Multiloop integrals in dimensional regularization made simple

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based on

- PRL 110 (2013) [arXiv:1304.1806],
- JHEP 1307 (2013) 128 [arXiv:1306.2799] with A.V. and V.A. Smirnov
- arXiv: 1307.4083 with V.A. Smirnov

supported in part by the Department of Energy grant DE-SC0009988
and the IAS AMIAS fund

Why study Feynman loop integrals?

- Important ingredient in cross-section calculations for collider experiments

- Many important processes depend on various scales. What are the multi-variable functions needed to describe them? Analytic evaluation is challenging, but often feasible and very interesting!

- Connections to and new tools from mathematics: iterated integrals, multiple zeta values; symbols; QFT provides interesting mathematical problems -- e.g. what are the functions needed to describe Feynman integrals?

Integrands versus integrals

- Integrands similar properties to tree-level amplitudes
- Good methods for obtaining loop integrands
  - from analytic properties:

  unitarity cuts, recursion relations [Bern, Dixon, Dunbar, Kosower, 90s]
  [Britto, Cachazo, Feng, Witten, 2004]
  [Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka, 2010]

  - using Feynman diagrams e.g. GQRAF: [Nogueira, 1993], FeynArts: [Hahn, 1999] ...

  - ongoing work on automation at two loops
  Mastrolia, Mirabella, Ossola, Peraro, Reiter, Tramontano, Johansson, Kosower, Larsen, Caron-Huot, Badger, Frellesvig, Zhang, ...

- Integrals: Good control at NLO: One-loop integrals under analytic control; Can be readily evaluated numerically.

Can we reach a similar state for integrals needed at NNLO?

Analytic computation of (Feynman) loop integrals

- what functions are needed?
- how do they depend on the kinematical variables?

(e.g. asymptotic limits, singularities)

- how do they depend on D=4 - 2 ϵ ?
- how can we compute the integrals?

Outline

- Part 1: Introduction to differential equations (DE) for Feynman integrals
- Part 2: New strategy for solving DE

- choice of integral basis

- solution as Chen iterated integrals

- Part 3: Examples:

- massless 2 to 2 scattering

- Bhabha scattering

Integral functions: examples

Experience shows: many processes described by iterated integrals

- simple cases: logarithms $\log z = \int_1^z \frac{dt}{t}$, polylogarithms Li
- generalization: harmonic polylogarithms (HPLs)
- more general: Goncharov polylogarithms, Chen iterated integrals
- multiple masses -> Elliptic functions (not in this talk)
- ...

Harmonic polylogarithms (HPL)

- defined iteratively [Remiddi, Vermaseren, 1999]

$$H_1(x) = -\log(1-x)\,, \qquad H_0(x) = \log(x)\,, \qquad H_{-1}(x) = \log(1+x)\,.$$

$$H_{a_1,a_2,\ldots a_n}(x) = \int_0^x f_{a_1}(y) H_{a_2,\ldots,a_n}(y) dy$$

kernels: $f_1(y) = \dfrac{1}{1-y}\,, \qquad f_0(y) = \dfrac{1}{y}\,, \qquad f_{-1}(y) = \dfrac{1}{1+y}$

- naturally arise in differential equations
- 'transcendental' weight: number of integrations
- more general integration kernels: Goncharov polylogarithms; Chen iterated integrals

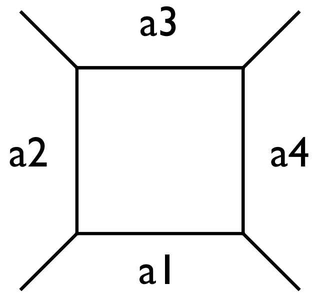
Integration by parts identities (IBP)

[Chetyrkin, Tkachov, 1981]

public computer codes [Anastasiou, Lazopoulos] [Smirnov, Smirnov] [Studerus, von Manteuffel]

- IBP relates integrals with different indices

'family' of integrals $F(a_1, a_2, a_3, a_4; D, s, t)$



$$\int d^{4-2\epsilon}k \frac{\partial}{\partial k^\mu} q^\mu \frac{1}{[k^2]^{a_1}[(k+p_1)^2]^{a_2}[(k+p_1+p_2)^2]^{a_3}[(k-p_4)^2]^{a_4}} = 0$$

- for a given topology, finite number of 'master' integrals needed

- how to compute the master integrals?

Differential equation (DE) technique

[Kotikov, 1991] [Gehrmann, Remiddi, 1999]
[in Feynman representation/one loop: Bern, Dixon, Koswer, 1993]

- differentiate master integrals w.r.t. momenta and masses
- use IBP to re-express RHS in terms of master integrals
- system of differential equations

$$\partial_i f(x_j, \epsilon) = A_i(x_j, \epsilon) f(x_j, \epsilon)$$

- very powerful, many integrals computed using this method

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- very powerful, many integrals computed using this method
- Some issues:

- A_i often complicated, physical properties not transparent (e.g. asymptotic behavior, singularities)
- multi-variable case can be complicated
- what integral functions are needed?
- coupled system of equations hard to solve

Part 2:
New strategy for solving DE
(for Feynman integrals)

How to choose a good integral basis?

What will the solution look like?

Change of basis in DE

- system of differential equations

$$\partial_i f(x_j, \epsilon) = A_i(x_j, \epsilon) f(x_j, \epsilon)$$

problem: A_i typically complicated

- change of integral basis:

$$f \longrightarrow B\, f$$
$$A_j \longrightarrow B^{-1} A_j B - B^{-1}(\partial_j B)$$

- idea: can we find an optimal integral basis that simplifies the system of DE?

- differential equations should make properties of the solution manifest

Pure functions of uniform weight

- uniform 'transcendental' weight $\mathcal{T}$

$$f_1(x) = \text{Li}_3(x) + \frac{1}{2}\log^3 x \qquad \mathcal{T}(f_1) = 3$$

$$f_2(x,y) = \text{Li}_4(x/y) + 3\log x\,\text{Li}_3(1-y) \qquad \mathcal{T}(f_2) = 4$$

- pure functions: derivative reduces weight

$$\mathcal{T}(f) = n \quad \longrightarrow \quad \mathcal{T}(d\,f) = n-1$$

f_1, f_2 are pure functions of uniform weight

$f_3 = \frac{1}{x}\log^2 x + \frac{1}{1+x}\text{Li}_2(1-x)$ has uniform weight 2, but is not pure

functions with unique normalization

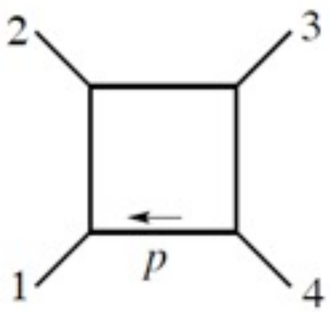
- dimensional regularization

$x^\epsilon = 1 + \epsilon\log(x) + \ldots$ assign weight -1 to ϵ

pure functions should obey simple differential equations!

