Symmetries of scattering amplitudes in $\mathcal{N} = 4$ SYM

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based on work with

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Johannes Henn and Theodor Schuster

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$\mathcal{N}=4$ super Yang Mills: Most symmetric interacting 4d QFT

- Field content: All fields in adjoint of $SU(N)$, $N \times N$ matrices
  - Gluons: A_μ
  - 6 real scalars: Φ_I
  - $4 \times \mathbf{4}$ real fermions: $\Psi_{\alpha\, A}$
- Action: Unique model completely fixed by SUSY

$$S = \frac{1}{{g_{\mathrm{YM}}}^2} \int d^4x \, \mathrm{Tr}\Big[\tfrac{1}{4}F_{\mu\nu}^2 + \tfrac{1}{2}(D_\mu \Phi_I)^2 - \tfrac{1}{4}[\Phi_I,\Phi_J][\Phi_I,\Phi_J] + \bar{\Psi}^A_{\dot\alpha}\sigma_\mu^{\dot\alpha\beta}\mathcal{D}^\mu\Psi_{\beta\, A} - \tfrac{i}{2}\Psi_{\alpha\, A}\sigma_I^{AB}\epsilon^{\alpha\beta}[\Phi^I,\Psi_{\beta\, B}] - \tfrac{i}{2}\bar{\Psi}_{\dot\alpha\, A}\sigma_I^{AB}\epsilon^{\dot\alpha\dot\beta}[\Phi^I,\bar{\Psi}_{\dot\beta\, B}]\Big]$$

- $\boxed{\beta_{g_{\mathrm{YM}}} = 0}$: Quantum Conformal Field Theory, 2 parameters: N & $\lambda = {g_{\mathrm{YM}}}^2 N$
- Shall consider 't Hooft planar limit: $N \to \infty$ with λ fixed.
- Is the 4d interacting QFT with highest degree of symmetry!

$\Rightarrow$ **"H-atom of gauge theories"**

Superconformal symmetry

- Symmetry: $\mathfrak{so}(2,4) \otimes \mathfrak{so}(6) \subset \mathfrak{psu}(2,2|4)$

  Poincaré: $p^{\alpha\dot{\alpha}} = p_\mu \, (\sigma^\mu)^{\dot{\alpha}\beta}, \quad l_{\alpha\beta}, \quad \bar{l}_{\dot{\alpha}\dot{\beta}}$

  Conformal: $k_{\alpha\dot{\alpha}}, \quad d$ (c : central charge)

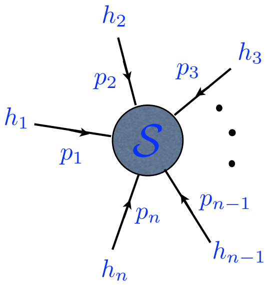
  R-symmetry: r_{AB}

  Poncaré Susy: $q^{\alpha\, A}, \bar{q}^{\dot{\alpha}}_A$

  Conformal Susy: $s_{\alpha\, A}, \bar{s}^A_{\dot{\alpha}}$

Scattering amplitudes in $\mathcal{N}=4$ SYM

- Consider n-particle scattering amplitude



- Planar amplitudes most conveniently expressed in color ordered formalism:

$$A_n(\{p_i, h_i, a_i\}) = \delta^{(4)}(\sum_{i=1}^{n} p_i) \sum_{\sigma \in S_n/Z_n} g^{n-2}\, \mathrm{tr}[t^{a_{\sigma_1}} \dots t^{a_{\sigma_n}}] \\ \times \mathcal{A}_n(\{p_{\sigma_1}, h_{\sigma_1}\}, \dots, \{p_{\sigma_1}, h_{\sigma_1}\}; \lambda = g^2 N)$$

$\mathcal{A}_n$: Color ordered amplitude. Color structure is stripped off.

Helicity of ith particle: $h_i = 0$ scalar, $h_i = \pm 1$ gluon, $h_i = \pm\frac{1}{2}$ gluino

Spinor helicity formalism

- Express momentum and polarizations via commuting spinors $\lambda^\alpha,\ \tilde{\lambda}^{\dot\alpha}$:

$$p^{\alpha\dot\alpha} = (\sigma^\mu)^{\alpha\dot\alpha}\, p_\mu = \lambda^\alpha \tilde{\lambda}^{\dot\alpha} \quad \Leftrightarrow \quad p_\mu\, p^\mu = \det p^{\alpha\dot\alpha} = 0$$

- Choice of helicity determines polarization vector ε^μ of external gluon

$$h = +1 \qquad \varepsilon^{\alpha\dot\alpha} = \frac{\lambda^\alpha \tilde{\mu}^{\dot\alpha}}{[\tilde{\lambda}\,\tilde{\mu}]} \qquad [\tilde{\lambda}\,\tilde{\mu}] := \epsilon^{\dot\alpha\dot\beta}\,\tilde{\lambda}_{\dot\alpha}\tilde{\mu}_{\dot\beta}$$

$$h = -1 \qquad \tilde{\varepsilon}^{\alpha\dot\alpha} = \frac{\mu^\alpha \tilde{\lambda}^{\dot\alpha}}{\langle\lambda\,\mu\rangle} \qquad \langle\lambda\,\mu\rangle := \epsilon_{\alpha\beta}\,\lambda^\alpha \mu^\beta$$

  $\mu, \bar{\mu}$ arbitrary reference spinors.

