Finite temperature field theory and heavy ion collisions

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Lecture 2: real-time quantum field theory

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Linear response theory

- To compute microscopic quantities in QFT, we must formulate questions as *correlation functions*
- At T=0 we learn to deal with *time-ordered* ones.:

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle$$

- At finite T these are neither *natural*, nor *meaningful*, nor *useful*, nor even *nice*

- A typical question:
  if we have e.g. an overdensity somewhere; how will the energy density evolve?

- Such questions can be answered as follows:

measurement

$$\langle \epsilon(y) \rangle_{\mathrm{perturbed}} = \mathrm{Tr}\, e^{-\beta H} e^{-i \int_{-\infty}^{y^0} (H+\delta H)} \epsilon(y) e^{i \int_{-\infty}^{y^0} (H+\delta H)}$$

unperturbed initial density matrix

evolution perturbed by source; the source creates the overdensity

Linear response theory

$$\langle \epsilon(y) \rangle_{\mathrm{perturbed}} = \mathrm{Tr}\; e^{-\beta H} e^{-i \int_{-\infty}^{y^0} (H+\delta H)} \epsilon(y) e^{i \int_{-\infty}^{y^0} (H+\delta H)}$$

- To linear order in the perturbation, given by retarded two-point function :

$$= -i \int d^4x \; \mathrm{Tr}\; e^{-\beta H} [\delta H(x), \epsilon(y)] \; \theta(y^0 - x^0)$$

$$:= -i \int d^4x \; G_R(x,y)$$

Retarded boundary conditions appear naturally

[Challenge: find one physical observable given by a T-ordered product!]

Path integrals

It is useful to represent correlators as path integrals:

$$\langle 0|T[\phi(x_1)\phi(x_2)]|0\rangle = \frac{1}{Z}\int [D\phi(x)]\phi(x_1)\phi(x_2)e^{iS[\phi]}$$

- Useful for perturbative expansion (write $S=S_0+S_{int}$)
- Useful for nonperturbative analysis (lattice,...)

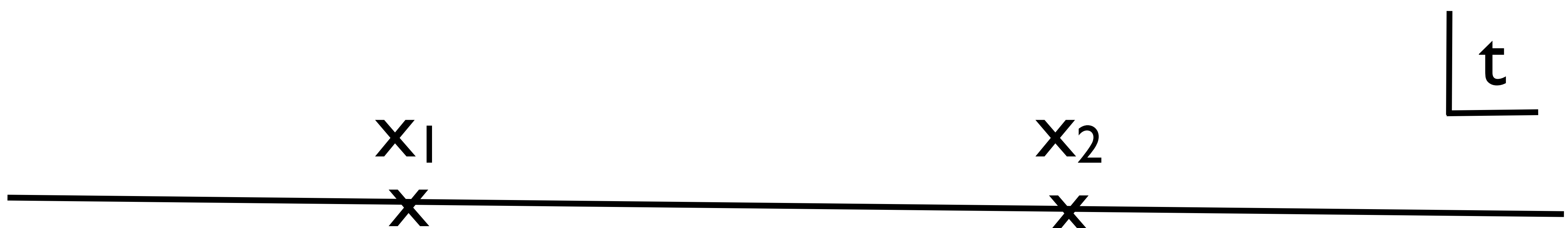
To get started and study QFT at finite T, we should find a path integral which computes retarded functions

First, let's answer two questions at T=0.

$$\langle 0|T[\phi(x_1)\phi(x_2)]|0\rangle = \frac{1}{Z}\int [D\phi(x)]\phi(x_1)\phi(x_2)e^{iS[\phi]}$$

Q: Why does RHS give the *time-ordered* product?

