Gauge Orbit Space, Supersymmetry & Mass Gap

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- Why is the gauge orbit space important?
  - Gauge orbit space ($\mathcal{C}$) = Space of physical field configurations in a gauge theory
  - Its geometry has implications for mass gap
  - An approach which is complementary to other analyses

We will discuss some aspects of the geometry of this space for Yang-Mills(2+1) and Yang-Mills (4)

A PLAN

- Generalities
- Yang-Mills in 2+1
  - Volume for $\mathcal{A}/\mathcal{G}_*$
  - Intuitive argument for mass gap
  - Volume for $\mathcal{A}/\mathcal{G}_*$, 3-d considerations
  - SUSY theories
    - $\mathcal{N} = 1$
    - $\mathcal{N} \geq 2$
- Yang-Mills in 4d
  - Parametrization for fields on $\mathbb{CP}^2$
  - Calculation of "leading term" of the volume
  - The 4d WZW action

- Consider $SU(N)$ Yang-Mills theory in 2+1 dimensions, i.e., on $\mathbb{R}^{2,1}$, or on a four-manifold $\mathbb{M}^4$.
- The gauge potential (connection) A is of the form $A = (-it^a)A^a_\mu dx^\mu$ where t^a are hermitian $N \times N$ matrices forming a basis for the Lie algebra of $SU(N)$
- The action for the theory is

$$
\begin{aligned}
S &= -\frac{1}{2\,e^2}\int d\mu\, \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = \frac{1}{4\,e^2}\int d\mu\, F^a_{\mu\nu}F^{a\mu\nu} \\
F &= dA + A \wedge A = \frac{1}{2}(-i\,t^a)\,F^a_{\mu\nu}\,dx^\mu \wedge dx^\nu
\end{aligned}
$$

- e^2 is the coupling constant, has the dimension of mass for YM(2+1) and has no dimension for YM(4)

- We use a Hamiltonian approach for YM(2+1) but a 4d-covariant approach for YM(4)
- For YM(2+1), set $A_0 = 0$, with the Hamiltonian given by

$$\mathcal{H} = \frac{1}{2} \int d^2x \left[e^2 E^2 + \frac{B^2}{e^2} \right] = \int d^2x \left[-\frac{e^2}{2} \frac{\delta^2}{\delta A_i^a \, \delta A_i^a} + \frac{B^2}{2\, e^2} \right]$$

- Define

$$\begin{aligned} \mathcal{A} &= \{\text{Space of all smooth gauge potentials (connections) on } \mathbb{R}^2\} \\ \mathcal{G}_* &= \{g(\vec{x}) : \mathbb{R}^2 \to SU(N),\ g \to 1 \text{ as } |\vec{x}| \to \infty\} \end{aligned}$$

- Gauge transformations act on A as

$$A \to A^g = g^{-1} A\, g + g^{-1} dg, \qquad g \in \mathcal{G}_*$$

- The physical configuration space (or gauge orbit space) is $\mathcal{C} = \mathcal{A}/\mathcal{G}_*$.
  - The kinetic term of the Hamiltonian is to be defined as a Laplacian on $\mathcal{C}$
  - Wave functions are functions on $\mathcal{C}$, because of the Gauss law

- As a fiber bundle, $(\mathcal{G}_*, \mathcal{A}, \mathcal{C})$ is nontrivial ($\exists$ Gribov problem)
- $\mathcal{A}$ is an affine space, but it is easy to see that $\mathcal{C}$ has very nontrivial topology and geometry.
- For example, $\Pi_2[\mathcal{C}] = \mathbb{Z}$. (There are many other nonzero Π_n, $n > 2$ as well.)
- The physically relevant metric for $\mathcal{A}$ is given by the Euclidean distance. For $\mathcal{C}$, we define

$$s^2(A, A') = -Inf_g \int d^2x \,\mathrm{Tr}(A' - A^g)^2$$

- We want to calculate the volume element for $\mathcal{C}$. For this, we need a good parametrization of the potentials
- With $A_0 = 0$ and we use complex coordinates $z = x_1 - ix_2$ with

$$\tfrac{1}{2}(A_1 + iA_2) = -\partial M\, M^{-1}, \qquad \tfrac{1}{2}(A_1 - iA_2) = M^{\dagger\,-1}\bar{\partial} M^{\dagger}$$

$M \in SL(N, \mathbb{C})$, for gauge group $SU(N)$. (More generally, $G \Rightarrow G^{\mathbb{C}}$.)

