Introduction to computed tomography

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Outline

Introduction

Reconstruction

Beam geometry

Sampling

Artifacts

Summary
Introduction
The problem

We have a solid item to investigate. . .

- For a first look of the outside

- Next step, use a transmission image

- Cut the item in pieces
Different sources to illuminate the sample

**X-rays**

- Electromagnetic radiation.
- Interaction with the electron shells.

**Neutrons**

- Neutral particle beam.
- Interaction with the nucleus.
**Introduction**

**Reconstruction**

**Beam geometry**

**Sampling**

**Artifacts**

**Summary**

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**Attenuation coefficients**

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**X-rays at 150keV**

**Thermal neutrons at 25meV/1.8Å**
Transmission image – the projection

A ray illuminates a semi-transparent medium
Transmission imaging – Radiography

A ray penetrating a medium is attenuated according to Beer-Lamberts law. The intensity is attenuated in the medium according to

\[ I = I_0 e^{\int_L k(x,y) \, dl} \]

- \( I \) - Intensity behind the sample
- \( I_0 \) - Incident intensity
- \( k \) - Attenuation coefficient,
  - \( \mu \) - Linear attenuation coefficient X-rays
  - \( \Sigma \) - Macroscopic cross-section for neutrons
- \( L \) - Line through the sample.
Computing an attenuation image

From Beer-Lamberts law we get

\[ p = -\log \left( \frac{r - r_{DC}}{r_{OB} - r_{DC}} \right) = -\log \left( \frac{r - r_{DC}}{I_0} \right) \]

- \( p \) Normed projection
- \( r \) Measured radiogram
- \( r_{DC} \) Dark current image (removes noise floor)
- \( r_{OB} \) Open beam image, measured \( I_0 \)

Each pixel represent the line integral \( \int_L k(x)dx \) through the sample.
Generalized attenuation law 1D

Piecewise constant sample

\[ I = I_0 e^{-\sum_i k_i x_i} \]

Let \( x_i = \Delta x \) and \( \Delta x \to 0 \)
We get the equation for a continuous sample

\[ I = I_0 e^{-\int_{L} k(x) \, dx} \]
What is tomography?

- Free translation is slice imaging from Greek: *Tomos* – 'a section' or 'a cutting' and Graph – write
- A method to capture three-dimensional images.
- An indirect method using projections (radiograms) to reconstruct the inner structure of a sample.
Inspecting the sample from different views
A first attempt to reconstruction: Algebraic solution

Observations

\[
\begin{align*}
2 & \quad 3 \quad \rightarrow \quad 5 \\
1 & \quad 4 \quad \rightarrow \quad 5 \\
\downarrow & \quad \downarrow \\
3 & \quad 7
\end{align*}
\]

Equation system

\[
\begin{align*}
a_{11}x_1 + a_{12}x_2 &= y_1 \\
a_{21}x_3 + a_{22}x_4 &= y_2 \\
a_{11}x_1 + a_{21}x_3 &= y_3 \\
a_{12}x_2 + a_{22}x_4 &= y_4 \\
&\vdots
\end{align*}
\]

\[\Rightarrow A\mathbf{x} = \mathbf{y}\]

solve the equation system for \(\mathbf{x}\)

Many equations, sparse matrix \(A\), no unique solution...
A first attempt to reconstruction: Basic back-projection

The solution is too smooth...something is missing!!!
History

Johann Radon
(1887–1956)
Developed the foundation for the inversion required by tomography in 1917. He found the analytical solution to the inverse of the projection of a sample.

Sir Godfrey N. Hounsfield
(1919–2004)
Constructed the first tomograph. He was awarded with the Nobel Prize in 1979
Reconstruction
Acquisition and rearranging the projection data

The sinogram is constructed by taking the same line from all projections.
The sinogram

A sinogram represents the information required to reconstruct a slice of the scanned sample.
The Radon Transform and the sinogram

A projection \( I \) acquired at angle \( \theta \)

\[
p = -\ln \left( \frac{I(u, \theta)}{I_0(u)} \right) = \int_{-\infty}^{\infty} k(x, y) \delta(x \cos \theta + y \sin \theta - u) \, dx \, dy
\]

Wanted Observation ray

This is the acquisition space.
The reverse process – reconstruction

The scanning provides projection data...

