Copenhagen, 26.8.2009

Relativistic Smooth Particle Hydrodynamics from a variational principle

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Astrophysics
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Literature: i) S.Rosswog, New Astronomy Reviews, in press (2009), arXiv:0903.5075
ii) S.Rosswog, subm. J. Comp. Phys. (2009), arXiv:0907:4890

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Think again!!!

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$$\mathrm{baryon\ number:} \qquad (\rho U^\mu)_{;\mu} = 0$$

$$\mathrm{energy-momentum:}\ T^{\mu\nu}{}_{;\nu} = 0$$

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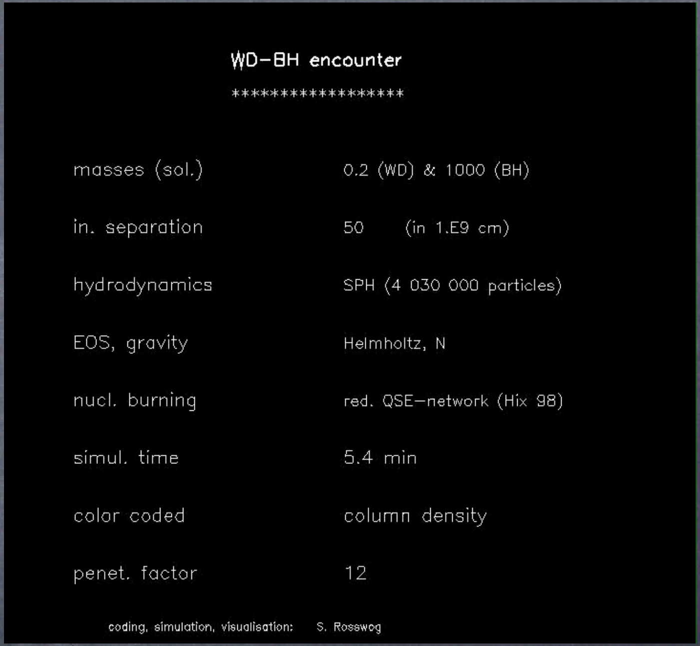

WD-BH encounter
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masses (sol.)	0.2 (WD) & 1000 (BH)
in. separation	50 (in 1.E9 cm)
hydrodynamics	SPH (4 030 000 particles)
EOS, gravity	Helmholtz, N
nucl. burning	red. QSE–network (Hix 98)
simul. time	5.4 min
color coded	column density
penet. factor	12

coding, simulation, visualisation: S. Rosswog

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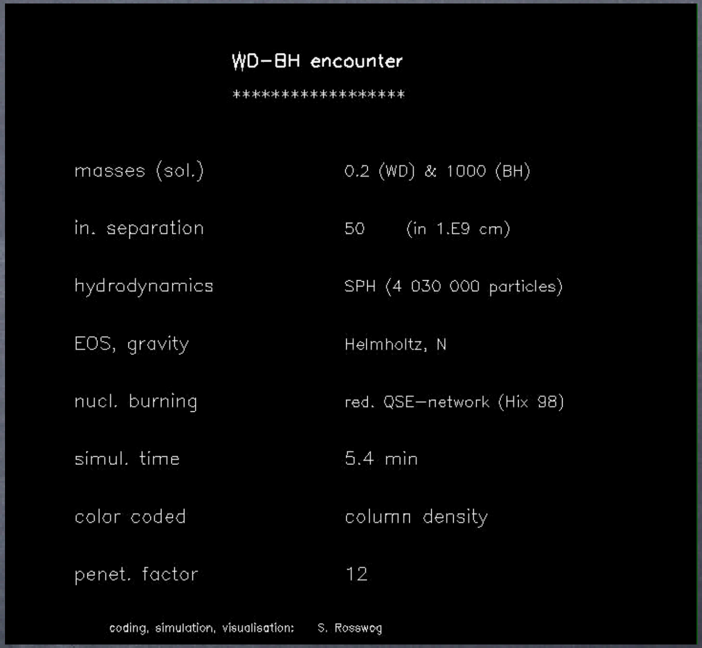

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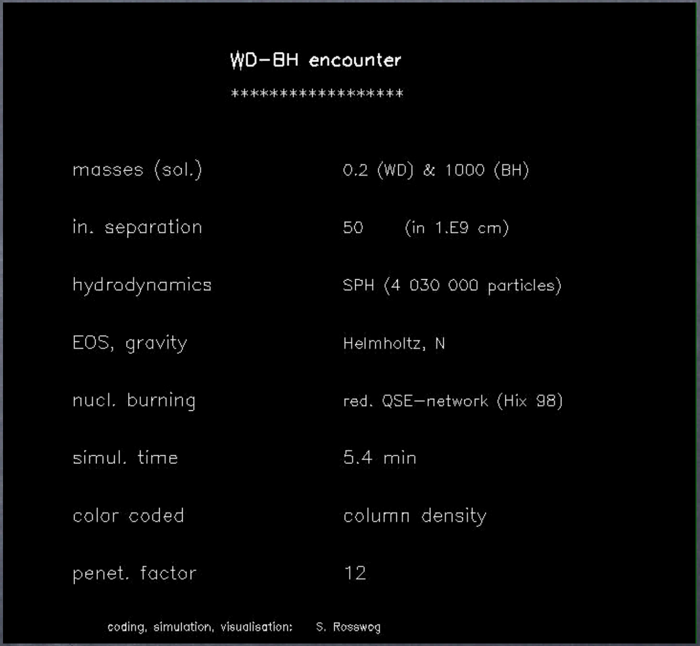

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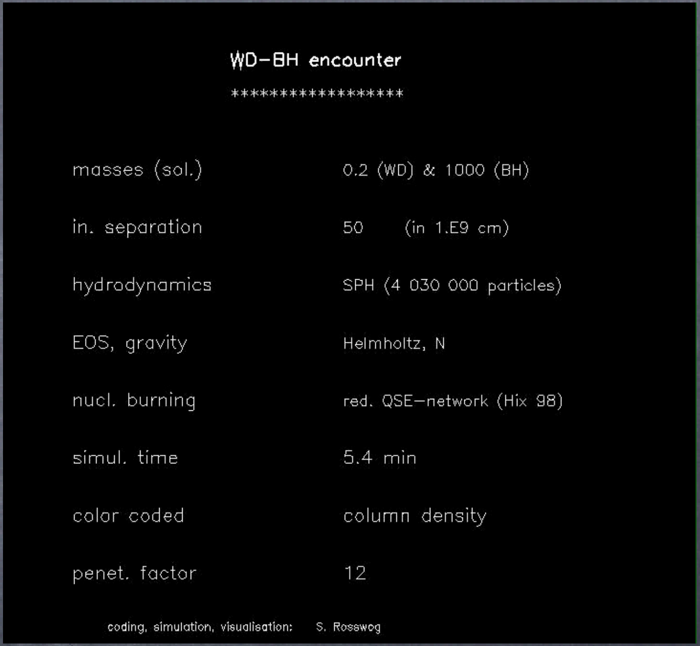

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- but also accurate advection !
- “hard-wired” conservation of physical invariants !

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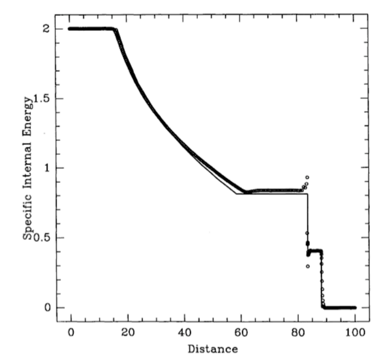

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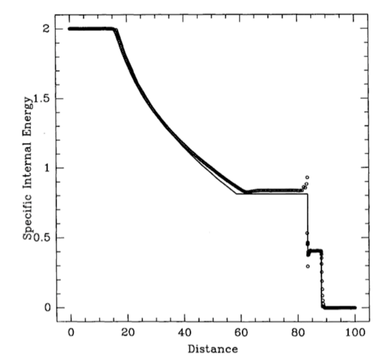

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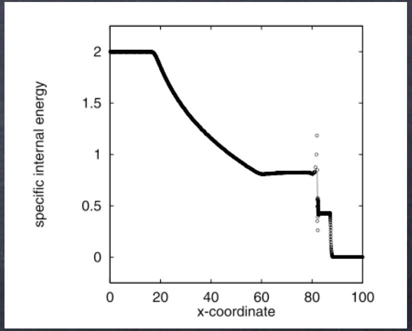

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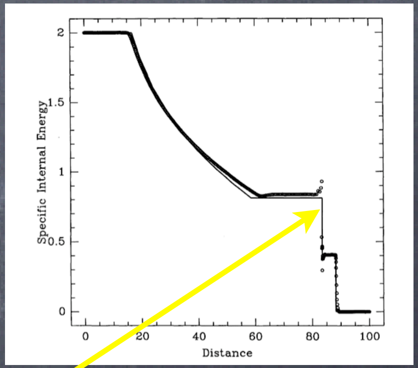

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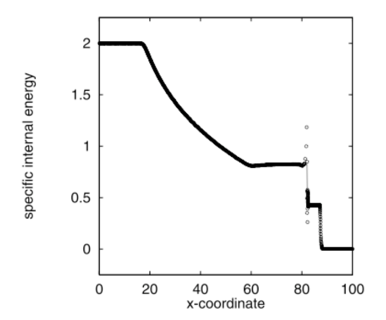

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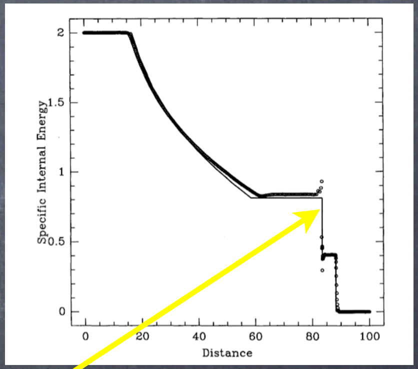

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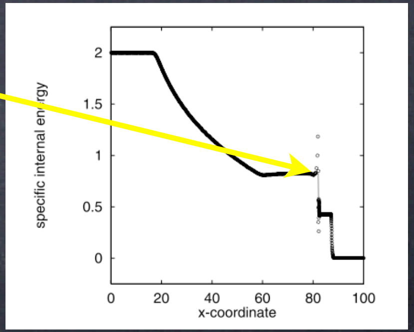

III. Conservative, special-relativistic SPH: consistent derivation from a Lagrangian

- our approach:
  - start from Lagrangian of ideal fluid
  - apply Euler-Lagrange equations + first law of thermodynamics
  - use canonical energy and momentum as guidance for numerical variables
  - use modern form of artificial viscosity

III.1 Lagrangian of an ideal, relativistic fluid

- Lagrangian perfect fluid: $$L_{\mathrm{pf,sr}} = -\int T^{\mu\nu} U_\mu U_\nu \, dV$$

  (Fock 1964)

- from now on: measure energies in m_0c^2 (baryon rest mass energy)

- choose frame in which computations are performed ("Computing Frame", CF)

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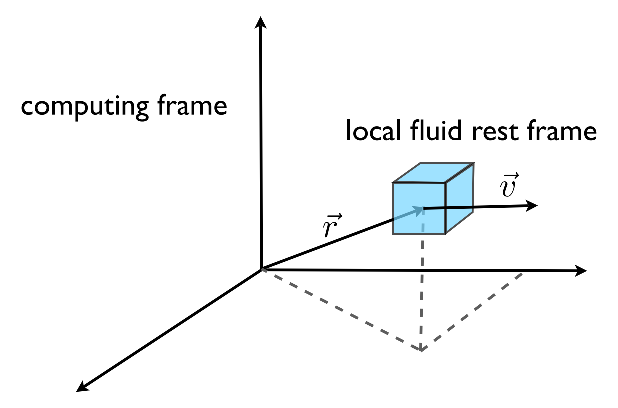

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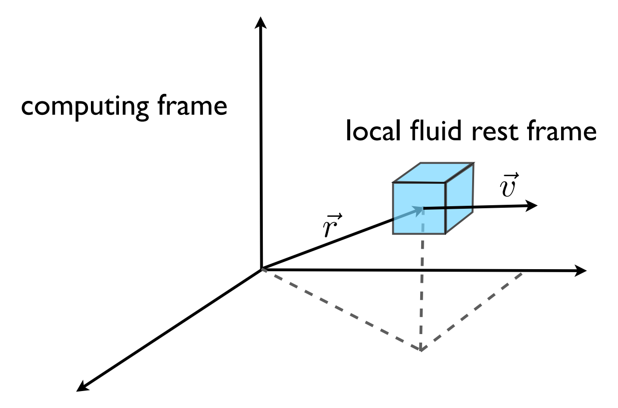

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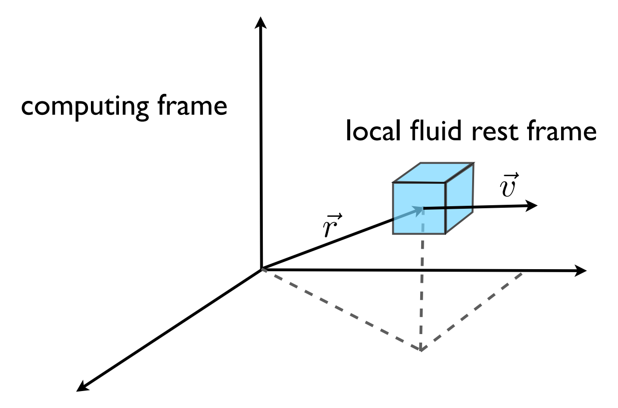

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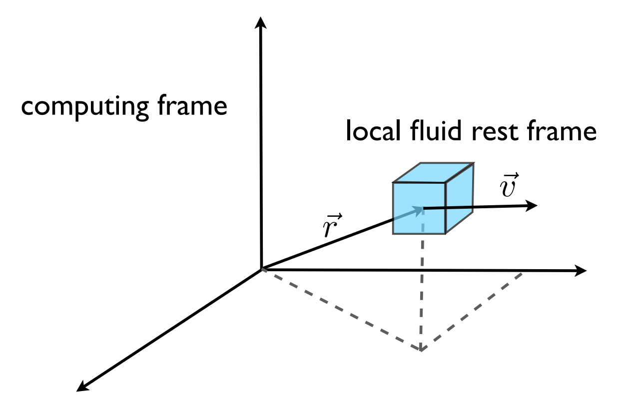
Number density in CF

number density in local rest frame



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Lorentz factor

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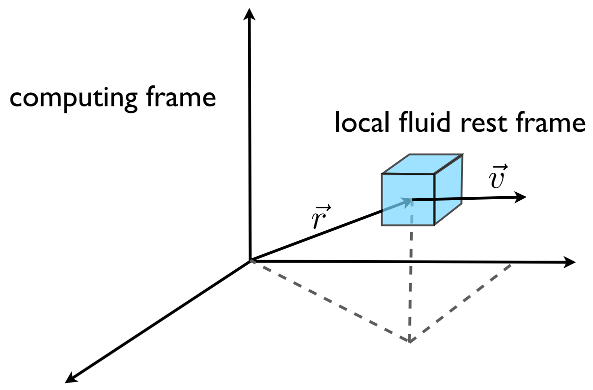