- $H = M^\dagger M \in SL(N, \mathbb{C}/SU(N)$ is the basic gauge-invariant variable we need.
- The variation of the potentials is given by

$$\delta A = -D(\delta M M^{-1}) \qquad \delta \bar{A} = \bar{D}(M^{\dagger -1} \delta M^\dagger)$$

- We then have

$$ds^2_{\mathcal{A}} = \int d^2x \, \mathrm{Tr}(\delta A \delta \bar{A}) = \int \mathrm{Tr} \left[(M^{\dagger -1} \delta M^\dagger)(-\bar{D}D)(\delta M M^{-1}) \right]$$

$$ds^2_{SL(N,\mathbf{C})} = \int \mathrm{Tr}(M^{\dagger -1} \delta M^\dagger \; \delta M M^{-1})$$

- For the volume element we get

$$d\mu_{\mathcal{A}} = \det(-\bar{D}D) \;\; d\mu(M, M^\dagger)$$

Haar measure for $SL(N, \mathbb{C})$

- One can do a polar decomposition $M = U\,\rho$, U unitary and ρ hermitian. Using this

$$d\mu_{SL(N,\mathbb{C})} = d\mu(H) \;\; d\mu(U)$$

Volume of $SL(N,\mathbb{C})/SU(N)$ — Volume of $SU(N)$

- For the gauge-orbit space

$$d\mu(\mathcal{C}) = \det(-\bar{D}D)\; d\mu(H)$$

$$= d\mu(H)\;\exp\left[2\, c_A\, S_{wzw}(H)\right]$$

$S_{wzw}(H)$ is the Wess-Zumino-Witten (WZW) action,

$$S_{wzw}(H) = \frac{1}{2\pi}\int \mathrm{Tr}(\partial H \bar{\partial} H^{-1}) - \frac{i}{12\pi}\int \mathrm{Tr}(H^{-1}dH)^3$$

$c_A\;\delta_{ab} = f_{amn} f_{bmn} = N\;\delta_{ab}$ for $SU(N)$.

- The inner product for wave functions is then

$$\langle 1|2\rangle = \int d\mu(H)\, \exp[2\, c_A\, S_{wzw}(H)]\; \Psi_1^* \Psi_2$$

- Remarks:
  - The essential step is an anomaly calculation, so this result is robust, independent of the regulators used.
  - Integration with $d\mu(\mathcal{C})$ is equivalent to the calculation of correlators in the hermitian WZW model, so it can be done unambiguously (GAWEDZKI & KUPIAINEN)
  - The same result is obtained by taking a suitable limit in the exact solutions of the 2-dimensional YM theory (WITTEN; GROSS & TAYLOR; SENGUPTA; ASHTEKAR *et al*)

- The Hamiltonian has the form

$$\mathcal{H} = \frac{1}{2} \int \left[e^2 E^2 + B^2/e^2 \right]$$

$[E, B] \sim p$ (in momentum space) $\Longrightarrow \Delta E \, \Delta B \sim p$, or $\Delta E \sim p/\Delta B$

$$\mathcal{E} = \langle \mathcal{H} \rangle \approx \frac{1}{2} \left[e^2 \frac{p^2}{(\Delta B)^2} + \frac{(\Delta B)^2}{e^2} \right]$$

Minimize with respect to $\Delta B \implies (\Delta B)^2 \sim p \Longrightarrow \mathcal{E} \sim p$. This is the photon.