...but we want to find the cross section which caused the projection.
Inversion – Fourier slice theorem

Theorem

The Fourier transform of a parallel projection \( p(x) \) of an object \( f(x, y) \) obtained at an angle \( \theta \) equals a line through origin in the 2D Fourier transform of \( f(x, y) \) at the same angle.

\[
f(x) = \int_{-\infty}^{\infty} f(x, y) \, dy
\]

\[
f(\xi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi \xi x} \, dx \, dy
\]
Analytical solution

Reconstruction in the frequency domain:

\[ k(x, y) = \int_0^\pi \int_{-\infty}^{\infty} |\omega| P(\omega, \theta) e^{j2\pi\omega(x \cos \theta + y \sin \theta)} d\omega d\theta \]

Reconstruction in the spatial domain:

\[ k(x, y) = \frac{1}{2\pi^2} \int_0^\pi \int_{-\infty}^{\infty} \frac{\partial p}{\partial u}(u, \theta) \left[ x \cos \theta + y \sin \theta - u \right]^{-1} dud\theta \]

Convolution
Rotation

\[ \mathbf{F}_{1D} \rightarrow \mathbf{H}(\omega) \rightarrow \mathbf{F}_{1D}^{-1} \rightarrow \text{Back projection} \]
Some line integrals in the sinogram

Sinogram

Cross section
The reconstruction filter

The filter has two parts:

- A derivative: \( \frac{\partial p}{\partial u}(u, \theta) \equiv \mathcal{F}^{-1}(|\omega| \cdot \mathcal{F}(p)) \)
- Apodization: Shepp-Logan, Hamming, etc
Reconstruction filter

The high level of discretization makes the inversion noise sensitive.

None (incomplete)  Ram-Lak $\equiv \partial p / \partial u$

Shepp-Logan  Hamming

Low-pass filters are used to smooth the solution.
When the analytical solution has problems

Few projections

Irregularly distributed

Limited view

Low SNR/Low contrast
Iterative methods overview

Algebraic methods

- ART
- SIRT
- TV
- etc...

Statistical methods

- Maximum likelihood
- Penalized ML
- etc..

Pros & cons

+ Sparse, irregularly sampled projection data
  - Limited angle
  - Few views
+ Physical model can be included
  - Requires prior information for best performance.
  - Time consuming
Introduction

Reconstruction

Beam geometry

Sampling

Artifacts

Summary

Algebraic Reconstruction Method (ART)

Problem to solve

We want to solve the equation $A x = y$, where $A$ is the forward projection operator, a large sparse matrix

Kaczmarz method (ART)

$$x^{k+1} = x^k + \lambda_k \frac{y_i - \langle a_i, x^k \rangle}{\|a_i\|^2} a_i$$

- $a_i$ the $i^{th}$ row of the system matrix $A$.
- $x^k$ the reconstructed image at the $k^{th}$ iteration.
- $y_i$ the $i^{th}$ element of the sinogram
- $\lambda_k$ relaxation parameter
Statistic reconstruction methods

Problem to solve
We want to solve the equation $A x = y + N$.

Iteration scheme

![Diagram showing iteration scheme]
Beam geometry
Different beamline configurations

Static beamline

Rotating beamline

Source
Sample
Detector

Source

Detector
Pencil-beam

- Simples beam geometry
- Single pixels are scanned
- The ’Hounsfield-approach’
Parallel beam

- Produces 2D projections
- No geometric unsharpness
- Simple reconstruction, filtered back projection [Kak and Slaney, 1988]
Fan beam

- Line-wise scan
- Beam incidence must be perpendicular to detector plane
- Magnifying in one direction
Cone beam

+ Uses 2D-projections.
+ Magnifying due to beam divergence.
- Non-trivial reconstruction using [Feldkamp et al., 1984].
- Only in the central slice is exact.
Problems with cone-beam

Parallel discs

FDK (cone angle 30°)
Helical scans

- Exact 3D solution
- Long objects
- Reconstruction using Katsevich [Katsevich, 2002]
Large samples – The problem

Requirement

Projections from at least $180^\circ$ + sample must always be visible.