A: Each segment gives time evolution from one point to the next



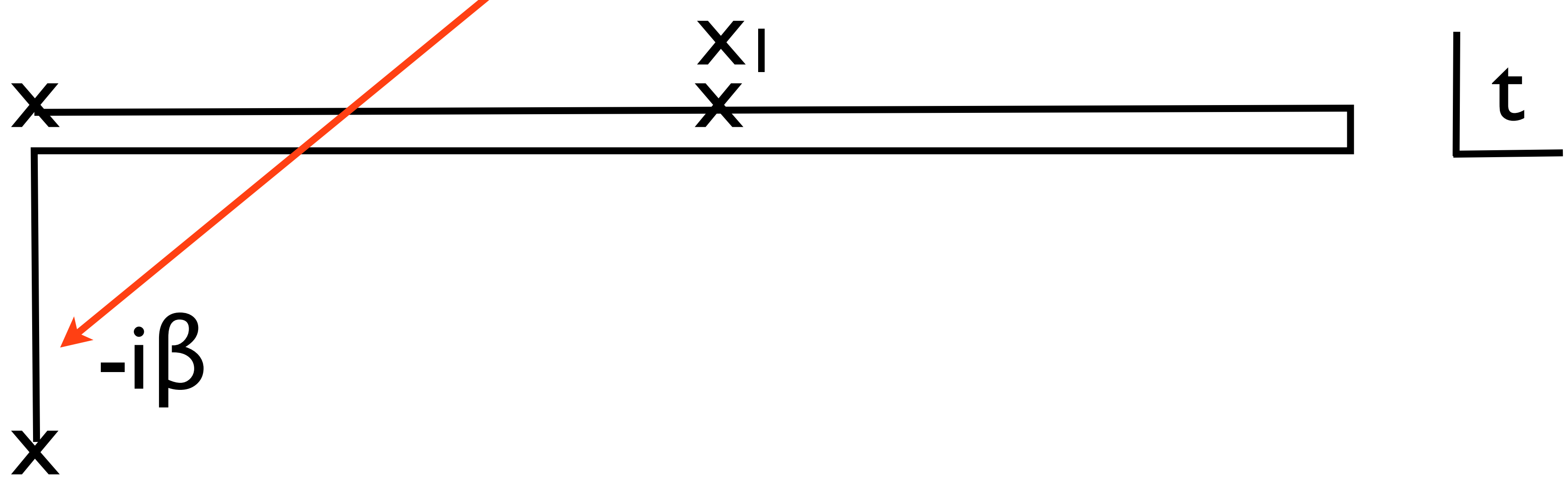
$$e^{-i\int_{-\infty}^{x_1} dtH} \qquad e^{-i\int_{x_1}^{x_2} dtH} \qquad e^{-i\int_{x_2}^{\infty} dtH}$$

The Schwinger-Keldysh contour

- We need to compute:

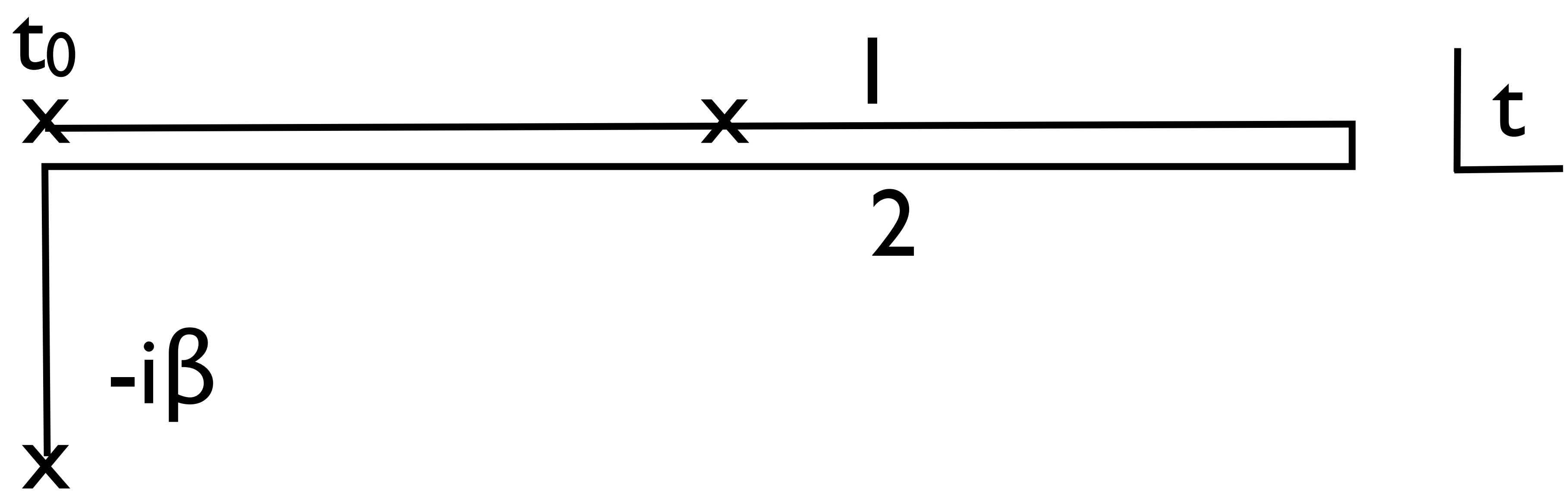
$$\langle \phi(x_1)\cdots\rangle_\beta := \frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}\phi(x_1)\cdots\right] \qquad (\beta=1/T)$$

