Integrability and Finite Size Effects of $(\mathrm{AdS}_5/\mathrm{CFT}_4)_\beta$

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Generalization of AdS/CFT integrability

After success of the original example $\mathrm{AdS}_5/\mathrm{CFT}_4$ based on Integrability, the accumulated formalism and techniques could be applied to other models. Here, if the one parameter generalization of some known models exist and still preserve "Integrability", it would be great.

- Classical String sigma model Integrability
- Exact S-matrix, Asymptotic Bethe Ansatz
- **Finite-size effects using Integrability**

Generalization of AdS/CFT integrability

Undeformed Theory	Deformed Theory
$N = 4$ SYM	β-deformed SYM ($N = 1$ SCFT)
$AdS_5 \times S^5$	TsT-transformed $AdS_5 \times S^5$
Beisert-Staudacher BAE	Beisert-Roiban BAE
$SU(2\|2)^2$ S-matrix	Twisted S-matrix with TBC (†)

- † : Alternatively, one can take the $SU(2|2)^2$ S-matrix with operatorial TBC.

β-deformed Super Yang-Mills Theory

$$S = \frac{2}{{g_{YM}}^2} \int d^4x \left(\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} (D_\mu \Phi_i)^2 - \frac{1}{4} \left| \Phi_i \Phi_j - e^{i\pi\beta} \Phi_j \Phi_i \right|^2 + \ldots \right)$$

1. It is still 4d Conformal field theory.
2. Susy is broken to $N = 1$.
3. "R-symmetry" is broken to $U(1) \times U(1)$.

Lunin-Maldacena background

$$\begin{aligned}
\frac{ds^2}{R^2} &= ds^2_{AdS_5} + \sum_{i=1}^{3}\left(d\rho_i^2 + G\rho_i^2 d\phi_i^2\right) + \hat{\gamma}^2 G\rho_1^2\rho_2^2\rho_3^2[d(\sum_{i=1}^{3}\phi_i)]^2 \\
B_2 &= -\hat{\gamma}R^2 G\epsilon_{ijk}\rho_j^2\rho_k^2 d\phi_j \wedge d\phi_k\,, \\
C_2 &= \hat{\gamma}R^2 G\sin^4\alpha\sin\theta\cos\theta d\theta \wedge \left(d\phi_1 + d\phi_2 + d\phi_3\right), \\
F_5 &= -4R^4\left(\omega_{AdS_5} + G\omega_{S^5}\right)\,, \\
e^{2\Phi} &= G \\
G^{-1} &= 1 + \hat{\gamma}^2(\rho_1^2\rho_2^2 + \rho_2^2\rho_3^2 + \rho_3^2\rho_1^2),\ \hat{\gamma} = \beta\sqrt{\lambda}
\end{aligned}$$

1. It is obtained by TsT-transformation.
2. The Lax pair of string σ-model is preserved under TsT-transf.

Twisted S-matrix with the twisted boundary condition

By Ahn, Bajnok, Bombardelli, Nepomechie (2010),

$$\tilde{\mathcal{S}}(p_1, p_2) = F\,\mathcal{S}(p_1, p_2)\,F$$

where the twist matrix F is given by

$$F = e^{2\pi\beta i(h\otimes\mathbb{I}\otimes\mathbb{I}\otimes h - \mathbb{I}\otimes h\otimes h\otimes\mathbb{I})}$$

and $\mathcal{S}(p_1, p_2)$ is the $su(2|2)^2$ S-matrix. Here h is the diagonal matrix

$$h = \mathrm{diag}(\frac{1}{2}, -\frac{1}{2}, 0, 0)$$

Besides, The twisted boundary condition have to be compensated for the correct description of the β-deformed SYM theory.