Pure functions of uniform weight

- example 1: box integral [Bern, Dixon, Smirnov, 2005]

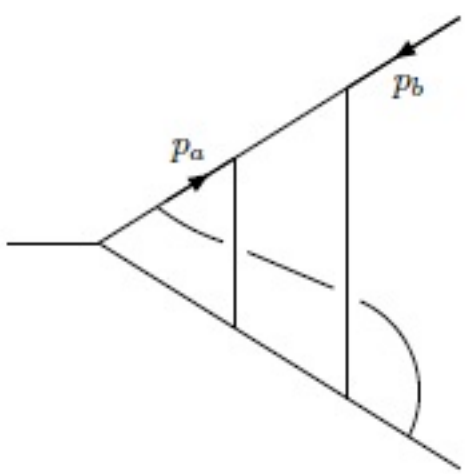


$$I_4^{(1)}(s,t) = -\frac{1}{(-s)^{1+\epsilon}t}\sum_{j=-4}^{2}\frac{c_j(x,L)}{\epsilon^j}\,, \qquad x=-t/s, L=\ln(s/t)$$

$$c_2 = 4\,, \qquad c_1 = 2L\,, \qquad c_0 = -\frac{4}{3}\pi^2\,,$$

$$c_{-1} = \pi^2 H_1(x) + 2H_{0,0,1}(x) - \frac{7}{6}\pi^2 L + 2H_{0,1}(x)L + H_1(x)L^2 - \frac{1}{3}L^3 - \frac{34}{3}\zeta_3\,,$$

- example 2: form factor integral [Gehrmann, J.M.H, Huber, 2011]



F_5

$$F_5 = S_\Gamma^3\left[-q^2 - i\eta\right]^{-2-3\epsilon}\cdot F_5^{\exp}\,,$$

$$F_5^{\exp} = +\frac{1}{12\epsilon^6} + \frac{\pi^2}{27\epsilon^4} + \frac{17\zeta_3}{9\epsilon^3} + \frac{71\pi^4}{540\epsilon^2} + \frac{1}{\epsilon}\left(\frac{71\pi^2\zeta_3}{54} + \frac{13\zeta_5}{3}\right) - \frac{679\zeta_3^2}{6} + \frac{3991\pi^6}{136080} + \epsilon\left(-\frac{2837\pi^4\zeta_3}{540} + \frac{205\pi^2\zeta_5}{9} - \frac{25135\zeta_7}{24}\right) + \epsilon^2\left(\frac{4006}{3}\zeta_{5,3} - 59\zeta_3\zeta_5 - \frac{10\pi^2\zeta_3^2}{27} - \frac{14156063\pi^8}{16329600}\right) + \mathcal{O}(\epsilon^3)\,.$$

note $\mathcal{T}(\zeta_n) = n$

Remark: results for pure functions are very compact and simple!

Example: choice of integral basis

three-loop N=4 SYM form factor

$$
\begin{aligned}
F_S^{(3)} = R_\epsilon^3 \Bigg[&+\frac{(3D-14)^2}{(D-4)(5D-22)} A_{9,1} - \frac{2(3D-14)}{5D-22} A_{9,2} - \frac{4(2D-9)(3D-14)}{(D-4)(5D-22)} A_{8,1} \\
&- \frac{20(3D-13)(D-3)}{(D-4)(2D-9)} A_{7,1} - \frac{40(D-3)}{D-4} A_{7,2} + \frac{8(D-4)}{(2D-9)(5D-22)} A_{7,3} \\
&- \frac{16(3D-13)(3D-11)}{(2D-9)(5D-22)} A_{7,4} - \frac{16(3D-13)(3D-11)}{(2D-9)(5D-22)} A_{7,5} \\
&- \frac{128(2D-7)(D-3)^2}{3(D-4)(3D-14)(5D-22)} A_{6,1} \\
&- \frac{16(2D-7)(5D-18)\left(52D^2-485D+1128\right)}{9(D-4)^2(2D-9)(5D-22)} A_{6,2} \\
&- \frac{16(2D-7)(3D-14)(3D-10)(D-3)}{(D-4)^3(5D-22)} A_{6,3} \\
&- \frac{128(2D-7)(3D-8)\left(91D^2-821D+1851\right)(D-3)^2}{3(D-4)^4(2D-9)(5D-22)} A_{5,1} \\
&- \frac{128(2D-7)\left(1497D^3-20423D^2+92824D-140556\right)(D-3)^3}{9(D-4)^4(2D-9)(3D-14)(5D-22)} A_{5,2} \\
&+ \frac{4(D-3)}{D-4} B_{8,1} + \frac{64(D-3)^3}{(D-4)^3} B_{6,1} + \frac{48(3D-10)(D-3)^2}{(D-4)^3} B_{6,2} \\
&- \frac{16(3D-10)(3D-8)\left(144D^2-1285D+2866\right)(D-3)^2}{(D-4)^4(2D-9)(5D-22)} B_{5,1} \\
&+ \frac{128(2D-7)\left(177D^2-1584D+3542\right)(D-3)^3}{3(D-4)^4(2D-9)(5D-22)} B_{5,2} \\
&+ \frac{64(2D-5)(3D-8)(D-3)}{9(D-4)^5(2D-9)(3D-14)(5D-22)} \\
&\quad \times \left(2502D^5-51273D^4+419539D^3-1713688D^2+3495112D-2848104\right) B_{4,1} \\
&+ \frac{4(D-3)}{D-4} C_{8,1} + \frac{48(3D-10)(D-3)^2}{(D-4)^3} C_{6,1} \Bigg] . \qquad \text{(B.1)}
\end{aligned}
$$