- Schwinger-Keldysh contour:



periodic identification (for Tr)

Notes.



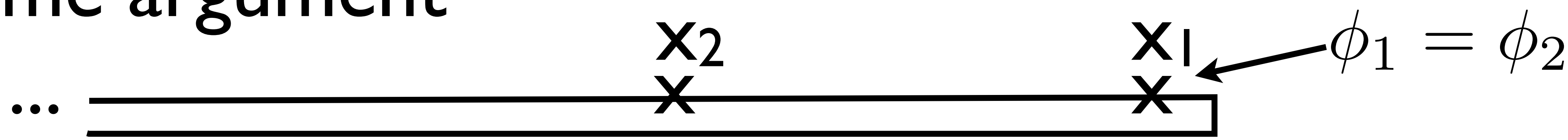
- We have two real branches; that's the minimum we can have (if we want some real-time correlator)
- That's enough to define a retarded correlator! *organize everything around it*

The Keldysh formalism

- Define average and difference fields:

$$\phi_r = \frac{\phi_1 + \phi_2}{2}, \quad \phi_a = \phi_1 - \phi_2$$

- a-fields vanish when they have the largest time argument



- $G_{aa} = 0$

- $G_{ra}(x_1, x_2) = \langle [\phi(x_1), \phi(x_2)] \rangle \theta(t_1 - t_2)$

$$:= G_R(x_1, x_2)$$

Fluctuation dissipation, I

(aka KMS relation, Kubo-Martin-Schwinger)

- Wightman functions (un-ordered) are related to each other, in momentum space
- $$\begin{aligned} G^{>}(p) &:= \frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}\phi(p)\phi(x=0)\right] \\ &= \frac{1}{Z}\mathrm{Tr}\left[\phi(p)e^{-\beta H}\phi(x=0)\right] e^{\beta p^0} \\ &= \frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}\phi(x=0)\phi(p)\right] e^{\beta p^0} \\ &= e^{\beta p^0} G^{<}(p) \end{aligned}$$

FLuctuation dissipation, 2

- From retarded functions, we can get the commutator (*spectral*) function:

$$G_R(x_1,x_2) - G_A(x_1,x_2) := \text{Disc } G_R(x_1,x_2)$$
$$= \langle [\phi(x_1), \phi(x_2)] \rangle_T$$

- Using that this is $G^> - G^<$ the previous slide, we get:

$$\underbrace{G^<(p)}_{\text{Fluctuation}} = \underbrace{\text{Disc } G_R}_{\text{Dissipation}} \frac{1}{e^{\beta p^0} - 1}$$