$$M_{a\dot{a}} = \mathbb{I}\otimes e^{2\pi iJ\beta h}$$

Beisert-Roiban Bethe Ansatz Equations

$$e^{i(\mathbf{AK})_0}\, U_0 \;=\; 1,\;\; e^{i(\mathbf{AK})_j}\, U_j(x_{j,k}) \prod_{j'=1}^{7} \prod_{\substack{k'=1 \\ (j',k')\neq(j,k)}}^{K_{j'}} \frac{u_{j,k}-u_{j',k'}+\frac{i}{2}M_{j,j'}}{u_{j,k}-u_{j',k'}-\frac{i}{2}M_{j,j'}} = 1$$

$$\mathrm{AK} \;=\; \begin{pmatrix} 2\pi\beta(-K_4+K_5+K_7) \\ 0 \\ 0 \\ 2\pi\beta(-K_4+K_5+K_7) \\ 2\pi\beta(J+K_4+K_3-K_5-2K_7) \\ 2\pi\beta(-J-K_3+K_7) \\ 0 \\ (-J-K_3+K_4-K_5)\beta \end{pmatrix}, \qquad J = L - K_4$$

Weak coupling side

1. Beisert-Roiban BAE can be reproduced from the twisted S-matrix with the TBC.
2. The Lüscher's formula based on the exact S-matrix [Ahn, Bajnok, Bombardelli, Nepomechie (2010)] is matched with the perturbative gauge theory computation by [Fiamberti, Santambrogio, Sieg, Zanon, (2008)].

$$\Delta E_{\text{wrapping}}/g^8 = -54(1+\Delta)^3(-5+3\Delta)\zeta(3) - 360(1+\Delta)^2\zeta(5) + \frac{81\,(1-3\,\Delta)^2\,(1+\Delta)^4}{(1+3\,\Delta)^2}$$

Disagreement between String direct computations

Finite-size effects of giant magnon for the deformed theory is computed by [Bykov, Frolov (2008)].

$$
\begin{aligned}
E - J &= 2g \sin\frac{p}{2} - \frac{8}{e^2} g \sin^3\frac{p}{2} \cos\Phi e^{-\frac{J}{g \sin p/2}} + \dots \, , \\
\Phi &= \frac{2\pi(n_2 - \beta J)}{2^{3/2}\cos^3\frac{p}{4}}
\end{aligned}
$$

But, it was inconsistent with the result by [Ahn, Bozhilov (2011)] for $AdS_4 \times \mathbb{CP}^3$. Here, Φ is just $2\pi(n_2 - \beta J) = 2\pi\beta J$.

Lüscher's formula

[Lüscher (1986)], [Klassen, Melzer (1991)], [Ambjørn, Janik, Kristjansen (2006)], [Janik, Lukowski (2007)]
The virtual particle process moved along the finite length direction have to considered. In this situation, we call the μ-term if there exist boundstate made of virtual particles. Actually, the boundstate is related with the pole of S-matrix and this is just residue computation. If there are only scattering between virtual particles and physical particles, it is called F-term.

$$\delta E_F = -\int \frac{dq}{2\pi} \left[1 - \frac{\varepsilon'_Q(p)}{\varepsilon'_1(q^*)}\right] e^{-iq^* L} \sum_b (-1)^{F_b} \left(S^{ba}_{ba}(q^*, p) - 1\right)$$

The both Lüscher's formulas can be perturbatively computed and the most leading is μ-term's leading and the next is F-term's leading.

Reanalysis using the Lüscher's μ-term at strong coupling

$$\delta E^{\mu}_{(a\dot{a})_Q} = -i\left(1-\frac{\epsilon'_Q(p)}{\epsilon'_1(\tilde{q}_\star)}\right)e^{-i\tilde{q}_\star J}\sum_{b,\dot{b},b',\dot{b}'}(-1)^{F_b+F_{\dot{b}}} \times \operatorname*{Res}_{q=\tilde{q}}\left[M^{b\dot{b}}_{b'\dot{b}'}\tilde{\mathcal{S}}^{(b'\dot{b}')(a\dot{a})_Q}_{(b\dot{b})(a\dot{a})_Q}(q_\star(q),p)\right]$$

$$\begin{aligned} E-J &= 2g\sin\frac{p}{2}+\delta E^{\mu}_{(1\dot{1})_Q} \\ &= -\frac{8g\sin^3\frac{p}{2}}{\cosh\frac{\theta}{2}}\mathrm{Re}\left\{e^{2\pi i\beta J}\ \exp\left[-\frac{J+\epsilon_Q(p)}{g\sin\frac{p-i\theta}{2}}\right]\right\} \\ &= -\frac{8g}{e^2}\sin^3\frac{p}{2}\cos(2\pi\beta J)\exp\left[-\frac{J}{2g\sin\left(\frac{p}{2}\right)}\right] \end{aligned}$$

