Solving the Einstein equations: why bother about the constraints?

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based on collaboration with
S. Bonazzola, P. Grandclément, E. Gourgoulhon and N. Vasset.

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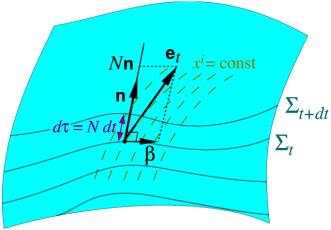
Summary (v1.0)

Constrained schemes lead to a stable evolution in the presence of matter, as they are a natural generalization of the conformally-flat approximation.

Decomposition of spacetime and of Einstein equations



Evolution equations:

$$\frac{\partial K_{ij}}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}} K_{ij} =$$
$$-D_i D_j N + N R_{ij} - 2N K_{ik} K^k{}_j +$$
$$N\left[K K_{ij} + 4\pi((S - E)\gamma_{ij} - 2S_{ij})\right]$$
$$K^{ij} = \frac{1}{2N}\left(\frac{\partial \gamma^{ij}}{\partial t} + D^i \beta^j + D^j \beta^i\right).$$

Constraint equations:

$$R + K^2 - K_{ij}K^{ij} = 16\pi E,$$
$$D_j K^{ij} - D^i K = 8\pi J^i.$$

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -N^2\, dt^2 + \gamma_{ij}\,(dx^i + \beta^i dt)\,(dx^j + \beta^j dt)$$

3+1 formalism

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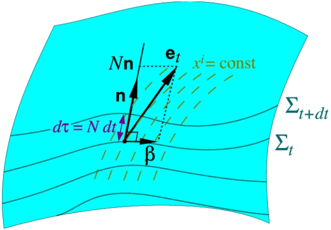

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3+1 FORMALISM

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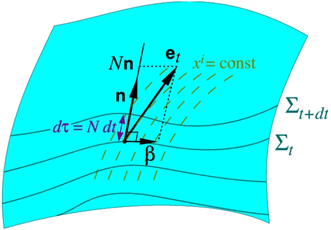

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Free vs. constrained formulations

As in electromagnetism, if the constraints are satisfied initially, they remain so for a solution of the evolution equations.

Free evolution

- start with initial data verifying the constraints,
- solve only the 6 evolution equations,
- recover a solution of all Einstein equations.

⇒apparition of constraint violating modes from round-off errors. Considered cures:

- Using of constraint damping terms and adapted gauges (many groups).
- Solving the constraints at every time-step (efficient elliptic solver?).

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Conformal flatness condition (CFC)

and

Fully constrained formulation (FCF)

Conformal flatness condition

Within 3+1 formalism, one imposes that :

$$\gamma_{ij} = \psi^4 f_{ij}$$

with f_{ij} the flat metric and $\psi(t, x^1, x^2, x^3)$ the conformal factor. First devised by Isenberg in 1978 as a waveless approximation to GR, it has been widely used for generating initial data, ...

Set of 5 non-linear elliptic PDEs ($K = 0$)

$$\Delta\psi = -2\pi\psi^{-1}\left(E^* + \frac{\psi^6 K_{ij}K^{ij}}{16\pi}\right),$$

$$\Delta(N\psi) = 2\pi N\psi^{-1}\left(E^* + 2S^* + \frac{7\psi^6 K_{ij}K^{ij}}{16\pi}\right),$$

$$\Delta\beta^i + \frac{1}{3}\nabla^i\nabla_j\beta^j = 16\pi N\psi^{-2}(S^*)^i + 2\psi^{10}K^{ij}\nabla_j\frac{N}{\psi^6}.$$

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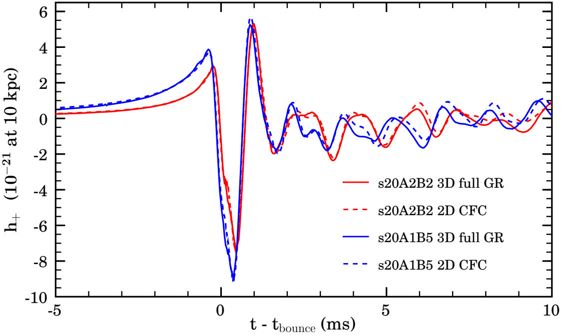
Core-collapse and CFC

Together with the use of a purely finite-differences code in full GR, first results of realistic collapse of rotating stellar iron cores in GR

- with finite temperature EOS;
- (approximate) treatment of deleptonization.