- We now have all two-point functions:

$$\begin{pmatrix} G_{rr} & G_{ra} \\ G_{ar} & G_{aa} \end{pmatrix} = \begin{pmatrix} G_{rr} & G_R \\ G_A & 0 \end{pmatrix}$$

$$G_{rr}(p) = \text{Disc } G_R(p) \left(\frac{1}{2} + n_B(p) \right)$$

- Example: free bosonic field

$$\begin{pmatrix} G_{rr} & G_{ra} \\ G_{ar} & G_{aa} \end{pmatrix} = \begin{pmatrix} 2\pi\delta(P^2) \left(\frac{1}{2} + n_B(|p^0|) \right) & \frac{-i}{P^2 - i\epsilon p^0} \\ \frac{-i}{P^2 + i\epsilon p^0} & 0 \end{pmatrix}$$

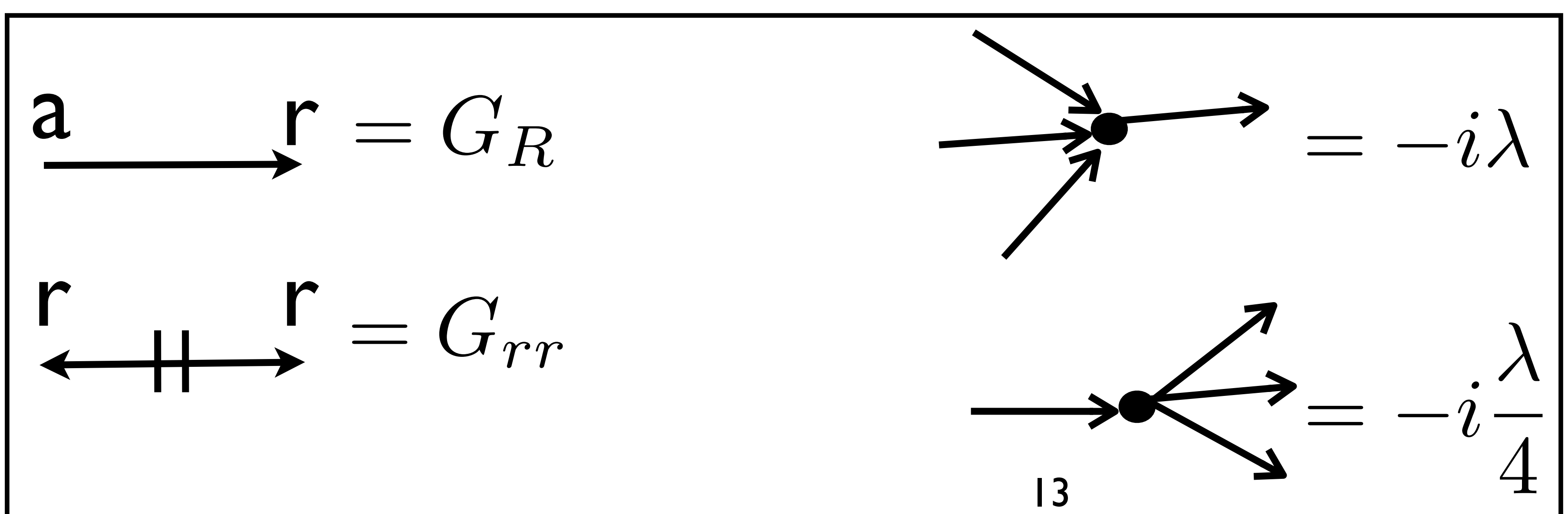
Let's compute some loops

- Consider the model (ϕ^4), at a finite T, $\lambda \ll 1$:

$$S = \int d^4x \left[-\frac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda}{24}\phi^4 \right]$$

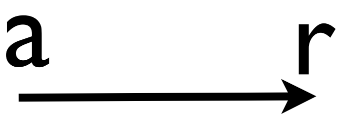
- Keldysh action:

$$S_1 - S_2 = \int d^4x \left[-\partial_\mu \phi_r \partial^\mu \phi_a - \frac{\lambda}{6}\phi_r^3 \phi_a - \frac{\lambda}{24}\phi_r \phi_a^3 \right]$$

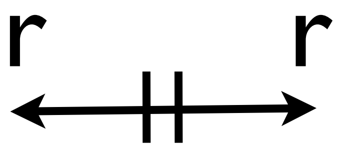


[notation devised w/G.D.Moore]