Quantum Finite-size effects from the Lüscher's $\mathbb{F}$-term

1. After studying the leading finite-size effects, one can also consider the leading quantum($\frac{1}{\sqrt{\lambda}}$) finite size effects.
2. This can be computed by the Lüscher's $\mathbb{F}$-term.

$$\delta E^F = -\int \frac{dq}{2\pi}\left(1-\frac{\epsilon'_Q(p)}{\epsilon'_1(q_\star)}\right)e^{-iq_\star J}\sum_{b,\dot{b},b',\dot{b}'}(-1)^{F_b+F_{\dot{b}}}\left[M^{b\dot{b}}_{b'\dot{b}'}\tilde{\mathcal{S}}^{(b'\dot{b}')(a\dot{a})_Q}_{(b\dot{b})(a\dot{a})_Q}\right]$$

Algebraic Curve Method

String Action can be written by coset. From this coset, we can construct the flat connection and the transfer matrix. The eigenvalues of the transfer matrix are called the quasi-momenta. The quasi-momenta define the Riemann sheets which have poles and cuts. The some string configuration mapped to the some algebraic curve in the Riemann sheets. In this situation, what is the semi-classics or quantum effects in this language?
In QFT, the quantum effects can be computed by field fluctuations. In the algebraic curve, this is to add the extra poles on the given curves. Although this looks little more complicated than usual field theory method, it is much more efficient and simpler.

Generic Twisted Algebraic Curve [Gromov, Vieira (2007)]

$$
\begin{aligned}
p_{\hat{1}}(x) &= \frac{\alpha x}{x^2-1} + \phi_{\hat{1}}, \;\; p_{\hat{2}}(x) = \frac{\alpha x}{x^2-1} + \phi_{\hat{2}} \\
p_{\hat{3}}(x) &= \frac{-\alpha x}{x^2-1} + \phi_{\hat{3}}, \;\; p_{\hat{4}}(x) = \frac{-\alpha x}{x^2-1} + \phi_{\hat{4}} \\
p_{\tilde{1}}(x) &= \frac{\alpha x}{x^2-1} + i\log\left(\frac{1/x - X^+}{1/x - X^-}\right) + \phi_{\tilde{1}} \\
p_{\tilde{2}}(x) &= \frac{\alpha x}{x^2-1} - i\log\left(\frac{x - X^+}{x - X^-}\right) + \phi_{\tilde{2}} \\
p_{\tilde{3}}(x) &= \frac{-\alpha x}{x^2-1} + i\log\left(\frac{x - X^+}{x - X^-}\right) + \phi_{\tilde{3}} \\
p_{\tilde{4}}(x) &= \frac{-\alpha x}{x^2-1} - i\log\left(\frac{1/x - X^+}{1/x - X^-}\right) + \phi_{\tilde{4}}
\end{aligned}
$$

Quantum finite-size effects from twisted quasi-momenta

One can easily obtain the twist factors from comparison between Gromov-Vieira BAE and Beisert-Roiban BAE.

$$\phi_{\tilde{1}} = p/2 + \pi\beta Q\,; \quad \phi_{\tilde{2}} = -p/2 - \pi\beta Q$$
$$\phi_{\tilde{3}} = p/2 + \pi\beta(2L - 3Q)\,; \quad \phi_{\tilde{4}} = -p/2 - \pi\beta(2L - 3Q)$$

As in undeformed theory, the one-loop quantum effects are the summation over all fluctuation frequencies.

$$\delta\Delta_{\mathrm{one-loop}} = \frac{1}{2}\sum_{ij}\sum_{n}(-1)^{F_{ij}}\,\Omega_{ij}^{n} = \int \frac{dx}{2\pi i}\partial_x\Omega(x)\sum_{ij}(-1)^{F_{ij}}\,e^{-i\left(p_i - p_j\right)}$$