Two options to handle samples larger than the field of view

- Translate the COR and use a $360^\circ$ orbit.
- Truncated reconstruction
Transcribed projections

Idea

• Translate the COR to the side of the projection
• Near doubled FOV

Requirements

• The projections must be stitched
• Projections must be acquired over 360°
• More voxels requires more projections
Truncated or Local tomography

A truncated tomography has incomplete data support.

Effects of truncation

1. Some attenuation information is missing → bias
   *The shadow contains more attenuation than the projection data shows.*

2. Truncation gives spikes on the edges.
   *The derivative in the reconstruction formula produce edge artifacts.*
Removing truncation artifacts

**Origin**  The derivative of the truncated edge is steep

**Solution**  Add a smooth transition from edge to zero

![Diagram showing the origin and solution of removing truncation artifacts](image-url)
Tilted acquisition axis

Along the beam

- Hard to correct
- Requires vector based reconstructor and geometry

Across the beam

Small angles corrected with COR shifts

Large angles corrected with rotation
Sampling
Discretizing the reconstruction formula

The inversion formula is impractical since it would require infinite amount of equations to solve.

- The projections are digital images
  - Intensity sampling [bits/pixel]
  - Spatial sampling [pixels/mm]
- The rotation is done in steps
- The reconstruction is done on a finite matrix
How many projections are needed?

The number of projections is determined by the sampling theorem [Kak and Slaney, 1988].

\[ N_{\text{projections}} = \frac{\pi}{2} N_u \]

\( N_u \)  Number of pixels in the direction perpendicular to the axis of rotation.
Intuitive proof of the sampling theorem

Basic idea  The unit circle in the Fourier domain must be filled.
Noise and Dose

Noise
Noise is an additive statistical phenomenon.

\[ R^{-1} \{ \text{image} \} + R^{-1} \{ \text{noise} \} = R^{-1} \{ \text{result} \} \]

Noise sources:
- Thermal noise from the electronics.
- Algorithmic, rounding errors, interpolation model etc.
- Noise induced by the radiation source.

Dose
The dose is the amount of radiation events hitting the detector. More events improve the SNR (the law of great numbers).
Noise, exposure time, and number of projections

The noise level of a slice is directly connected to the dose used. The dose is defined as

\[ Dose = Flux \times Time \]

The signal to noise ratio can be improved by increasing

- the beam intensity,
- the exposure time,
- the number of projections,
- detector exchange.
Contrast

What influences the contrast?

\[ C_{\text{slice}} \times W_{\text{sample}} = C_{\text{projection}} \times N_{\text{projections}} \]

- \( C_{\text{slice}} \): Slice contrast
- \( C_{\text{projection}} \): Projection contrast (Open beam - darkest region)
- \( N_{\text{projections}} \): Number of projections
- \( W_{\text{sample}} \): Largest width of the sample in pixels
Contrast experiment

The phantom

Parameters

- $w = 192$
- $N_{projections} = 288$
- $C_{projection} = 6, 7, 8, 9, 10, 11, 12, 13$ bits
- Contrast ratio: 1000:1, ..., 1:2
- Noise free
What can be seen?

Changing projection contrast with constant number of projections

The reconstruction noise decrease with increasing dynamics
Artifacts
Common artifacts

**Rings** are caused by stuck or dead pixels. They have the same value for all projections.

**Lines** are caused by single pixels or groups of pixels in a single projection.

**High contrast** these artifacts appear as starlike streaks originating from the high contrast object.

**Motion** when the sample changes during acquisition.

**Beam hardening** Polychromatic beam

**Scattering** The beam is scattered
Ring artifacts

- Ring artifacts are very common artifacts in tomography.
- They are caused by a stuck, dead, or hot pixels.
- They appear as:
  - Lines in the sinogram
  - Concentric rings in the CT slices
Correction in the Radon space

**Projections** Identify and remove spots that persist through projections.

**Sinograms** Identify lines parallel to the $\theta$-axis

- Subtract first derivative of average projection form sinogram.
- Filter sinogram in Fourier domain (notch filter or wavelet filter).