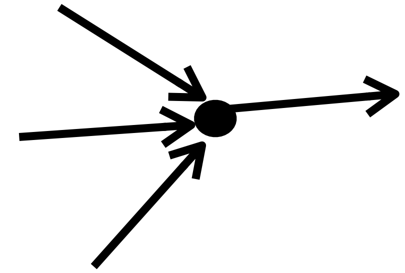
(seldom used)



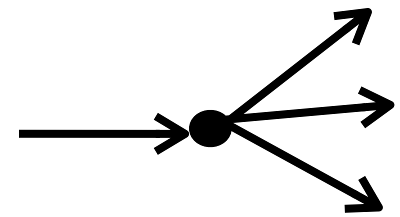
$= G_R$



$= G_{rr}$

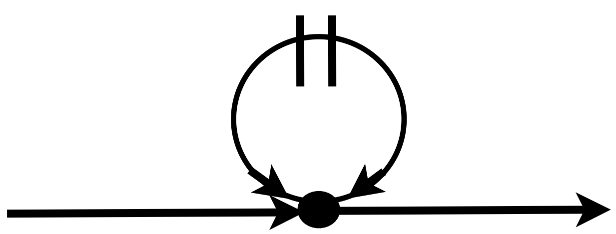


$= -i\lambda$



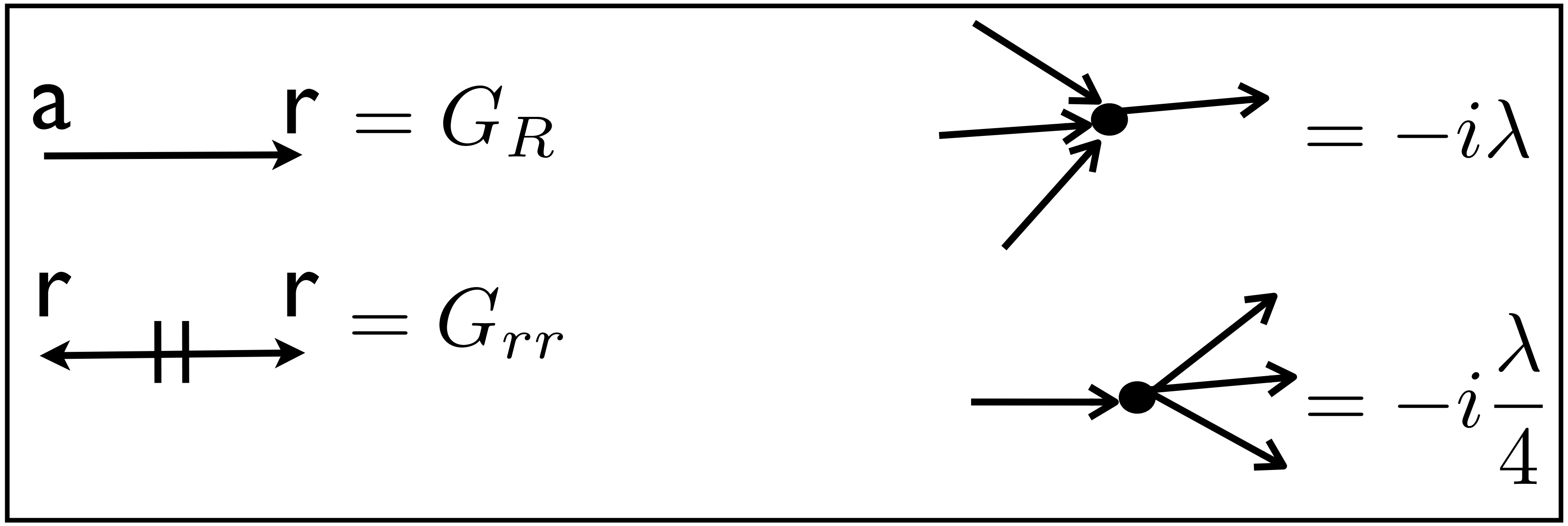
$= -i\frac{\lambda}{4}$

- The one-loop retarded self-energy:



$$\Pi_R = \frac{\lambda}{2}\int \frac{d^4p}{(2\pi)^4}\; G_{rr}(p)$$

$$= \frac{\lambda}{2}\int \frac{d^3p}{(2\pi)^4}\frac{\frac{1}{2}+n(p)}{p} = \frac{\lambda T^2}{24}$$



