Analytic 2-loop Form factor in N=4 SYM

Gang Yang
University of Hamburg

Nordic String Theory Meeting
Feb 20-21, 2012, NBI

Based on the work

- Brandhuber, Travaglini, GY 1201.4170 [hep-th]
- Brandhuber, Gurdogan, Mooney, Travaglini, GY 1107.5067 [hep-th]
- Brandhuber, Spence, Travaglini, GY 1011.1899 [hep-th]

See also some other work on form factors in N=4:

Bork, Kazakov, Vartanov 2010, 2011
Gehrmann, Henn, Huber 2011
Henn, Moch, Naculich 2011
Maldacena, Zhiboedov 2010
Alday, Maldacena 2007
van Neerven 1986
……

Outline

- Motivation. Why form factor?
- A pre-two-loop summary of form factor
- A non-trivial two-loop computation

Unwanted complexity

Text book method by traditional Feynman diagrams

n	2	3	4	5	6	7	8
# of diagrams	4	25	220	2485	34300	559405	10525900

$$A(1^{\pm}, 2^{+}, ..., n^{+}) =$$

$$= 0$$

Final results are simple !

MHV (maximally-helicity-violating) Parke-Taylor formula :

$$A_{\mathrm{MHV}}(1^{+}, .., i^{-}, .., j^{-}, .., n^{+}) = \frac{\langle i \ j \rangle^4}{\langle 1 \ 2 \rangle \cdots \langle n \ 1 \rangle}$$

$$p_\mu \to p_\mu \sigma^{\mu}_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} \ , \quad \langle i \ j \rangle = \lambda_i^{\alpha} \lambda_{j\alpha}$$

Spinor helicity formalism

Unwanted complexity

Six-point MHV amplitude (or WL) in N=4 SYM:

(Del Duca, Duhr, Smirnov 2010)

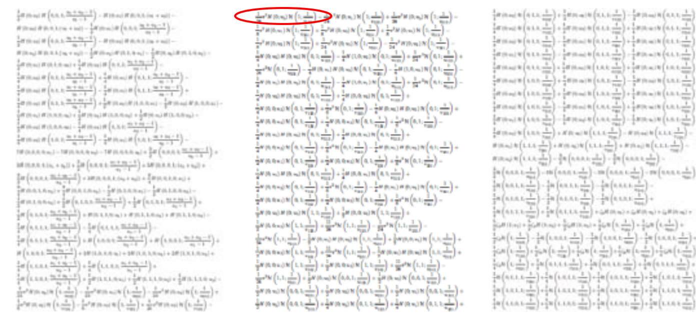
Heroic computation by evaluating Feynman diagrams loop integrals:

Goncharov PolyLog

$$\frac{1}{4}\mathcal{G}\left(v_{132}, 1, \frac{1}{1-u_1}; 1\right) H(0; u_2)$$

Other 10 pages

••••••



Using **symbol** technique (Goncharov, Spradlin, Vergu, Volovich 2010)

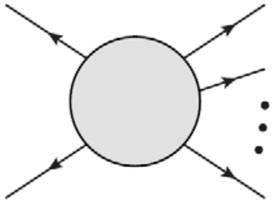
$$R_6^{(2)}(u_1,u_2,u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+, x_i^-) - \frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}.$$

Simple combination of classical PolyLog functions !

Progress

Significant progress for scattering amplitudes in past years.

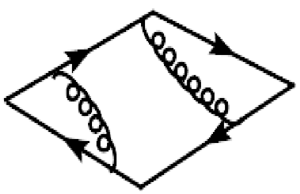
$A_n = \langle 0 | p_1 p_2 \cdots p_n \rangle$

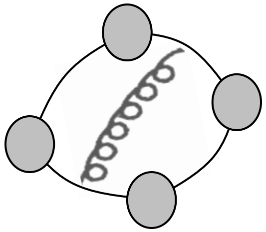


More powerful computational techniques:
MHV, BCFW, Unitarity, DCS…

Surprising relations between
different observables
(in N=4 SYM)

Dual conformal
symmetry (DCS)
Integrability (Yangian)
AdS/CFT





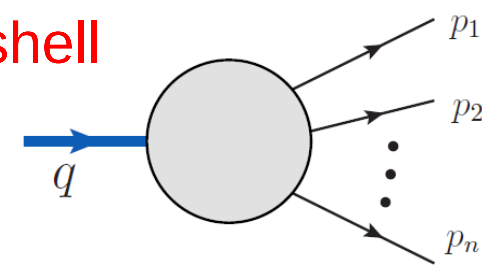
$\langle W(C_n) \rangle = \left\langle 0 \left| \mathrm{TrP} \exp \left(ig \oint_{C_n} dx^\mu A_\mu(x) \right) \right| 0 \right\rangle$

$\langle 0 | \mathcal{O}(x_1) \mathcal{O}(x_2) \ldots \mathcal{O}(x_n) | 0 \rangle \Big|_{x^2_{ii+1} \to 0}$

Most of these developments are focused on “on-shell” quantities.
Can we go beyond this ?

Why form factor ?

Form factor : partially on-shell, partially off-shell



$$q = \sum_i p_i, \quad q^2 \neq 0$$

$$F = \int d^4x \, e^{iq\cdot x} \, \langle 0|\mathcal{O}(x)|\text{states}\rangle$$

$$= \langle 0|\mathcal{O}(q)|p_1 p_2 \cdots p_n\rangle$$

Scattering amplitudes

$$\langle 0|p_1 p_2 \cdots p_n\rangle$$

Correlation functions

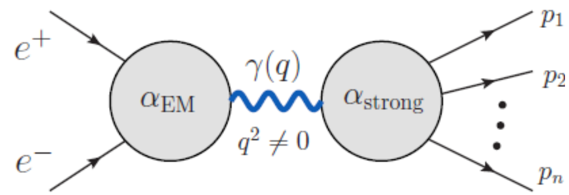
$$\langle 0\,|\mathcal{O}(x_1)\mathcal{O}(x_2)...\mathcal{O}(x_n)|\,0\rangle$$

Some examples

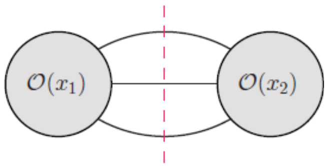
- Two-point: Sudakov form factor

- $e^+e^- \to$ hadrons

$$\mathcal{O}(q) \to J_\mu^{\text{e.m}}(q) = \overline{\psi}\gamma_\mu\psi(q)$$

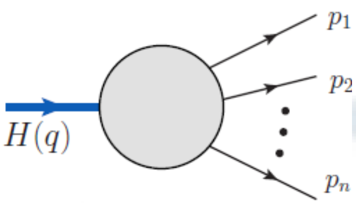


- "cut" of correlators



- Higgs to jets

$$\mathcal{L}_{\text{eff.int.}} = -\frac{\lambda}{4} H \ \text{tr}\,(F_{\mu\nu}F^{\mu\nu})$$ (integrate over quark field)



$$= \langle p_1 p_2 \cdots p_n | \text{tr}(F_{\mu\nu}F^{\mu\nu})(q) | 0 \rangle$$

Close phenomenological relations,
and surprising observation (talk later)!

