A new $\mathcal{N} = (2, 2)$ CFT at $c = 3$

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Based on work in collaboration with Stefan Fredenhagen and Rui Sun

Vocabulary: CFT_2 are our concern

- CFT_2 enjoy 2d chiral symmetry. Virasoro, superVirasoro, W-algebras. Central charge c: $T(z)T(w) \sim \frac{c/2}{(z-w)^4}$
- unitary reps: conserved probability $\Rightarrow$ vectors must have strictly positive norm
- rational CFT: the algebra of the conserved charges is large enough to decompose the Hilbert space into a FINITE number of reps

(super(extended))Virasoro + unitarity $\Rightarrow$

- if $c < c_0$ only discrete series of allowed values: RATIONAL MINIMAL MODELS
- if $c > c_0$ DIFFICULT continuum of fields NON-RATIONAL LIOUVILLE/TODA

WHAT ABOUT $c = c_0$?

WE WANT TO ANSWER THIS QUESTION FOR $\mathcal{N} = 2$ MINIMAL MODELS

Why this limit of $\mathcal{N} = 2$ Minimal Models?

2 dimensional world

- this procedure is a tested way to produce likely consistent non-rational CFTs. Very short list of known non-free non-rational CFTs: $\mathcal{N} = 2$ non trivial generalisation of the known examples

3++ dimensional world

- AdS_3/CFT_2 holographic minimal models Gaberdiel,Gopakumar 10. Different limits of $\frac{\mathfrak{su}(N)_k \oplus \mathfrak{su}(N)_1}{\mathfrak{su}(N)_{k+1}}$: supersymmetric generalisation Creutzig et al 11 small 't Hooft coupling $\lambda = \frac{N}{N+k} \to 0$
- super AGT: hint on conformal blocks for super-Liouville?

Outline

Things we need to define a bulk theory

to define a CFT (to answer physical questions) we do not need a Lagrangian

BOOTSTRAP APPROACH: symmetries constrain the theory

Bulk CFT data

- central charge
- spectrum
- 3pt correlators on the sphere

CONSISTENCY CONDITION: crossing symmetry ($\sim$ OPE associativity)

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- spectrum + modular invariance $\rightarrow$ torus partition fct
- 3-pt fct + state/operator correspondence $\rightarrow$ n-pt fct

In this talk $\mathcal{N} = 2$: some details on the data, no crossing symmetry.

[Fredenhagen,CR,Sun],[Fredenhagen,CR]

Known examples: Runkel-Watts and generalisations

- CFT_2 conformal covariance in 2dim: minimal series parametrised by positive integer parameter k. We are interested in $k \to \infty$, where the theory cease to be rational

Unitary representations, rational conformal field theories:

- $\mathcal{N} = 0 \to T(z) \to$ Virasoro $\to$ minimal models $c < 1$ Runkel,Watts 01
- $\mathcal{N} = 1 \to T(z), G(z) \to$ super-Virasoro $\to \mathcal{N} = 1$ minimal models $c < 3/2$
- W_N and SW_N: higher spin generalisations (technically very involved)

They are described by diagonal cosets $\frac{\mathfrak{su}(N)_k \oplus \mathfrak{su}(N)_l}{\mathfrak{su}(N)_{k+l}}$

algebra	coset	central charge	limit central ch.
$\mathcal{N} = 0$	$\frac{\mathfrak{su}(2)_k \oplus \mathfrak{su}(2)_1}{\mathfrak{su}(2)_{k+1}}$	$c = k\frac{k+5}{(k+2)(k+3)}$	$c \to 1$
$\mathcal{N} = 1$	$\frac{\mathfrak{su}(2)_k \oplus \mathfrak{su}(2)_2}{\mathfrak{su}(2)_{k+2}}$	$c = \frac{3}{2}k\frac{k+6}{(k+2)(k+4)}$	$c \to \frac{3}{2}$
$\mathcal{W}_N$	$\frac{\mathfrak{su}(N)_k \oplus \mathfrak{su}(N)_1}{\mathfrak{su}(N)_{k+1}}$	$c = (N-1)k\frac{(k+2N+1)}{(k+N)(k+N+1)}$	$c \to N-1$

In the $k \to \infty$ limit they are likely continuous orbifold Gaberdiel,Suchanek 11

$$\frac{\mathfrak{su}(N)_l}{SU(N)}$$

$\mathcal{N} = 2$ minimal models: spectrum and central charge

- The $\mathcal{N} = 2$ case is very different: the OPE between the two superpartners of T gives a $U(1)$ current $\Rightarrow$ we have $T(z), G^{\pm}(z), J(z)$
- The irreps are labeled by (h, q) eigenvalues of the zero modes of T and J
- It is useful a coset description (NON DIAGONAL) to organise repr

