Fefferman–Graham Expansions for Asymptotically Schroedinger Space-Times

Jelle Hartong

Niels Bohr Institute

Nordic String Theory Meeting

Copenhagen, February 20, 2012

Based on work in progress in collaboration with Blaise Rollier

Introduction

- Many systems in nature exhibit critical points with non-relativistic scale invariance. Such systems typically have Lifshitz symmetries:

$$
\begin{aligned}
D_z &: \quad \vec{x} \to \lambda \vec{x} \qquad t \to \lambda^z t\,,\\
H &: \quad t \to t + c\,,\\
P_i &: \quad x^i \to x^i + a^i\,,\\
M_{ij} &: \quad x^i \to R^i{}_j x^j\,.
\end{aligned}
$$

- Lifshitz algebra (only nonzero commutators shown, left out M_{ij} and $z \neq 1$):

$$
[D_z, H] = zH\,, \qquad [D_z, P_i] = P_i\,.
$$

- An example of a symmetry group that also displays non-relativistic scale invariance but which is larger than Lifshitz is the Schroedinger group.
- Additional symmetries are Galilean boosts V_i ($x^i \to x^i + v^i t$) and a particle number symmetry N.
- Schroedinger algebra (only nonzero commutators shown, left out M_{ij} and $z \neq 1, 2$):

$$[D_z, H] = zH\,, \qquad [D_z, P_i] = P_i\,, \qquad [D_z, N] = (2-z)N\,,$$
$$[D_z, V_i] = (1-z)V_i\,, \qquad [H, V_i] = -2P_i\,, \qquad [P_i, V_j] = N\delta_{ij}\,.$$

- When $z = 2$ there is an additional special conformal symmetry C.

- Aim: to construct holographic techniques for (strongly coupled) systems with NR symmetries.
- From a different perspective, Schroedinger space-times form interesting examples of non-AdS space-times for which it appears to be possible to construct explicit holographic techniques.
- Schroedinger holography initiated by: [Son, 2008] [Balasubramanian, McGreevy, 2008].

Outline Talk

- Geometric definition of $z = 2$ Schroedinger space-times
- Causal structure
- The Schroedinger boundary
- Asymptotically Schroedinger (ASch) space-times:
  - The model
  - From AAdS to ASch
  - The FG expansions: what we know so far

Schroedinger space-time

- A $(d+3)$-dimensional Schroedinger space-time:

$$ds^2 = -\frac{\gamma^2}{r^{2z}} dt^2 + \frac{1}{r^2}\left(-2dt d\xi + dr^2 + d\vec{x}^2\right) \, .$$

- $\xi =$ cst slices are Lifshitz space-times.
- $\frac{dt d\xi}{r^2}$ preserves Lifshitz symmetries.
- Extra symmetries: N ($\xi \to \xi + c$) and Galilean boost invariance V_i ($x^i \to x^i + v^i\, t\, , \quad \xi \to \xi + \frac{1}{2}\vec{v}^2 t + \vec{v}\cdot\vec{x}$).
- $\mathtt{sch}_z(d+3) \subset \mathtt{so}(2, d+2)$: the deformation term breaks all the symmetries of AdS that are not in $\mathtt{sch}_z(d+3)$.
- For $z = 2$ all tidal forces are bounded and the space-time is completely regular.

Geometric Definition

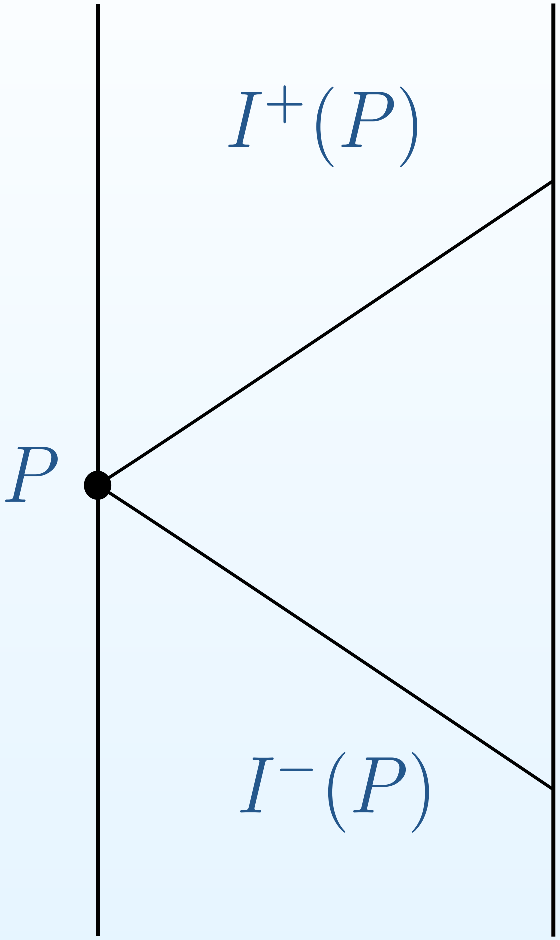
- The metric of a $z = 2$ Schroedinger space-time can be written as