Form factor in N=4 SYM

We will mainly consider planar form factor in N=4 SYM with half BPS operators in the stress tensor supermultiplet.

Full stress-tensor supermultiplet (using harmonic superspace) :

$$\begin{aligned}\mathcal{T}(x,\theta^+,\bar\theta_-) &= e^{Q_+\theta^+ + \bar{Q}^-\bar\theta_-}\mathrm{Tr}(\phi^{++}\phi^{++})(x)\\ &= \mathrm{Tr}(\phi^{++}\phi^{++}) + (\theta^+)^4\mathcal{L}_{SD} + (\tilde\theta_-)^4\tilde{\mathcal{L}}_{ASD} + (\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta)T_{\mu\nu} + \cdots\end{aligned}$$

We mostly focus on : $\mathrm{Tr}\left(\phi_{AB}^2\right)$ and $\mathrm{Tr}\left(F_{\mathrm{SD}}^2\right)$ (related to QCD)

$$F_{AB} = \langle p_1 p_2 \cdots p_n|\mathrm{Tr}(\phi_{AB}^2)(q)|0\rangle$$

$$F_{\mathrm{SD}} = \langle p_1 p_2 \cdots p_n|\mathrm{Tr}(F_{\mathrm{SD}}^2)(q)|0\rangle$$

New feature of Form factor

- Not fully on-shell, there is one off-shell leg q.
- The operator is color singlet, so the position of q is not fixed.
- At two and higher loops, there are non-planar integrals.
- No dual super conformal symmetry.

Despite these differences, there are still many nice properties for form factors. The simplicity we still have.

Outline

- Motivation. Why form factor?

- A pre-two-loop summary of form factor

  - MHV form factor
  - Super form factor
  - Form factor / periodic Wilson line correspondence

- A non-trivial two-loop computation

MHV Form factor

MHV amplitudes:

$$A_{\mathrm{MHV}}(1^+, .., i^-, .., j^-, .., n^+) = \frac{\langle i\ j\rangle^4}{\langle 1\ 2\rangle \cdots \langle n\ 1\rangle}$$

$$p_\mu \to p_\mu \sigma^\mu_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}\ , \quad \langle i\ j\rangle = \lambda_i^\alpha \lambda_{j\alpha}$$

MHV form factor:

$$F_{AB}^{\mathrm{MHV}}(1^+, .., i_\phi, .., j_\phi, .., n^+; q) = \frac{\langle i\ j\rangle^2}{\langle 1\ 2\rangle \cdots \langle n\ 1\rangle}$$

$$F_{\mathrm{SD}}^{\mathrm{MHV}}(1^+, .., i^-, .., j^-, .., n^+; q) = \frac{\langle i\ j\rangle^4}{\langle 1\ 2\rangle \cdots \langle n\ 1\rangle}$$

$\mathrm{Tr}\left(\phi_{AB}^2\right)$ and $\mathrm{Tr}\left(F_{\mathrm{SD}}^2\right)$

The simple expression implies the underlying simplicity of form factor. Efficient computational methods, such as MHV rules, BCFW recursion relation…

Supersymmetric generalization

Super amplitudes:

$$\mathcal{A}_{\text{MHV}}(\Phi(1), \Phi(2), \cdots, \Phi(n)) = \frac{\delta^{(8)}(\sum_i \lambda_i \eta_i)}{\langle 1\ 2\rangle\langle 2\ 3\rangle \cdots \langle n\ 1\rangle}$$

(Nair 1988)

Super states:

$$\Phi(p,\eta) := g^+(p) + \eta_A \lambda^A(p) + \frac{\eta_A \eta_B}{2!} \phi^{AB}(p) + \epsilon^{ABCD} \frac{\eta_A \eta_B \eta_C}{3!} \bar{\lambda}_D(p) + \eta_1 \eta_2 \eta_3 \eta_4\, g^-(p)$$

η expansion ⟷ **different external states**

The power of using supersymmetry.

- New identities or constraints from supersymmetry
- Greatly simplify the computations $\sum_{\text{states}} \rightarrow \int d^4\eta$

Super Form factor

$$\mathcal{F}^{\text{MHV}}_{\mathcal{T}(x,\theta^{3,4},0)} = \frac{\delta^{(4)}_{34}(\gamma - \sum_i \lambda_i \eta_i)\ \delta^{(4)}_{12}(\sum_i \lambda_i \eta_i)}{\langle 1\ 2\rangle\langle 2\ 3\rangle \cdots \langle n\ 1\rangle}$$

η expansion ⟷ **different external states**

γ expansion ⟷ **different operators**

Chiral supermultiplet : $\mathcal{T}(x,\theta^{3,4},0) = e^{Q_3\theta^3 + Q_4\theta^4}\text{Tr}(\phi_{12}\phi_{12})(x)$

$$\begin{aligned} \rightarrow \quad & \text{Tr}(\phi_{12}\phi_{12})(x) \\ & + [\ \text{Tr}(\psi_{123}\psi_{123}) + \text{Tr}(\phi_{12}\phi_{23}\phi_{31})(x)\ ] \times (\theta^3)^2 + \ldots \\ & + [\ \text{Tr}(F^2_{\text{SD}}) + \text{Tr}(\psi_{ACD}\psi_{BCD}\phi_{AB}) + \text{Tr}(\phi_{AB}\phi_{CD}\phi^{AB}\phi^{CD})(x)\ ] \times (\theta^3)^2(\theta^4)^2 \end{aligned}$$

$$F^{\text{MHV}}_{\text{SD}}(1^+, .., i^-, j^-, .., n^+) = \frac{\langle ij\rangle^4}{\langle 12\rangle \cdots \langle n1\rangle}$$

$$F^{\text{max-NMHV}}_{\text{SD}}(1^-, \ldots, n^-) = \frac{q^4}{[1\ 2][2\ 3]\cdots[n\ 1]}$$

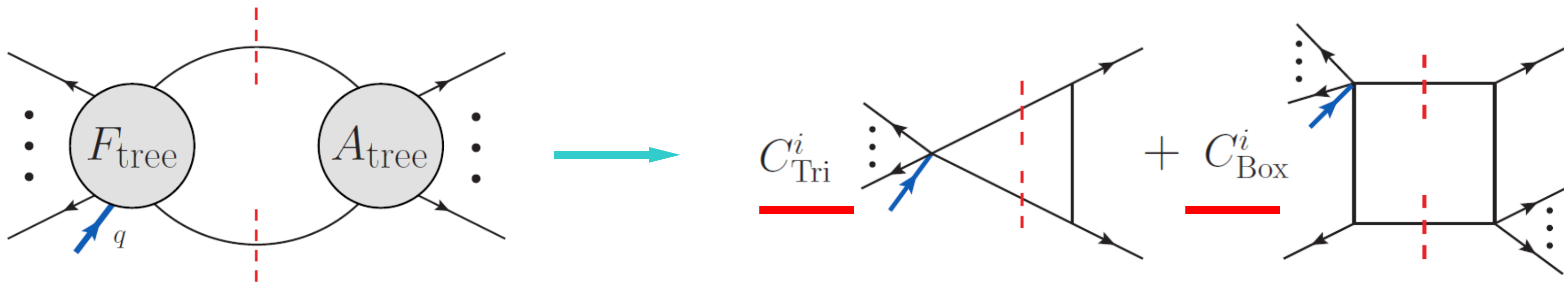
Related by supersymmetry!