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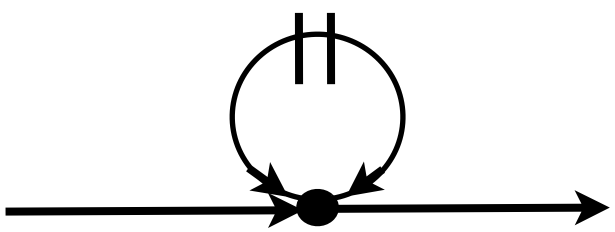


$$\Pi_R = \frac{\lambda}{2}\int \frac{d^4p}{(2\pi)^4}\; G_{rr}(p)$$

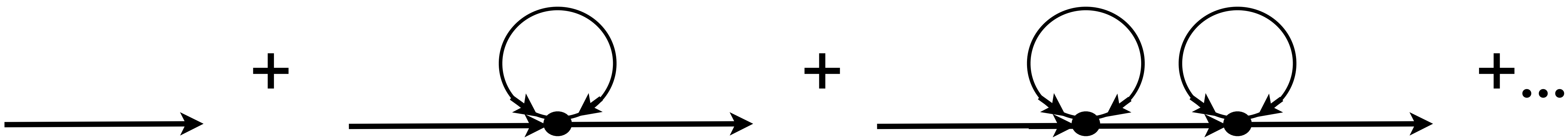
$$= \frac{\lambda}{2}\int \frac{d^3p}{(2\pi)^4}\frac{\frac{1}{2}+n(p)}{p} = \frac{\lambda T^2}{24}$$