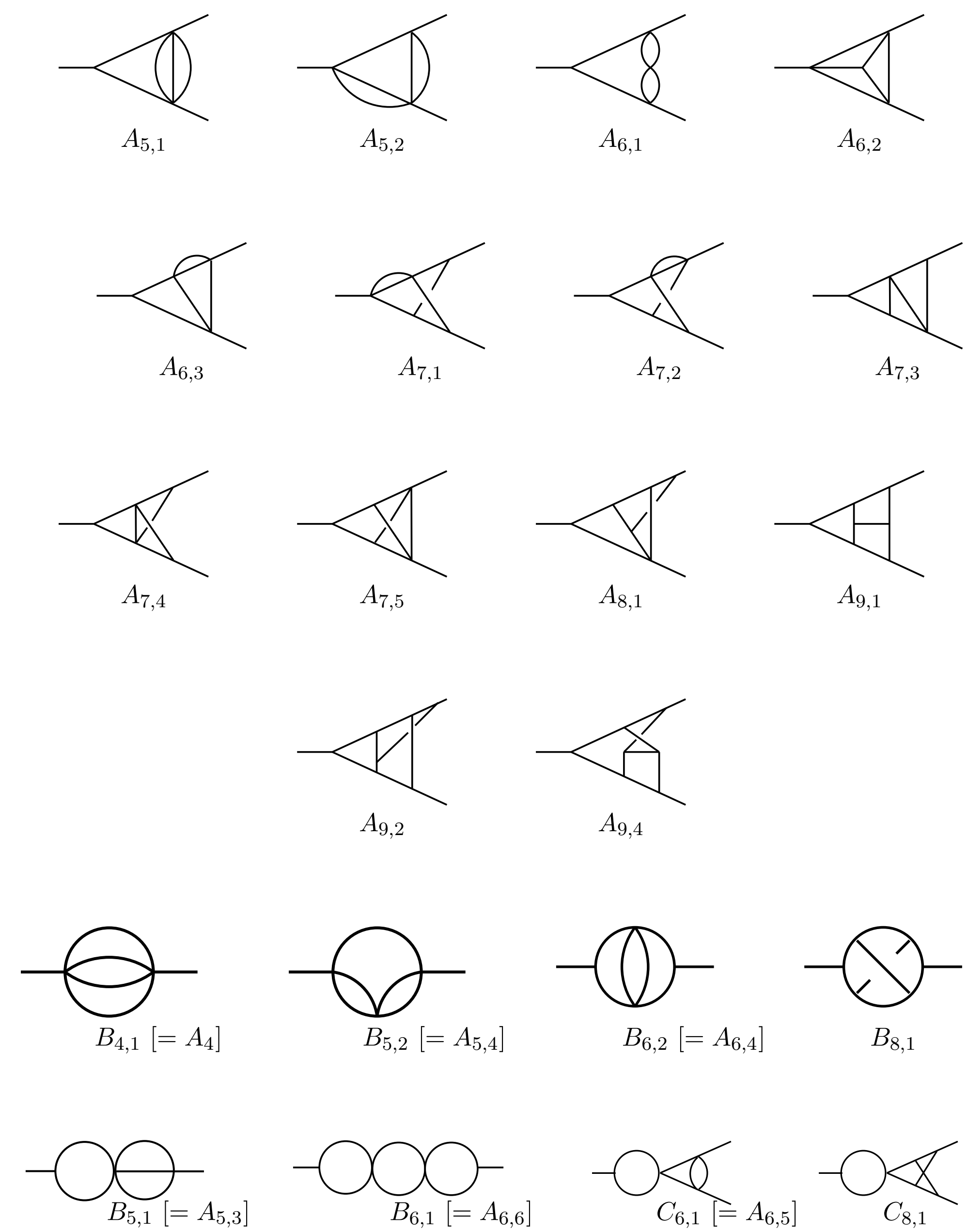


Figure 1: Master integrals for the three-loop form factors. Labels in brackets indicate the naming convention of Ref. [25].

Gehrmann, J.M.H., Huber (2011)

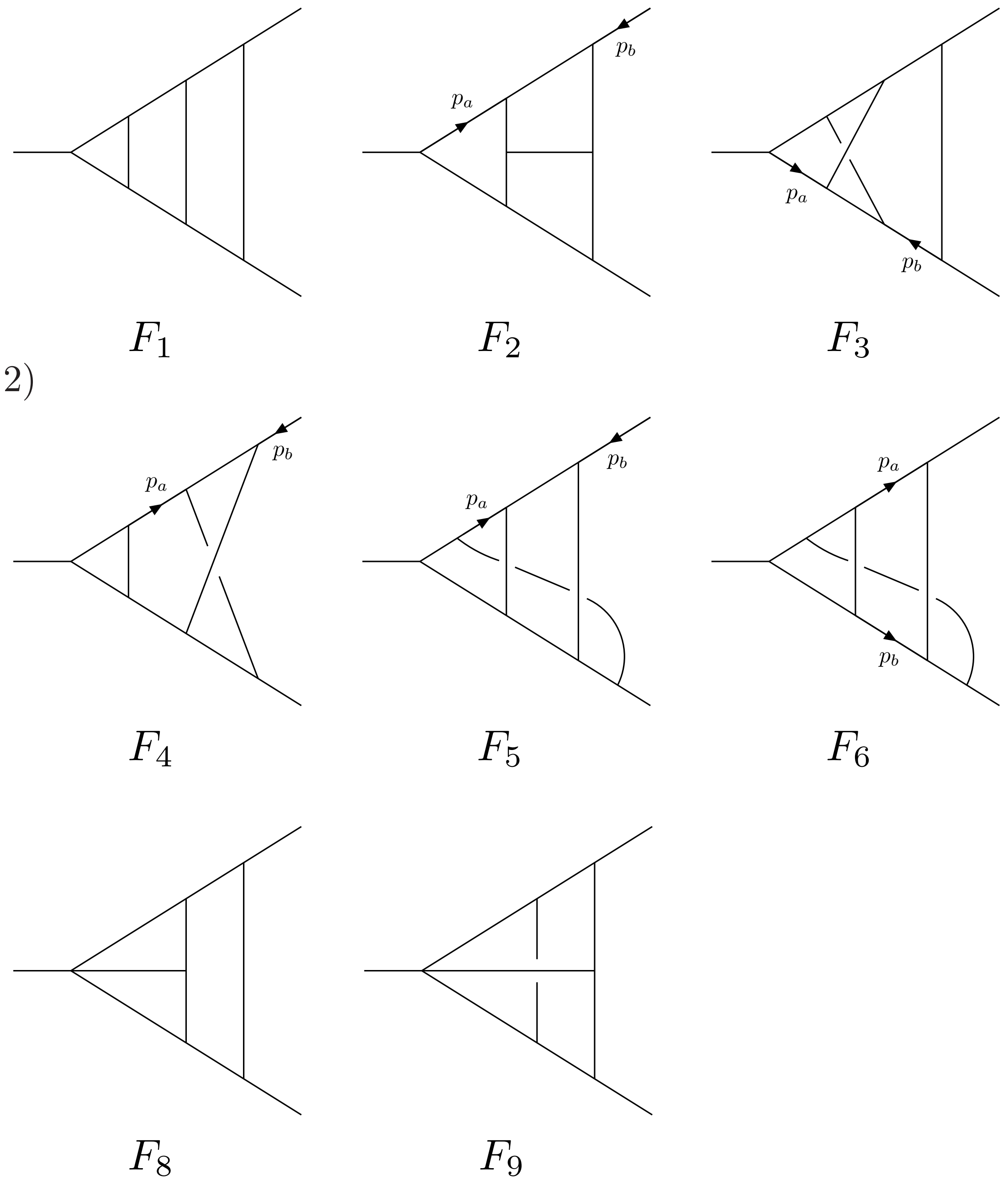
Gehrmann, Glover, Huber, Ikizlerli, Studerus; Lee, Smirnov & Smirnov