(hard to understand otherwise)

One-loop MHV Form factor

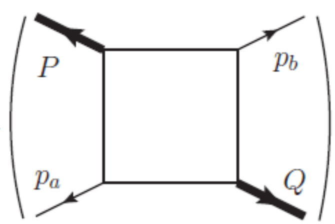
Unitarity method: do cuts and compute the coefficient of integrals

(Bern, Dixon, Dunbar, Kosower)



General MHV one-loop result:

$$F^{(1)}_{\text{MHV}} = F^{(0)}_{\text{MHV}}\left[-\sum_{i=1}^{n}\frac{(-s_{ii+1})^{-\epsilon}}{\epsilon^2} + \sum_{a,\,b}\text{Fin}\left(\quad\right)\right]$$

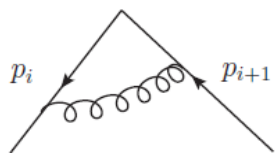


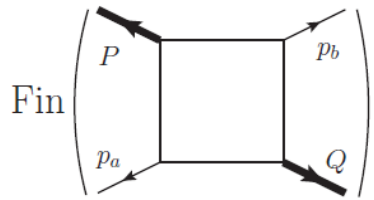
same structure as MHV amplitudes!

Corresponding to periodic Wilson line

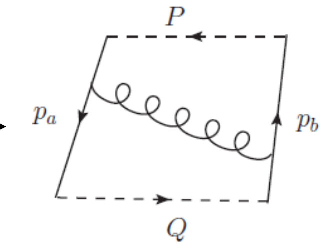
Correspondence (in Feynman gauge):

$$\frac{(-s_{ii+1})^{-\epsilon}}{\epsilon^2} \longleftrightarrow$$

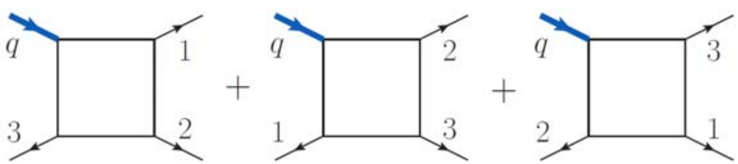




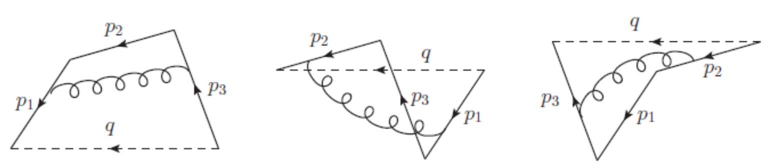
$\longleftrightarrow$



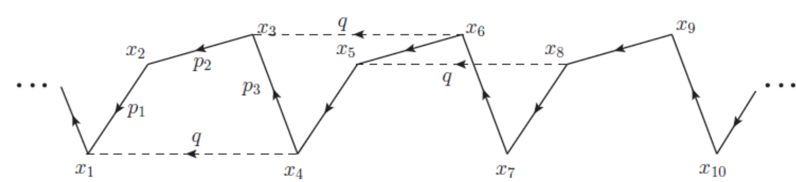
3-point example:



Dual picture:



Unified in a periodic WL

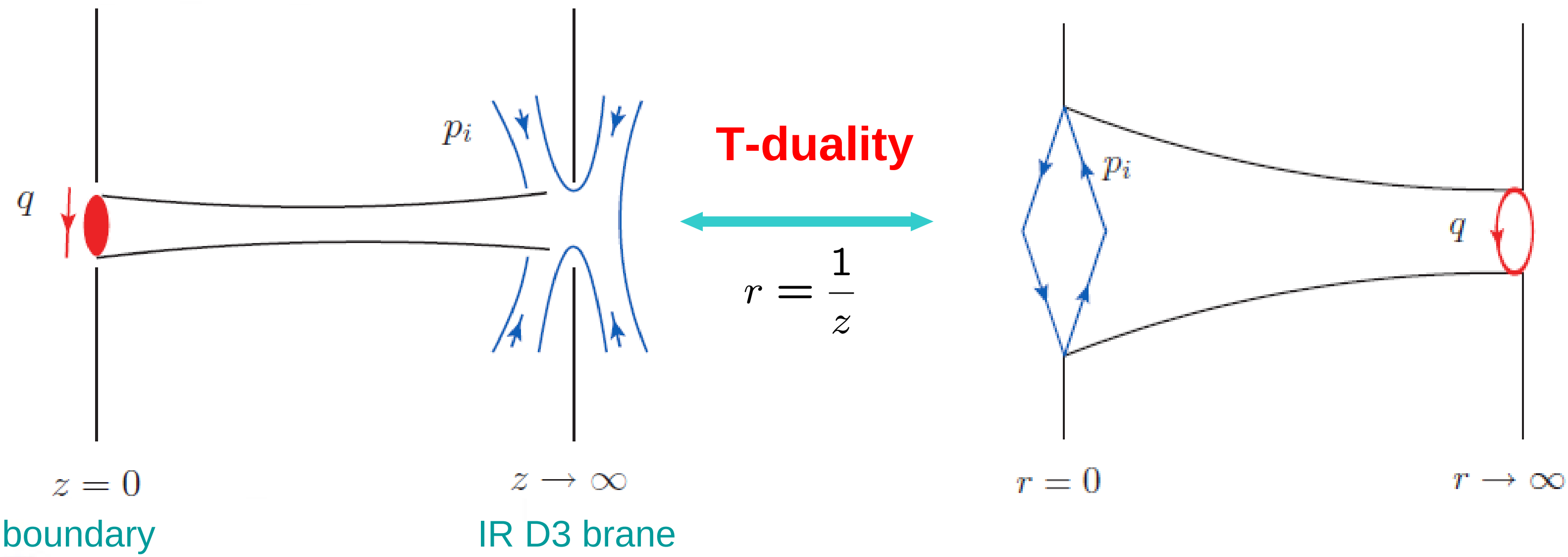


The periodic structure is necessary:

there is no fixed position of q

In dual string theory

N=4 SYM ⟷ AdS/CFT duality ⟷ Type IIB superstring in AdS5 x S5



Momenta of strings

Winding of strings

(Alday, Maldacena; Maldacena, Zhiboedov)