- E.g. scalar products: $2\,p_1 \cdot p_2 = \langle\lambda_1, \lambda_2\rangle\,[\tilde{\lambda}_2, \tilde{\lambda}_1] = \langle 1, 2\rangle\,[2, 1]$
- Helicity assignments:

$$h(\lambda^\alpha) = -1/2 \qquad h(\tilde{\lambda}^{\dot\alpha}) = +1/2$$

Trees

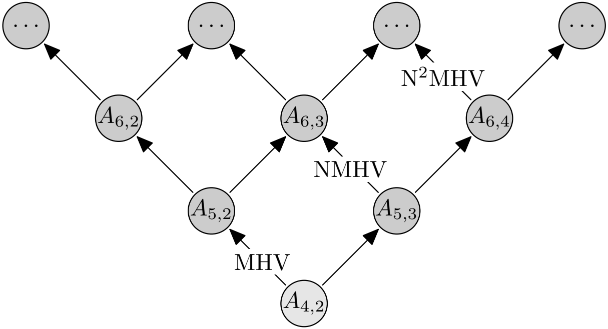
Gluon Amplitudes and Helicity Classification

Classify gluon amplitudes by # of helicity flips

- By SUSY Ward identities: $\mathcal{A}_n(1^+, 2^+, \ldots, n^+) = 0 = \mathcal{A}_n(1^-, 2^+, \ldots, n^+)$ true to all loops
- Maximally helicity violating (MHV) amplitudes

$$\mathcal{A}_n(1^+, \ldots, i^-, \ldots, j^-, \ldots n^+) = \delta^{(4)}(\sum_i p_i) \frac{\langle i, j \rangle^4}{\langle 1, 2 \rangle \langle 2, 3 \rangle \ldots \langle n, 1 \rangle}$$ [Parke, Taylor]

- Next-to-maximally helicity amplitudes (N^{k}MHV) have more involved structure!



$$A_{n,m} : g_+^{n-m} g_-^m$$

[Picture from T. McLoughlin]

On-shell superspace

- Augment λ_i^{α} and $\tilde{\lambda}_i^{\dot{\alpha}}$ by Grassmann variables $\eta_i^A \quad A = 1, 2, 3, 4$ [Nair]
- On-shell superspace $(\lambda_i^{\alpha}, \tilde{\lambda}^{\dot{\alpha}}, \eta_i^A)$ with on-shell superfield:

$$\varphi(p,\eta) = G^+(p) + \eta^A \Gamma_A(p) + \frac{1}{2}\eta^A\eta^B S_{AB}(p) + \frac{1}{3!}\eta^A\eta^B\eta^C \epsilon_{ABCD}\bar{\Gamma}^D(p) + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D \, \epsilon_{ABCD} G^-(p)$$

- Superamplitudes: $\Big\langle \varphi(\lambda_1, \tilde{\lambda}_1, \eta_1)\, \varphi(\lambda_2, \tilde{\lambda}_2, \eta_2) \ldots \varphi(\lambda_n, \tilde{\lambda}_n, \eta_n) \Big\rangle$

  Packages all n-parton gluon$^{\pm}$-gluino$^{\pm 1/2}$-scalar amplitudes

- General form of **tree superamplitudes**:

$$\mathbb{A}_n = \frac{\delta^{(4)}(\sum_i \lambda_i \tilde{\lambda}_i)\, \delta^{(8)}(\sum_i \lambda_i \eta_i)}{\langle 1,2\rangle \langle 2,3\rangle \ldots \langle n,1\rangle} \mathcal{P}_n(\{\lambda_i, \tilde{\lambda}_i, \eta_i\})$$

  Conservation of super-momentum: $\delta^{(8)}(\sum_i \lambda^{\alpha}\eta_i^A) = (\sum_i \lambda^{\alpha}\eta_i^A)^8$