- For us

$$\langle \mathcal{H} \rangle = \int d\mu(H) \exp\left[2\, c_A S_{wzw}(H)\right] \int \tfrac{1}{2} \left[e^2 E^2 + B^2/e^2 \right]$$

- Expanding the WZW action

$$\langle \mathcal{H} \rangle \approx \int d\mu(H) \exp\left[-\frac{c_A}{2\pi} \int B \frac{1}{p^2} B + \dots \right] \int \tfrac{1}{2} \left[e^2 E^2 + \tfrac{B^2}{e^2} \right]$$

Gaussian distribution for B

- Width $(\Delta B)^2 \sim \pi p^2 / c_A$, very narrow for small momenta $\Longrightarrow$ mass gap $\sim e^2 c_A / 2\pi$.

- More detailed analysis shows similar results

- The Hamiltonian and the wave functions can be expressed as functions of the (scaled version of the) current $J = (c_A/\pi)\, \partial H\, H^{-1}$.
- The Wilson loop operator is given by

$$W(C) = \mathrm{Tr}_R\, \mathcal{P} e^{-\oint_C A} = \mathrm{Tr}\, \mathcal{P} \exp\left(\frac{\pi}{c_A} \oint_C J \right)$$

All gauge-invariant quantities can be made from J.

- The Hamiltonian is given by

$$\mathcal{H} = m \left[\int_u J^a(\vec{u}) \frac{\delta}{\delta J^a(\vec{u})} + \int_{u,v} \left(\frac{c_A}{\pi^2} \frac{\delta_{ab}}{(u-v)^2} - i\, \frac{f_{abc}\, J^c(\vec{v})}{\pi(u-v)} \right) \frac{\delta}{\delta J^a(\vec{u})} \frac{\delta}{\delta J^b(\vec{v})} \right] + \frac{\pi}{m c_A} \int_u : \bar{\partial} J_a(u)\, \bar{\partial} J_a(u) :$$

where $m = e^2 c_A / 2\pi$.

- For the calculation of the measure, the action is not important, so we can consider the Chern-Simons theory

$$S_{CS} = -\frac{k}{4\pi}\int \mathrm{Tr}\left[A\, dA + \frac{2}{3}A^3\right] = -\frac{k}{4\pi}\int d^3x\, \epsilon^{\mu\nu\alpha}\,\mathrm{Tr}\left[\left(A_\mu \partial_\nu A_\alpha + \frac{2}{3}A_\mu A_\nu A_\alpha\right)\right]$$

We can take the level number $k = 0$ at the end.

- In a Hamiltonian quantization, the wave functions obey the Gauss law

$$\left[D\,\frac{\delta}{\delta A} - \frac{k}{2\pi}\bar{\partial}A - \sum_r (-i\, t^a)_{(r)}\,\delta^{(2)}(x - x_r)\right]\,\Psi = 0$$

- The state with no charges and the state with two (conjugate) charges are given by

$$\begin{aligned}\Psi_0 &= \chi_0\, \exp(k\, S_{wzw}(M))\\ \Psi_2 &= \chi(z_1, z_2)\, M(1)\, M^{-1}(2)\; \Psi_0\end{aligned}$$

- The normalizations are determined by

$$\begin{aligned} \mathcal{I}_0 &= |\chi_0|^2 \int d\mu(H)\, \exp[\bar{k}\, S_{wzw}(H)] \;= 1 \\ \mathcal{I}_2 &= |\chi(z_1, z_2)|^2 \int d\mu(H)\, \exp[\bar{k}\, S_{wzw}(H)]\;\; H(1)\, H(2)^{-1} \;= 1 \end{aligned}$$

- We know from previous result that $\bar{k} = k + 2\, c_A$, but for the moment, we will pretend that it is not known.
- The correlators are determined by the Schrödinger equation (identical to the Knizhnik-Zamolochdikov (KZ) equation) and shows that
  - $\chi(z_1, z_2)$ obeys the KZ equation for level k $SU(N)$ WZW model, i.e., with parameter $\kappa = k + c_A$
  - The H-correlators obey the same KZ equation with parameter $-\bar{k} + c_A$
  - The z-dependence of the $|\chi(z_1, z_2)|^2$ must be canceled by the H-correlator, so we get