Outline

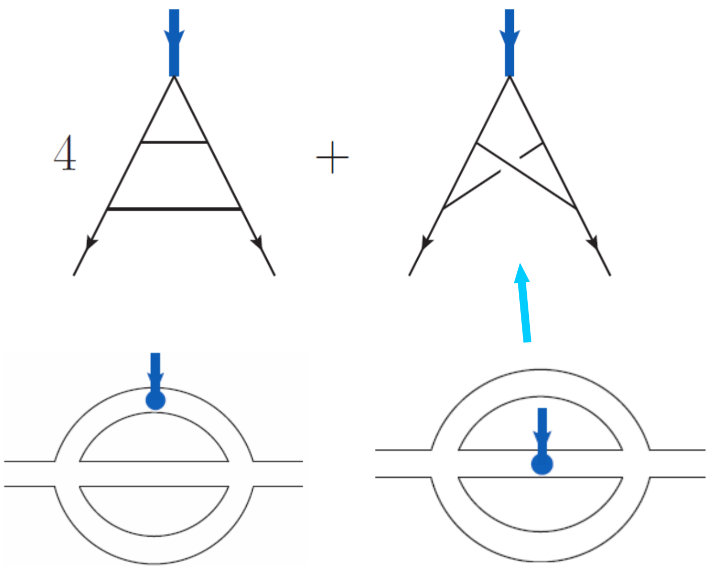
- Motivation. Why form factor?
- A pre-two-loop summary of form factor
- A non-trivial two-loop computation
  - Honest unitarity computation
  - Symbol technique
  - Surprising relation to QCD

Two-loop form factor

New feature starting from two loops.

Two-point planar form factor: (van Neerven 1986)

$$F_2^{2-\text{loop}} =$$



Non-planar topology !

Diagrammatic origin:

(In double line picture)

(Two-point three-loop recently computed by Gehrmann, Henn, Huber)

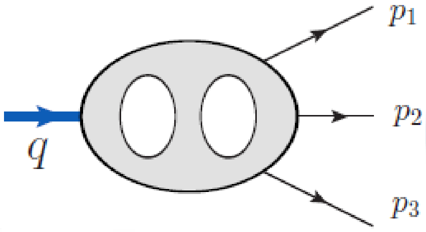
Higher-point are more interesting

New feature starting from three-point two-loop.

The two-point case is special: trivial dependence on the single kinematic variable s. For higher point, there will be non-trivial kinematic dependent functions.

We consider two-loop three-point planar form factor.

- First, honest computation by unitarity method
- Second, Analytic expression obtained by physical constraints based on symbol technique



$$u = \frac{s_{12}}{q^2}, \quad v = \frac{s_{23}}{q^2}, \quad w = \frac{s_{31}}{q^2}$$

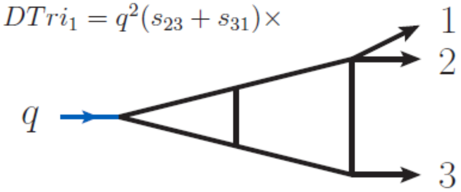
$$q^2 = s_{12} + s_{23} + s_{31}$$

$$u + v + w = 1$$

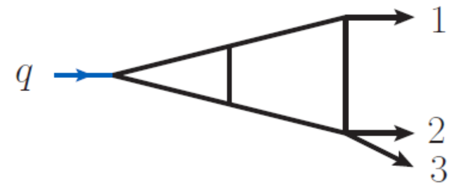
A look at the final result

$$F_3^{2-\text{loop}}/F_3^{\text{tree}} = \sum_{i=1}^{2}(DTri_i + DBox_i) + TriPent + NBox + NTri + \text{cyclic}$$

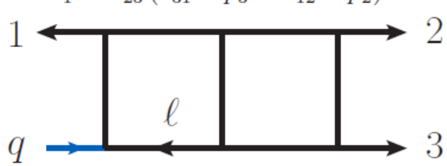
$DTri_1 = q^2(s_{23}+s_{31})\times$



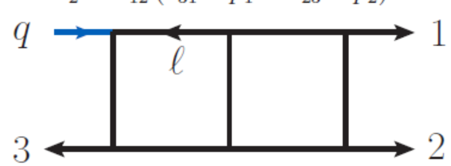
$DTri_2 = q^2(s_{12}+s_{31})\times$



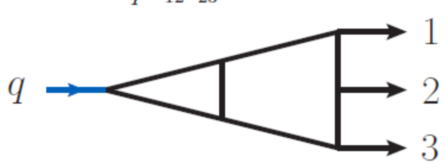
$DBox_1 = s_{23}\,(s_{31}\ell\cdot p_3 - s_{12}\ell\cdot p_2)\times$



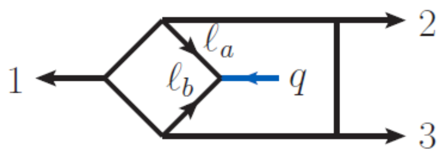
$DBox_2 = s_{12}\,(s_{31}\ell\cdot p_1 - s_{23}\ell\cdot p_2)\times$



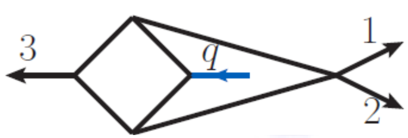
$TriPent = q^2 s_{12}s_{23}\times$



$NBox = s_{23}\left(\frac{1}{2}s_{12}s_{31} - s_{12}\ell_a\cdot p_2 - s_{31}\ell_b\cdot p_3\right)\times$



$NTri = \frac{1}{2}q^2(s_{23}+s_{31})\times$



Computed by (generalized) unitarity method:

Apply unitarity cuts, do tensor reductions, find integrals and coefficients.

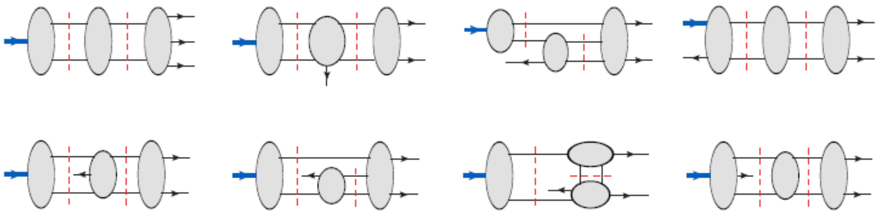
(Bern, Dixon, Dunbar, Kosower 1994)

(Britto, Cachazo, Feng 2004)

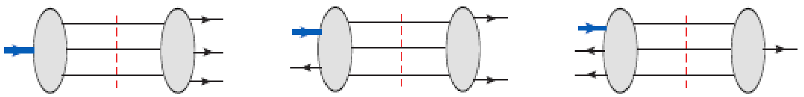
Generalized unitarity method

Our strategy:

First apply all possible double two-particle cuts to detect the integrals and coefficients.