Example: choice of integral basis

three-loop N=4 SYM form factor

$$F_S^{(3)} = R_\epsilon^3 \cdot [8\,F_1^{\rm exp} - 2\,F_2^{\rm exp} + 4\,F_3^{\rm exp} + 4\,F_4^{\rm exp} - 4\,F_5^{\rm exp} - 4\,F_6^{\rm exp} - 4\,F_8^{\rm exp} + 2\,F_9^{\rm exp}]$$

$$\begin{aligned} F_S^{(3)} &= R_\epsilon^3 \cdot [8\,F_1^{\rm exp} - 2\,F_2^{\rm exp} + 4\,F_3^{\rm exp} + 4\,F_4^{\rm exp} - 4\,F_5^{\rm exp} - 4\,F_6^{\rm exp} - 4\,F_8^{\rm exp} + 2\,F_9^{\rm exp}] \\ &= -\frac{1}{6\epsilon^6} + \frac{11\zeta_3}{12\epsilon^3} + \frac{247\pi^4}{25920\epsilon^2} + \frac{1}{\epsilon}\left(-\frac{85\pi^2\zeta_3}{432} - \frac{439\zeta_5}{60}\right) \\ &\quad -\frac{883\zeta_3^2}{36} - \frac{22523\pi^6}{466560} + \epsilon\left(-\frac{47803\pi^4\zeta_3}{51840} + \frac{2449\pi^2\zeta_5}{432} - \frac{385579\zeta_7}{1008}\right) \\ &\quad +\epsilon^2\left(\frac{1549}{45}\zeta_{5,3} - \frac{22499\zeta_3\zeta_5}{30} + \frac{496\pi^2\zeta_3^2}{27} - \frac{1183759981\pi^8}{7838208000}\right) + \mathcal{O}(\epsilon^3)\,. \end{aligned} \tag{5.2}$$



- each integral has uniform (and maximal) `transcendental' weight
  T[Zeta[n]] = n
  T[eps^-n] = n
  T[A B] = T[A] + T[B]

- for theories with less susy, other integrals also needed

Gehrmann, J.M.H., Huber (2011)

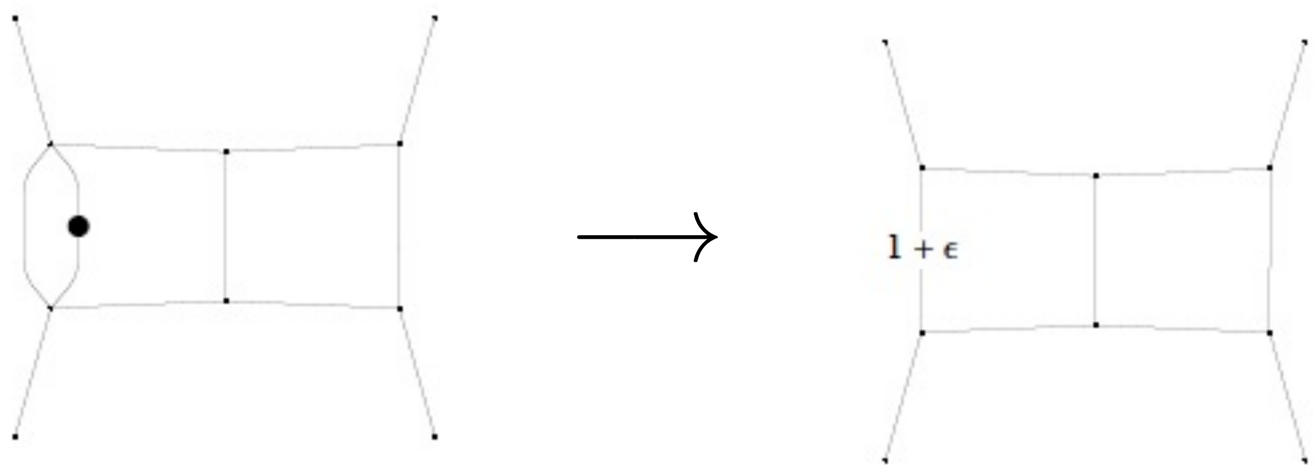
Optimal choice of integral basis

- idea: use transcendental weight as guiding principle [J.M.H., 2013]

- how to find such integrals?
  - unitarity cuts, leading singularities
  - 'd-log' representations
  - explicit parameter integrals

[Cachazo]
[Arkani-Hamed et al.]