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- comparison to "standard SPH":

$$f(\vec{r}) = \sum_b f_b \frac{m_b}{\rho_b} W(|\vec{r} - \vec{r}_b|, h)$$

- the discretization applied to Lagrangian:

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specific energy number density

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- further strategy:

  i) apply Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L_{\mathrm{SPH,sr}}}{\partial \vec{v}_a} - \frac{\partial L_{\mathrm{SPH,sr}}}{\partial \vec{v}_a} = 0$$

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  iii) canonical energy and momentum per baryon as numerical variables

- canonical momentum:

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- canonical momentum:

$$\begin{aligned}\vec{p}_a &\equiv \frac{\partial L_{\rm SPH,sr}}{\partial \vec{v}_a} = \dots \\ &= \nu_a \gamma_a \vec{v}_a \left(1 + u_a + \frac{P_a}{n_a}\right)\end{aligned}$$

→ introduce

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$$E \equiv \sum_a \frac{\partial L_{\rm SPH,sr}}{\partial \vec{v}_a} \cdot \vec{v}_a - L_{\rm SPH,sr} = ..$$

$$= \sum_a \nu_a \left(\vec{v}_a \cdot \vec{S}_a + \frac{1 + u_a}{\gamma_a}\right)$$

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$$\hat{\epsilon}_a \equiv \vec{v}_a \cdot \vec{S}_a + \frac{1 + u_a}{\gamma_a}$$

- resulting **SPH equation set:**

baryon number:

$$\left.\begin{aligned} N_b &= \sum_k \nu_k W(|\vec{r}_b - \vec{r}_k|, h_b) \\ h_b &= \eta N_b^{-1/D} \end{aligned}\right\} \text{iteration!}$$

momentum:

$$\frac{d\vec{S}_a}{dt} = -\sum_b \nu_b \left(\frac{P_a}{N_a^2 \tilde{\Omega}_a} \nabla_a W_{ab}(h_a) + \frac{P_b}{N_b^2 \tilde{\Omega}_b} \nabla_a W_{ab}(h_b) \right),$$

$$\vec{S}_a \equiv \gamma_a \vec{v}_a \left(1 + u_a + \frac{P_a}{n_a} \right) \qquad \text{can. momentum per baryon}$$

energy:

$$\frac{d\epsilon_a}{dt} = -\sum_b \nu_b \left(\frac{P_a \vec{v}_b}{N_a^2 \tilde{\Omega}_a} \cdot \nabla_a W_{ab}(h_a) + \frac{P_b \vec{v}_a}{N_b^2 \tilde{\Omega}_b} \cdot \nabla_a W_{ab}(h_b) \right)$$

$$\epsilon_a \equiv \gamma_a \left(1 + u_a + \frac{P_a}{n_a} \right) - \frac{P_a}{N_a} = \vec{v}_a \cdot \vec{S}_a + \frac{1 + u_a}{\gamma_a}$$

can. energy per baryon

- comments:

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  - equations include "corrective terms" from derivatives of kernels with resp. to smoothing length h:

$$\tilde{\Omega}_b \equiv 1 - \frac{\partial h_b}{\partial N_b} \sum_k \frac{\partial W_{bk}(h_b)}{\partial h_b}$$

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relativistic "grad-h-terms"

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$$\tilde{\Omega}_b \equiv 1 - \frac{\partial h_b}{\partial N_b} \sum_k \frac{\partial W_{bk}(h_b)}{\partial h_b}$$

relativistic "grad-h-terms"

  - like in relativistic grid-based methods: conversion between "numerical" and "physical variables" required at each time step

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projection along particle line of sight

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dissipation only where needed!

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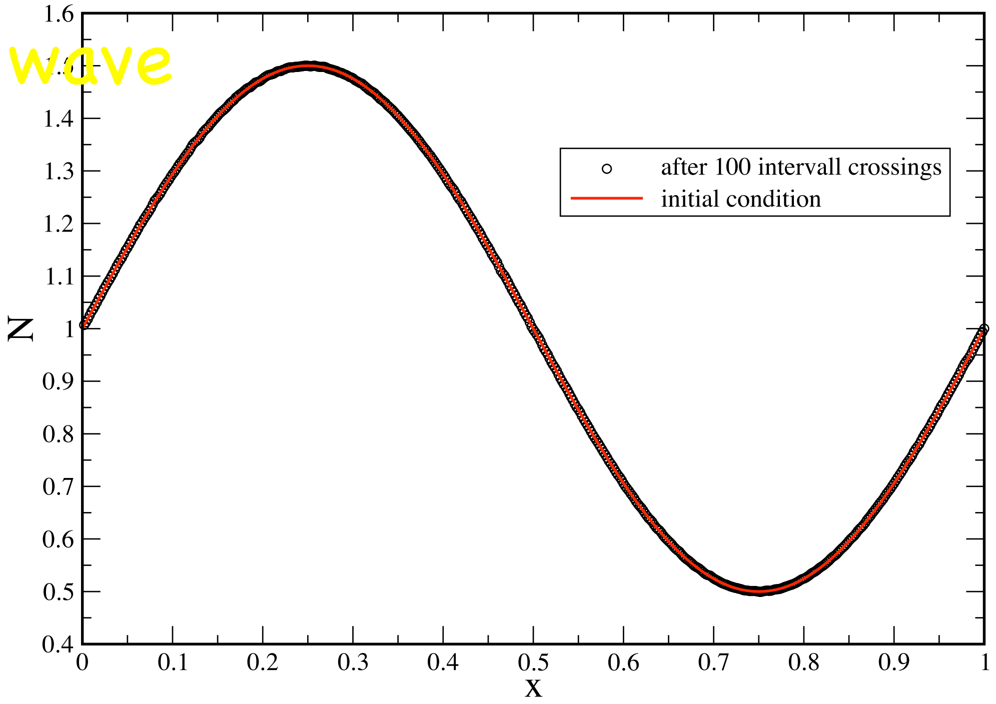

500 particles

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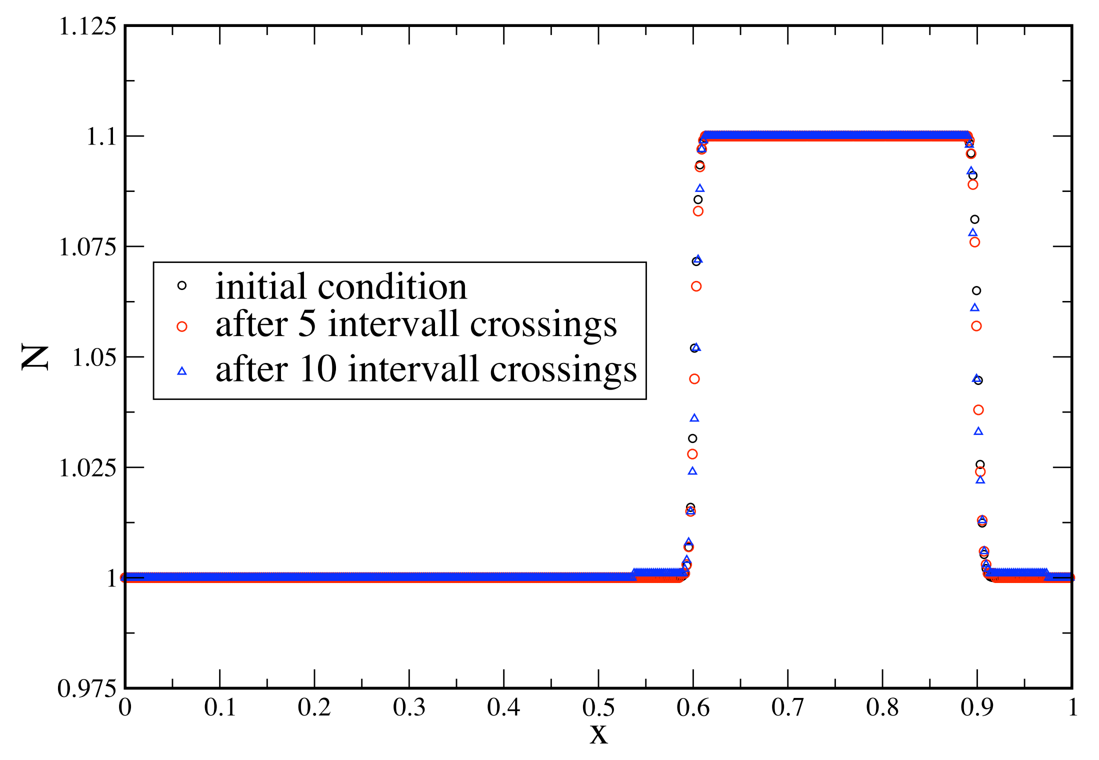

V.II “Shock tests”

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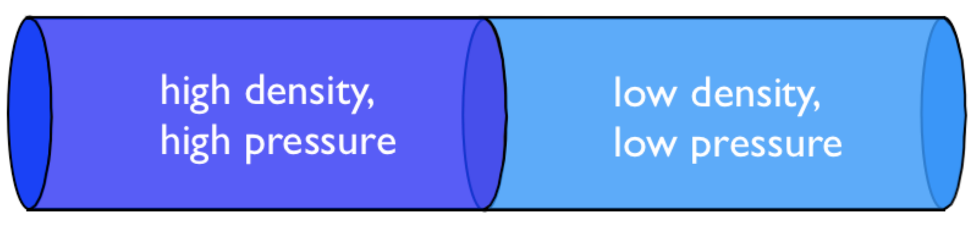

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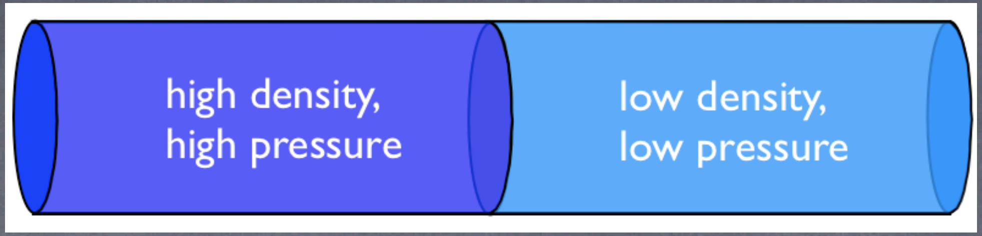

- Test 3: mildly relativistic shock tube

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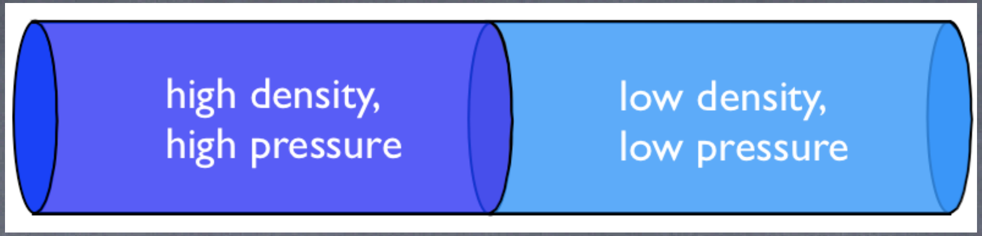

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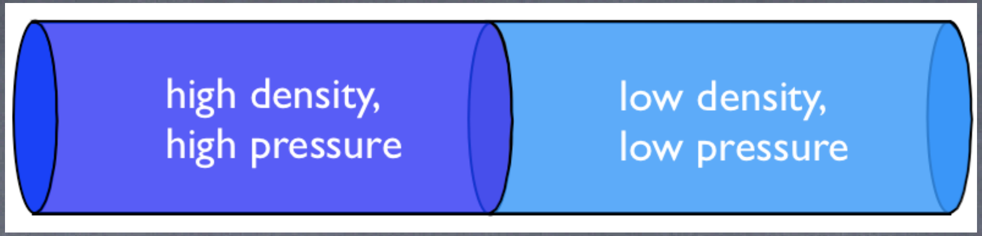

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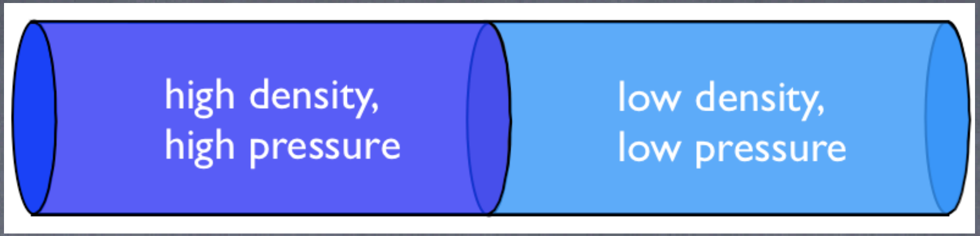

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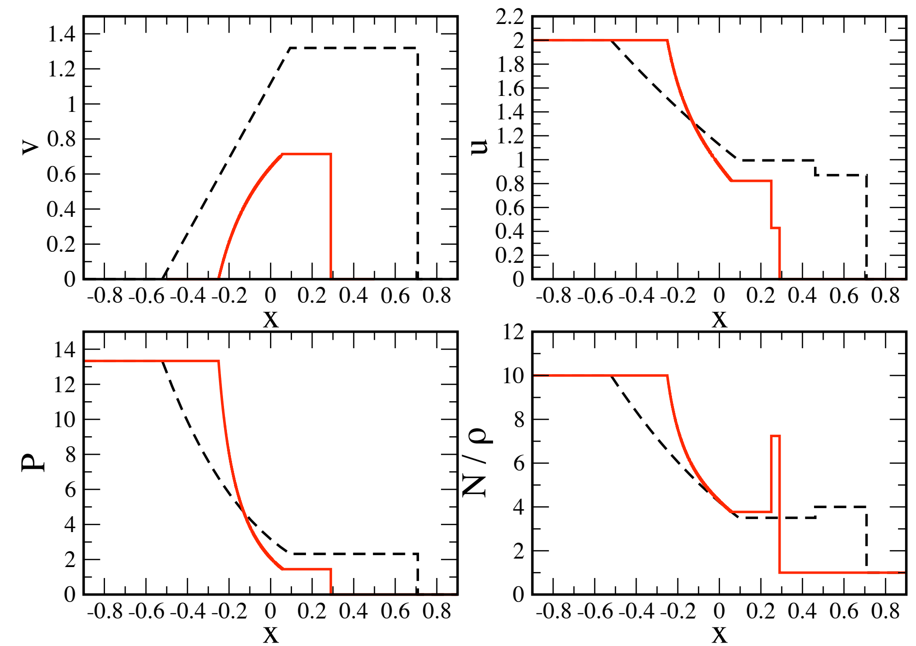

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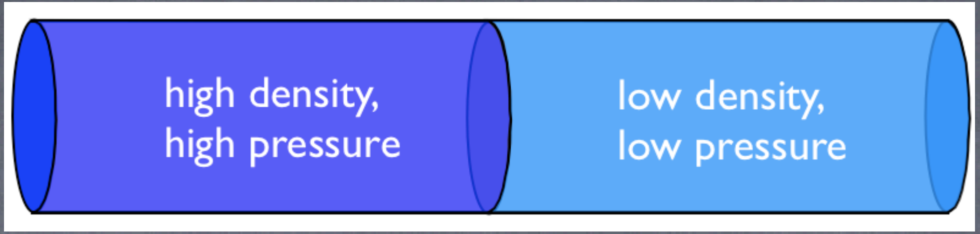

