Interfaces

in conformal field theories and Landau-Ginzburg models

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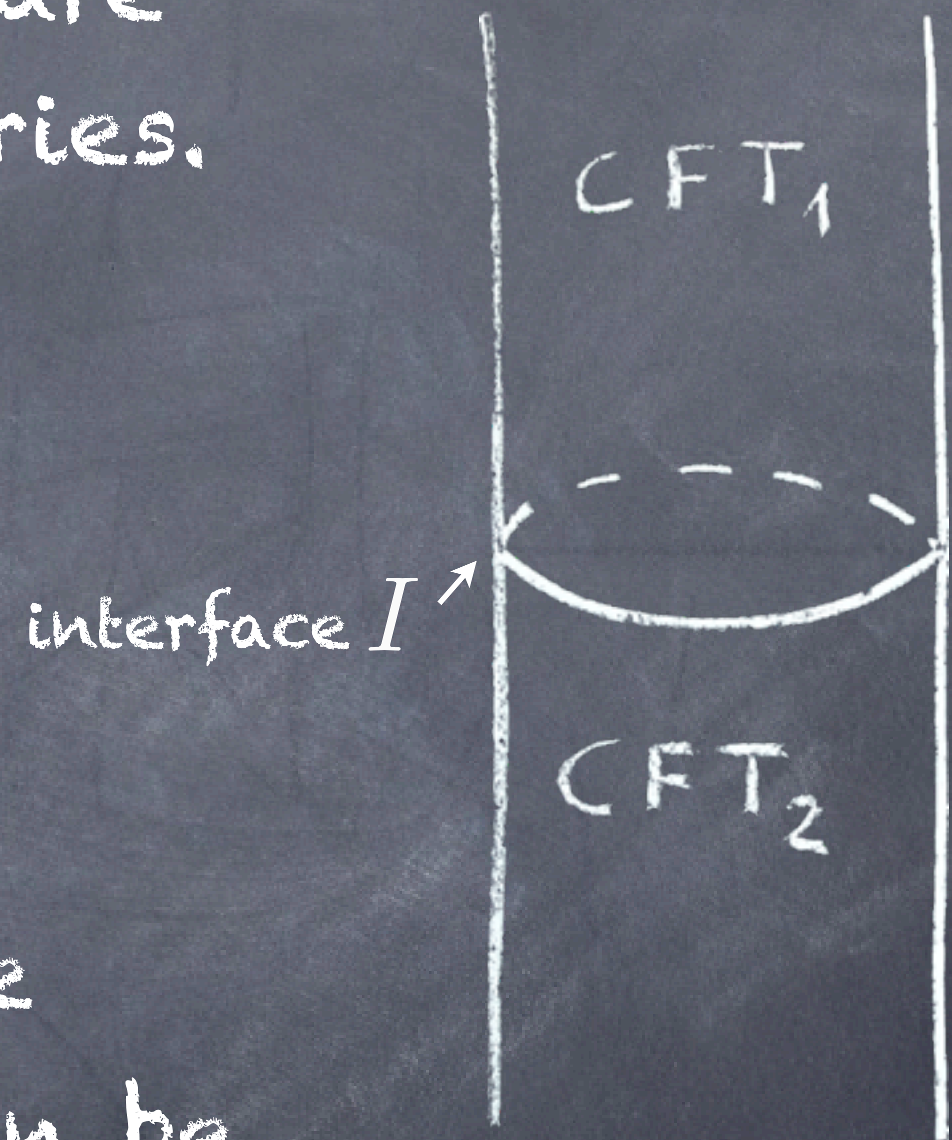
What are interfaces?

Interfaces in 2 dimensions are junctions of two field theories.



Conformal interface between conformal FTs:

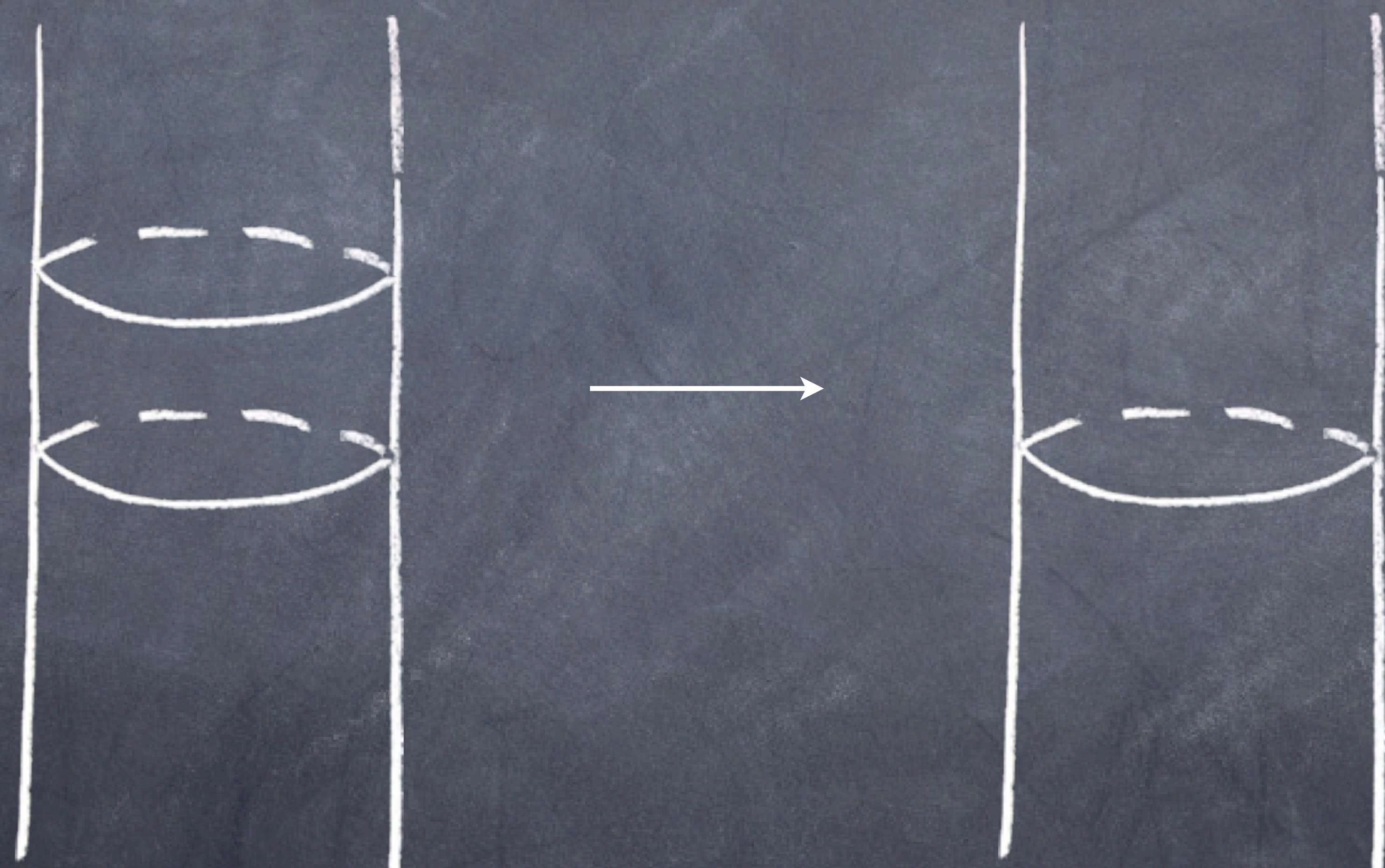
T_{01} continuous

If all components of T are continuous, the interface can be arbitrarily deformed (tensionless line).

⟶ topological interface [Petkova, Zuber '00]

Fusion

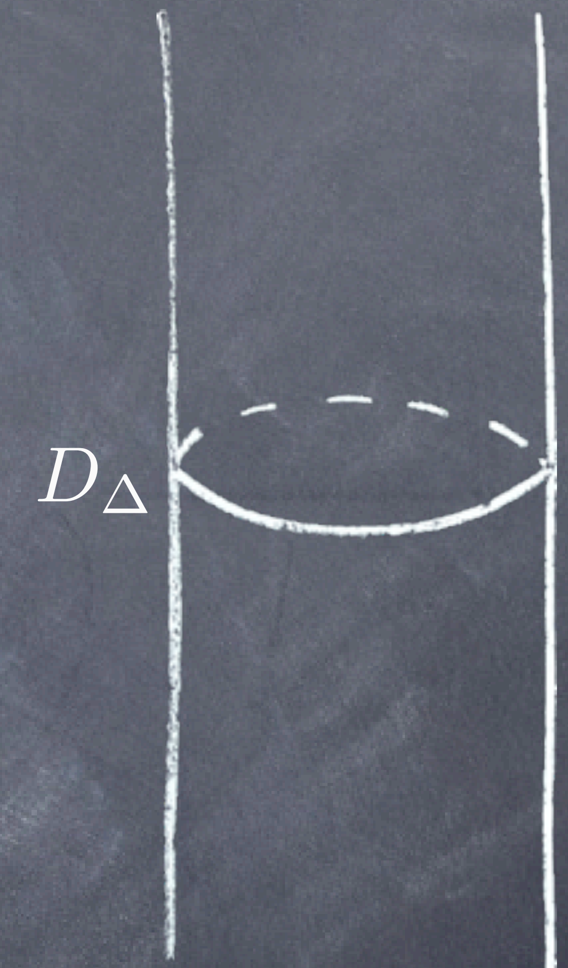
Interfaces can be fused:



Most remarkable property of interfaces!

Example: free boson $X(\tau,\sigma)$

Introduce a defect that shifts the field, $X \to X + \Delta$.



This is achieved via the defect operator

$$D_\Delta = e^{-ip\Delta}$$

D_Δ generate symmetries, and they fuse as

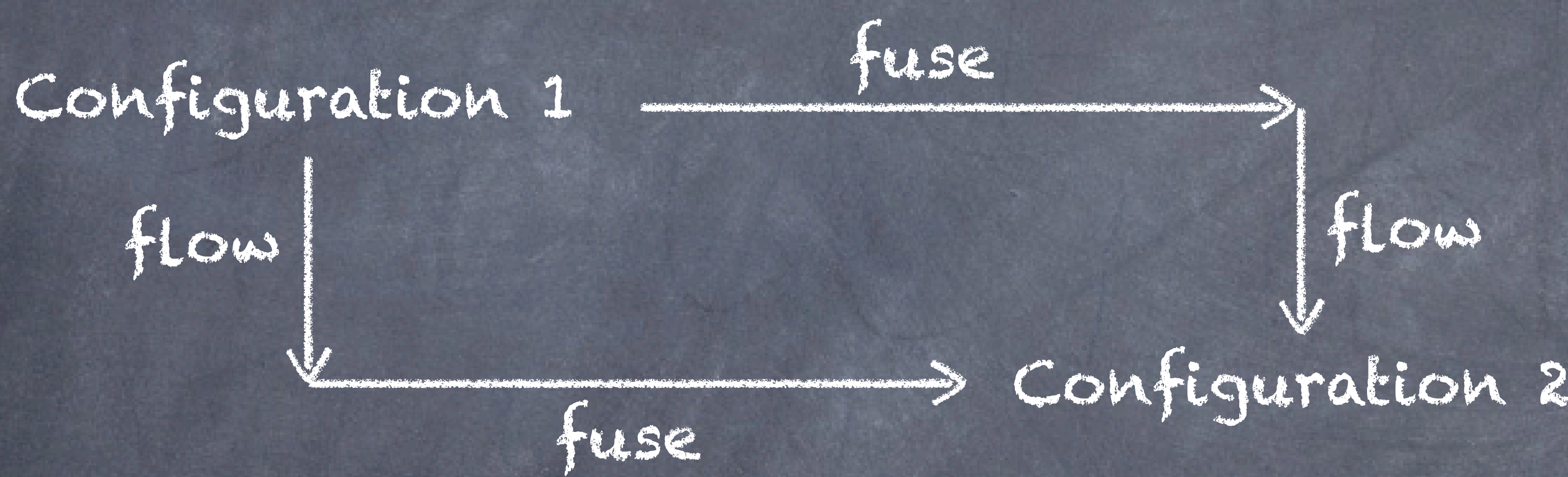
$$D_\Delta \cdot D_{\Delta'} = D_{\Delta+\Delta'}$$

Applications of interfaces

Structures (symmetries, dualities)

[Fröhlich, Fuchs, Runkel, Schweigert, ...]

Renormalisation group flows:



- boundary/defect flows [Graham, Watts; Bachas, Gaberdiel]
- coupled bulk-boundary flows [S.F., Gaberdiel, Schmidt-Colinet]
- encode bulk flows in interfaces [Brunner, Roggenkamp; Gaiotto]

Formulations

(R)CFT Interfaces: operators

Fusion: multiplication [Petkova, Zuber]

(Rational topol. defects -> fusion ring)

NLSM Interfaces: bibranes $Q \subset M_1 \times M_2$

Fusion: "intersection" [Fuchs, Schweigert, Waldorf]

Landau-Ginzburg Interfaces(B): matrix factorisations

Fusion: tensor product ← hard to compute!