Then use triple-cut to fix remaining ambiguities.



(Only algebraic operations)

The complexity comparing to planar amplitudes:

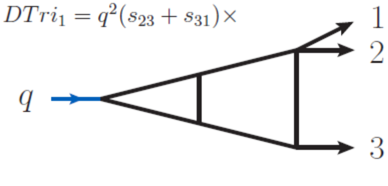
There is no dual conformal symmetry here, we don't know the integrals and therefore need to do honest tensor reduction to find the integrals.

Unitarity computation

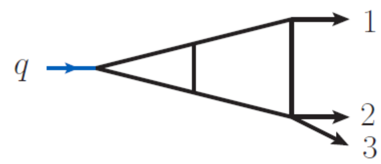
Result given in terms of integrals (with very simple coefficients):

$$\sum_{i=1}^{2}(DTri_i + DBox_i) + TriPent + NBox + NTri + \text{cyclic}$$

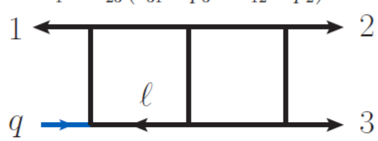
$DTri_1 = q^2(s_{23}+s_{31})\times$



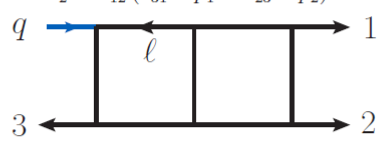
$DTri_2 = q^2(s_{12}+s_{31})\times$



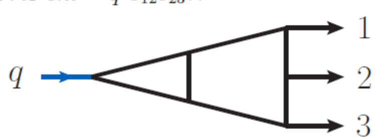
$DBox_1 = s_{23}\,(s_{31}\ell\cdot p_3 - s_{12}\ell\cdot p_2)\times$



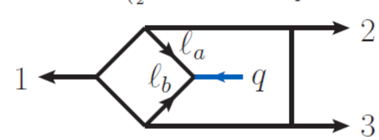
$DBox_2 = s_{12}\,(s_{31}\ell\cdot p_1 - s_{23}\ell\cdot p_2)\times$



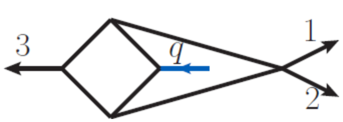
$TriPent = q^2 s_{12}s_{23}\times$



$NBox = s_{23}\left(\frac{1}{2}s_{12}s_{31} - s_{12}\ell_a\cdot p_2 - s_{31}\ell_b\cdot p_3\right)\times$



$NTri = \frac{1}{2}q^2(s_{23}+s_{31})\times$



There are no analytic expressions for all of the integrals, we have to evaluate them numerically.

(**MB.m** code by Czakon)

It is convenient to consider some divergence extracted function:

Remainder function!

Remainder function

Gauge theory amplitudes have well understood universal infrared and collinear behavior.

ABDK/BDS expansion: (Anastasiou, Bern, Dixon, Kosower; Bern, Dixon, Smirnov)

$$\mathcal{M}_n^{(2)}(\epsilon) = \underbrace{\frac{1}{2}(\mathcal{M}_n^{(1)}(\epsilon))^2 + f^{(2)}(\epsilon)\,\mathcal{M}_n^{(1)}(2\epsilon)}_{\text{divergence}} + \underbrace{\mathcal{R}_n^{(2)}}_{\text{finite remainder function (scheme indep.)}} + \mathcal{O}(\epsilon)$$

Important property in the collinear limit: $\mathcal{R}_n^{(2)} \to \mathcal{R}_{n-1}^{(2)}$

In particular, three-point remainder function $\mathcal{G}_n^{(L)} := F_n^{(L)}/F_n^{(0)}$

$$\mathcal{R}_3^{(2)} := \mathcal{G}_3^{(2)}(\epsilon) - \frac{1}{2}(\mathcal{G}_3^{(1)}(\epsilon))^2 - f^{(2)}(\epsilon)\,\mathcal{G}_3^{(1)}(2\epsilon) - C^{(2)} + \mathcal{O}(\epsilon)$$

$$\boxed{\mathcal{R}_3^{(2)}(u,v,w)\Big|_{u\to 0} \to 0}$$

$$u = \frac{s_{12}}{q^2}, \quad v = \frac{s_{23}}{q^2}, \quad w = \frac{s_{31}}{q^2}$$

Construct analytic expression ?

Symbol technique !

A brief introduction of symbol

Loop results can be given in terms of transcendental functions such as Log or PolyLog or more complicated functions.

$$f_k(x) = \int^1 dt_1 \int^{t_1} dt_2 \ldots \int^{t_{k-1}} dt_k \, R(x; t_1, \ldots, t_k)$$

Goncharov polylogarithms:

$$G(a_k, a_{k-1}, \ldots, a_1; z) = \int_0^z G(a_{k-1}, \ldots, a_1; t) \frac{dt}{t - a_k}, \quad G(z) = 1$$

Recursive definition of symbol:

$$\mathrm{d} f_k = \sum_i f_{k-1}^i \, \mathrm{dLog}(R_i), \qquad \mathrm{Symbol}(f_k) = \sum_i \mathrm{Symbol}(f_{k-1}^i) \otimes R_i$$

$$\mathrm{Symbol}(f_k) = R_1 \otimes \ldots \otimes R_k$$

A brief introduction of symbol

Simple example:

$$f_0 = \mathsf{R}(u), \qquad \mathsf{Symbol}(f_0) = 0$$

$$f_1 = \mathsf{Log}[R(u)], \qquad \mathsf{Symbol}(f_1) = R(u)$$

$$f_2 = \mathsf{Log}(R_1)\mathsf{Log}(R_2), \qquad \mathsf{Symbol}(f_2) = R_1 \otimes R_2 + R_2 \otimes R_1$$

$$f_2 = \mathsf{Li}_2(R), \qquad \mathsf{Symbol}(f_2) = -(1-R) \otimes R$$

Basic operations:

$$R_1 \otimes \ldots \otimes (c\, R_i) \otimes \ldots \otimes R_n = R_1 \otimes \ldots \otimes R_i \otimes \ldots \otimes R_n$$

$$R_1 \otimes \ldots \otimes (R_i\, R_j) \otimes \ldots \otimes R_n = R_1 \otimes \ldots \otimes R_i \otimes \ldots \otimes R_n + R_1 \otimes \ldots \otimes R_j \otimes \ldots \otimes R_n$$