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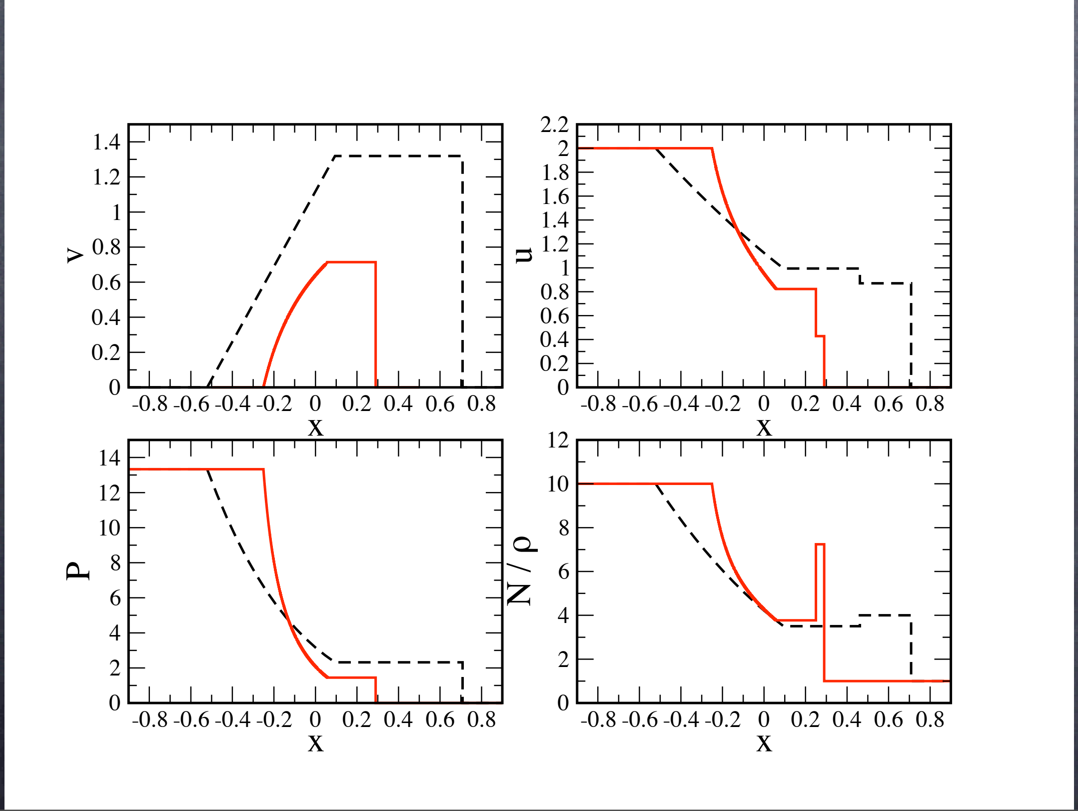

red: special-relativistic

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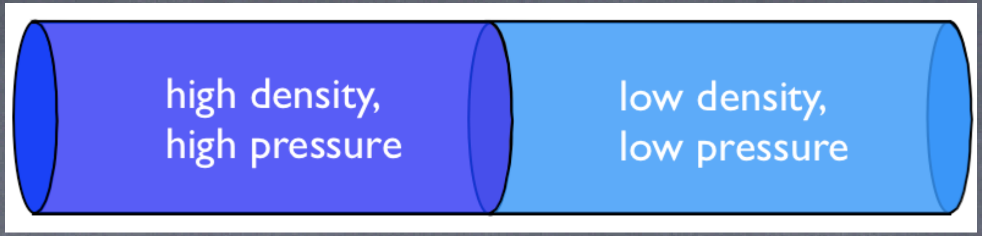

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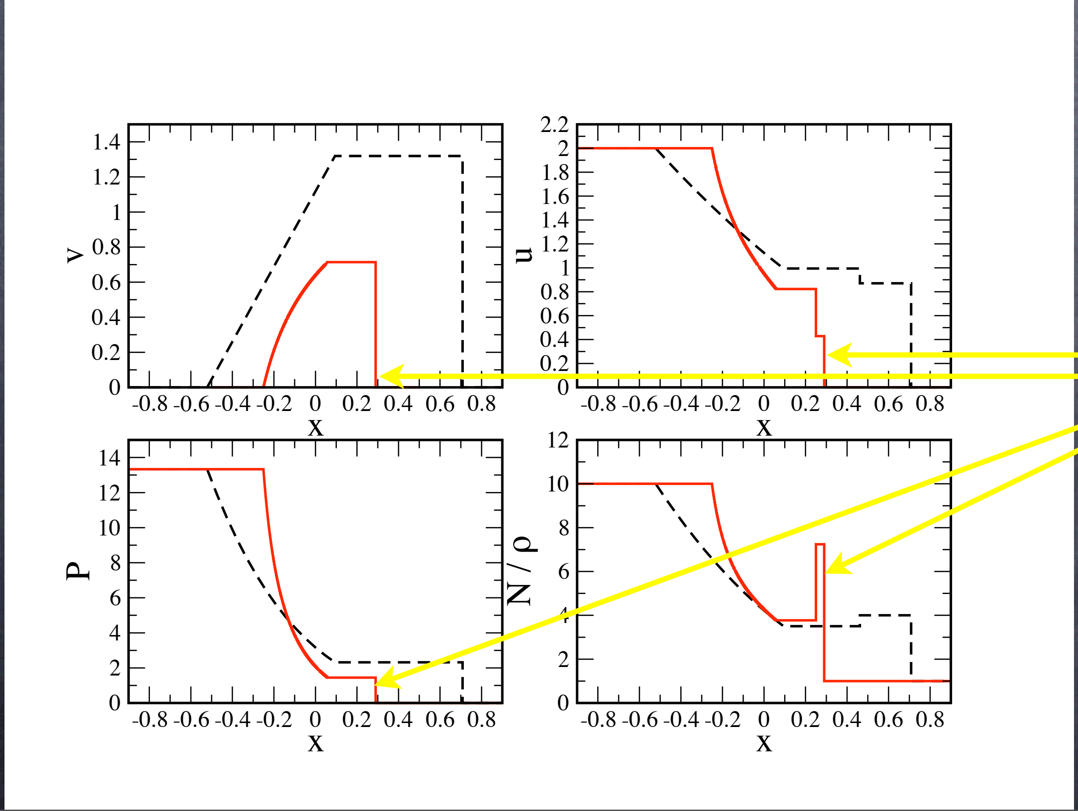

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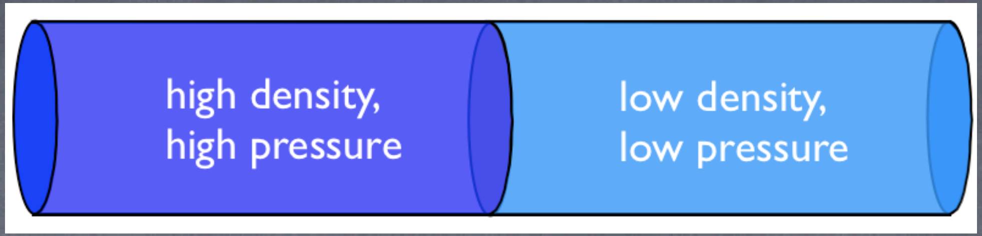

high density,
high pressure

low density,
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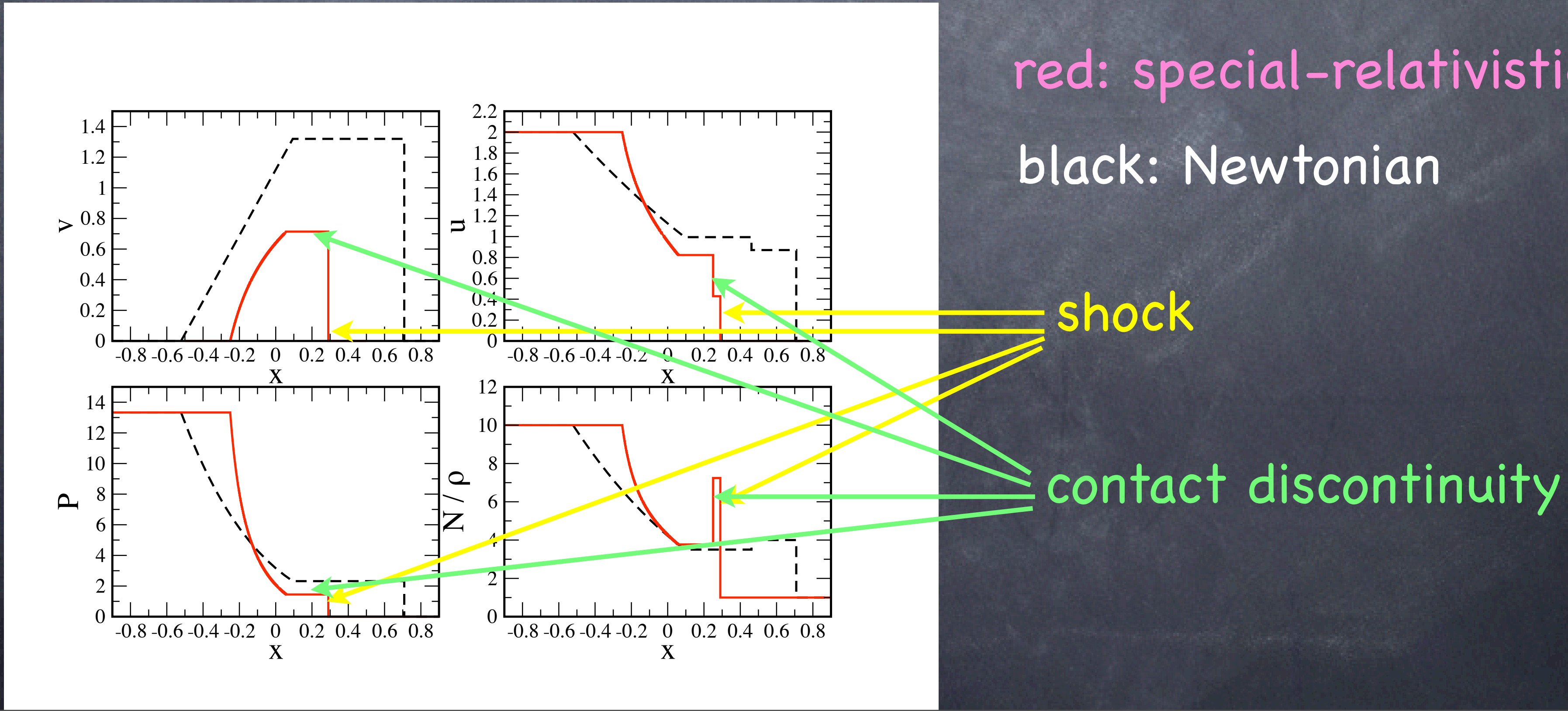

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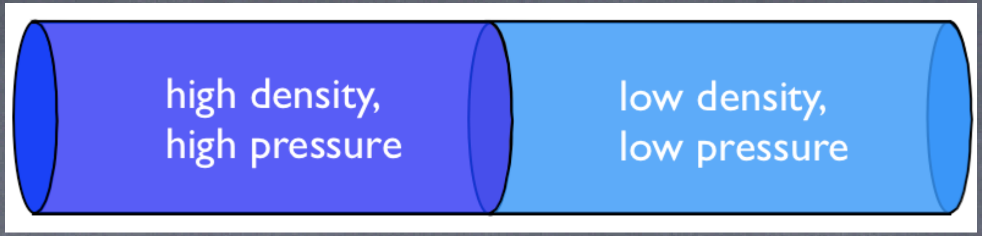

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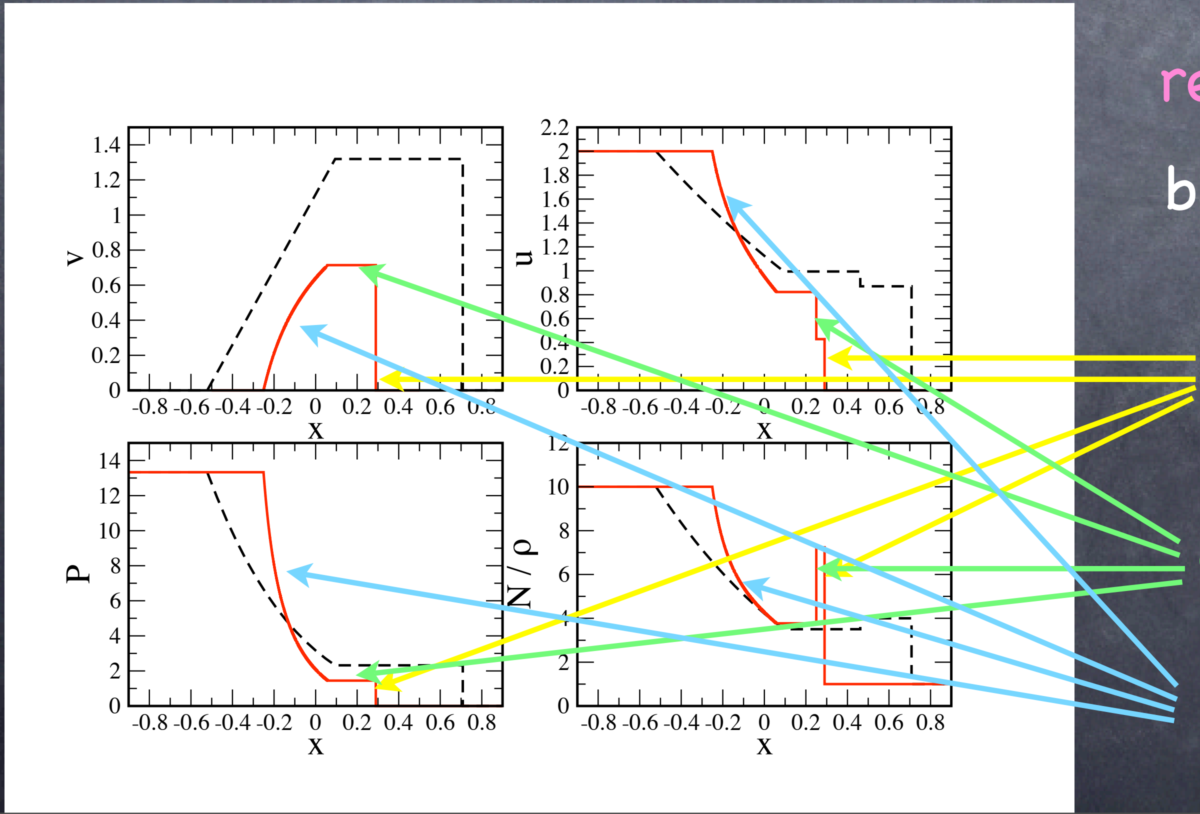