Coset construction for $\mathcal{N} = 2$ minimal models

$\frac{\hat{\mathfrak{su}}(2)_k \oplus \hat{\mathfrak{u}}(1)_4}{\hat{\mathfrak{u}}(1)_{2(k+2)}}$ central charge: $c = 3 - \frac{6}{k+2}$, $k \in \mathbb{Z}_{\geq 0}$

irreps: $\mathcal{H}_{(l,m,s)}$ $l = 0 \dots k$ $m \in \mathbb{Z}/2(k+2)$ $s \in \mathbb{Z}/4$

$$\begin{cases} h^s_{l,m} = \frac{l(l+2)-m^2}{4(k+2)} + \frac{s^2}{8} \\ q_{m,s} = -\frac{m}{k+2} + \frac{s}{2} \end{cases}$$

standard range: $|m - s| \leq l$

selection rules: $l + m + s \in 2\mathbb{Z}$

The central charge in the limit:

$$\boxed{k \to \infty \text{ gives } c \to 3}$$

$\mathcal{N} = 2$ Minimal Models spectrum

- the coset parameters are not physical though: the repr are labeled by h (eigenvalue of L_0) and q (eigenvalue of J_0)

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- the spectral parameter s labels different sectors (defined mod 4) ($s = 0, 2$ NS, $s = \pm 1$ R) connected by half integer spectral flow
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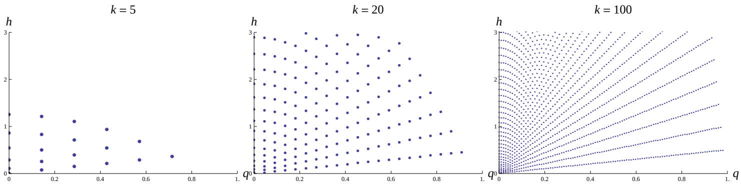
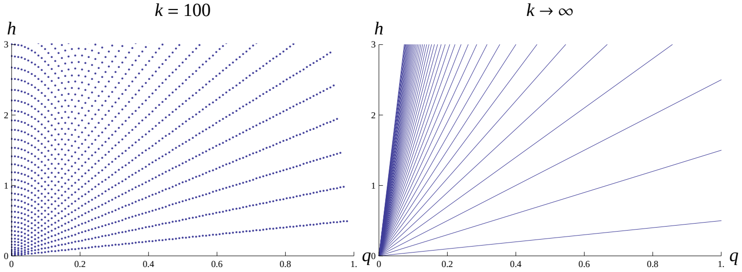


Spectrum of the limit theory



$$h^s_{l,m} = \frac{l(l+2)-m^2}{4(k+2)}$$
$$q_{m,s} = -\frac{m}{k+2}$$

- in the limit m must scale with k to keep q finite
- $l = |m| + \frac{2h}{|q|} + \mathcal{O}(\frac{1}{k+2})$ and $l \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z} \Rightarrow l = |m| + 2n$, with $n \in \mathbb{Z}_{\geq 0}$

Spectrum of the limit theory

$$(n, q) : \quad n \in \mathbb{Z}_{\geq 0} \,, \quad q \in (-1, 0) \cup (0, 1) \,; \quad h_n(q) = |q|(n + \tfrac{1}{2})$$

same limit discrete representations $\mathcal{N} = 2$ Liouville, like Schomerus 03 for $\mathcal{N} = 0$

Averaged fields and correlators

- Our spectrum is a continuum of primary fields labeled by n and with real $U(1)$ charge q
- To compute correlators we need then a notion of "smeared" fields with labels that approach asymptotically the continuous ones

$$\phi^{\epsilon,k}_{n,q} = \frac{1}{|N(q,k,\epsilon)|} \sum_{\substack{m\in N(q,k,\epsilon)\\ l=|m|+2n}} \phi_{l,m}$$

$N(q,k,\epsilon) := \left\{ m : \left| \frac{m}{k+2} + q \right| < \frac{\epsilon}{2} \right\}$ m such that the charge q_m is close to q

- define correlators:

$$\langle \phi_{n_1 q_1} \dots \phi_{n_r q_r} \rangle \equiv \lim_{\epsilon\to 0} \lim_{k\to\infty} \underbrace{\beta^2(k)}_{\text{vacuum}} \underbrace{\alpha^r(k)}_{\text{fields}} \underbrace{\langle \phi^{\epsilon,k}_{n_1,q_1} \dots \phi^{\epsilon,k}_{n_r,q_r} \rangle}_{\text{from MM}}$$

fix the two normalisations $\beta(k)$ and $\alpha(k)$ through 2pt and 3pt functions in the limit such that they are finite

Survey of the technicalities

- Impose limit of 2pt function of the minimal model is finite $\rightarrow$

$$\boxed{\alpha(k)\beta(k) = \sqrt{k+2}}$$

- the 3pt function is $\propto C(\{l_i, m_i\})$ with $m_1 + m_2 + m_3 = 0$ and

$$C(\{l_i, m_i\}) = \begin{pmatrix} \frac{l_1}{2} & \frac{l_2}{2} & \frac{l_3}{2} \\ \frac{m_1}{2} & \frac{m_2}{2} & \frac{m_3}{2} \end{pmatrix}^2 \sqrt{(l_1+1)(l_2+1)(l_3+1)} \underbrace{d_{l_1,l_2,l_3}}_{\text{product of } \Gamma\text{s}}$$