- η-expansion of $\mathcal{P}_n$ yields N^{k}MHV-classification of superamps as $h(\eta) = -1/2$

$$\mathcal{P}_n = \mathcal{P}_n^{\text{MHV}} + \eta^4\, \mathcal{P}_n^{\text{NMHV}} + \eta^8\, \mathcal{P}_n^{\text{NNMHV}} + \ldots + \eta^{4n-8}\, \mathcal{P}_n^{\overline{\text{MHV}}}$$

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Symmetries

$\mathfrak{su}(2,2|4)$ invariance

- Superamplitude: $(i = 1, \ldots, n)$

$$\boxed{\mathbb{A}_n^{\text{tree}}(\{\lambda_i, \tilde\lambda_i, \eta_i\}) = \frac{\delta^{(4)}(\sum_i \lambda_i^\alpha \tilde\lambda_i^{\dot\alpha})\, \delta^{(8)}(\sum_i \lambda_i^\alpha \eta_i^A)}{\langle 1,2\rangle \langle 2,3\rangle \ldots \langle n,1\rangle}\, \mathcal{P}_n(\{\lambda_i, \tilde\lambda_i, \eta_i\})}$$

- Realization of $\mathfrak{psu}(2,2|4)$ generators in **on-shell superspace**, e.g. [Witten]

$$p^{\alpha\dot\alpha} = \sum_{i=1}^n \lambda_i^\alpha\, \tilde\lambda_i^{\dot\alpha} \qquad q^{\alpha\, A} = \sum_{i=1}^n \lambda_i^\alpha\, \eta_i^A \qquad \Rightarrow \text{ obvious symmetries}$$

$$k_{\alpha\dot\alpha} = \sum_{i=1}^n \frac{\partial}{\partial\lambda_i^\alpha}\, \frac{\partial}{\partial\tilde\lambda_i^{\dot\alpha}} \qquad s_{\alpha\, A} = \sum_{i=1}^n \frac{\partial}{\partial\lambda_i^\alpha}\, \frac{\partial}{\partial\eta_i^A} \qquad \Rightarrow \text{ less obvious sym}$$

- Invariance: $\boxed{\{\, p, k, l, \bar l, d, r, q, \bar q, s, \bar s, c_i \,\}\, \mathbb{A}_n^{\text{tree}}(\{\lambda_i^\alpha, \tilde\lambda_i^{\dot\alpha}, \eta_i^A\}) = 0}$

- N.B.: **Local** invariance $\quad h_i\, \mathbb{A}_n = 1 \cdot \mathbb{A}_n$

Helicity operator: $\quad h_i = -\tfrac{1}{2}\, \lambda_i^\alpha\, \partial_{i\,\alpha} + \tfrac{1}{2}\, \tilde\lambda_i^{\dot\alpha}\, \partial_{i\,\dot\alpha} + \tfrac{1}{2}\, \eta_i^A\, \partial_{i\,A} = 1 - c_i$

$\mathfrak{su}(2,2|4)$ invariance

- The $\mathfrak{su}(2,2|4)$ generators acting in on-shell superspace $(\lambda_i^\alpha, \tilde\lambda_i^{\dot\alpha}, \eta_i^A)$:

$$p^{\dot\alpha\alpha} = \sum_i \tilde\lambda_i^{\dot\alpha}\lambda_i^\alpha\,, \qquad k_{\alpha\dot\alpha} = \sum_i \partial_{i\alpha}\partial_{i\dot\alpha}\,,$$

$$\overline{m}_{\dot\alpha\dot\beta} = \sum_i \tilde\lambda_{i(\dot\alpha}\partial_{i\dot\beta)}, \qquad m_{\alpha\beta} = \sum_i \lambda_{i(\alpha}\partial_{i\beta)}\,,$$

$$d = \sum_i [\tfrac{1}{2}\lambda_i^\alpha\partial_{i\alpha} + \tfrac{1}{2}\tilde\lambda_i^{\dot\alpha}\partial_{i\dot\alpha} + 1], \qquad r^A{}_B = \sum_i [-\eta_i^A\partial_{iB} + \tfrac{1}{4}\delta_B^A\eta_i^C\partial_{iC}]\,,$$

$$q^{\alpha A} = \sum_i \lambda_i^\alpha\eta_i^A\,, \qquad \bar q_A^{\dot\alpha} = \sum_i \tilde\lambda_i^{\dot\alpha}\partial_{iA}\,,$$

$$s_{\alpha A} = \sum_i \partial_{i\alpha}\partial_{iA}, \qquad \bar s_{\dot\alpha}^A = \sum_i \eta_i^A\partial_{i\dot\alpha}\,,$$

$$\partial_{i\alpha} := \frac{\partial}{\partial\lambda_i^\alpha} \qquad \partial_{iA} := \frac{\partial}{\partial\eta_i^A}\,.$$