Applications

Easy to prove some identities:

$$\mathrm{Li}_2\left(\frac{x}{1-y}\right)+\mathrm{Li}_2\left(\frac{y}{1-x}\right)-\mathrm{Li}_2(x)-\mathrm{Li}_2(y)-\mathrm{Li}_2\left(\frac{xy}{(1-x)(1-y)}\right) = \mathrm{Log}(1-x)\mathrm{Log}(1-y)$$

$(x < 1 \text{ and } y < 1)$

$$\mathrm{Symbol}\,(\mathrm{LHS}) = (1-x)\otimes(1-y) + (1-y)\otimes(1-x)$$

Ambiguity about lower degree piece and branch cuts:

$$\begin{aligned}\mathrm{Li}_2(x) + \mathrm{Li}_2\left(\frac{1}{x}\right) &= -\frac{\pi^2}{6} - \frac{1}{2}\left[\mathrm{Log}(-x)\right]^2 && (x \notin [0,1]) \\ &= \frac{\pi^2}{3} - \frac{1}{2}\left[\mathrm{Log}(x)\right]^2 - i\pi\mathrm{Log}(x) && (x \geq 1)\end{aligned}$$

Applications

Simplify complicated expressions:

1) Compute the symbol of some known function

2) Simplify the symbol (algebraic operations)

3) Reconstruct a simpler function giving the same symbol

Ambiguity about lower degree piece and branch cuts are usually much less complicated, and may be fixed by other physical constraints, such as collinear limit.

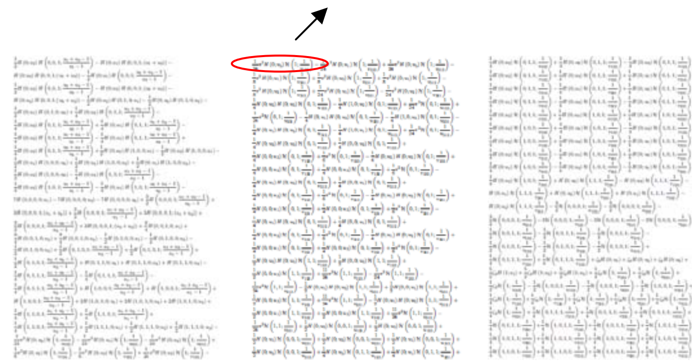
In this way, as we showed before,

(Del Duca, Duhr, Smirnov 2010)

Other 10 pages

• • • • • •

Goncharov PolyLog



Becomes one line formula !

$$R_6^{(2)}(u_1,u_2,u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}.$$

(Goncharov, Spradlin, Vergu, Volovich 2010)

Can we apply symbol technique without knowing the result first ?

Compute symbol directly

Back to three-point form factor, the remainder function.

$$\mathcal{R}_3^{(2)} := \mathcal{G}_3^{(2)}(\epsilon) - \frac{1}{2}(\mathcal{G}_3^{(1)}(\epsilon))^2 - f^{(2)}(\epsilon)\,\mathcal{G}_3^{(1)}(2\epsilon) - C^{(2)} + \mathcal{O}(\epsilon)$$

Compute its symbol directly, without knowing the result first.

Constraints:

- Variables in symbol : $\{u, v, w; 1-u, 1-v, 1-w\}$
- Entry conditions: restriction on the position of variables
- Collinear limit : $\text{Symbol} \to 0$
- Totally symmetric in kinematics
- Integrability condition $\sum dw_i \wedge dw_{i+1}(w_1 \otimes \cdots \otimes w_{i-1} \otimes w_{i+2} \otimes \cdots \otimes w_n) = 0$

Solution of the symbol

There is a
unique solution !

$$
\begin{aligned}
\mathcal{S}^{(2)} \;=\; & -2u \otimes (1-u) \otimes (1-u) \otimes \frac{1-u}{u} + u \otimes (1-u) \otimes u \otimes \frac{1-u}{u} \\
& -u \otimes (1-u) \otimes v \otimes \frac{1-v}{v} - u \otimes (1-u) \otimes w \otimes \frac{1-w}{w} \\
& -u \otimes v \otimes (1-u) \otimes \frac{1-v}{v} - u \otimes v \otimes (1-v) \otimes \frac{1-u}{u} \\
& +u \otimes v \otimes w \otimes \frac{1-u}{u} + u \otimes v \otimes w \otimes \frac{1-v}{v} \\
& +u \otimes v \otimes w \otimes \frac{1-w}{w} - u \otimes w \otimes (1-u) \otimes \frac{1-w}{w} \\
& +u \otimes w \otimes v \otimes \frac{1-u}{u} + u \otimes w \otimes v \otimes \frac{1-v}{v} \\
& +u \otimes w \otimes v \otimes \frac{1-w}{w} - u \otimes w \otimes (1-w) \otimes \frac{1-u}{u} \\
& + \text{cyclic permutations}\,.
\end{aligned}
$$

It satisfies $\mathcal{S}^{(2)}_{abcd} - \mathcal{S}^{(2)}_{bacd} - \mathcal{S}^{(2)}_{abdc} + \mathcal{S}^{(2)}_{badc} - (a \leftrightarrow c\,,\, b \leftrightarrow d) \;=\; 0$

therefore can be obtained from a function involving only classical polylog functions:

$\log x_1 \log x_2 \log x_3 \log x_4\ ,\ \mathrm{Li}_2(x_1) \log x_2 \log x_3\ ,\ \mathrm{Li}_2(x_1)\mathrm{Li}_2(x_2)\ ,\ \mathrm{Li}_3(x_1) \log x_2$ and $\mathrm{Li}_4(x_i)$

Analytic functions

Reconstruct the function (plus collinear constraint) :

$$\mathcal{R}_3^{(2)} = -2\left[\mathrm{J}_4\left(-\frac{uv}{w}\right)+\mathrm{J}_4\left(-\frac{vw}{u}\right)+\mathrm{J}_4\left(-\frac{wu}{v}\right)\right]-8\sum_{i=1}^{3}\left[\mathrm{Li}_4\left(1-u_i^{-1}\right)+\frac{\log^4 u_i}{4!}\right]$$
$$-2\left[\sum_{i=1}^{3}\mathrm{Li}_2(1-u_i)+\frac{\log^2 u_i}{2!}\right]^2+\frac{1}{2}\left[\sum_{i=1}^{3}\log^2 u_i\right]^2-\frac{\log^4(uvw)}{4!}-\frac{23}{2}\zeta_4$$

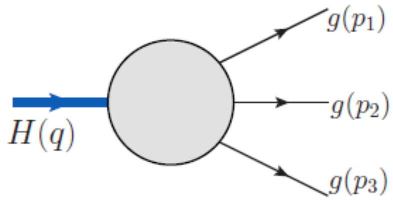
$$\mathrm{J}_4(z) := \mathrm{Li}_4(z)-\log(-z)\mathrm{Li}_3(z)+\frac{\log^2(-z)}{2!}\mathrm{Li}_2(z)-\frac{\log^3(-z)}{3!}\mathrm{Li}_1(z)-\frac{\log^4(-z)}{48}\,.$$

Simple combination of classical polylog functions !