  - asymptotics of a bunch of Γ functions (see upcoming paper)
  - certain asymptotic expansion of 3j-Wigner symbols $\sim$ classical limit of addition of angular momenta in a specific regime: nice old problem!

$$\boxed{\alpha(k) = (k+2)^{-1/2} \quad , \quad \beta(k) = (k+2)}$$

Highlight of the computations: asymptotics of 3j-symbols

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \qquad \text{with} \quad l_i = m_i + n_i \qquad \text{large} \quad l, m$$

Q: can $\{l_i, m_i\}$ can be interpreted classically? What is the asymptotic limit otherwise?

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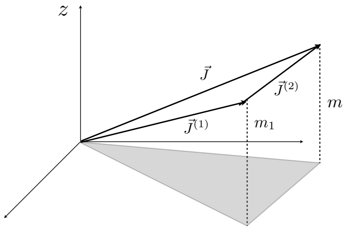

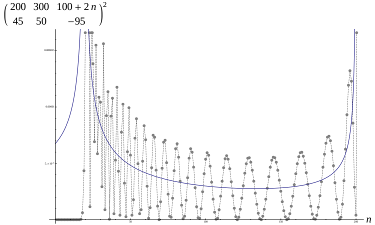

- Wigner: "allowed" region → solution "Wigner's estimate"
- our region between the allowed and forbidden → no use of the known approximations

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- complicated expression: in general not well behaved in "transition region"

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix} = (-1)^{j_1 - j_2 - \mu_3} \left[(j_1+j_2-j_3)!(j_1-j_2+j_3)!(-j_1+j_2+j_3)!/(j_1+j_2+j_3+1)!\right]^{1/2}$$
$$\times \left[(j_1+\mu_1)!(j_1-\mu_1)!(j_2+\mu_2)!(j_2-\mu_2)!(j_3+\mu_3)!(j_3-\mu_3)!\right]^{1/2}$$
$$\times \sum_z \frac{(-1)^z}{z!(j_1+j_2-j_3-z)!(j_1-\mu_1-z)!(j_2+\mu_2-z)!(j_3-j_2+\mu_1+z)!(j_3-j_1-\mu_2+z)!}$$

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- we are lucky (this is a good sign that the theory makes sense): in our specific range

$$\text{3j-symb} \propto {}_2F_1\left(-n_3, -n_2, n_1 - n_3 + 1; -\frac{|m_2|}{|m1|}\right)(1 + \mathcal{O}(1/k))$$

Limit of $\mathcal{N} = (2, 2)$ Minimal Models

We have everything we need to define the bulk theory
(missing consistency check... but Liouville gives us good hope (see next slide!))

- central charge

$$c = 3$$

- spectrum

$$h(q) = |q| \left(n + \frac{1}{2}\right) , \quad n \in \mathbb{Z}_{\geq 0} , \quad q \in (-1, 0) \cup (0, 1)$$

- 3pt function

$$\langle \phi_{q_1,n_1} \phi_{q_2,n_2} \phi_{q_3,n_3} \rangle = \delta(\textstyle\sum_i q_i) |z_{12}|^{2(h_3-h_1-h_2)} |z_{13}|^{2(h_2-h_1-h_3)} |z_{23}|^{2(h_1-h_2-h_3)}$$

$$\left(\frac{|q_1 q_2|}{|q_3|}\right)^{1/2} (d^J_{M',M}(\beta))^2 \left(\prod_{j=1}^{3} \frac{\Gamma(1+q_j)}{\Gamma(1-q_j)}\right)^{n_1+n_2-n_3+\frac{1}{2}}$$

$$\cos\beta = \frac{|m_1| - |m_2|}{|m_1| + |m_2|} , \; J = \frac{n_1+n_2}{2} , \; M' = -\frac{n_1+n_2}{2} + n_3 , \; M = \frac{n_1+n_2}{2} - n_2$$

d Wigner matrix expressable as hypergeometric functions

Additional material and ongoing work

- More data collected
  - ✓ characters for the spectrum
  - ✓ disk amplitudes
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- Tests from $\mathcal{N} = 2$ Liouville side Fredenhagen, Schomerus 04
  - ✓ spectrum of primaries in the discrete series
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- Continuous orbifold proposal ("physical" interpretation) Gaberdiel, Suchanek 11
  - √ q related to continuous twisting parameter
  - √ concrete proposal and checks

Questions, open problems, future perspectives

- Characteristics similar to Runkel-Watts (one more quantum label) but simpler correlators
- New playground to test "asymptotic fixed points" in space of theories Douglas 10. Perturbation by relevant operators $\rightarrow$ flows with fixed points. Compared to Virasoro MM non-decreasing Zamolodchikov metric
- Good example to understand new Continuous Orbifolds
- Classification of $c = 3$ theories?

Thank you for the attention!