- N.B: For collinear momenta ($\lambda_i \sim \lambda_{i+1}$) important additional length changing terms, due to holomorphic anomaly $\frac{\partial}{\partial\tilde\lambda^{\dot\alpha}}\frac{1}{\langle\lambda,\mu\rangle} = 2\pi\tilde\mu_{\dot\alpha}\,\delta^2(\langle\lambda,\mu\rangle)$

[Bargheer, Beisert, Galleas, Loebbert, McLoughlin] [Korchemsky, Sokatchev] [Skinner, Mason][Arkani-Hamed, Cachazo, Kaplan]

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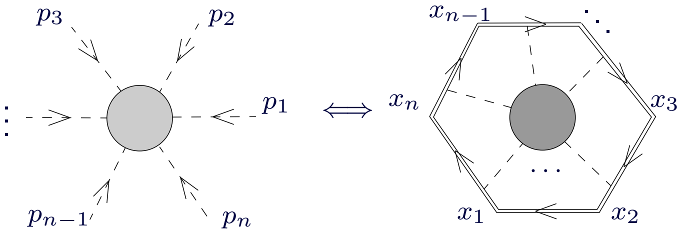
Dual conformal symmetry



$$x^{\mu}_{i+1} - x^{\mu}_{i} = p^{\mu}_{i}$$

- Trees are dual superconformal invariant: $(\theta_i - \theta_{i+1})^{\alpha\, A} = \lambda^{\alpha}_i\, \eta^A_i$

  [Drummond, Henn, Korchemsky, Sokatchev]

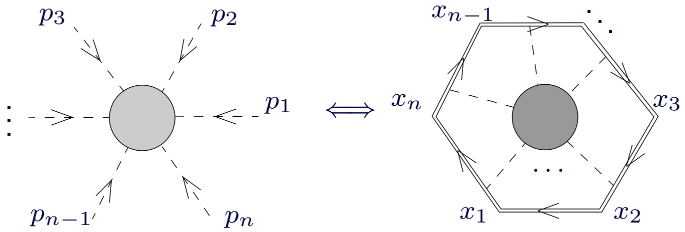
- Dual conformal generator: $K^{\mu} = \sum_i \left(2x^{\mu}_i\, x^{\nu}_i - x^2_i\, \eta^{\mu\nu}\right) \frac{\partial}{\partial x^{\nu}_i} + \text{ferms}$

- Translate to original on-shell superspace: $x^{\alpha\dot\alpha}_i = \sum_{j=1}^{i-1} \lambda^{\alpha}_j\, \tilde\lambda^{\dot\alpha}_j + x^{\alpha\dot\alpha}_1$ etc.

$$\Rightarrow \quad K^{\alpha\dot\alpha} = \sum_{i=1}^{n} x^{\dot\alpha\beta}_i\, \lambda^{\alpha}_i\, \frac{\partial}{\partial \lambda^{\beta}_i} + x^{\alpha\dot\beta}_{i+1}\, \tilde\lambda^{\dot\alpha}_i\, \frac{\partial}{\partial \tilde\lambda^{\dot\beta}_i} + \tilde\lambda^{\dot\alpha}_i\, \theta^{\alpha B}_{i+1}\, \frac{\partial}{\partial \eta^B_i} + x^{\alpha\dot\alpha}_i$$

Nonlocal structure!

Dual conformal symmetry



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Nonlocal structure!

Yangian symmetry of scattering amplitudes in $\mathcal{N} = 4$ SYM

- Superconformal + Dual superconformal algebra = Yangian $Y[\mathfrak{psu}(2,2|4)]$

[Drummond, Henn, JP]

$$[J_a^{(0)}, J_b^{(0)}\} = f_{ab}{}^c \, J_c^{(0)}$$ conventional superconformal symmetry

$$[J_a^{(0)}, J_b^{(1)}\} = f_{ab}{}^c \, J_c^{(1)}$$ from dual conformal symmetry

with nonlocal generators

$$J_a^{(1)} = f^{cb}{}_a \sum_{1<j<i<n} J_{i,b}^{(0)} \, J_{j,c}^{(0)}$$

and super Serre relations (representation dependent).