Ott *et al.* (2007)

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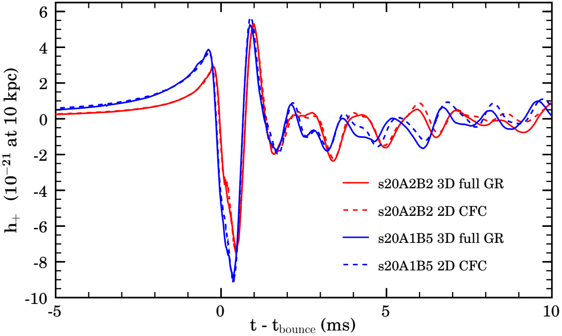

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Fully constrained formulation

Bonazzola *et al.* (2004)

With no approximation: $\tilde{\gamma}^{ij} = \psi^4 \gamma^{ij}$ and the choice of generalized Dirac gauge (and maximal slicing)

$$\nabla_j \tilde{\gamma}^{ij} = \nabla_j h^{ij} = 0. \qquad (\tilde{\gamma}^{ij} = f^{ij} + h^{ij})$$

$\Rightarrow$very similar equations to the CFC system + evolution equations for $\tilde{\gamma}^{ij}$:

$$\frac{\partial K^{ij}}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}} K^{ij} = N D_k D^k h^{ij} - D^i D^j N + \mathcal{S}^{ij},$$

$$\frac{\partial h^{ij}}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}} h^{ij} = 2 N K^{ij}.$$

When combined, reduce to a wave-like (strongly hyperbolic) operator on h^{ij}, with no incoming characteristics from a black hole excision boundary (Cordero-Carrión *et al.* (2008)).

Fully constrained formulation

Motivations for the FCF:

- Easy to use CFC initial data for an evolution using the constrained formulation,
- Evolution of two scalar fields: the rest of the tensor h^{ij} can be reconstructed using the gauge conditions.
  $\Longleftrightarrow$ dynamical degrees of freedom of the gravitational field.
- Elliptic systems have good stability properties (what about uniqueness?).
- Newtonian limit obtained without difficulty.
- Constraints are verified!

+ the generalized Dirac gauge gives the property that h^{ij} is asymptotically transverse-traceless
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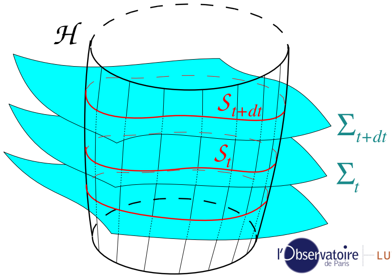
Excision techniques

apparent horizons as a boundary

- Remove a neighborhood of the central singularity from computational domain;
- Replace it with boundary conditions on this newly obtained boundary (usually, a sphere),
- Until now, imposition of apparent horizon / isolated horizon properties: zero expansion of outgoing light rays.

⇒New views on the concept of black hole, following works by Hayward, Ashtekar and Krishnan: dynamical horizon

- Quasi-local approach, making the black hole a causal object;
- For hydrodynamic, electromagnetic and gravitational waves (Dirac gauge): no incoming characteristics.



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Summary (v2.0)

Constrained schemes, together with excision techniques can be valuable tools for the modeling of a collapse to a black hole.

Summary - Perspectives

Conformally-flat condition:

- Is a reasonable approximate theory of gravity for core-collapse simulations.
- With a good Poisson solver, not too difficult to implement CFC equations.

Constrained scheme:

- Can the FCF provide better accuracy on the gravitational field for long-term astrophysical simulations?
- Does FCF need too much CPU to solve elliptic PDEs?

Black holes:

- Implement excision technique within collapse code (CoCoNuT).
- Study the dynamical interplay between black holes and surrounding environment.

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