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rarefaction fan

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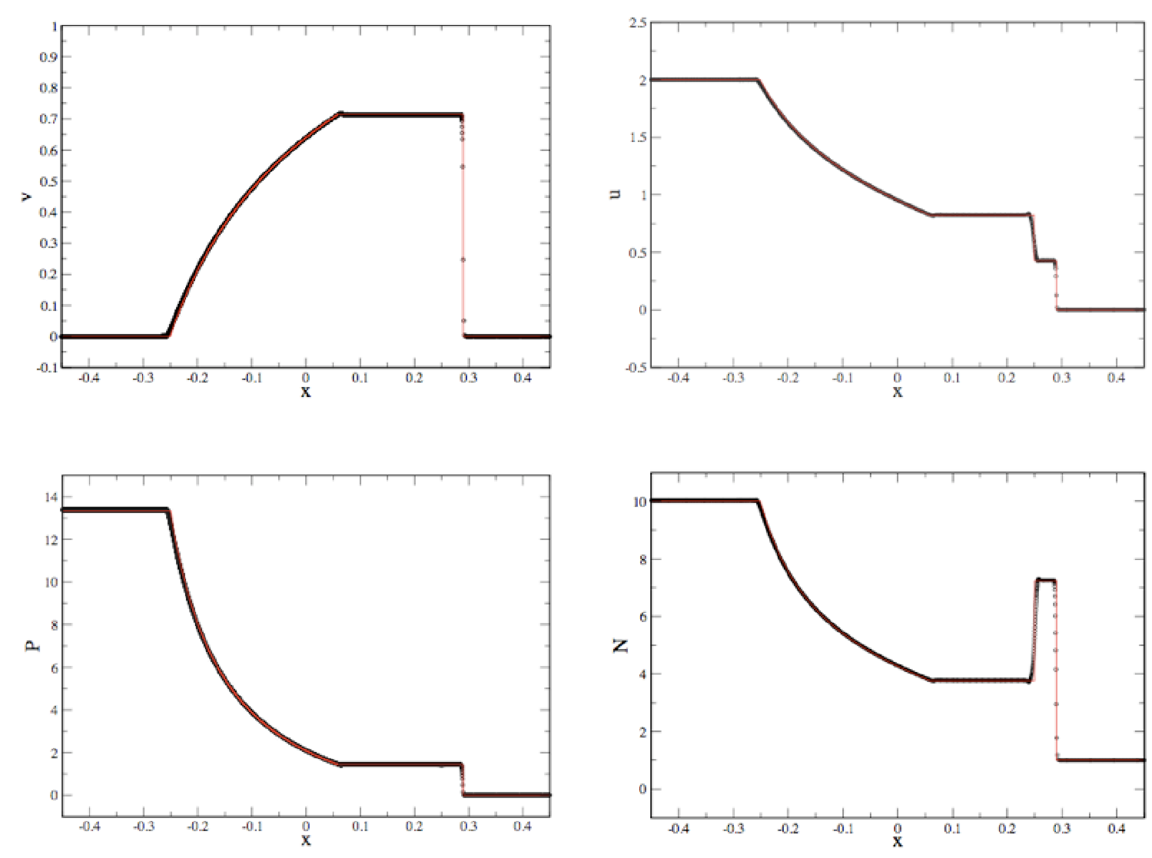

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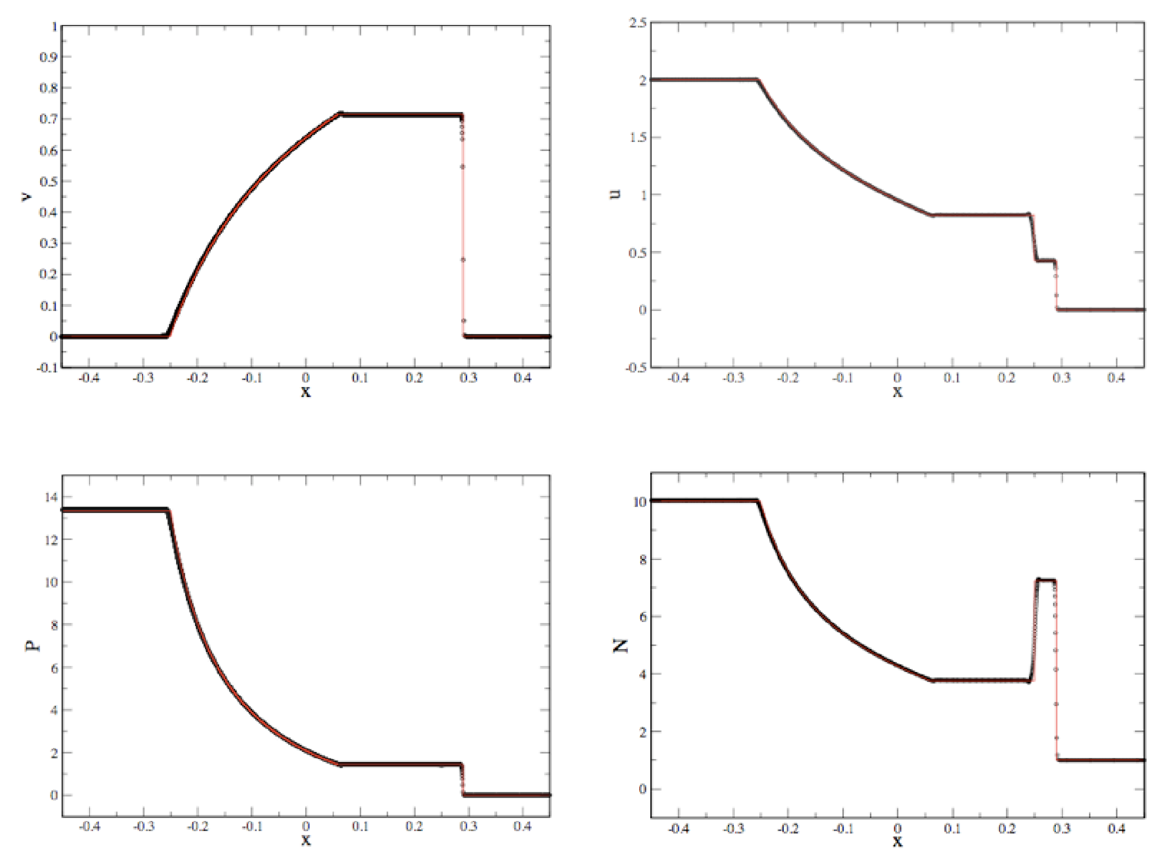

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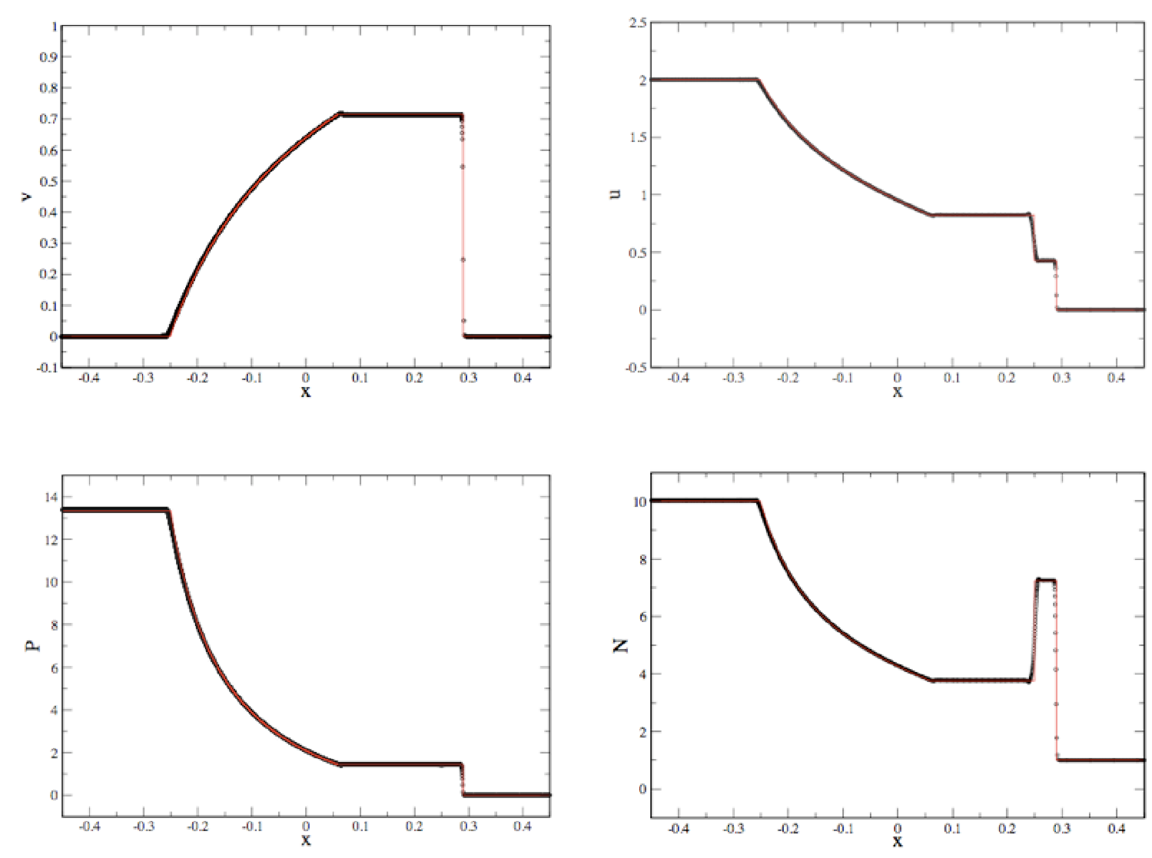

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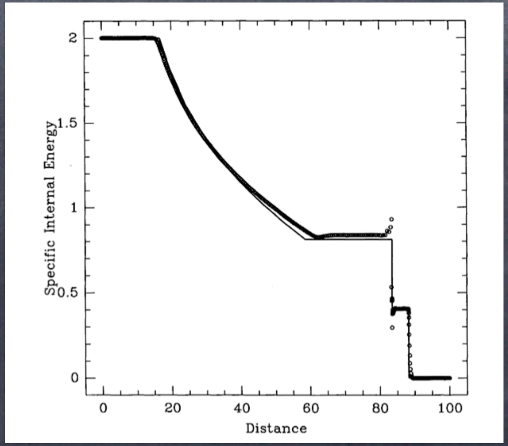
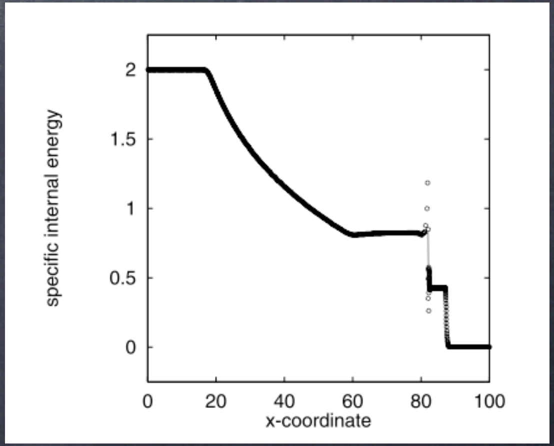
Laguna et al. (1993)



Siegler & Riffert (2000)



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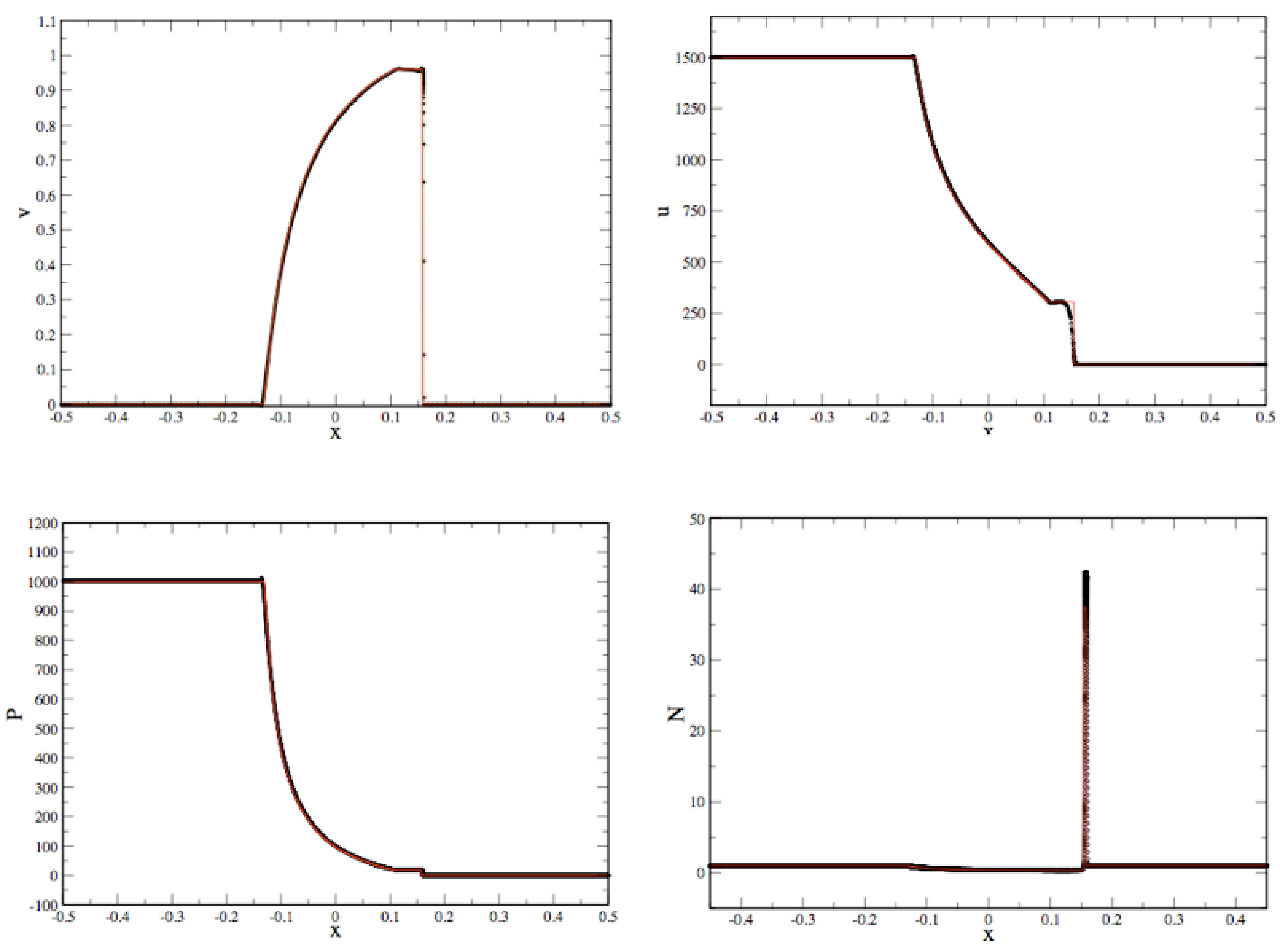

(Dolezal & Wong 1995)