Also, the fluctuation frequencies are the same for all polarizations and also same with the undeformed's.

$$\Omega_{ij}(x) = \frac{2}{x^2 - 1}\left(1 - x\frac{X^+ + X^-}{X^+X^- + 1}\right)$$

Summation for all polarizations

Here, $\sum_{ij}(-1)^{F_{ij}}e^{-i(p_i-p_j)}$ is

$$\left(e^{i\pi\beta(2J-Q)}\frac{x-X^-}{x-X^+}\sqrt{\frac{X^+}{X^-}}+e^{-i\pi\beta(2J-Q)}\frac{xX^+-1}{xX^--1}\sqrt{\frac{X^-}{X^+}}-2\right)$$
$$\times\quad\left(e^{i\pi\beta Q}\frac{x-X^-}{x-X^+}\sqrt{\frac{X^+}{X^-}}+e^{-i\pi\beta Q}\frac{xX^+-1}{xX^--1}\sqrt{\frac{X^-}{X^+}}-2\right)e^{-i\frac{2\alpha x}{x^2-1}}$$

where α is $\Delta/2g$.

The integration is nothing but the Lüscher's $\mathbb{F}$-term. One can compute the explicit form using the saddle point method ($x=i$).

Twisted Boundary conditions and the relevant background - In progress

By [Frolov(2005)], the IIB superstring on TsT-transformed $AdS_5 \times S^5$ is just the superstring on $AdS_5 \times S^5$ with twisted boundary condition. Frolov's TBC is the same with the operatorial twisted boundary conditions in the transfer matrix language. Here, the exact S-matrix is just Beisert's $SU(2|2)^2$ S-matrix. Alternatively, there was the twisted S-matrix with usual twisted boundary conditions (charge independent). This usual TBC is something like subset of the operatorial TBC. Also, this usual TBC in the worldsheet field sense is less twisted contrast with Frolov's TBC. When we only kept Frolov's TBC, the background became $AdS_5 \times S^5$. Therefore, if we consider less twisted TBC, the resulting background would be little deformed from $AdS_5 \times S^5$. As analyzed the $U(1)$ currents, we found such a background.

Near BMN limit and light-cone gauge fixing - In progress

The pp-wave limit give us the quadratic Lagrangian after gauge fixing. This is just free theory. For non-trivial scattering events, we have to consider the curvature correction - called the near BMN limit.
The obtained quartic interactions are the undeformed piece plus the deformed piece. The tree amplitudes from the deformed interaction can be easily computed.

Comparison with the twisted exact S-matrix - In progress

The strong coupling limit of the twisted S-matrix can be compared with the T-matrix from the quartic interaction.

$$\begin{aligned} F &= e^{\frac{2\pi i\hat{\gamma}}{\sqrt{\lambda}}} = (\mathbb{I} + 2\pi i\hat{\gamma}\frac{M}{\sqrt{\lambda}}) \\ \tilde{\mathcal{S}}_{strong} &= F\,\mathcal{S}_{strong}\,F = (\mathbb{I} + 2\pi i\hat{\gamma}\frac{M}{\sqrt{\lambda}})\,(1 + 2\pi i\frac{T}{\sqrt{\lambda}})\,(\mathbb{I} + 2\pi i\hat{\gamma}\frac{M}{\sqrt{\lambda}}) \\ &= 1 + 2\pi i\frac{(2M\gamma + T)}{\sqrt{\lambda}} \end{aligned}$$

It seems to be matched well.

Summary

- We obtained the leading finite-size effects for giant magnon from μ-term.
- It resolved a controversy over classical string computation.
- We also computed the leading quantum finite-size effects from both $\mathbb{F}$-term and algebraic curve.
- One can perturbatively compute WS scattering from the near BMN limit and it can be compared with the strong coupling limit of the conjectured exact S-matrix.

Future Works

- Non-supersymmetric three parameter deformation
- Deformed Konishi at strong coupling
- Deformation of Open integrability
- Many new nonperturbative results will appear soon.