$$g^{\text{Sch}}_{\mu\nu} = g^{\text{AdS}}_{\mu\nu} - A_\mu A_\nu$$

where A_μ is any AdS null Killing vector [Duval, Hassaïne, Horváthy, 2008].

- The isometries of $g^{\text{Sch}}_{\mu\nu}$ are all AdS Killing vectors that commute with A^μ.
- All AdS null Killing vectors are hypersurface orthogonal.
- A^μ is also hypersurface orthogonal with respect to $g^{\text{Sch}}_{\mu\nu}$.

Causal structure for $g^{\text{Sch}}_{\mu\nu} = g^{\text{AdS}}_{\mu\nu} - A_\mu A_\nu$



- All points inside $I^\pm(P)$ can be connected to P via timelike curves (both for AdS and Sch).
- On AdS points outside $I^\pm(P)$ can be connected to P via null and spacelike geodesics. These correspond to curves on Sch with tangent u^μ s.t. $g^{\text{Sch}}_{\mu\nu} u^\mu u^\nu = \kappa = k - (P_+)^2$ with $u^\mu A_\mu = -P_+$.
- On AdS the geodesic parameter $P_+ \neq 0$ can be boosted without affecting the geodesic curve s.t. $\kappa < 0$.

Causal structure

- Hence on Sch the only points that cannot be connected by a timelike curve are separated by null and spacelike geodesics with $P_+ = 0$, i.e. the lightlike hypersurfaces generated by A^μ.
- All points on such a lightlike hypersurface have the same chronological past and future: Sch is a non-distinguishing space-time with a Galilean-like causal structure.
- On Sch the only achronal sets are the lightlike hypersurfaces generated by A^μ.

The Sch boundary (barred: Sch, unbarred: AdS)

- Let Ω be a defining function for the AdS space, i.e. $\Omega > 0$ in the bulk and $\Omega = 0$ at the boundary and

$$g^{\mu\nu}\frac{\partial_\mu \Omega}{\Omega}\frac{\partial_\nu \Omega}{\Omega}|_{\Omega=0} = 1\,.$$

- The Riemann tensor of a Sch space-time (metric $\bar{g}_{\mu\nu} = g_{\mu\nu} - A_\mu A_\nu$) satisfies

$$\bar{R}_{\mu\nu\rho\sigma}A^\sigma = \left(-g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}\right)A^\sigma\,.$$

- Conformally rescaling $\bar{g}_{\mu\nu} = \Omega^{-2}\tilde{\bar{g}}_{\mu\nu}$, using that

$$\bar{R}_{\mu\nu\rho\sigma} = -\Omega^{-4}\tilde{\bar{g}}^{\kappa\tau}\partial_\kappa\Omega\partial_\tau\Omega\left(\tilde{\bar{g}}_{\mu\rho}\tilde{\bar{g}}_{\nu\sigma} - \tilde{\bar{g}}_{\mu\sigma}\tilde{\bar{g}}_{\nu\rho}\right) + \ldots$$

contracting with A^σ we get $\bar{g}^{\mu\nu}\frac{\partial_\mu \Omega}{\Omega}\frac{\partial_\nu \Omega}{\Omega}|_{\Omega=0} = 1\,.$

The Sch boundary

$$\bar{g}^{\mu\nu}\frac{\partial_\mu\Omega}{\Omega}\frac{\partial_\nu\Omega}{\Omega}|_{\Omega=0}-g^{\mu\nu}\frac{\partial_\mu\Omega}{\Omega}\frac{\partial_\nu\Omega}{\Omega}|_{\Omega=0}=\left(A^\mu\frac{\partial_\mu\Omega}{\Omega}\right)^2|_{\Omega=0}=1-1=0$$

since $\bar{g}^{\mu\nu}=g^{\mu\nu}+A^\mu A^\nu$.