- Example: [J.M.H., A.V. Smirnov, V.A. Smirnov, 2013]



Conjecture/Observation

[J.M.H., 2013]

- (in many cases) all basis integrals can be chosen to be pure functions of uniform weight
- leads to simplified form of differential equations

$$\partial_i f(x_j, \epsilon) = \epsilon\, A_i(x_j) f(x_j, \epsilon)$$

key step: equation(s) in differential form

$$d\, f(\epsilon, x_n) = \epsilon\, d\, \tilde{A}(x_n) f(\epsilon, x_n)$$

$\tilde{A}$ makes properties of answer manifest:

- specifies class of iterated integrals needed ('symbol' as corollary)
- singularities
- asymptotic behavior

Solution of differential equations

- equation(s) in differential form

$$d\, f(\epsilon, x_n) = \epsilon\, d\, \tilde{A}(x_n) f(\epsilon, x_n)$$

- observation: often $\tilde{A}$ contains only logarithms

-> weight properties manifest

- solution in terms of Chen iterated integrals

[Chen, 1997]
[Goncharov; Brown]

$$f = \sum_{k\geq 0} \epsilon^k f^{(k)}$$

$$f^{(0)} = \text{const} \qquad f^{(1)} = \int d\tilde{A}\, f^{(0)} \qquad f^{(2)} = \int d\tilde{A}\, f^{(1)}$$

higher orders in ϵ trivial to obtain

In general:

$$f = P\, e^{\epsilon \int_{\mathcal{C}} d\, \tilde{A}}\, g(\epsilon)$$

here $\mathcal{C}$ is a contour in the kinematical space

boundary conditions at base point from physical limits

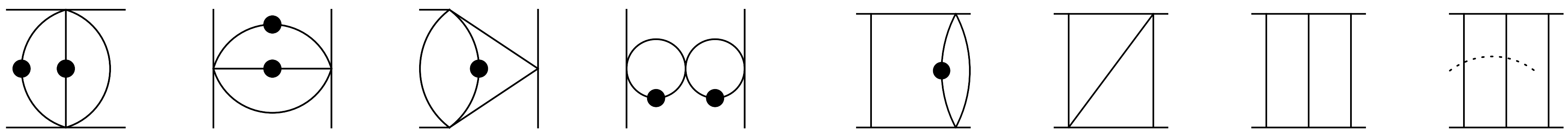
Part 3: Examples

- massless 2 to 2 scattering
- two-loop Bhabha scattering integrals

Example: massless 2 to 2 scattering

[Smirnov, 1999][Gehrmann, Remiddi, 1999]

- good choice of master integrals [J.M.H., 2013]



- Knizhnik-Zamolodchikov equations

$$\partial_x f = \epsilon \left[\frac{a}{x} + \frac{b}{1+x} \right] f$$

$$x = t/s$$

$$a == \begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{2} & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & -2 & 0 & 0 \\ -3 & -3 & 0 & 0 & 4 & 12 & -2 & 0 \\ \frac{9}{2} & 3 & -3 & -1 & -4 & -18 & 1 & 1 \end{pmatrix} \quad b == \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{2} & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 3 & 6 & 6 & 2 & -4 & -12 & 2 & 2 \\ -\frac{9}{2} & -3 & 3 & -1 & 4 & 18 & -1 & -1 \end{pmatrix}$$

- (regular) singular points

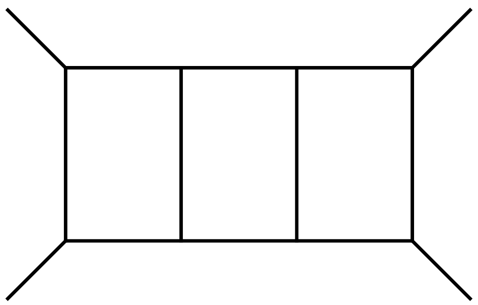
$$s = 0\,, \quad t = 0\,, \quad u = -s - t = 0$$

- boundary conditions: finiteness as $u \to 0$

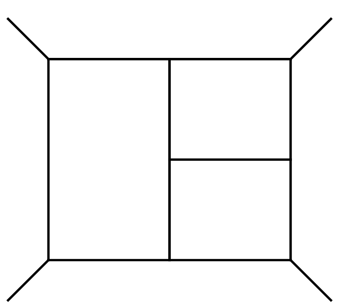
Generalization to 3 loops

- all planar 2 to 2 three-loop master integrals

[J.M.H., A.V. Smirnov, V.A. Smirnov, 2013]