- UV divergence *exactly* as in vacuum; T correction finite

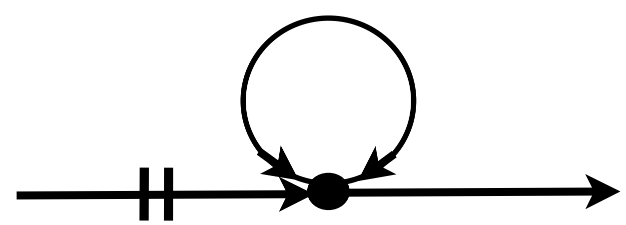
- Thermal mass^2 is positive

Keldysh formalism

- Series for the retarded function well-behaved

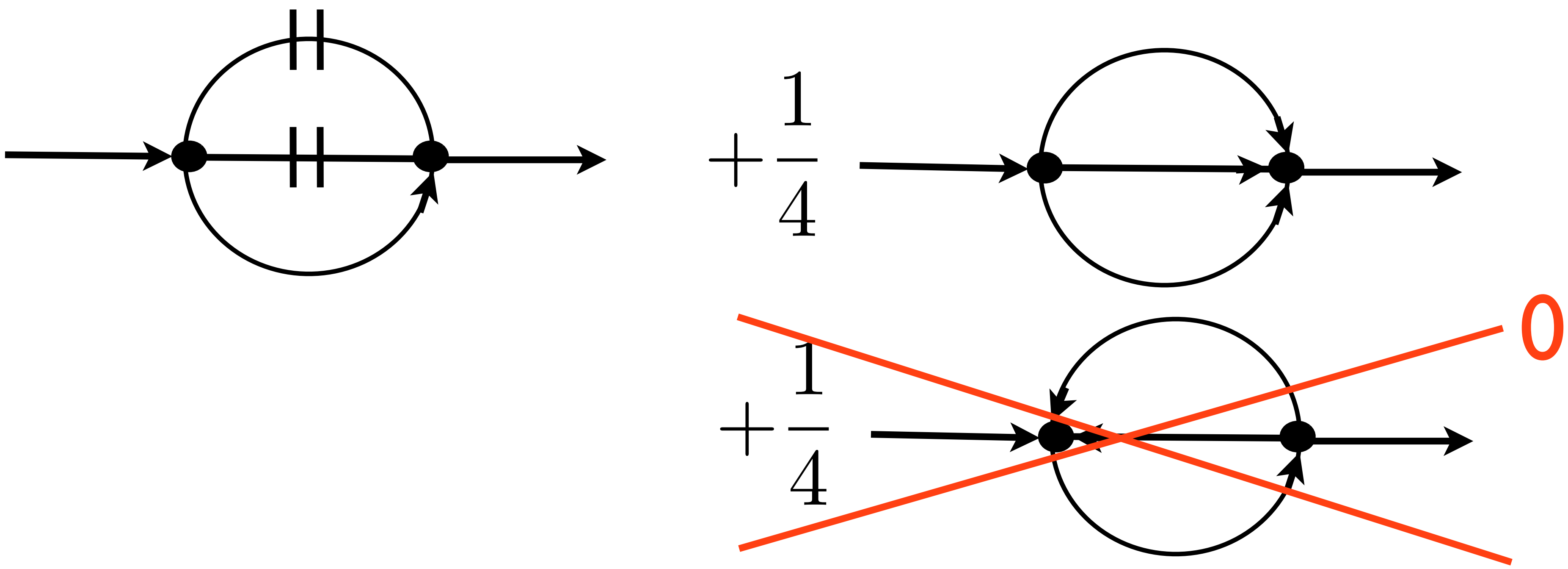


$$G_R = \frac{-i}{p^2 - m_{\rm th}^2 - i\epsilon p^0}$$

- not so for G_{rr}:  $\propto \delta(p^2)\frac{1}{p^2} = $ weird

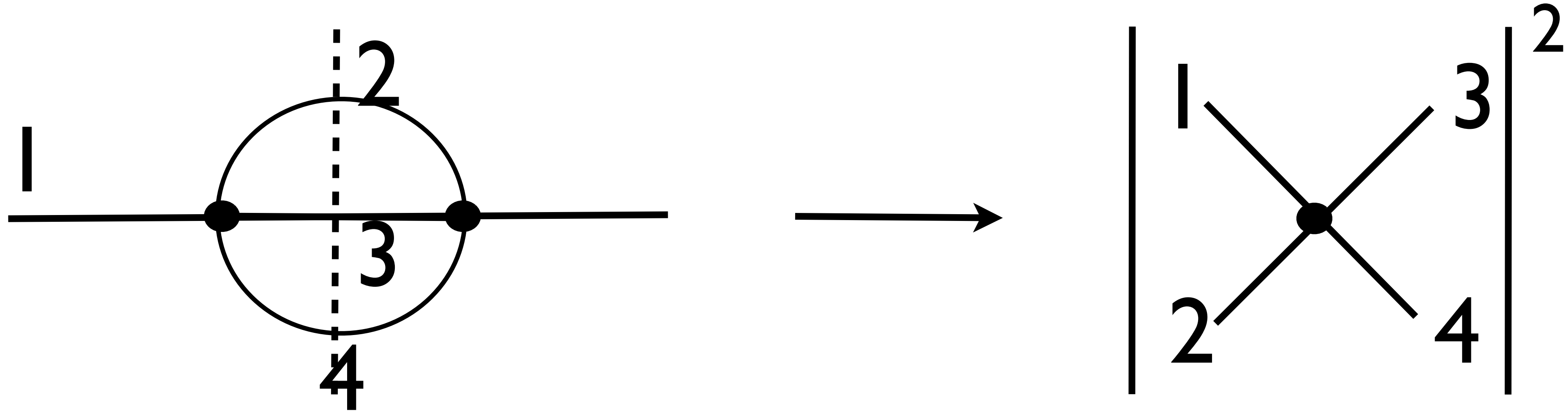
- Keldysh's solution: compute only *retarded* guys, get others by fluctuation-dissipation

- Let's keep going... 2-loop:



Let's just look at the T-dependent imaginary part, on-shell.