- We thus find that the Sch boundary is at $\Omega=0$ and that Ω satisfies the same conditions as on AdS with the additional condition that

$$A^\mu\frac{\partial_\mu\Omega}{\Omega}|_{\Omega=0}=0\,.$$

- A^μ is tangential to the boundary. Since furthermore A^μ is a null Killing vector in the bulk of AdS it is also a null Killing vector of the AdS boundary metric.

- The fact that A^μ is tangent to the Sch boundary suggests that the Sch boundary inherits the non-relativistic causal structure of the Sch space-time.
- Since the only achronal sets are the lightlike hypersurface generated by A^μ expanding away from the boundary along a normal achronal curve, so that radial and time dependence do not mix, is only possible when A^μ is tangential to the boundary.
- We have not defined a Sch boundary metric. This will not be needed for the construction of FG expansions.

Asymptotically Schroedinger space-times

- For simplicity consider ASch solutions of the massive vector model

$$S = \int d^{d+3}x\sqrt{-\bar{g}}\left(\bar{R} - \frac{1}{4}\bar{F}^2 - (d+2)\bar{A}^2 + (d+1)(d+2)\right)$$

- In string theory ASch space-times are solutions to such Lagrangians that also have scalars. Setting these scalars to constants typically enforces two constraints

$$\bar{A}^2 = 0\,, \qquad \bar{F}^2 = 0\,.$$

- Goal: to solve the equations of motion of the massive vector model subject to these two constraints such that the solutions are ASch.

From AAdS to ASch

- We write again $\bar{g}_{\mu\nu} = g_{\mu\nu} - A_\mu A_\nu$ where $\bar{g}_{\mu\nu}$ is now ASch and $g_{\mu\nu}$ an AAdS space admitting a defining function satisfying the same conditons as for a pure Sch space-time.
- The equations of motion for A^μ and $g_{\mu\nu}$ are

$$
\begin{aligned}
R_{\mu\nu} + (d+2) g_{\mu\nu} &= \frac{1}{2}\left(\mathcal{L}_A S_{\mu\nu} - S_{\mu\rho} S^\rho{}_\nu\right) , \\
\nabla_\mu S^{\mu\nu} &= 0 , \\
\nabla_\mu A^\mu &= 0 , \\
A_\mu A^\mu &= 0 , \\
F_{\mu\nu} F^{\mu\nu} &= -2 \left(A^\mu \nabla_\mu A_\nu\right) A^\rho \nabla_\rho A^\nu ,
\end{aligned}
$$

where $S_{\mu\nu} = \nabla_\mu A_\nu + \nabla_\nu A_\mu$ and $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$.

Fefferman–Graham expansions

$$
\begin{aligned}
g_{\mu\nu}dx^\mu dx^\nu &= \frac{dr^2}{r^2} + \frac{1}{r^2}h_{ab}dx^a dx^b\,, \qquad A^\mu\partial_\mu = rA'^r\partial_r + A^a\partial_a\,,\\
h_{ab} &= g_{(0)ab} + \ldots\,,\\
A^a &= A^a_{(0)} + \ldots\,,\\
A'^r &= 0 + \ldots\,.
\end{aligned}
$$

- The expansion for h_{ab} is identical to the AAdS case (without matter) as long as $\mathcal{L}_A S_{\mu\nu} - S_{\mu\rho}S^\rho{}_\nu = 0$. (This is a generalization of FG expansions for ASch spaces that can be obtained via TsT where $S_{\mu\nu} = 0$.)
- For a pure Sch space $A^a_{(0)}$ is a boundary hypersurface orthogonal (HSO) null Killing vector. For ASch spaces this is relaxed to $A^a_{(0)}$ being tangent to a HSO, expansion and shear free, null geodesic congruence.

Conclusions and future work

- We have defined Sch spaces in terms of AdS quantities.
- This allowed for a definition of the Sch boundary in terms of a defining function.
- This sets the boundary conditions for the FG construction for ASch spaces.
- In progress: working out the details of the FG expansion.
- Potential applications:
  - Holographic renormalization
  - The asymptotic symmetry group