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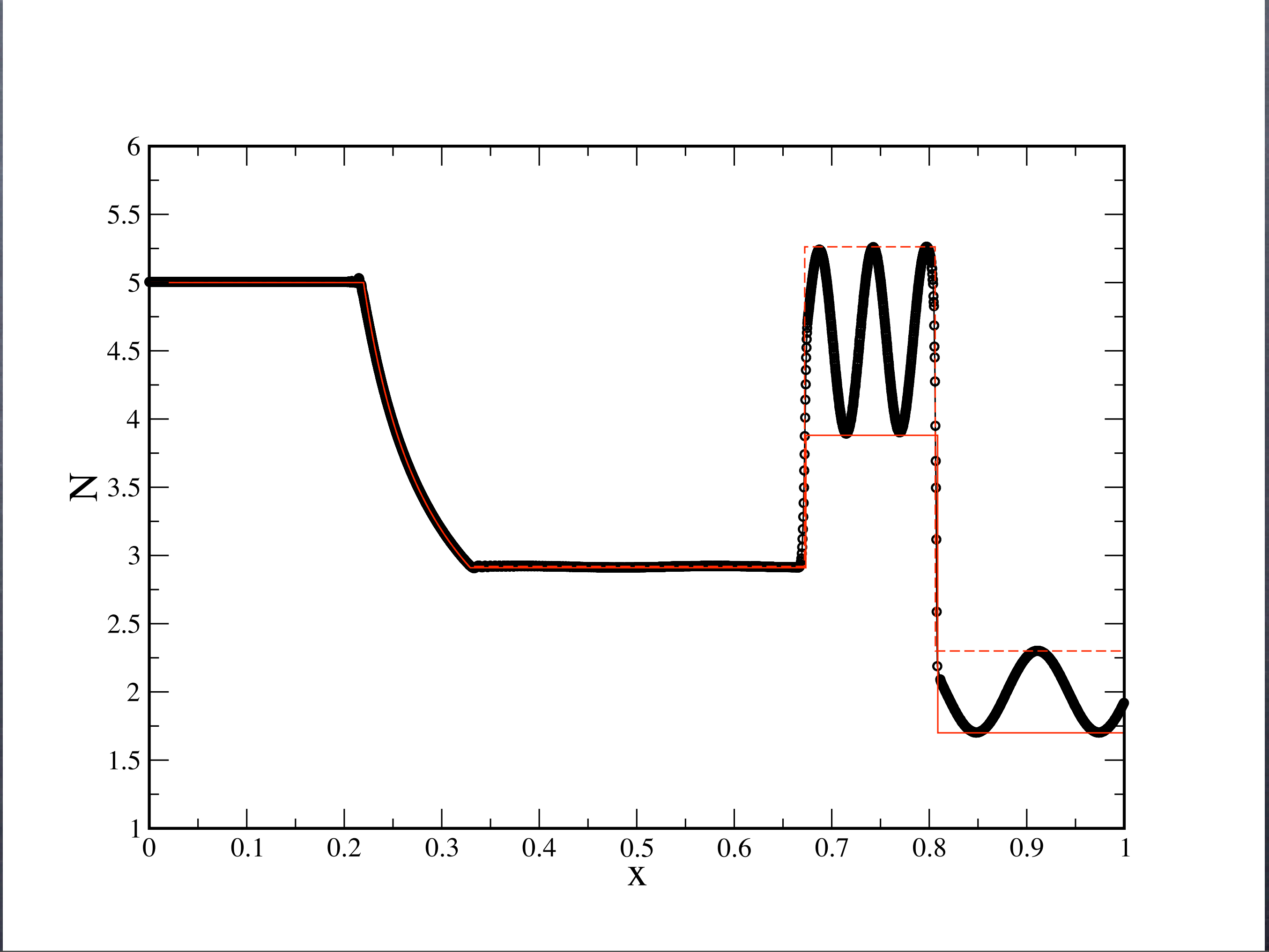

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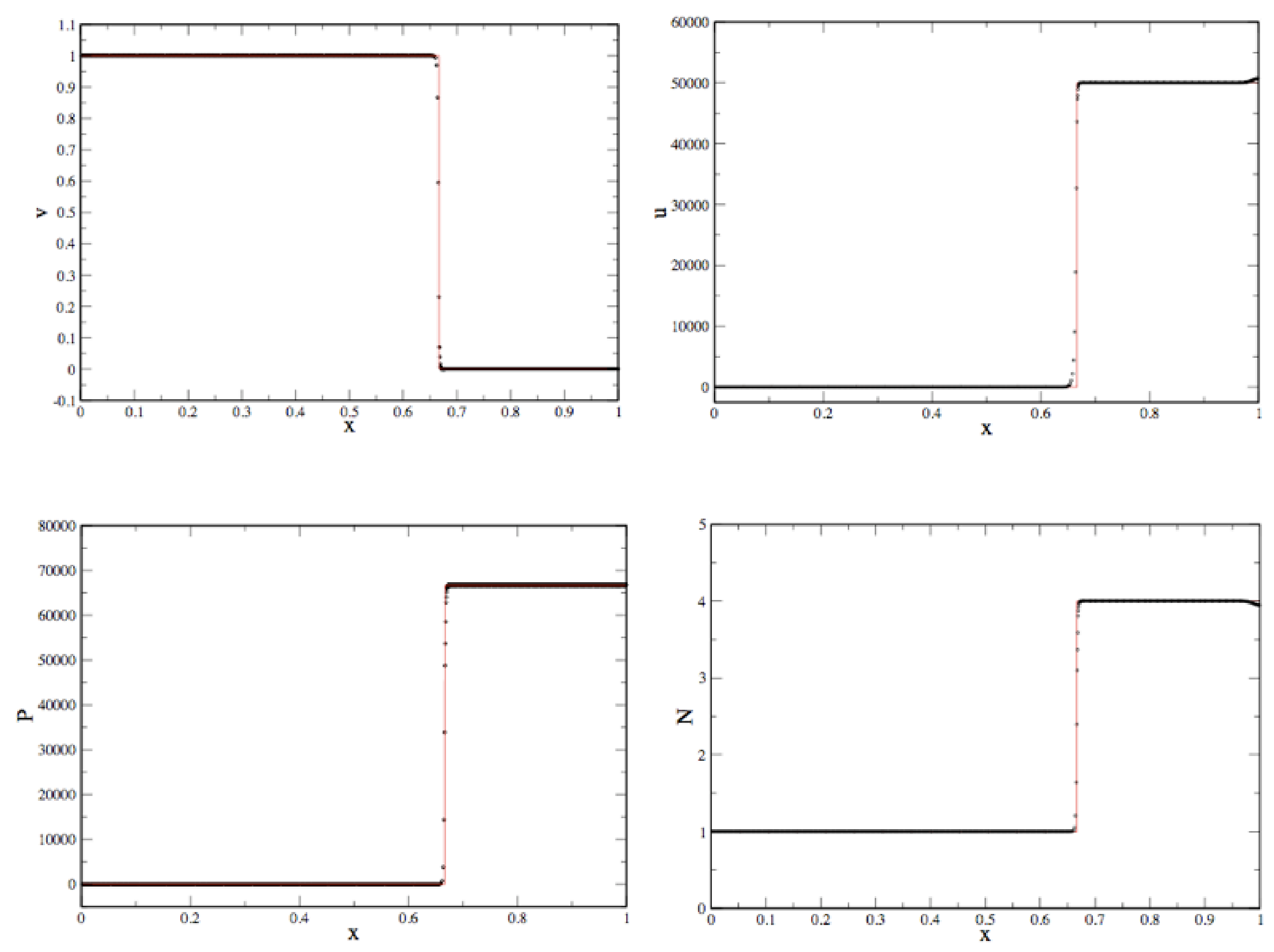

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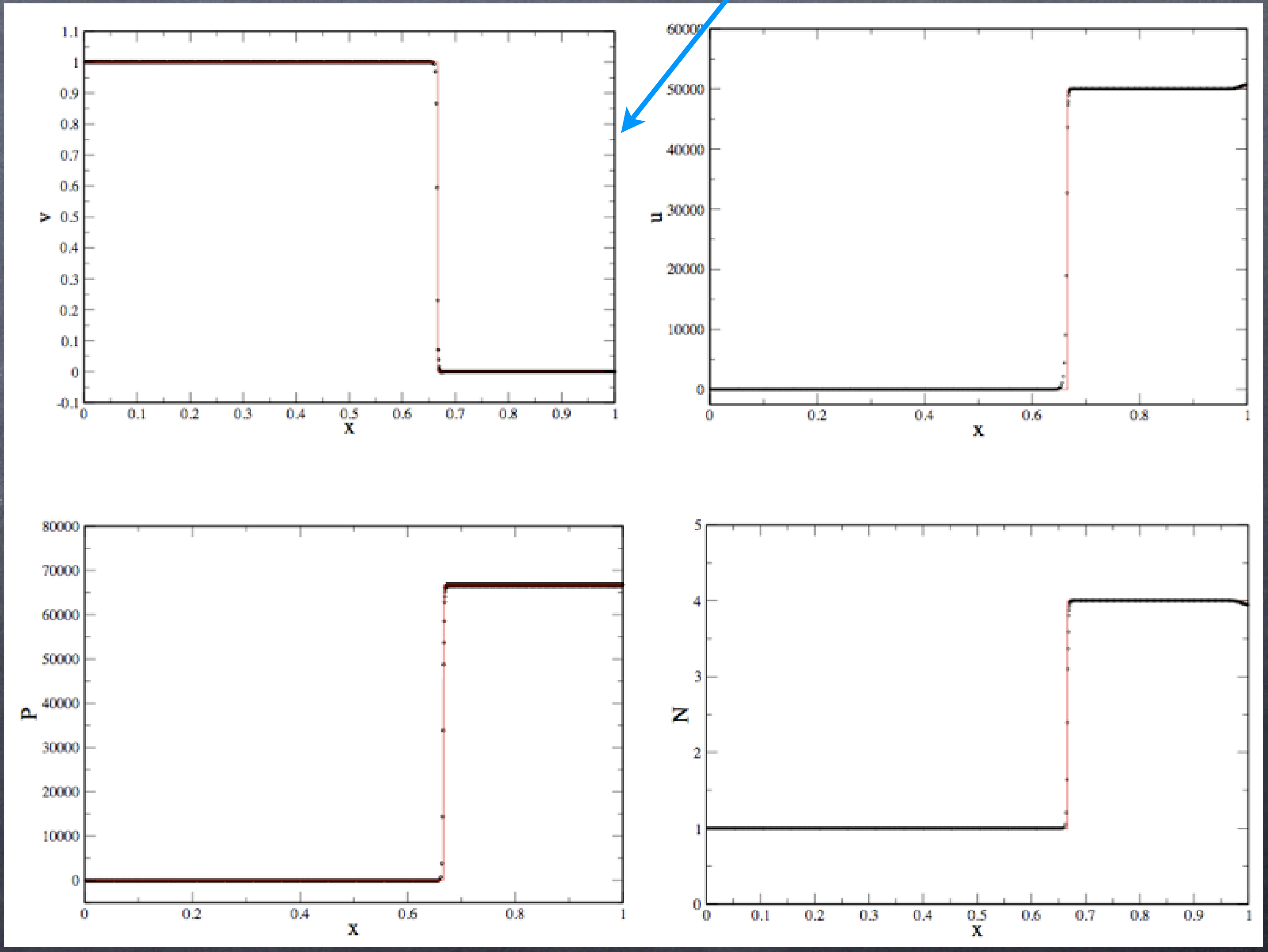

“wall” modeled by “ghost particles”

- Test 7: evolution of relativistic simple wave
  - rel. simple wave: spatial and temporal constancy of 2 of 3 Riemann invariants
  - here: specific entropy + J-; $J_{\pm} = \ln(\gamma + U) \pm \int \frac{c_{\mathrm{s}}}{\rho} d\rho$
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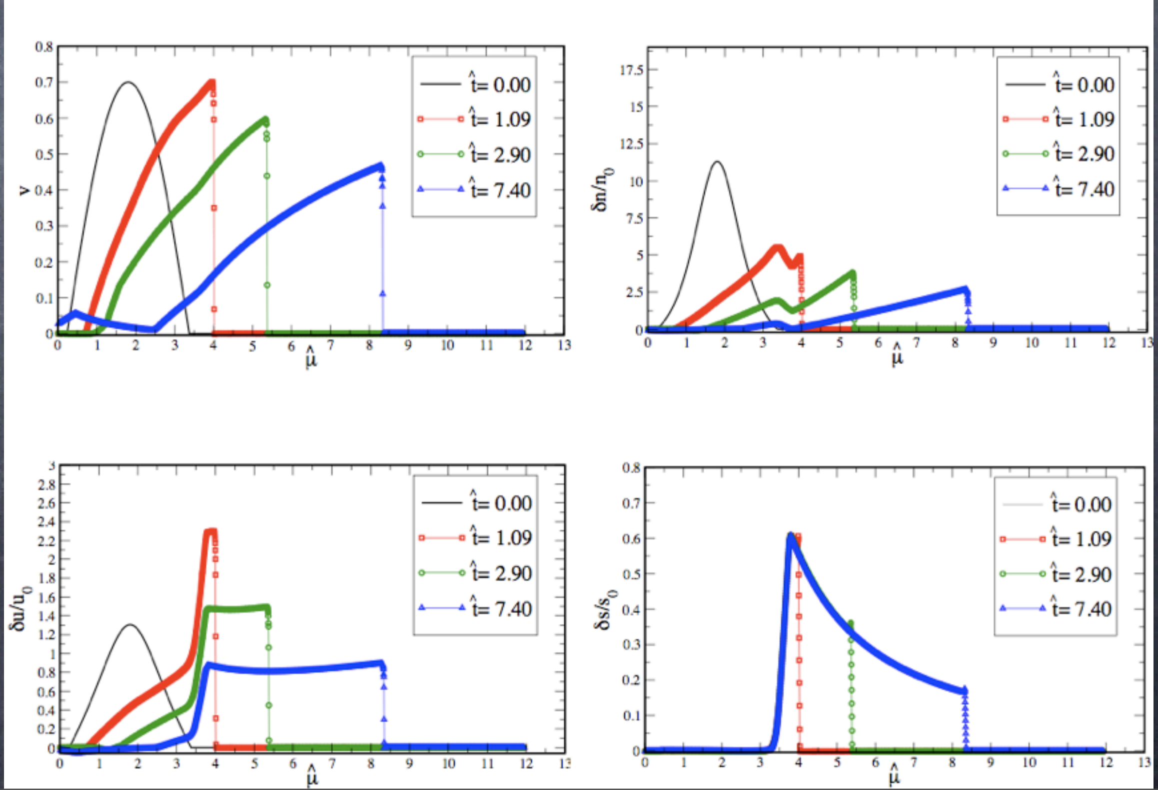