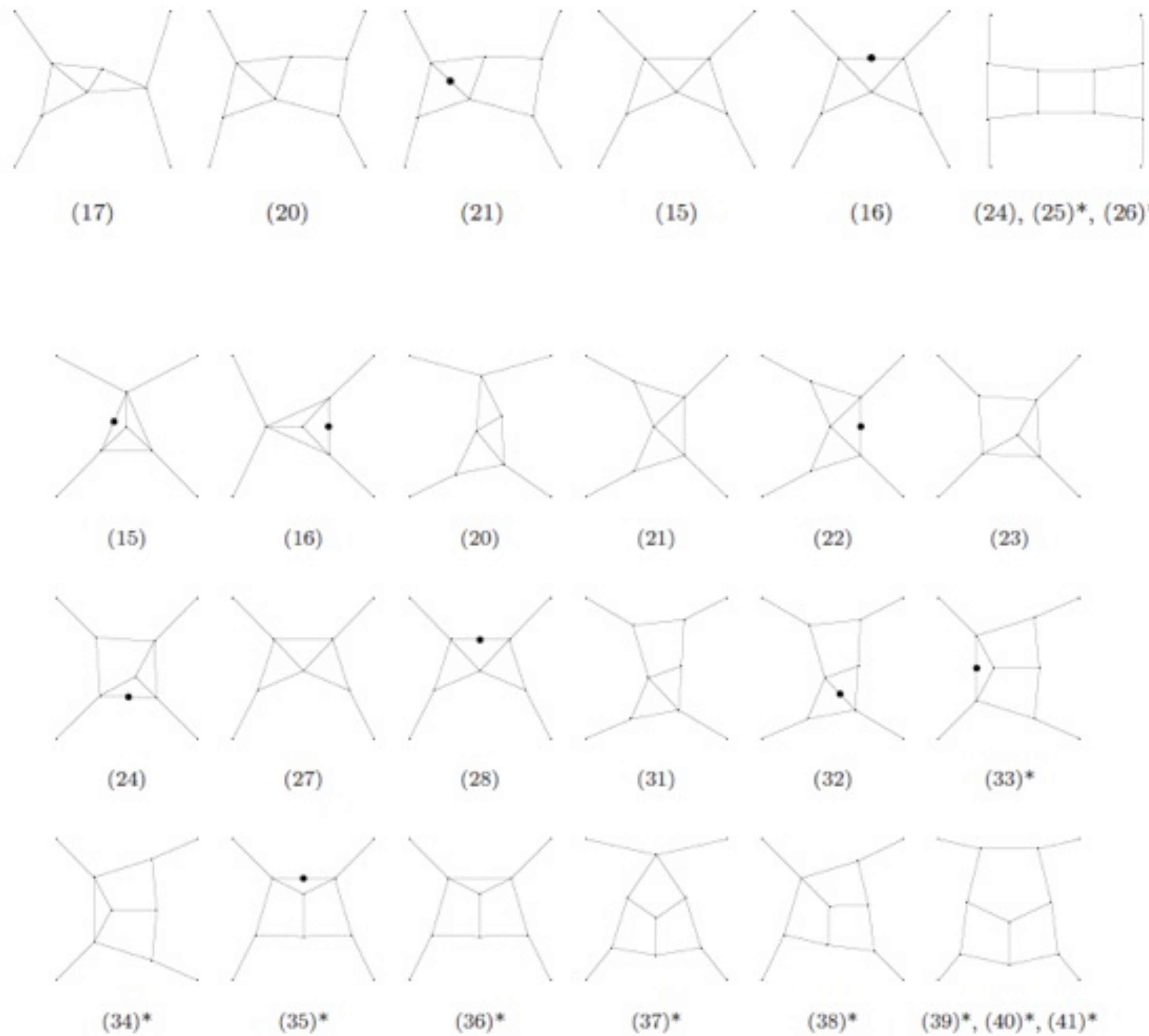


(26 master integrals)



(41 master integrals)

integrals without bubble subintegrals:

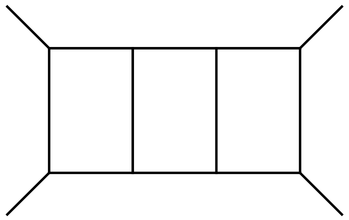
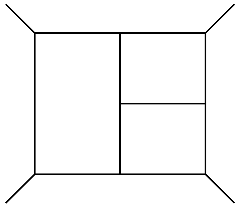


Generalization to 3 loops

- all planar 2 to 2 three-loop master integrals

[J.M.H., A.V. Smirnov, V.A. Smirnov, 2013]

$$\partial_x f = \epsilon \left[\frac{a}{x} + \frac{b}{1+x} \right] f \qquad x = t/s$$

- boundary conditions from finiteness at $x = -1$
- solution to arbitrary order ϵ^k in terms of HPLs of weight k by simple algebra
- singularity and monodromy structure manifest

Generalization to 3 loops

- all planar 2 to 2 three-loop master integrals

[J.M.H., A.V. Smirnov, V.A. Smirnov, 2013]

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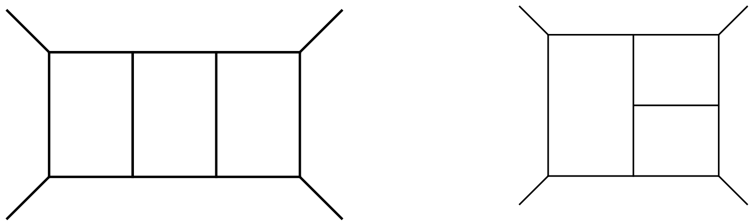
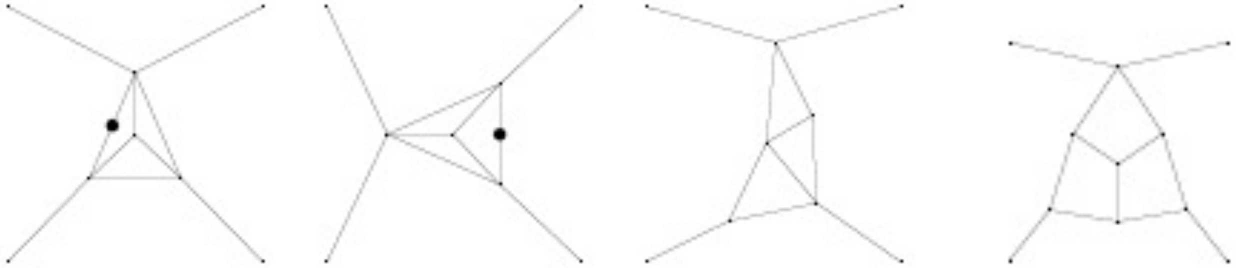


- boundary conditions from finiteness at $x = -1$
- solution to arbitrary order ϵ^k in terms of HPLs of weight k by simple algebra
- singularity and monodromy structure manifest

Note: also single-scale integrals completely determined!

E.g.



cf. [Heinrich, Huber, Kosower, Smirnov, 2009]

No additional computation needed for boundary conditions!

Example 2: Bhabha integrals

- previous results (using DE or Mellin-Barnes)

[Smirnov, 2001] [Heinrich, Smirnov, 2004]
[Czakon, Gluza, (Kajda) Riemann, 2004-2006]
[Bonciani, Ferroglia, Mastrolia, Remiddi, van der Bij, 2003, 2004]

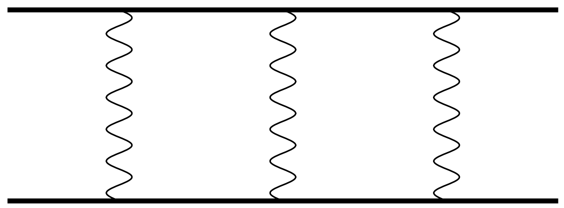
many individual integrals computed, to some order in ϵ
problems with coupled differential equations

- for phenomenology, approximate limits were computed

e.g. [Actis, Czakon, Gluza, Riemann, 2007-2008]

