) \\ \hline \tilde{\psi}_m(\mathbf{k}) & \end{array}\right) \qquad
\bar{\xi}_{12} = \left(\begin{array}{c|c} & \bar{\tilde{\psi}}_{12}(\mathbf{k}) \\ \hline \bar{\psi}_{12}(\mathbf{k}) & \end{array}\right)
\end{cases}
$$



Strategy

1) Substitute the matrices into the action of mother theory

2) Shift $\{z_m(\mathbf{k}), \bar{z}_m(\mathbf{k})\}$ as well as $\{\hat{z}_m(\mathbf{k}), \bar{\hat{z}}_m(\mathbf{k})\}$ by $1/a$:

$$z_m(\mathbf{k}) \to 1/a + z_m(\mathbf{k}) \qquad \hat{z}_m(\mathbf{k}) \to 1/a + \hat{z}_m(\mathbf{k})$$
$$\bar{z}_m(\mathbf{k}) \to 1/a + \bar{z}_m(\mathbf{k}) \qquad \bar{\hat{z}}_m(\mathbf{k}) \to 1/a + \bar{\hat{z}}_m(\mathbf{k})$$

3) Fix the fields with “hat” to be zero with keeping Q-symmetry:

$$\hat{z}_m(\mathbf{k}) = \hat{\bar{z}}_m(\mathbf{k}) = \hat{d}(\mathbf{k}) = 0,$$
$$\hat{\lambda}(\mathbf{k}) = \hat{\lambda}_m(\mathbf{k}) = \hat{\lambda}_{12}(\mathbf{k}) = 0$$

$$
\begin{aligned}
&S_{\text{matter}} = \\
&-\frac{1}{2}\bar{\phi}(\mathbf{k})[(\nabla_m^- + z_m(\mathbf{k}))(\nabla_m^+ - \bar{z}_m(\mathbf{k})) + (\nabla_m^+ - \bar{z}_m(\mathbf{k}-\hat{m}))(\nabla_m^- + z_m(\mathbf{k}-\hat{m}))]\phi(\mathbf{k}) \\
&-\frac{1}{2}\tilde{\phi}(\mathbf{k})[(\nabla_m^- + z_m(\mathbf{k}))(\nabla_m^+ - \bar{z}_m(\mathbf{k})) + (\nabla_m^+ - \bar{z}_m(\mathbf{k}-\hat{m}))(\nabla_m^- + z_m(\mathbf{k}-\hat{m}))]\bar{\bar{\phi}}(\mathbf{k}) \\
&+\frac{1}{2}\mathrm{Tr}\,(\phi(\mathbf{k})\bar{\phi}(\mathbf{k}) - \bar{\bar{\phi}}(\mathbf{k})\tilde{\phi}(\mathbf{k}))^2 \\
&+\bar{\psi}(\mathbf{k})(\nabla_m^- + z_m(\mathbf{k}))\psi_m(\mathbf{k}+\hat{m}) \\
&+\tilde{\psi}(\mathbf{k})(\nabla_m^- + z_m(\mathbf{k}))\bar{\bar{\psi}}_m(\mathbf{k}+\hat{m}) \\
&-\frac{1}{2}\bar{\psi}_{mn}(\mathbf{k}+\hat{m}+\hat{n})\Big[(\nabla_m^+ - \bar{z}_m(\mathbf{k}+\hat{n}))\psi_n(\mathbf{k}+\hat{n}) - (\nabla_n^+ - \bar{z}_n(\mathbf{k}+\hat{m}))\psi_m(\mathbf{k}+\hat{m})\Big] \\
&-\frac{1}{2}\Big[\tilde{\psi}_n(\mathbf{k}+\hat{m})(\nabla_m^+ - \bar{z}_m(\mathbf{k}))\bar{\bar{\psi}}_{mn}(\mathbf{k}) - \tilde{\psi}_m(\mathbf{k}+\hat{n})(\nabla_n^+ - \bar{z}_n(\mathbf{k}))\bar{\bar{\psi}}_{mn}(\mathbf{k})\Big] \\
&+\sqrt{2}i\Big(\bar{\psi}(\mathbf{k})\lambda(\mathbf{k})\phi(\mathbf{k}) - \tilde{\phi}(\mathbf{k})\lambda(\mathbf{k})\bar{\bar{\psi}}(\mathbf{k}) - \tilde{\psi}_m(\mathbf{k})\lambda_m(\mathbf{k})\bar{\bar{\phi}}(\mathbf{k}+\hat{m}) + \bar{\phi}(\mathbf{k})\lambda_m(\mathbf{k})\psi_m(\mathbf{k}+\hat{m}) \\
&\qquad + \frac{1}{2}\bar{\psi}_{mn}(\mathbf{k}+\hat{m}+\hat{n})\lambda_{mn}(\mathbf{k})\phi(\mathbf{k}) - \frac{1}{2}\tilde{\phi}(\mathbf{k}+\hat{m}+\hat{n})\lambda_{mn}(\mathbf{k})\bar{\bar{\psi}}_{mn}(\mathbf{k})\Big)
\end{aligned}
$$

properties

- The continuum limit gives the matter action of two-dimensional N=(2,2) Theory:

dimensionally reduced theory of

$$\mathcal{L}_{4D} = \int d^2\theta d^2\bar{\theta}^2 \left(\bar{\Phi} e^{2V} \Phi + \tilde{\Phi} e^{-2V} \bar{\tilde{\Phi}} \right)$$

- Q-symmetry is preserved.

- There is no fermion/boson doubler.

(kinetic term: $\phi \nabla_+ \nabla_- \phi$)

- Chiral symmetry is explicitly broken.