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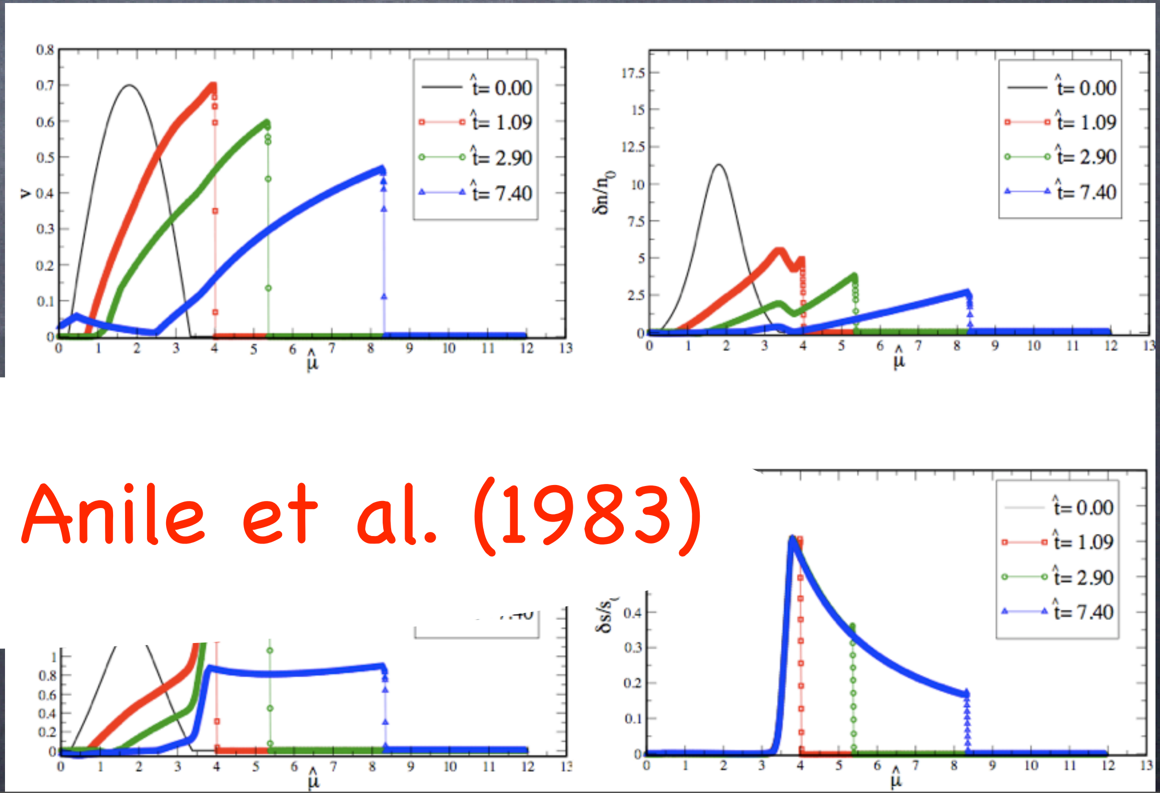
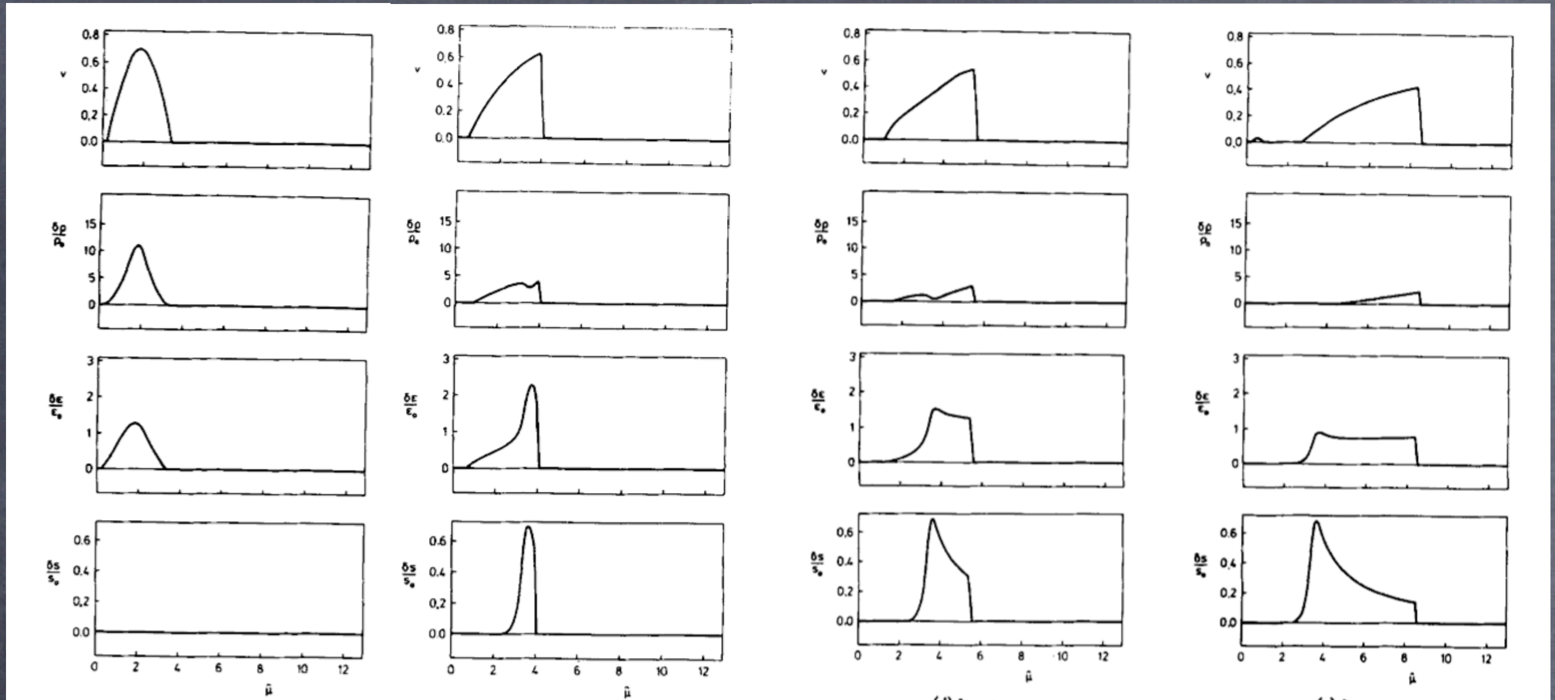


close agreement with Anile et al. (1983)

from Anile, Miller, Motta, Physics of Fluids, 26, 1450, 1983



(a) $\hat{t} = 0$ (c) $\hat{t} = 1.09$ (d) $\hat{t} = 2.90$ (e) $\hat{t} = 7.40$

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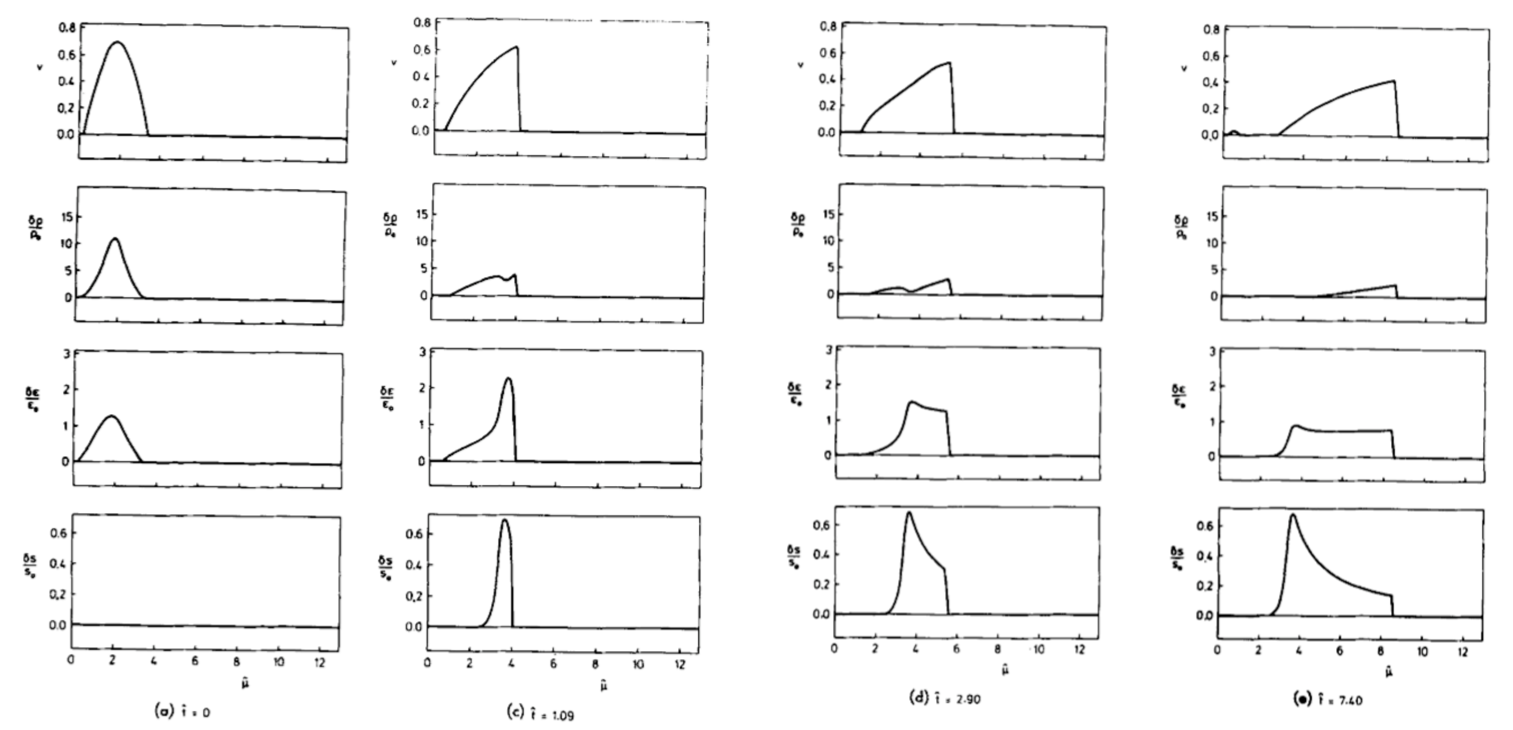
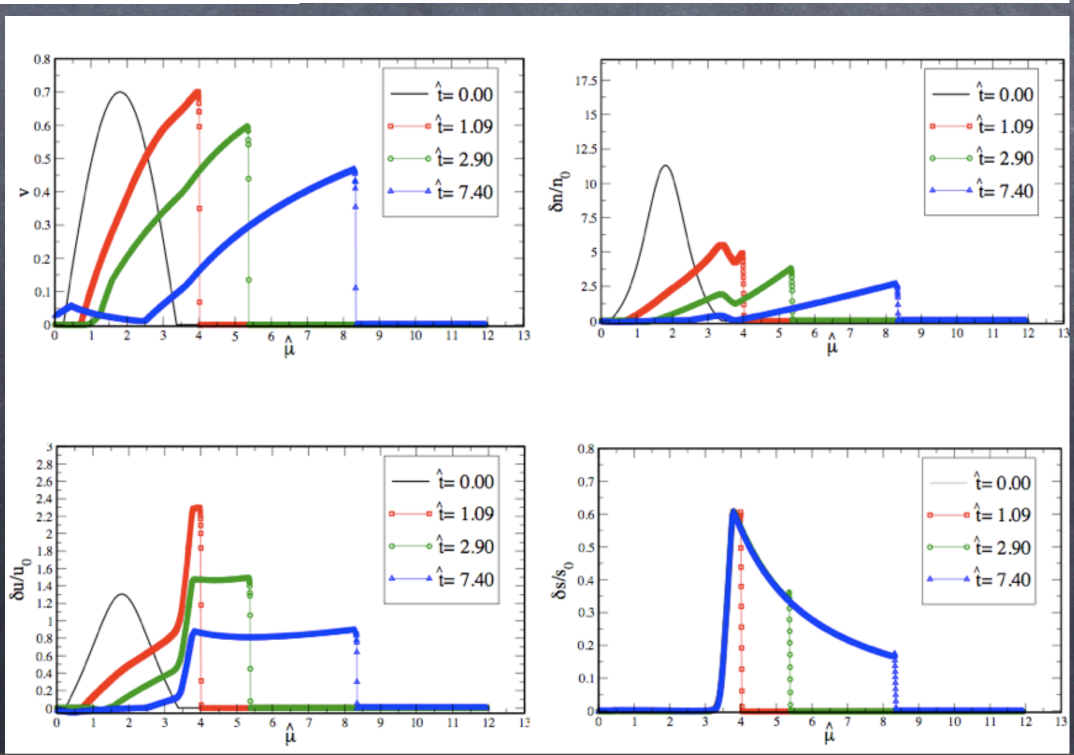


(a) t̂ = 0 (c) t̂ = 1.09 (d) t̂ = 2.90 (e) t̂ = 7.40



VI. Outlook to general relativity

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- General-relativistic Lagrangian

$$L_{\mathrm{pf,GR}} = -\int T^{\mu\nu} U_\mu U_\nu \sqrt{-g}\, dV,$$

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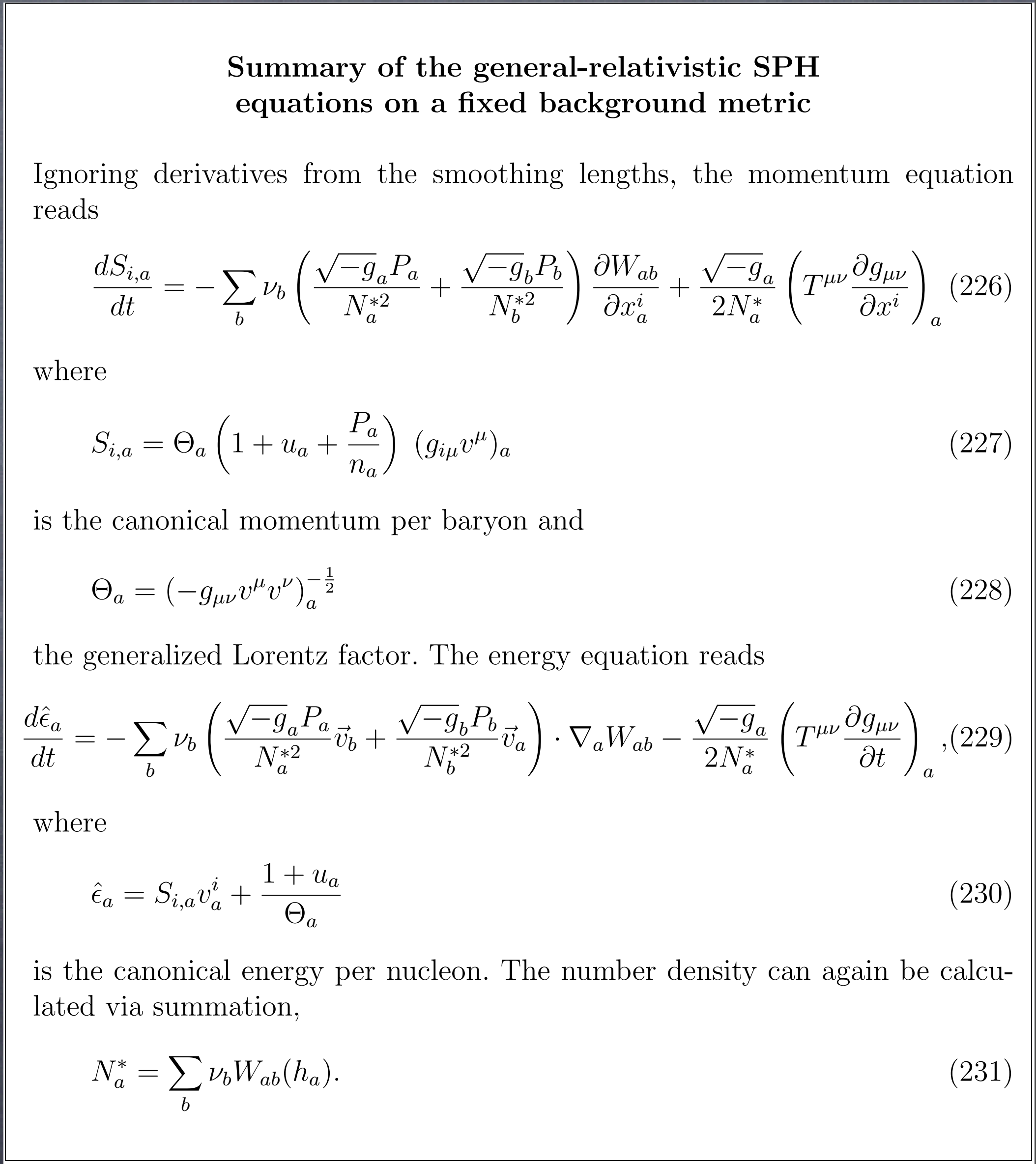