- imaginary part



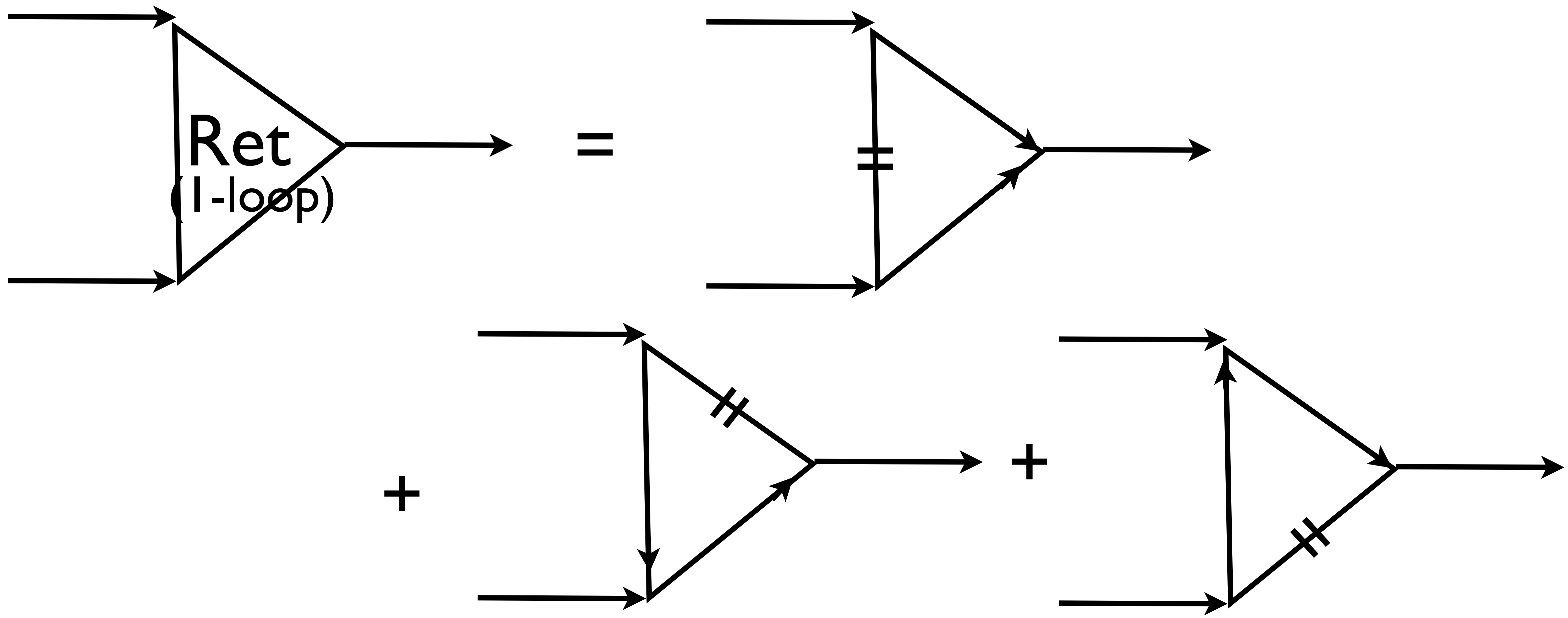
Kinematics require one particle in the loop to have negative energy, two have positive energy

$$-2\mathrm{Im}\Pi_R = \int \frac{d^3p_3}{(2\pi)^3 2p_3}\frac{d^3p_4}{(2\pi)^3 2p_4}|\lambda^2|\ [n_2(1+n_3)(1+n_4)-(1+n_2)n_3n_4]$$

Imaginary part measures *scattering rate* (*not* decay)!

$$G_R \sim \frac{-i}{p^2 - m_{\mathrm{th}}^2 - i\Gamma}$$

Other examples



Notice the simple time flow.
Can you find a classical statistical interpretation?