[Khovanov, Rozansky; Brunner, Roggenkamp; Carqueville, Runkel]

Now: - Interfaces in LG models

- Variable transformation interfaces

Boundary conditions in LG models

Some N=(2,2) CFTs are realised as IR limit of Landau-Ginzburg models.

$$S = \int dz^2 d\theta^4 \bar{\Phi}\Phi + \int dz^2 d\theta^2 W(\Phi) + c.c.$$

F-term not invariant under B-type SUSY variation in the presence of a boundary:

$$\delta_B \int_\Sigma dz^2 d\theta^2 \ W(\Phi) \sim \bar{\epsilon} \int_{\partial\Sigma} dx d\theta \ W(\Phi)$$

[Warner]

Add boundary fermion Π with $\bar{D}\Pi = E(\Phi)$

and potential $\int_{\partial\Sigma} dx d\theta J(\Phi)\Pi$

SUSY if $J \cdot E = W$

[Kapustin, Li; Brunner, Herbst, Lerche, Scheuner]

Matrix factorisations

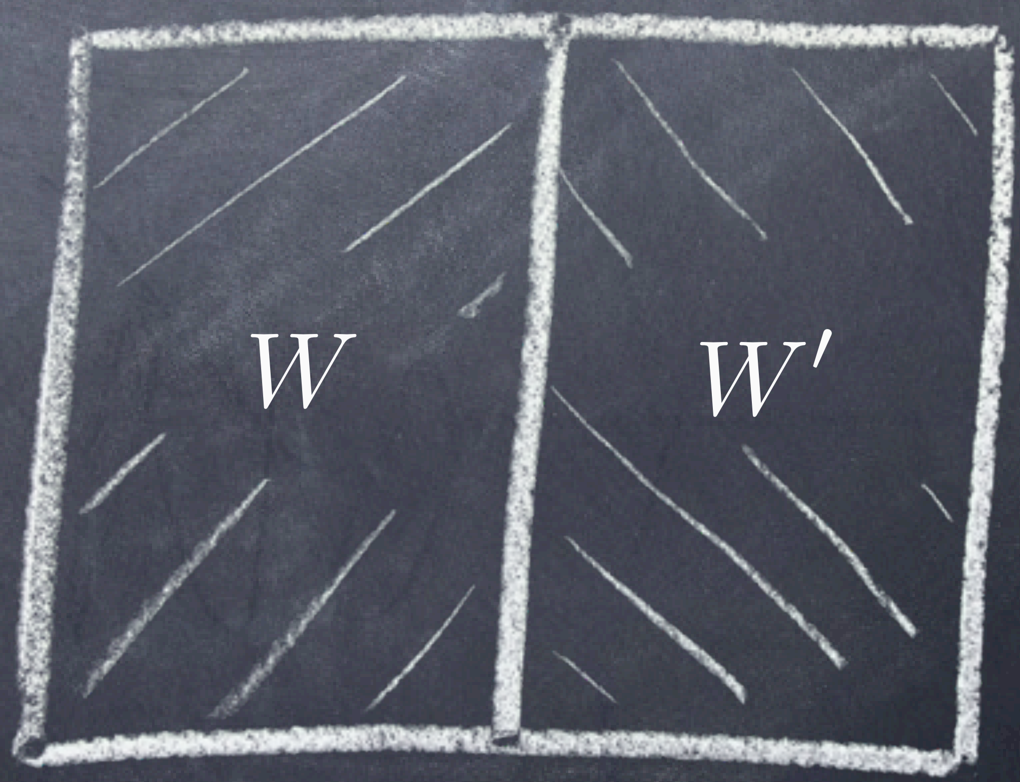
in general: E and J are matrices

Combine E and J : $Q = \begin{pmatrix} 0 & E \\ J & 0 \end{pmatrix}$

$$EJ = JE = W \cdot \mathbf{1} \quad \Leftrightarrow \quad Q^2 = W \cdot \mathbf{1}$$

For an interface:

$$({}_xQ_{x'})^2 = W(x) - W'(x')$$



[Brunner, Roggenkamp]

Example: minimal models $W = x^k$

Boundary: Elementary factorisations

$$x^{\ell+1} \cdot x^{k-\ell-1} = x^k$$

Identified with CFT boundary states.