Summary of the general-relativistic SPH equations on a fixed background metric

Ignoring derivatives from the smoothing lengths, the momentum equation reads

$$\frac{dS_{i,a}}{dt} = -\sum_b \nu_b \left(\frac{\sqrt{-g_a} P_a}{N_a^{*2}} + \frac{\sqrt{-g_b} P_b}{N_b^{*2}} \right) \frac{\partial W_{ab}}{\partial x_a^i} + \frac{\sqrt{-g_a}}{2N_a^*} \left(T^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x^i} \right)_a \quad (226)$$

where

$$S_{i,a} = \Theta_a \left(1 + u_a + \frac{P_a}{n_a} \right) (g_{i\mu} v^\mu)_a \quad (227)$$

is the canonical momentum per baryon and

$$\Theta_a = (-g_{\mu\nu} v^\mu v^\nu)_a^{-\frac{1}{2}} \quad (228)$$

the generalized Lorentz factor. The energy equation reads

$$\frac{d\hat{\epsilon}_a}{dt} = -\sum_b \nu_b \left(\frac{\sqrt{-g_a} P_a}{N_a^{*2}} \vec{v}_b + \frac{\sqrt{-g_b} P_b}{N_b^{*2}} \vec{v}_a \right) \cdot \nabla_a W_{ab} - \frac{\sqrt{-g_a}}{2N_a^*} \left(T^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial t} \right)_a, \quad (229)$$

where

$$\hat{\epsilon}_a = S_{i,a} v_a^i + \frac{1 + u_a}{\Theta_a} \quad (230)$$

is the canonical energy per nucleon. The number density can again be calculated via summation,

$$N_a^* = \sum_b \nu_b W_{ab}(h_a). \quad (231)$$

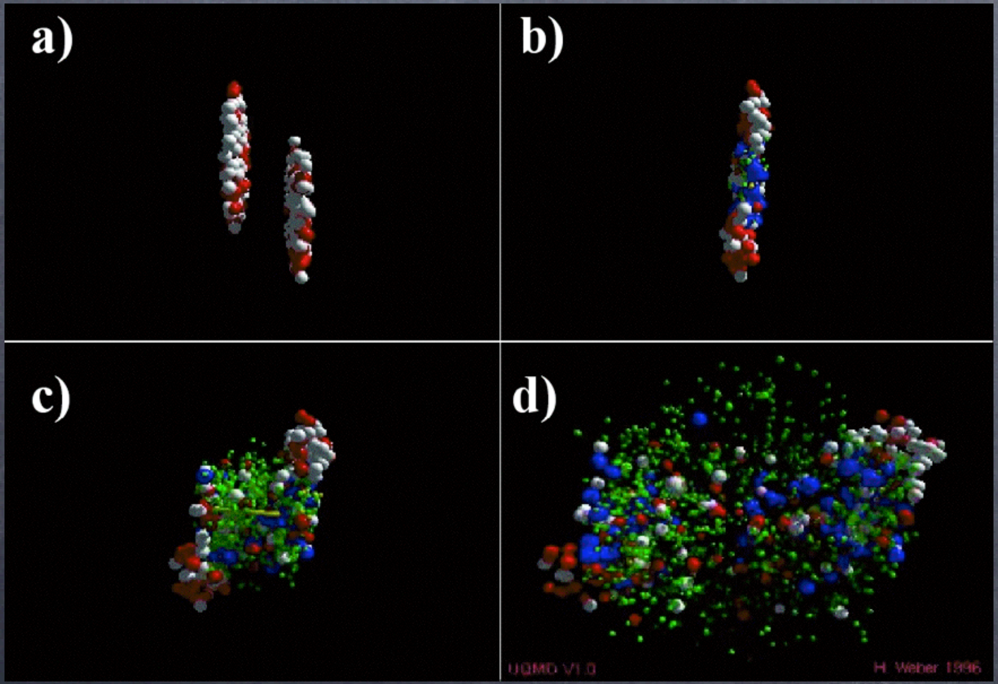
from Rosswog (2009), New Astronomy Reviews

VI. Summary

- new formulation of special-relativistic SPH
- features:
  - derived Lagrangian of perfect fluid + first law of thermodynamics
  - no ambiguity in symmetrization
  - artificial viscosity motivated by Riemann solvers, time-dependent parameters
- convincing performance in both advection and strong, relativistic shocks

I. Where is special-relativistic Hydrodynamics used?

- Heavy ion collisions:



Lorentz factors up to

$$\gamma = \sqrt{\frac{1}{1 - v/c}} \sim 30$$

i.e.

$$v \approx 0.9994\, c$$

- Astrophysics:
    - Jets from Active Galactic Nuclei

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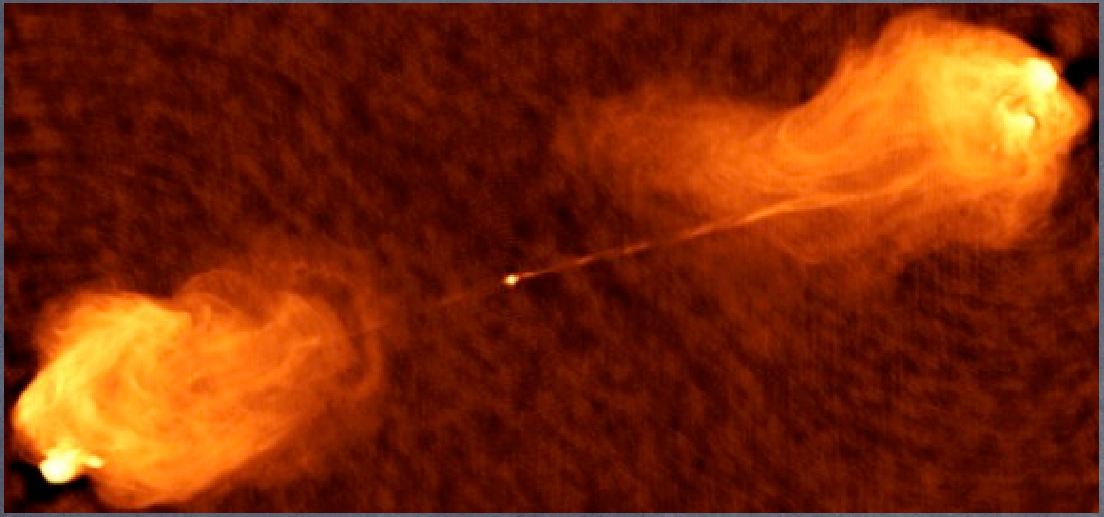

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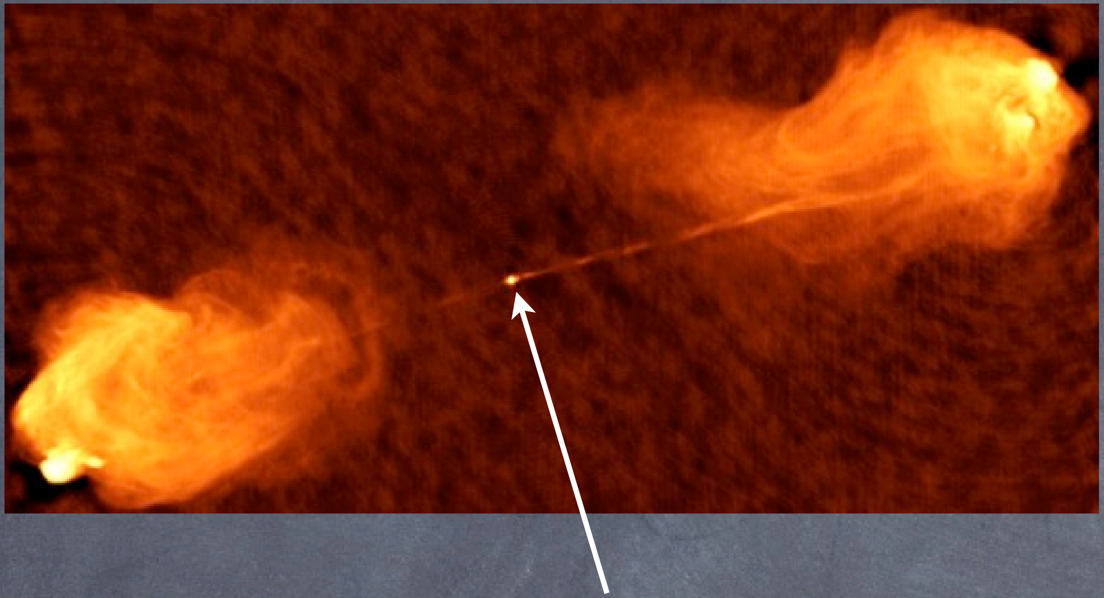
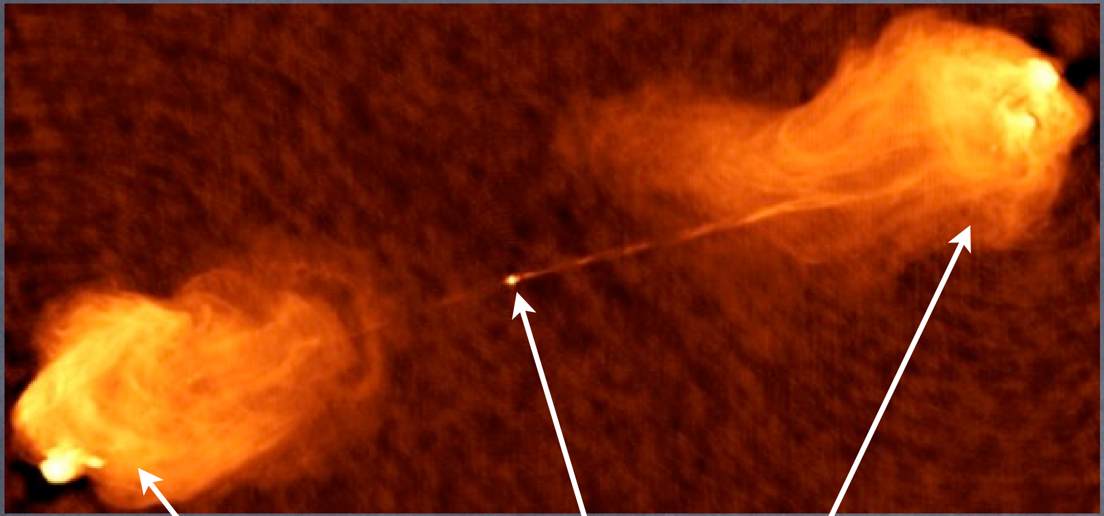


Black hole at the centre of a galaxy

- Astrophysics:
    - Jets from Active Galactic Nuclei



Black hole at the centre
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relativistic outflows, “jets”

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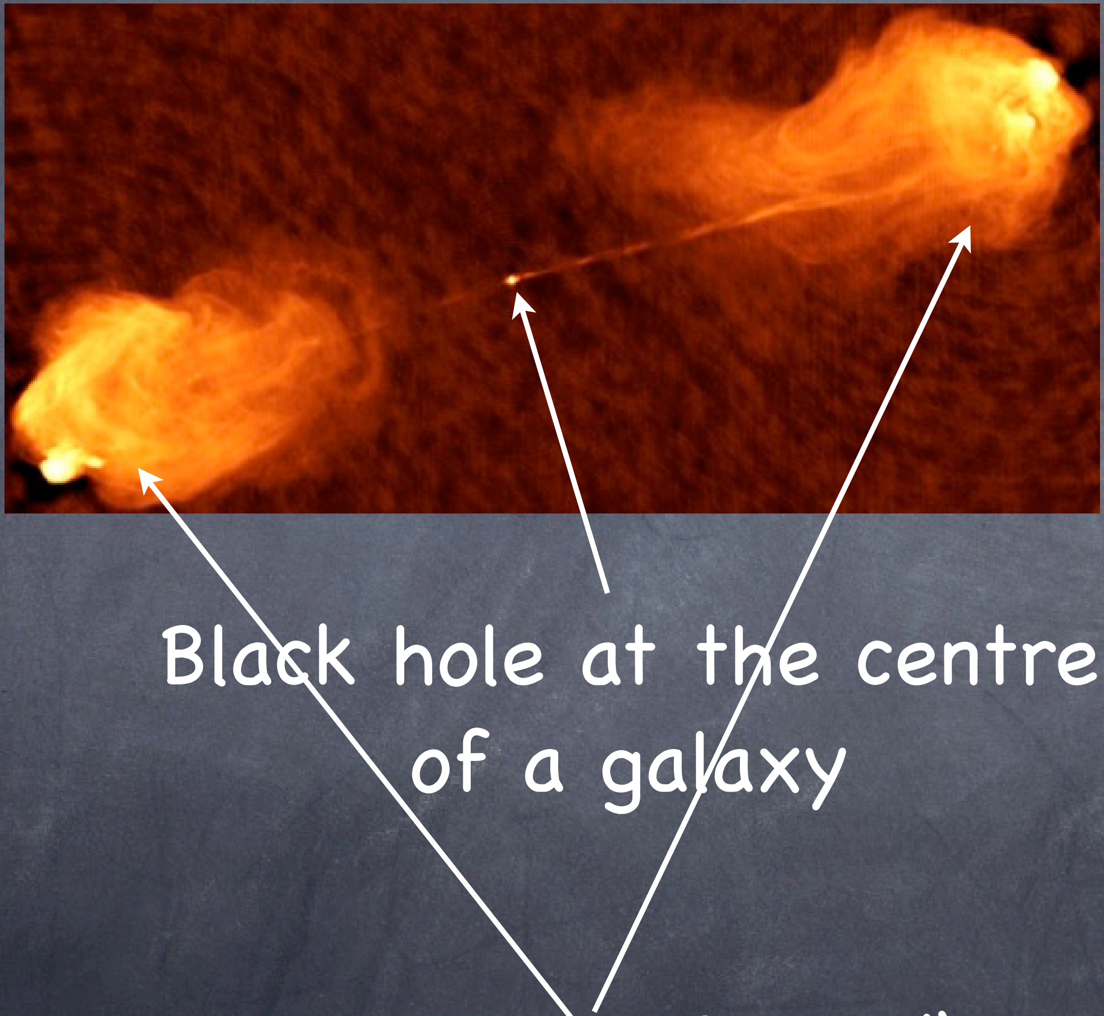

Black hole at the centre of a galaxy

relativistic outflows, “jets”

Lorentz factors up to $\gamma \sim 20$

i.e.