The result is also consistent with the numerical evaluation.

Relation to QCD

$$\mathcal{L}_{\text{eff.int.}} = -\frac{\lambda}{4} H \ \text{tr}\,(F_{\mu\nu}F^{\mu\nu})$$



$$= \ \langle p_1 p_2 \cdots p_n | \text{tr}(F_{\mu\nu}F^{\mu\nu})(q)|0\rangle$$

Feynman diagram two-loop computation (Gehrmann, Jaquier, Glover, Koukoutsakis)

(leading transcendental planar piece)

Goncharov PolyLog

$$
\begin{aligned}
&-2G(0,0,1,0,u) + G(0,0,1-v,1-v,u) + 2G(0,0,-v,1-v,u) - G(0,1,0,1-v,u) + 4G(0,1,1,0,u) - G(0,1,1-v,0,u) + G(0,1-v,0,1-v,u)\\
&+G(0,1-v,1-v,0,u) - G(0,1-v,-v,1-v,u) + 2G(0,-v,0,1-v,u) + 2G(0,-v,1-v,0,u) - 2G(0,-v,1-v,1-v,u) - 2G(1,0,0,1-v,u)\\
&-2G(1,0,1-v,0,u) + 4G(1,1,0,0,u) - 4G(1,1,1,0,u) - 2G(1,1-v,0,0,u) + G(1-v,0,0,1-v,u) - G(1-v,0,1,0,u) - 2G(-v,1-v,1-v,u)H(0,v)\\
&-2G(1-v,1,0,0,u) + 2G(1-v,1,0,1-v,u) + 2G(1-v,1,1-v,0,u) + G(1-v,1-v,0,0,u) + 2G(1-v,1-v,1,0,u) - 2G(1-v,1-v,-v,1-v,u)\\
&-G(1-v,-v,1-v,0,u) + 4G(1-v,-v,-v,1-v,u) - 2G(-v,0,1-v,1-v,u) - 2G(-v,1-v,0,1-v,u) - 2G(-v,1-v,1-v,0,u) + 4G(1,0,1,0,u)\\
&+4G(-v,-v,1-v,1-v,u) - 4G(-v,-v,-v,1-v,u) - G(0,0,1-v,u)H(0,v) - G(0,1,0,u)H(0,v) - G(0,1-v,0,u)H(0,v) + G(0,1-v,1-v,u)H(0,v)\\
&-G(0,-v,1-v,u)H(0,v) - 2G(1,0,0,u)H(0,v) + G(1,0,1-v,u)H(0,v) + G(1,1-v,0,u)H(0,v) + G(1-v,0,0,u)H(0,v) - G(1-v,0,1-v,u)H(0,v)\\
&-G(1-v,1,0,u)H(0,v) - G(1-v,1-v,0,u)H(0,v) - G(1-v,-v,1-v,u)H(0,v) + G(-v,0,1-v,u)H(0,v) + G(-v,1-v,0,u)H(0,v) + H(1,0,0,1,v)\\
&-G(0,0,1-v,u)H(1,v) - G(0,0,-v,u)H(1,v) + G(0,1,0,u)H(1,v) - G(0,1-v,0,u)H(1,v) + G(0,1-v,-v,u)H(1,v) - 2G(0,-v,0,u)H(1,v)\\
&+2G(0,-v,1-v,u)H(1,v) + 2G(1,0,0,u)H(1,v) - G(1-v,0,0,u)H(1,v) + G(1-v,0,-v,u)H(1,v) - 2G(1-v,1,0,u)H(1,v) - G(1-v,0,-v,1-v,u)\\
&+G(1-v,-v,0,u)H(1,v) - 4G(1-v,-v,-v,u)H(1,v) + 2G(-v,0,1-v,u)H(1,v) + 2G(-v,1-v,0,u)H(1,v) - 4G(-v,1-v,-v,u)H(1,v)\\
&-4G(-v,-v,1-v,u)H(1,v) + 4G(-v,-v,-v,u)H(1,v) + G(0,0,u)H(0,0,v) + G(0,1-v,u)H(0,0,v) + G(1-v,0,u)H(0,0,v) + H(1,0,1,0,v)\\
&-G(0,0,u)H(0,1,v) + G(0,-v,u)H(0,1,v) - G(1,0,u)H(0,1,v) + 2G(1-v,0,u)H(0,1,v) + 2G(1-v,1-v,u)H(0,1,v) - 3G(1-v,-v,u)H(0,1,v)\\
&-G(-v,0,u)H(0,1,v) - 2G(-v,1-v,u)H(0,1,v) + 4G(-v,-v,u)H(0,1,v) - G(0,0,u)H(1,0,v) + G(0,-v,u)H(1,0,v) - G(1,0,u)H(1,0,v)\\
&+2G(1-v,0,u)H(1,0,v) - 2G(1-v,1-v,u)H(1,0,v) + G(1-v,-v,u)H(1,0,v) - G(-v,0,u)H(1,0,v) + 2G(-v,1-v,u)H(1,0,v) + G(0,0,u)H(1,1,v)\\
&-2G(0,-v,u)H(1,1,v) - 2G(-v,0,u)H(1,1,v) + 4G(-v,-v,u)H(1,1,v) + G(0,u)H(0,0,1,v) - 3G(1-v,u)H(0,0,1,v) + 4G(-v,u)H(0,0,1,v)\\
&+G(0,u)H(0,1,0,v) + G(1-v,u)H(0,1,0,v) - G(0,u)H(0,1,1,v) + 2G(-v,u)H(0,1,1,v) + G(0,u)H(1,0,0,v) + G(1-v,u)H(1,0,0,v) + H(1,1,0,0,v)\\
&-G(0,u)H(1,0,1,v) + 2G(-v,u)H(1,0,1,v) - G(0,u)H(1,1,0,v) + 4G(1-v,u)H(1,1,0,v) - 2G(-v,u)H(1,1,0,v) + H(0,0,1,1,v) + H(0,1,0,1,v)\\
&+G(1-v,1-v,u)H(0,0,v) + 2G(1-v,1-v,-v,u)H(1,v) - G(1-v,-v,0,1-v,u) + H(0,1,1,0,v) + G(1-v,0,1-v,0,u) - G(0,1-v,1,0,u)\\
&+4G(-v,1-v,-v,1-v,u)
\end{aligned}
$$

Surprising observation

The symbol is exactly the same as form factors !