[Dolan,Nappi,Witten]

- In particular: Bosonic invariance $p^{(1)}_{\alpha\dot\alpha} \, \mathbb{A}_n = 0$ with

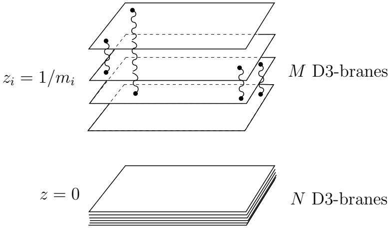
$$p^{(1)}_{\alpha\dot\alpha} = K_{\alpha\dot\alpha} + \Delta K_{\alpha\dot\alpha}$$
$$= \frac{1}{2} \sum_{i<j} (l_{i,\,\alpha}{}^{\gamma} \delta^{\dot\gamma}_{\dot\alpha} + \bar{l}_{i,\,\dot\alpha}{}^{\dot\gamma} \delta^{\gamma}_{\alpha} - d_i \, \delta^{\gamma}_{\alpha} \delta^{\dot\gamma}_{\dot\alpha}) \, p_{j,\,\gamma\dot\gamma} + \bar{q}_{i,\,\dot\alpha C} \, q^{C}_{j,\alpha} - (i \leftrightarrow j)$$

- In fact $J_a^{(0)}$ and $p^{(1)}$ generate all of $Y[\mathfrak{psu}(2,2|4)]$

Loops

Higher loops and Higgs regulator

- Beyond tree-level: Need of regularization (IR divergences) a priori breaks conformal and dual conformal symmetry
- Standard regularization method Dim reduction $10 \to 4 - \epsilon$
- Alternative method: Massive or Higgs regulator [Alday, Henn, JP, Schuster]
- Close connection to 6d amplitudes in $\mathcal{N} = (1,1)$ super Yang-Mills
- String picture serious:



(a)

- Field Theory: Higgsing $U(N+M) \to U(N) \times U(1)^M$.
  One brane for every scattered particle, $N \gg M$.

Higgsing $\mathcal{N} = 4$ Super Yang-Mills

- Action

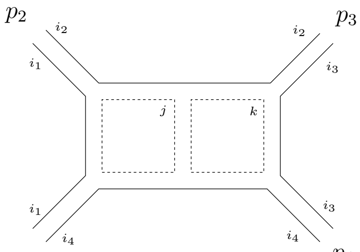
$$\hat{S}^{U(N+M)}_{\mathcal{N}=4} = \int d^4x \, \mathrm{Tr}\Big(-\frac{1}{4}\hat{F}^2_{\mu\nu} - \frac{1}{2}(D_\mu \hat{\Phi}_I)^2 + \frac{g^2}{4}[\hat{\Phi}_I, \hat{\Phi}_J]^2 + \text{ferms}\Big),$$

- Decompose into $N + M$ blocks

$$\hat{A}_\mu = \begin{pmatrix} (A_\mu)_{ab} & (A_\mu)_{aj} \\ (A_\mu)_{ia} & (A_\mu)_{ij} \end{pmatrix}, \qquad \hat{\Phi}_I = \begin{pmatrix} (\Phi_I)_{ab} & (\Phi_I)_{aj} \\ (\Phi_I)_{ia} & \boldsymbol{\delta_{I9}\,\frac{m_i}{g}\,\delta_{ij}} + (\Phi_I)_{ij} \end{pmatrix}$$

$$a, b = 1, \ldots, N, \quad i, j = N+1, \ldots, N+M,$$

- Yields mass terms and novel bosonic 3-point interactions proportional to m_i
- Renders amplitudes IR finite. Has 'light' $(m_i - m_j)$ and 'heavy' m_i W-bosons

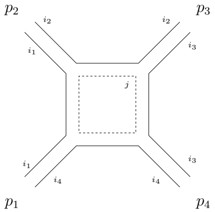


One loop test of extended dual conformal symmetry 1

- Consider the (special) purely scalar amplitude:

$$A_4 = \langle \Phi_4(p_1)\, \Phi_5(p_2)\, \Phi_4(p_3)\, \Phi_5(p_4) \rangle = i g_{\mathrm{YM}}^2 \Big(1 + \lambda\, I^{(1)}(s,t,m_i) + O(\lambda^2) \Big)$$

$I^{(1)}(s,t,m_i)$: Massive box integral in dual variables ($p_i = x_i - x_{i+1}$)



$$= \int d^4 x_5 \frac{(x_{13}^2 + (m_1 - m_3)^2)(x_{24}^2 + (m_2 - m_4)^2)}{(x_{15}^2 + m_1^2)(x_{25}^2 + m_2^2)(x_{35}^2 + m_3^2)(x_{45}^2 + m_4^2)}$$

- Reexpressed in 5d variables $\hat{x}^M$: $\hat{x}_i^\mu := x_i^\mu\,, \quad \hat{x}_i^4 := m_i\,,$ $\qquad i = 1 \ldots 4$

$$I^{(1)}(s,t,m_i) = \hat{x}_{13}^2 \hat{x}_{24}^2 \int d^5 \hat{x}_5 \frac{\delta(\hat{x}_5^{M=4})}{\hat{x}_{15}^2 \hat{x}_{25}^2 \hat{x}_{35}^2 \hat{x}_{45}^2}$$

- $I^{(1)}(s,t,m_i)$ is extended dual conformal invariant: $\hat{K}_\mu I^{(1)}(s,t,m_i) = 0$