\[
s(u,\theta) - 1^T \cdot (E_u[s] - \text{median}_N(E_u[s])) = \]

\[
\text{Correction in the Radon space}
\]
Correction in the matrix

Correction procedure:

• Transform matrix to polar coordinates
• Detect lines
• Make replacement map
• Transform map to Carthesian coordinates
• Correct matrix

Advantage  Good for testing different strengths

Disadvantage  The coordinate transformations
Line artifacts

- Line artifacts are more common with neutrons.
- The origin of a line is a local spot in the sinogram.
- The orientation and position depends on when the spot was registered.
Correction of the lines

Correction method

- Detect the spots on the projections – compute local variances
- Replacement e.g.

\[ p_{corrected} = w(\sigma) \cdot p + (1 - w(\sigma)) \cdot p_{median} \text{ with } 0 \leq w \leq 1 \]
Motion artifacts

Sequential acquisition

Golden ratio acquisition
Suppressing the effect of motion

Dynamic processes are hard to observe with CT

- CT needs long scan times.
- If the interfaces move more than 1 pixel during the scan motion artifacts will appear.

The solution

- Increment the acquisition angle by the Golden ratio \( \phi = \frac{1+\sqrt{5}}{2} \)
- The sample will always be observed at ’orthogonal’ angles.

[Köhler, 2004][Kaestner et al., 2011]
Beam hardening

Monochromatic

Polychromatic

measured attenuation

length

measured attenuation

length

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Attenuation for neutrons

The attenuation law assumes the intensity to be absorbed...

This is not true for neutrons!!!
**Scatter**

Scatter is bad for

- Quantitative imaging
- Segmentation algorithms

\[
\text{Uncorrected} \quad + \quad \text{Corrected by QNI [Hassanein, 2006]}
\]
Summary
Summary

- Tomography is an indirect acquisition method
- Different sources can be used
- The perfect tomography needs
  - many projections
  - well illuminated projections
- Artifacts may and will appear but can mostly be corrected.
Filtered back-projection (Proof)

Image function:

\[ f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\xi_1, \xi_2) e^{j2\pi(x\xi_1 + y\xi_2)} d\xi_1 d\xi_2 \]

Coordinate transform \( \{\xi_1, \xi_2\} = \{\omega \cos \theta, \omega \sin \theta\} \)

\[ f(x, y) = \int_{0}^{2\pi} \int_{-\infty}^{\infty} F(\omega \cos \theta, \omega \sin \theta) e^{j2\pi\omega(x \cos \theta + y \sin \theta)} d\omega d\theta \]

Fourier slice theorem:

\[ f(x, y) = \int_{0}^{2\pi} \int_{-\infty}^{\infty} P(\omega, \theta) e^{j2\pi\omega(x \cos \theta + y \sin \theta)} \omega d\omega d\theta \]

Symmetry properties:

\[ P(\omega, \theta + \pi) = P(-\omega, \theta) \]

Rotated coordinates:

\[ f(x, y) = \int_{0}^{\pi} \int_{-\infty}^{\infty} P(\omega, \theta) e^{j2\pi\omega(x \cos \theta + y \sin \theta)} |\omega| d\omega d\theta \]
A basic back-projection algorithm

\[ p_{\text{Proj}} : \text{pointer to line in sinogram} \]
\[ p_{\text{Slice}} : \text{pointer to slice matrix} \]

```c
for (float line = 0; line < nProjections; line++) {
    for (size_t y = 0; y < SizeY; y++) {
        const size_t cfStartX = mask[y].first; // Get x-coordinates
        const size_t cfStopX = mask[y].second;
        fStartU += cos(theta[line]); // Compute first proj. pos.
        float fPosU = fStartU - sin(theta[line]) * cfStartX;
        for (size_t x = cfStartX; x < cfStopX; x++) {
            // Loop over matrix in x
            int nPosU = static_cast<int>(fPosU - sin(theta[line])); // Compute position
            const float interpB = fPosU - nPosU; // Interpolation weight right
            const float interpA = 1.0f - interpB; // Interpolation weight left
            pSlice[x + y * sizeX] += interpA * pProj[nPosU] + interpB * pProj[nPosU + 1];
        }
    }
}
```

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Practical cone-beam algorithm.

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