[Kapustin, Li]

Defects: Elementary factorisations from

$$\prod_{\eta} (x_1 - \eta x_2) = x_1^k - x_2^k \qquad (\eta^k = 1)$$

Some identified with CFT defects.

[Brunner, Roggenkamp]

Fusion

Fusion described by tensor product:

$$
{}_xQ_{x'} \otimes {}_{x'}\tilde{Q}_{x''} := \begin{pmatrix} 0 & & J\otimes\mathbf{1} & \mathbf{1}\otimes\tilde{J} \\ & & -\mathbf{1}\otimes\tilde{E} & E\otimes\mathbf{1} \\ E\otimes\mathbf{1} & -\mathbf{1}\otimes\tilde{J} & 0 & \\ \mathbf{1}\otimes\tilde{E} & J\otimes\mathbf{1} & & \end{pmatrix}
$$

is a factorisation of

$$
({}_xQ_{x'})^2 + ({}_{x'}\tilde{Q}_{x''})^2 = W(x) - W''(x'')
$$

Problem: this MF still depends on x' !

Effectively the MF has infinite size.

Its reduction to finite size is in general a difficult problem.

Variable transformations

LG models related by a variable transformation:

$$\phi : Y = \mathbb{C}[y_i] \to X = \mathbb{C}[x_j]$$

$$\phi(W_y) = W_x$$

Example:

$$y_1 \to x_1 + x_2 \qquad y_2 \to x_1 x_2$$

$$W_x = x_1^k + x_2^k \qquad W_y = \dots$$

W_x : product of two minimal models

W_y : $SU(3)/U(2)$ Kazama-Suzuki model

Variable transformations

How to relate MFs in the models?

One obvious way: " y -> x "

Take Q_y and replace variables: $\phi(Q_y)$

$$\phi(Q_y)\phi(Q_y) = \phi(W_y) = W_x$$

What about the other direction " x -> y " ?

X can be seen as a Y-module: ${}_yX$

Multiplication by $p_y \in Y$ is done via $\phi : Y \to X$

$$p_y \cdot p_x := \phi(p_y)p_x \quad \text{for} \quad p_x \in X$$

Example: $y_1 = x_1 + x_2 \qquad y_2 = x_1 x_2$

$$p(x_1, x_2) = p_1(y_1, y_2) + (x_1 - x_2) p_2(y_1, y_2)$$

${}_yX$ is a free module: $\rho : {}_yX \xrightarrow{\sim} {}_yY \oplus {}_yY$

$$X \xrightarrow{p} X \quad \longrightarrow \quad Y \oplus Y \xrightarrow{\rho \circ p \circ \rho^{-1}} Y \oplus Y$$

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \xrightarrow{\rho^{-1}} p_1 + (x_1 - x_2) p_2$$

$$\xrightarrow{p} p p_1 + (x_1 - x_2) p p_2$$

$$\xrightarrow{\rho} \begin{pmatrix} p_S p_1 + (x_1 - x_2) p_A p_2 \\ \frac{p_A}{x_1 - x_2} p_1 + p_S p_2 \end{pmatrix}$$

$$= \begin{pmatrix} p_S & (x_1 - x_2) p_A \\ \frac{p_A}{x_1 - x_2} & p_S \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

Interfaces

These two natural maps on MFs are realised by fusion of an interface ${}_y\mathcal{I}_x$.

[Behr, S.F.]

$Q_y \otimes {}_y\mathcal{I}_x \cong \phi^*(Q_y)$ (replace y_i)

${}_y\mathcal{I}_x \otimes {}_xQ \cong \phi_*({}_xQ)$ (conjugate matrix elements by ρ)

Simple fusion behaviour!

By fusing those we can construct more interfaces with simple fusion.

Application

Kazama-Suzuki models $SU(3)/U(2)$ [Behr, S.F.]

- ${}_y\mathcal{I}_x$ can be used to obtain MFs for rational boundary states
- also rational topological defects are obtained as MFs
- rational fusion semi-ring of defects (should be) realised by MFs

Khovanov-Rozansky link homology:
MFs appearing there are of that type!

Summary

- Interfaces are powerful tools in two-dimensional theories, because they can be <u>fused</u>
- In LG models, B-type interfaces are described by matrix factorisations
- Fusion of MF interfaces is in general hard
- Variable transformation interfaces have a simple fusion behaviour