Extended dual conformal invariance

- Extended dual conformal invariance

$$\hat{K}_\mu \, I^{(1)}(s,t,m_i) := \sum_{i=1}^{4} \left[2x_{i\mu} \left(x_i^\nu \frac{\partial}{\partial x_i^\nu} + m_i \frac{\partial}{\partial m_i} \right) - (x_i^2 + m_i^2) \frac{\partial}{\partial x_i^\mu} \right] I^{(1)}(s,t,m_i) = 0$$

- m_i is the fifth coordinate $x^M = (x^\mu, m)$.
- Triangle and bubble graphs are forbidden by extended conformal symmetry!
- Indeed an explicit one-loop calculation shows the cancelation of triangles.
- Has been extended to higher loops & higher multiplicities as well as Regge limit

  [Henn, Naculich, Schnitzer, Drummond]

Symmetries of massively regulated loop amplitudes

[JP, Schuster]

On shell superspace for 6d $\mathcal{N} = (1,1)$ sYM

4d massively regulated $\mathcal{N} = 4$ SYM amplitudes	$\overset{\text{KK red.}}{=}$	6d massless $\mathcal{N} = (1,1)$ SYM amps with $p^5 + ip^6 = m_{4d}$ & internal loop momenta in 4d

On shell 6d superspace:

- Lorentz group $SO(1,5) \simeq SU(4)^*$ Little group $SO(4) \simeq SU(2) \times SU(2)$
- Helicity spinors: λ^{Aa} & $\tilde{\lambda}_{A\dot{a}}$ $A = 1,2,3,4$ $a = 1,2$ $\dot{a} = 1,2$
- 6d momentum: $p^{AB} = \lambda^{Aa}\,\lambda^{Bb}\,\epsilon_{ab}$ $p_{AB} = \tilde{\lambda}_{A\dot{a}}\,\lambda_{B\dot{b}}\,\epsilon^{\dot{a}\dot{b}} = \frac{1}{2}\,\epsilon_{ABCD}\,p^{AB}$
- SUSY partners: η_a & $\tilde{\eta}^{\dot{a}}$
- Supertranslations: $q^A = \lambda^{Aa}\,\eta_a$ & $\tilde{q}_A = \tilde{\lambda}_{A\dot{a}}\,\tilde{\eta}^{\dot{a}}$
- On-shell superfield as η expansion:

$$\Omega(p,\eta,\tilde{\eta}) = \phi + \psi^a\eta_a + \tilde{\psi}_a\tilde{\eta}^a + \phi'\,\eta^2 + \phi''\,\tilde{\eta}^2 + g_a^{\dot{a}}\,\eta^a\,\tilde{\eta}_{\dot{a}} \ldots + \phi''\,\eta^2\,\bar{\eta}^2$$

- Perturbative results:
  - 3,4,5 point amplitudes @ tree-level [Cheung,O'Connell]
  - Super BCFW recursion [Dennen,Huang,Siegel]
  - 4pt amplitude @ 1 & 2 loops [Bern,Carrasco,Dennen,Huang,Ita]

Dual conformal symmetry for 6d $\mathcal{N} = (1,1)$ SYM

- Introduce dual on-shell superspace

$$x_i^{AB} - x_{i+1}^{AB} = p_i^{AB} \quad \theta_i^A - \theta_{i+1}^A = q_i^A \quad \tilde{\theta}_{i\,A} - \tilde{\theta}_{i+1\,A} = \tilde{q}_{iA}$$

- Dual conformal generator:

$$K^\mu = \sum_i (2x_i^\mu\, x_i^\nu - x_i^2\, \eta^{\mu\nu})\, \frac{\partial}{\partial x_i^\nu} + \theta_i^A (\sigma^\mu)_{AB}\, x_i^{BC}\, \frac{\partial}{\partial \theta_i^C} + \dots$$

- Statement: K^μ is symmetry of 6d δ-fct. stripped super amplitudes [Dennen,Huang]

$$\boxed{(K^\mu + 2\sum_i x_i^\mu)\, \frac{\mathbb{A}_n^{L\text{-loop}}}{\delta^{(6)}(p)\, \delta^{(4)}(q)\, \delta^{(4)}(\tilde{q})} = 0}$$

- True also for higher loops $L > 0$ iff loop integration is in 4d $\Rightarrow$

Proof of extended dual conformal symmetry of Higgs-regulated $\mathcal{N} = 4$ SYM!

From 6 to 4

Question:
May we interpret dual conformal K_μ as the level one Yangian generator $p_\mu^{(1)}$ of Higgs regulated $\mathcal{N}=4$ SYM upon dim reduction to 4d?