$$
\begin{aligned}
&-2G(0,0,1,0,u)+G(0,0,1-v,1-v,u)+2G(0,0,-v,1-v,u)-G(0,1,0,1-v,u)+4G(0,1,1,0,u)-G(0,1,1-v,0,u)+G(0,1-v,0,1-v,u)\\
&+G(0,1-v,1-v,0,u)-G(0,1-v,-v,1-v,u)+2G(0,-v,0,1-v,u)+2G(0,-v,1-v,0,u)-2G(0,-v,1-v,1-v,u)-2G(1,0,0,1-v,u)\\
&-2G(1,0,1-v,0,u)+4G(1,1,0,0,u)-4G(1,1,1,0,u)-2G(1,1-v,0,0,u)+G(1-v,0,0,1-v,u)-G(1-v,0,1,0,u)-2G(-v,1-v,1-v,u)H(0,v)\\
&-2G(1-v,1,0,0,u)+2G(1-v,1,0,1-v,u)+2G(1-v,1,1-v,0,u)+G(1-v,1-v,0,0,u)+2G(1-v,1-v,1,0,u)-2G(1-v,1-v,-v,1-v,u)\\
&-G(1-v,-v,1-v,0,u)+4G(1-v,-v,-v,1-v,u)-2G(-v,0,1-v,1-v,u)-2G(-v,1-v,0,1-v,u)-2G(-v,1-v,1-v,0,u)+4G(1,0,1,0,u)\\
&+4G(-v,-v,1-v,1-v,u)-4G(-v,-v,-v,1-v,u)-G(0,0,1-v,u)H(0,v)-G(0,1,0,u)H(0,v)-G(0,1-v,0,u)H(0,v)+G(0,1-v,1-v,u)H(0,v)\\
&-G(0,-v,1-v,u)H(0,v)-2G(1,0,0,u)H(0,v)+G(1,0,1-v,u)H(0,v)+G(1,1-v,0,u)H(0,v)+G(1-v,0,0,u)H(0,v)-G(1-v,0,1-v,u)H(0,v)\\
&-G(1-v,1,0,u)H(0,v)-G(1-v,1-v,0,u)H(0,v)-G(1-v,-v,1-v,u)H(0,v)+G(-v,0,1-v,u)H(0,v)+G(-v,1-v,0,u)H(0,v)+H(1,0,0,1,v)\\
&-G(0,0,1-v,u)H(1,v)-G(0,0,-v,u)H(1,v)+G(0,1,0,u)H(1,v)-G(0,1-v,0,u)H(1,v)+G(0,1-v,-v,u)H(1,v)-2G(0,-v,0,u)H(1,v)\\
&+2G(0,-v,1-v,u)H(1,v)+2G(1,0,0,u)H(1,v)-G(1-v,0,0,u)H(1,v)+G(1-v,0,-v,u)H(1,v)-2G(1-v,1,0,u)H(1,v)-G(1-v,0,-v,1-v,u)\\
&+G(1-v,-v,0,u)H(1,v)-4G(1-v,-v,-v,u)H(1,v)+2G(-v,0,1-v,u)H(1,v)+2G(-v,1-v,0,u)H(1,v)-4G(-v,1-v,-v,u)H(1,v)\\
&-4G(-v,-v,1-v,u)H(1,v)+4G(-v,-v,-v,u)H(1,v)+G(0,0,u)H(0,0,v)+G(0,1-v,u)H(0,0,v)+G(1-v,0,u)H(0,0,v)+H(1,0,1,0,v)\\
&-G(0,0,u)H(0,1,v)+G(0,-v,u)H(0,1,v)-G(1,0,u)H(0,1,v)+2G(1-v,0,u)H(0,1,v)+2G(1-v,1-v,u)H(0,1,v)-3G(1-v,-v,u)H(0,1,v)\\
&-G(-v,0,u)H(0,1,v)-2G(-v,1-v,u)H(0,1,v)+4G(-v,-v,u)H(0,1,v)-G(0,0,u)H(1,0,v)+G(0,-v,u)H(1,0,v)-G(1,0,u)H(1,0,v)\\
&+2G(1-v,0,u)H(1,0,v)-2G(1-v,1-v,u)H(1,0,v)+G(1-v,-v,u)H(1,0,v)-G(-v,0,u)H(1,0,v)+2G(-v,1-v,u)H(1,0,v)+G(0,0,u)H(1,1,v)\\
&-2G(0,-v,u)H(1,1,v)-2G(-v,0,u)H(1,1,v)+4G(-v,-v,u)H(1,1,v)+G(0,u)H(0,0,1,v)-3G(1-v,u)H(0,0,1,v)+4G(-v,u)H(0,0,1,v)\\
&+G(0,u)H(0,1,0,v)+G(1-v,u)H(0,1,0,v)-G(0,u)H(0,1,1,v)+2G(-v,u)H(0,1,1,v)+G(0,u)H(1,0,0,v)+G(1-v,u)H(1,0,0,v)+H(1,1,0,0,v)\\
&-G(0,u)H(1,0,1,v)+2G(-v,u)H(1,0,1,v)-G(0,u)H(1,1,0,v)+4G(1-v,u)H(1,1,0,v)-2G(-v,u)H(1,1,0,v)+H(0,0,1,1,v)+H(0,1,0,1,v)\\
&+G(1-v,1-v,u)H(0,0,v)+2G(1-v,1-v,-v,u)H(1,v)-G(1-v,-v,0,1-v,u)+H(0,1,1,0,v)+G(1-v,0,1-v,0,u)-G(0,1-v,1,0,u)\\
&+4G(-v,1-v,-v,1-v,u)
\end{aligned}
$$

QCD

$\Updownarrow$

$$
\begin{aligned}
\mathcal{R}_3^{(2)} \;=\; & -2\left[J_4\left(-\frac{uv}{w}\right)+J_4\left(-\frac{vw}{u}\right)+J_4\left(-\frac{wu}{v}\right)\right]-8\sum_{i=1}^{3}\left[\mathrm{Li}_4\left(1-u_i^{-1}\right)+\frac{\log^4 u_i}{4!}\right]\\
& -2\left[\sum_{i=1}^{3}\mathrm{Li}_2(1-u_i)+\frac{\log^2 u_i}{2!}\right]^2+\frac{1}{2}\left[\sum_{i=1}^{3}\log^2 u_i\right]^2-\frac{\log^4(uvw)}{4!}-\frac{23}{2}\zeta_4
\end{aligned}
$$

N=4

Possible explanations

It is known before that anomalous dimension of N=4 is equal to leading transcendental QCD result. “Principle of Maximal Transcendentality”

N=4 = maximal transcendental piece of QCD

This is a first example for non-trivial kinematic dependent functions.

It is also possible that this is accidental for three-point case, due to the highly constraints, if QCD also have similar collinear behavior.

(We need more data. QCD two-loop computation is a much harder challenge.)

Implications

N=4 SYM may have closer relation to QCD than we expected.

Power of symbol technique.

Old philosophy: $\text{result} = \sum \text{coeff} \times \text{Integral}$

New philosophy:

compute the final expression directly, in a simpler way !

Thank you.