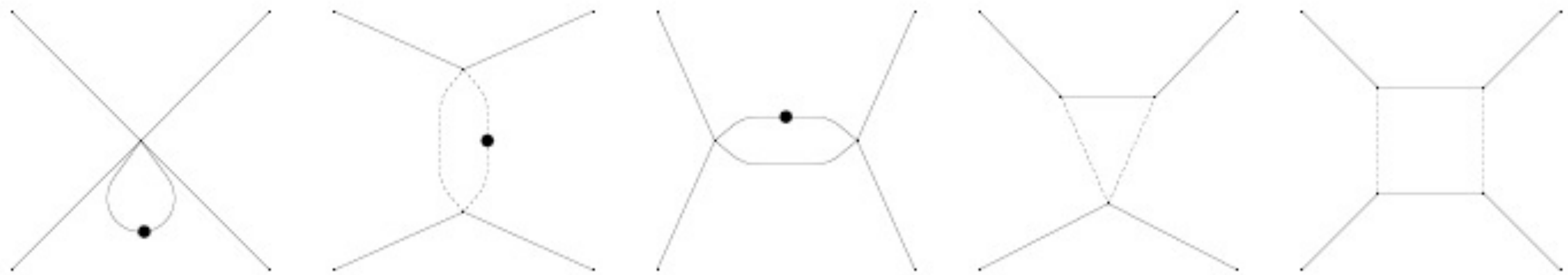
- complete analytic answer for integral family I

[J.M.H., V.A. Smirnov, 2013]



One-loop warm-up

- choice of master integrals



(1) (2) (3) (4) (5)

$$f_i = (m^2)^\epsilon \, e^{\epsilon\gamma_E} \, g_i$$

$$\begin{aligned} g_1 &= \epsilon \, G_{2,0,0,0} \,, \\ g_2 &= \epsilon t \, G_{0,2,0,1} \,, \\ g_3 &= \epsilon \sqrt{(-s)(4m^2 - s)} \, G_{2,0,1,0} \,, \\ g_4 &= -2\epsilon^2 \, (4m^2 - t)(-t) \, G_{1,1,0,1} \,, \\ g_5 &= -2\epsilon^2 \sqrt{(-s)(4m^2 - s)} t \, G_{1,1,1,1} \,. \end{aligned}$$

- kinematics

$$\frac{-s}{m^2} = \frac{(1-x)^2}{x} \,, \qquad \frac{-t}{m^2} = \frac{(1-y)^2}{y} \,.$$

One-loop warm-up

- differential equations [J.M.H., V.A. Smirnov, 2013]

$$d\,f = \epsilon\, d\,\tilde{A}\, f$$

$$\tilde{A} = \Bigg[\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \end{pmatrix} \log x + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -8 & 0 & -2 \end{pmatrix} \log(1+x) + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \end{pmatrix} \log y +$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \log(1+y) + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix} \log(1-y) +$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 1 \end{pmatrix} \log(x+y) + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -2 & 1 \end{pmatrix} \log(1+xy) \Bigg] . \qquad (2.14)$$

- symbol alphabet $\{x\,, 1 \pm x\,, y\,, 1 \pm y\,, x+y\,, 1+xy\}$

(natural generalization of) two-dimensional HPLs

One-loop warm-up

- kinematics

[J.M.H., V.A. Smirnov, 2013]

$$\frac{-s}{m^2} = \frac{(1-x)^2}{x}, \qquad \frac{-t}{m^2} = \frac{(1-y)^2}{y}.$$

note: inversion symmetry $x \quad \leftrightarrow \quad 1/x$ and $\quad y \quad \leftrightarrow \quad 1/y$

- (regular) singular points of DE have physical meaning, e.g.:

$$x = 1 \quad \leftrightarrow \quad s = 0$$

$$x = 0 \quad \leftrightarrow \quad s = \infty$$

$$x = -1 \quad \leftrightarrow \quad s = 4m^2$$

$$x = -y \quad \leftrightarrow \quad u = 0$$

One-loop warm-up

- Symbol follows as corollary of DE [J.M.H., V.A. Smirnov, 2013]

Boundary condition: $f^{(0)} = \{1,1,0,0,0\}$

Symbol iteratively defined by DE: $d\, f = \epsilon\, d\, \tilde{A}\, f$

Symbol:

$$
\begin{aligned}
\mathcal{S}(f_5^{(0)}) &= 0\,, \\
\mathcal{S}(f_5^{(1)}) &= 4[x]\,, \\
\mathcal{S}(f_5^{(2)}) &= -8[x,1-y] + 4[x,y] - 8[1-y,x] + 4[y,x]\,, \\
\mathcal{S}(f_5^{(3)}) &= 8[x,x,1+x] + 4[x,x,y] - 4[x,x,x+y] - 4[x,x,1+xy] \\
&\quad - 16[x,1+x,1+x] - 8[x,1+x,y] + 8[x,1+x,x+y] \\
&\quad + 8[x,1+x,1+xy] + 16[x,1-y,1+x] + 16[x,1-y,1-y] \\
&\quad - 8[x,1-y,x+y] - 8[x,1-y,1+xy] - 8[x,y,1+x] \\
&\quad - 8[x,y,1-y] + 4[x,y,x+y] + 4[x,y,1+xy] + 16[1-y,x,1+x] \\
&\quad + 16[1-y,x,1-y] - 8[1-y,x,x+y] - 8[1-y,x,1+xy] \\
&\quad + 16[1-y,1-y,x] - 8[1-y,y,x] + 8[1-y,y,x+y] \\
&\quad - 8[1-y,y,1+xy] - 8[y,x,1+x] - 8[y,x,1-y] \\
&\quad + 4[y,x,x+y] + 4[y,x,1+xy] - 8[y,1-y,x]x + 4[y,y,x] \\
&\quad - 4[y,y,x+y] + 4[y,y,1+xy]\,.
\end{aligned}
$$

- Analytic answer to any order in terms of 2d HPLs

Two-loop differential equations

- differential equations [J.M.H., V.A. Smirnov, 2013]

$$d\,f = \epsilon\, d\,\tilde{A}\, f$$

$$\begin{aligned}\tilde{A} =& B_1 \log(x) + B_2 \log(1+x) + B_3 \log(1-x) + B_4 \log(y) + B_5 \log(1+y) \\ &+ B_6 \log(1-y) \ + B_7 \log(x+y) + B_8 \log(1+xy) \\ &+ B_9 \log(x+y-4xy+x^2y+xy^2) + B_{10} \log\left(\frac{1+Q}{1-Q}\right) \\ &+ B_{11} \log\left(\frac{(1+x)+(1-x)Q}{(1+x)-(1-x)Q}\right) + B_{12} \log\left(\frac{(1+y)+(1-y)Q}{(1+y)-(1-y)Q}\right) ,\end{aligned}$$