$v \approx 0.99875$

- “Gamma-ray burst”:

  relativistic outflow from a dying, massive star

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(artist’s view)

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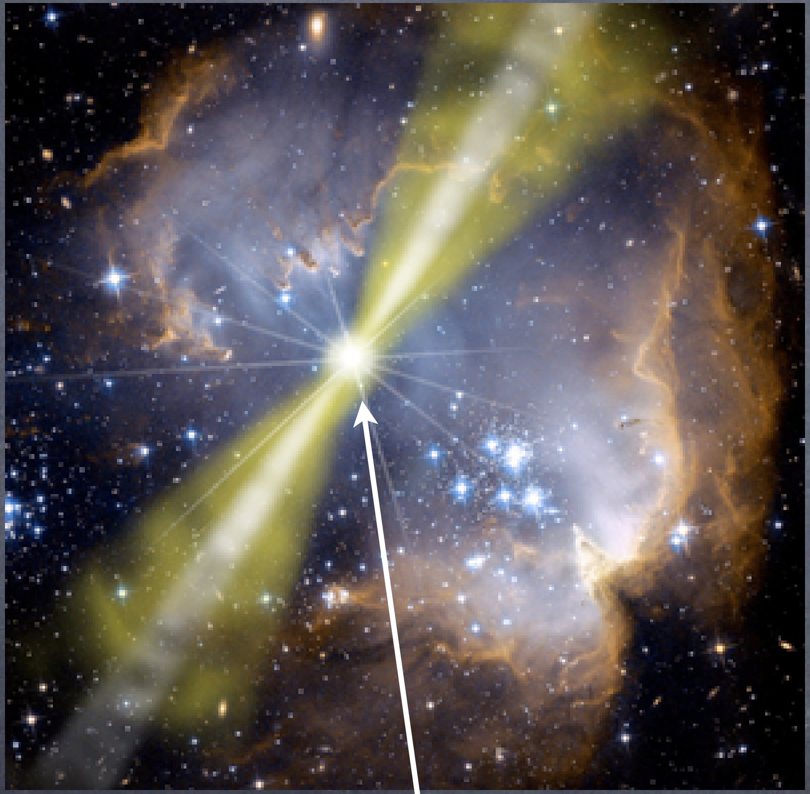
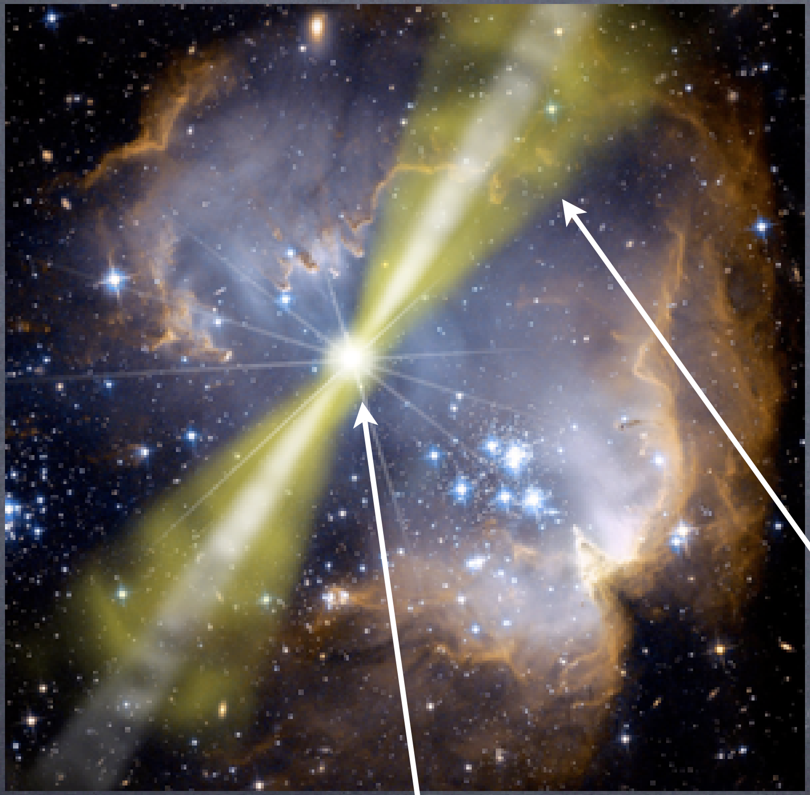


black hole formation
inside a dying star

- “Gamma-ray burst”:

relativistic outflow from a dying, massive star

(artist’s view)



black hole formation inside a dying star

jetted, relativistic outflow

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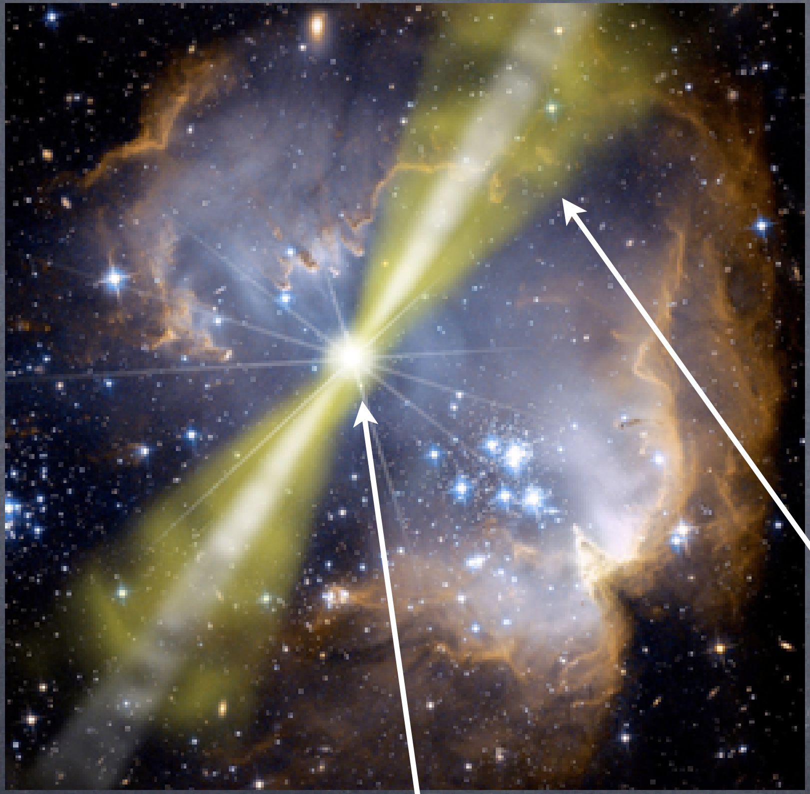

(artist’s view)

typical Lorentz factors

$\gamma \sim 300$

i.e.

$v \approx 0.99999444$

black hole formation inside a dying star

jetted, relativistic outflow

Energy- momentum tensor $T^{\mu\nu}$

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- describes density and flux of energy and momentum in spacetime:

  $T^{\mu\nu}$ = “flux of 4-momentum component μ

  across surface with constant ν-coordinate”

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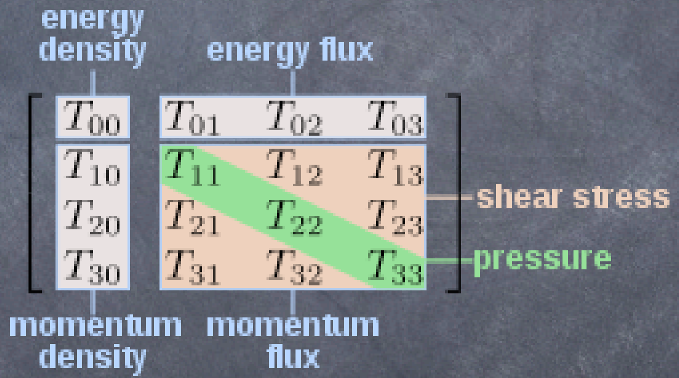

$$\begin{bmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{bmatrix}$$

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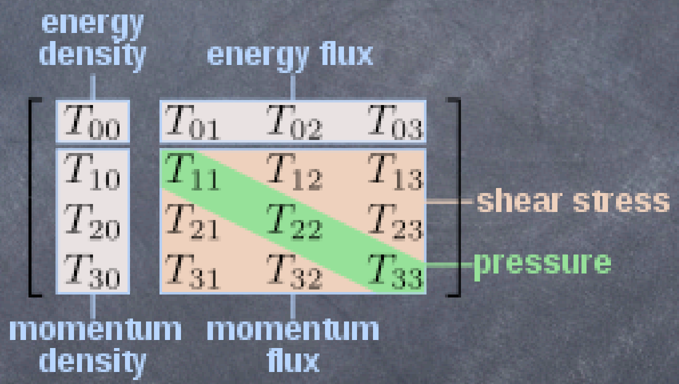

- for an ideal fluid:

Energy- momentum tensor $T^{\mu\nu}$

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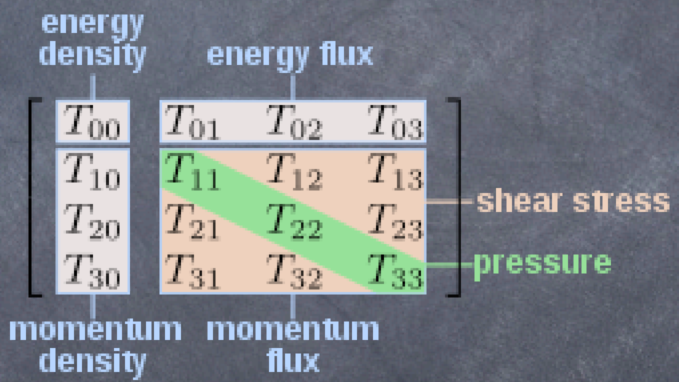

- for an ideal fluid: $T^{\mu\nu} = (\rho + P)U^{\mu}U^{\nu} + Pg^{\mu\nu}$

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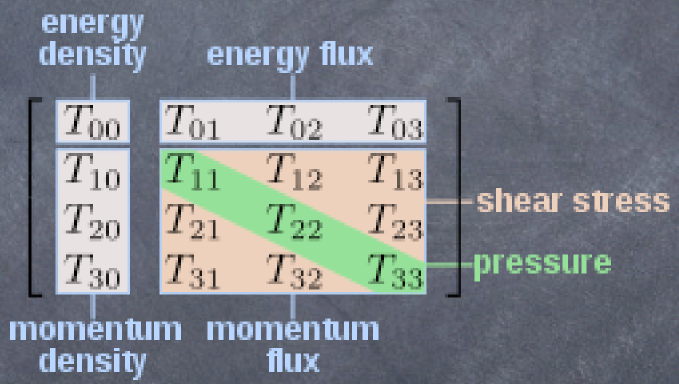

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rest mass density in comoving frame

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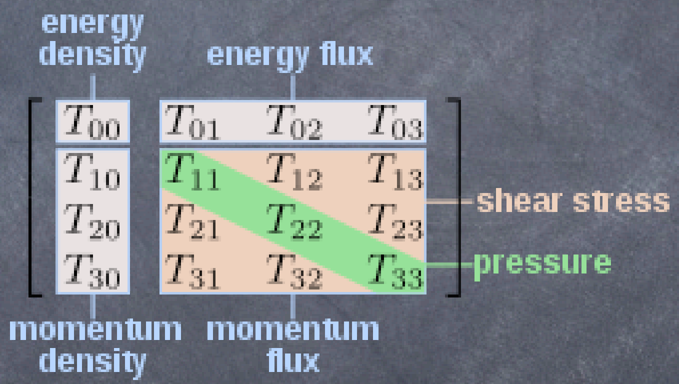

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rest mass density in comoving frame

pressure

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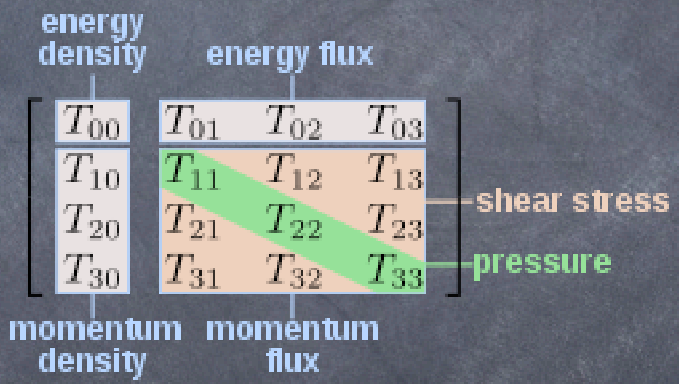

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rest mass density in comoving frame pressure 4-velocity $U^{\mu} = \frac{dx^{\mu}}{d\tau}$

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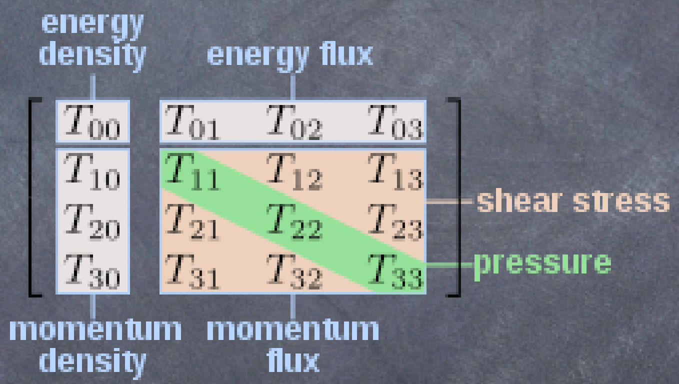

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rest mass density in comoving frame   pressure   4-velocity $U^\mu = \frac{dx^\mu}{d\tau}$ pressure

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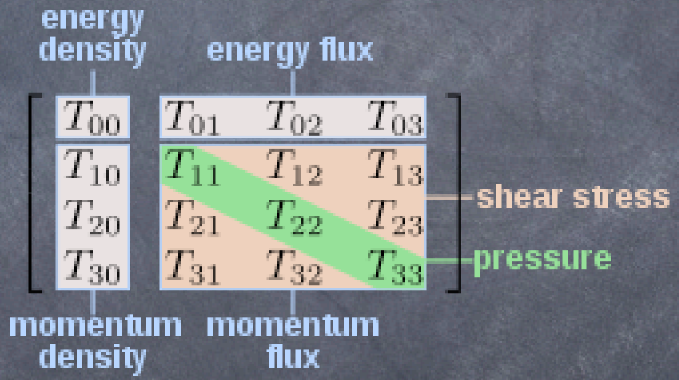

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metric tensor

- for an ideal fluid: $T^{\mu\nu} = (\rho + P)U^{\mu}U^{\nu} + Pg^{\mu\nu}$

rest mass density in comoving frame

pressure

4-velocity $U^{\mu} = \frac{dx^{\mu}}{d\tau}$

pressure

- “thermokinetic energy equation”:

$$\frac{d\hat{e}_a}{dt} \quad = \quad -\sum_b m_b \left(\frac{P_a \vec{v}_b}{\rho_a^2} + \frac{P_b \vec{v}_a}{\rho_b^2} \right) \cdot \nabla_a W_{ab}.$$