- Needs to compactify to 4d: $p^{AB} = \begin{pmatrix} m\,\epsilon_{\alpha\beta} & -p_\alpha{}^{\dot\beta} \\ p^{\dot\alpha}{}_\beta & -\bar m\,\epsilon^{\dot\alpha\dot\beta} \end{pmatrix}$
- Helicity spinors for massive 4d particles:

$$p^{\alpha\dot\alpha} = \lambda^\alpha\tilde\lambda^{\dot\alpha} + \mu^\alpha\tilde\mu^{\dot\alpha} \qquad \text{(two sets of spinors)}$$

with constraints $\langle\lambda\,\mu\rangle = m$ and $[\tilde\mu\,\tilde\lambda] = \bar m$. Reality condition: $m=\bar m$.
- Inherited Lorentz symmetries of massively regulated $\mathcal{N}=4$ amplitudes

$$l_{\alpha\beta} = \sum_i \lambda_{i\,(\alpha}\partial_{i\,\beta)} + \mu_{i\,(\alpha}\delta_{i\,\beta)} \qquad \bar l_{\dot\alpha\dot\beta} = \ldots \qquad \Leftrightarrow \qquad l_{\mu\nu}$$

$$h_{\alpha\dot\alpha} = \sum_i \tilde\mu_{i\,\dot\alpha}\partial_{i\,\alpha} - \tilde\lambda_{i\,\dot\alpha}\delta_{i\,\alpha} + \mu_{i\,\alpha}\tilde\partial_{i\,\dot\alpha} - \lambda_{i\,\alpha}\tilde\delta_{i\,\dot\alpha} \qquad \Leftrightarrow \qquad l_{\mu 5} + i l_{\mu 6}$$

$$d = \sum_i \tfrac{1}{2}(\lambda_i^\alpha\partial_{i\,\alpha} + \tilde\lambda_i^{\dot\alpha}\tilde\partial_{i\,\dot\alpha} + \mu_i^\alpha\delta_{i\,\alpha} + \tilde\mu_i^{\dot\alpha}\tilde\delta_{i\,\dot\alpha}) + 1 \qquad \Leftrightarrow \qquad l_{56}$$

with $\delta_{i\alpha} := \frac{\partial}{\partial\mu_i^\beta}$ $\quad \partial_{i\alpha} := \frac{\partial}{\partial\lambda_i^\beta}$. Generators commute with $m=\bar m$ constraint

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From 6 to 4

- Inherited supersymmetries of massively regulated $\mathcal{N}=4$ amplitudes

$$Q^a_{i\,\alpha} = \lambda_{i\,\alpha}\bar{\xi}^a_i - \mu_{i\,\alpha}\xi^a_i \qquad Q^a_{i\,\dot\alpha} = \tilde\lambda_{i\,\dot\alpha}\xi^a_i + \tilde\mu_{i\,\dot\alpha}\bar{\xi}^a_i$$

$$\bar{Q}_{i\,\dot\alpha\,a} = \tilde\lambda_{i\,\dot\alpha}\frac{\partial}{\partial\bar{\xi}^a_i} - \tilde\mu_{i\,\dot\alpha}\frac{\partial}{\partial\xi^a_i} \qquad \bar{Q}_{i\,\alpha\,a} = \lambda_{i\,\alpha}\frac{\partial}{\partial\xi^a_i} + \mu_{i\,\alpha}\frac{\partial}{\partial\bar{\xi}^a_i}$$

with $\xi^a = \begin{pmatrix}\eta_1\\ -\tilde\eta^{\dot 1}\end{pmatrix} \quad \bar{\xi}^a = \begin{pmatrix}\eta_2\\ \tilde\eta^{\dot 2}\end{pmatrix}$

- Bosonic algebra: Standard 4d Poincaré plus:

$$[\,h_{\alpha\dot\alpha}, p_{\beta\dot\beta}\,] = 2\,\epsilon_{\alpha\beta}\,\epsilon_{\dot\alpha\dot\beta}\,m\,, \qquad [\,h_{\alpha\dot\alpha}, m\,] = p_{\alpha\dot\alpha}\,,$$

$$[\,h_{\alpha\dot\alpha}, h_{\beta\dot\beta}\,] = 2\epsilon_{\alpha\beta}\,\bar{l}_{\dot\alpha\dot\beta} + 2\epsilon_{\dot\alpha\dot\beta}\,l_{\alpha\beta}\,, \qquad [\,l_{\beta\gamma}, h_{\alpha\dot\alpha}\,] = \epsilon_{\alpha(\beta}\,h_{\gamma)\dot\alpha}\,,$$