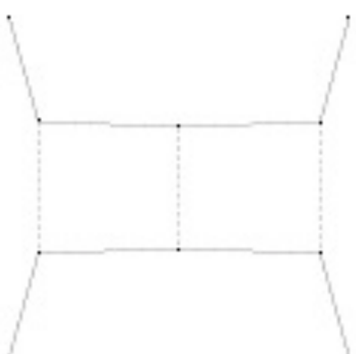
$$Q = \sqrt{\frac{(x+y)(1+xy)}{x+y-4xy+x^2y+xy^2}} ,$$

- larger, 12-letter symbol alphabet
- solution to any order in terms of Chen iterated integrals
- observation: except for one integral, up to weight 4, only the one-loop symbol alphabet is needed!

Two-loop differential equations

- Example

[J.M.H., V.A. Smirnov, 2013]



(22),(23)*

$$
\begin{aligned}
f_{23} =\,& \epsilon^2\Big[-12H_{0,0}(x)\Big] + \epsilon^3\Big[-16G_0(y)H_{0,0}(x) + 32G_1(y)H_{0,0}(x) + 8H_{2,0}(x) \\
&+ 16H_{-1,0,0}(x) - 4H_{0,0,0}(x) + \frac{4}{3}\pi^2 H_0(x) + 4\zeta_3\Big] + \epsilon^4\Big[32G_0(y)H_{-2,0}(x) \\
&- 32H_{-2,0}(x)G_{-\frac{1}{x}}(y) - 32H_{-2,0}(x)G_{-x}(y) + 64G_{1,0}(y)H_{0,0}(x) - 128G_{1,1}(y)H_{0,0}(x) \\
&- 32H_{0,0}(x)G_{-\frac{1}{x},0}(y) + 64H_{0,0}(x)G_{-\frac{1}{x},1}(y) - 32H_{0,0}(x)G_{-x,0}(y) \\
&+ 64H_{0,0}(x)G_{-x,1}(y) - 16H_0(x)G_{-\frac{1}{x},0,0}(y) + 32H_0(x)G_{-\frac{1}{x},0,1}(y) \\
&+ 16H_0(x)G_{-x,0,0}(y) - 32H_0(x)G_{-x,0,1}(y) + 64G_0(y)H_{-1,0,0}(x) \\
&- 64H_{-1,0,0}(x)G_{-\frac{1}{x}}(y) - 64H_{-1,0,0}(x)G_{-x}(y) \\
&- 48G_0(y)H_{0,0,0}(x) + 48H_{0,0,0}(x)G_{-\frac{1}{x}}(y) + 48H_{0,0,0}(x)G_{-x}(y) - 120H_{-3,0}(x) \\
&+ \frac{52}{3}\pi^2 H_{0,0}(x) + 48H_{3,0}(x) + 128H_{-2,-1,0}(x) - 120H_{-2,0,0}(x) - 48H_{-2,1,0}(x) \\
&+ 64H_{-1,-2,0}(x) - 32H_{-1,2,0}(x) - 48H_{2,-1,0}(x) + 32H_{2,0,0}(x) + 16H_{2,1,0}(x) \\
&+ 64H_{-1,-1,0,0}(x) - 80H_{-1,0,0,0}(x) + 76H_{0,0,0,0}(x) + \frac{8}{3}\pi^2 G_0(y)H_0(x) \\
&- \frac{40}{3}\pi^2 H_0(x)G_{-\frac{1}{x}}(y) + 8\pi^2 H_0(x)G_{-x}(y) - 16\zeta_3 H_{-1}(x) - 28\zeta_3 H_0(x) \\
&+ \frac{8}{3}\pi^2 H_{-2}(x) - \frac{4}{3}\pi^2 H_2(x) - \frac{4\pi^4}{15}\Big] + \mathcal{O}(\epsilon^5) \qquad (3.28)
\end{aligned}
$$

Two-loop differential equations

- Example for iterated integral [J.M.H., V.A. Smirnov, 2013]

$$d\,f_{11}^{(4)} = g_1\,d\log\left(\frac{1-Q}{1+Q}\right) + g_2\,d\log\left(\frac{(1+x)+(1-x)Q}{(1+x)-(1-x)Q}\right) + g_3\,d\log\left(\frac{(1+y)+(1-y)Q}{(1+y)-(1-y)Q}\right), \tag{3.29}$$

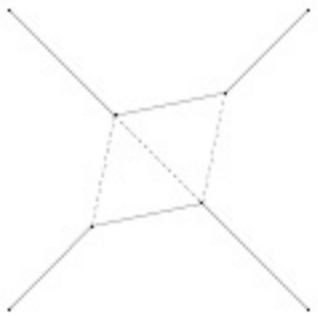
$$f_{11}^{(4)}(x=1, y=1) = 0.$$

$$f_{11}^{(4)}(x_f, y_f) = \int_{\mathcal{C}} d\,f_{11}^{(4)}\,.$$

e.g. parametrization: $x(s) = (1-s) + s x_f\,, \qquad y(s) = (1-s) + s x_f\,.$

sample numerical values:

$$f_{11}^{(4)}(1/3, 2/5) = 108.10505928259012\,,$$
$$f_{11}^{(4)}(6/7, 3/5) = 46.46787346208666\,,$$
$$f_{11}^{(4)}(3/8, 7/9) = 130.0624797134173\,.$$


(11)

numerical checks in Euclidean as well as physical region.

Conclusions and outlook

- criteria for finding optimal loop integral basis
  - important concept: pure functions of uniform weight
- new form of differential equations
  - determine functions needed to all orders in ϵ
  - make properties of functions manifest (analytic structure, discontinuities, homotopy invariance)
  - trivial to solve in terms of Chen iterated integrals
  - symbol/coproduct structure helpful for rewriting answer (e.g. for faster numerical evaluation)
- further applications:
  - other physical processes (different mass configurations/many external legs)
  - non-planar integrals
  - phase space integrals
  - integrals in other dimensions $D = D_0 - 2\epsilon$