- SUSY algebra

$$\{Q^a_\alpha, \bar{Q}_{\dot\alpha\,b}\} = p_{\alpha\dot\alpha}\,\delta^a_b \qquad \{Q^a_{\dot\alpha}, \bar{Q}_{\alpha\,b}\} = p_{\alpha\dot\alpha}\,\delta^a_b$$

$$\{Q^a_\alpha, \bar{Q}_{\beta\,b}\} = m\,\epsilon_{\alpha\beta}\,\delta^a_b \qquad \{Q^a_{\dot\alpha}, \bar{Q}_{\dot\beta\,b}\} = -m\,\epsilon_{\dot\alpha\dot\beta}\,\delta^a_b$$

Nonlocal symmetries of massively regulates $\mathcal{N} = 4$ SYM

- Rewriting the 6d dual conformal K^μ $(\mu = 0, 1, 2, 3)$ in the massive on-shell 4d superspace $\{\lambda_\alpha, \tilde{\lambda}_{\dot\alpha}; \mu_\alpha, \tilde{\mu}_{\dot\alpha}; \xi^a, \bar{\xi}^a\}$ yields

$$K_{\alpha\dot\alpha} + \Delta K_{\alpha\dot\alpha} + 2\sum_i x_{i\,\alpha\dot\alpha} - x_1\text{-terms}$$

$$= \sum_{i<j}\Big[(\epsilon_{\dot\alpha\dot\beta}\, l_{i\,\alpha\beta} + \epsilon_{\alpha\beta}\,\bar{l}_{i\,\dot\alpha\dot\beta} + \epsilon_{\alpha\beta}\,\epsilon_{\dot\alpha\dot\beta}\, d_i\,)\, p_j^{\beta\dot\beta} + h_{i\,\alpha\dot\alpha}\, m_j$$

$$- \bar{Q}_{i\,\alpha a} Q^a_{j\,\dot\alpha} - \bar{Q}_{i\,\dot\alpha a}\, Q^a_{j\,\alpha} - (i \leftrightarrow j)\Big] = \mathbf{p}^{(\mathbf{1})}_{\alpha\dot\alpha}$$

Is level-one Yangian like extension of translational part of Poincaré algebra

$$[\,h_{\alpha\dot\alpha}, p^{(1)}_{\beta\dot\beta}\,] = 2\,\epsilon_{\alpha\beta}\,\epsilon_{\dot\alpha\dot\beta}\, m^{(1)}$$

with $\quad m^{(1)} = \frac{1}{2}\sum_{j<i}\Big[h_{i\,\gamma\dot\gamma}\, p_j^{\gamma\dot\gamma} + 2d_i\, m_j - \bar{Q}_{i\,\gamma a}\, Q_j^{a\gamma} - \bar{Q}^{\dot\gamma}_{i\;a}\, Q^a_{j\,\dot\gamma} - (i \leftrightarrow j)\Big]$

Nonlocal symmetries of massively regulated $\mathcal{N}=4$ 4d SYM

- However, no ∞-dim symmetry structure emerges as $p^{(1)}_{\alpha\dot{\alpha}}$ and $m^{(1)}$ form an ideal of the algebra:

$$\mathfrak{i} = \{p, m\} \subset \mathfrak{a} \qquad [\mathfrak{a}^{(0)}, \mathfrak{i}^{(0)}] = \mathfrak{i}^{(0)} \qquad [\mathfrak{a}^{(0)}, \mathfrak{i}^{(1)}] = \mathfrak{i}^{(1)}$$

- Conclusion:

$$\boxed{\{p, m, l, \bar{l}, h, d; q, \tilde{q}\,; p^{(1)}, m^{(1)}\} \frac{\mathbb{A}_n^{L\text{-loop},\, \mathcal{N}=4 \text{ SYM}}}{\delta^{(6)}(p)\,\delta^{(4)}(q)\,\delta^{(4)}(\tilde{q})} = 0}$$

- Level 1 SUSY generators do not seem to exist
- Analysis suggests ∞-dim symmetry structure hinges upon conformal symmetry at level 0 as there

$$[k, p] \sim l + \bar{l} + d$$

Summary and Outlook

- Tree level amplitudes are invariant under an infinite dimensional Yangian symmetry
- Challenge at weak coupling: Does Yangian symmetry extend to the loop level?
- Breaking of dual conformal invariance at loop level under control: Best seen in massive (Higgs) regulator
- Restriction of possible integrals at higher loops.
- Established symmetry structure of massively regulated $\mathcal{N}=4$ SYM:

  $\text{Superpoincaré} + h_\mu + d + p_\mu^{(1)} + m^{(1)}$

- Can breaking of standard conformal invariance at loop level be controlled?
  $\exists$ perturbative construction in dim. regularization [Sever,Vieira][Beisert,Henn,McLoughlin, JP]
  Recent all loop claim [Caron-Huot,He]
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