$$-(k + c_A) = -\bar{k} + c_A, \qquad \text{or} \qquad \bar{k} = k + 2\, c_A$$

- Finally, we know that we can determine the χ's and Wilson loop expectation value by a covariant 3-dim calculation via the effective action for the CS theory.
- One-loop calculations (with no further renormalizations) give the shift $k \rightarrow k + c_A$ going from S_{CS} to the effective action.
- Combining everything, we can find the shift from the Feynman diagrams and then use the arguments above to fix the volume element.
- For supersymmetric theories, the known shifts from the Feynman diagrams are (PISARSKI & RAO; KAO, LEE & LEE)

$$k \rightarrow \begin{cases} k + c_A & \mathcal{N} = 0 \\ k + c_A/2 & \mathcal{N} = 1 \\ k & \mathcal{N} \geq 2 \end{cases}$$

- Correspondingly, the volume elements should be

$$d\mu(\mathcal{C}) = d\mu(H)\ \exp\left(\bar{k}\, S_{wzw}(H)\right)\ d\,[\text{Fermions}]$$

$$\bar{k} = \begin{cases} 2\,c_A & \mathcal{N} = 0 \\ c_A & \mathcal{N} = 1 \\ 0 & \mathcal{N} \geq 2 \end{cases}$$

- The expectations for SUSY YM are then:
  - We can have mass gap for $\mathcal{N} = 0, 1$, no gap expected for $\mathcal{N} \geq 2$
  - For $\mathcal{N} = 1$, one has to add a CS term for the YM theory for consistency (because of the parity anomaly), so there is always a mass gap (WITTEN; ELLIOTT & MOORE)
  - For $\mathcal{N} = 2$, one expects zero gap from other considerations, but a stable suspersymmetric vacuum may not exist. (SEIBERG & WITTEN; GOMIS & RUSSO; UNSAL; AHARONY *et al*)
  - For $\mathcal{N} = 4$, there is no mass term with unbroken supersymmetry (SEIBERG & WITTEN)
  - $\mathcal{N} = 8$ is expected to flow to a CFT (SEIBERG; AHARONY *et al*; HERZOG *et al*)

- We can further check these expectations by explicit calculations
- Consider a Majorana field $\Psi = (\psi, \psi^\dagger)$ in the adjoint representation of $SU(N)$.
- There are two ways to go to gauge-invariant variables:

Choice I

$$\begin{pmatrix} \chi^a \\ \chi^{a\dagger} \end{pmatrix} = \begin{pmatrix} (M^{-1})^{ab}\psi^b \\ (M^\dagger)^{ab}\psi^{b\dagger} \end{pmatrix}$$

Choice II

$$\begin{pmatrix} \chi^a \\ \chi^{a\dagger} \end{pmatrix} = \begin{pmatrix} (M^\dagger)^{ab}\psi^b \\ (M^{-1})^{ab}\psi^{b\dagger} \end{pmatrix}$$

where $M^{ab} = 2\,\mathrm{Tr}(t^a\, M\, t^b\, M^{-1})$ is the adjoint representative of M.

- Calculate the Jacobian of the transformation to the new variables χ by integrating small variations in M to obtain

$$[d\psi\, d\psi^\dagger] = [d\chi\, d\chi^\dagger]\ \exp\left(\pm c_A S_{wzw}(H)\right)$$

(Upper plus sign for Choice I, lower minus sign for Choice II.)

- The action is

$$S=\int\left[-\frac{1}{4\,e^2}F^a_{\mu\nu}F^{a\mu\nu}-\frac{i}{2e^2}\bar{\Psi}^a(\gamma^\mu D_\mu\Psi)^a-\frac{k}{4\pi}\,\mathrm{Tr}\left(A_\mu\partial_\nu A_\alpha-\frac{2}{3}A_\mu A_\nu A_\alpha\right)\epsilon^{\mu\nu\alpha}\right.$$
$$\left.+i\,e^2\mathrm{Tr}\,\bar{\Psi}\Psi\right]$$

- The supercharges are given by

$$Q^\dagger=\int(i\,\Psi^\dagger\gamma^i\frac{\delta}{\delta A^i}+\frac{1}{e^2}\psi^\dagger B),\qquad Q=\int(i\,\gamma^i\Psi\frac{\delta}{\delta A^i}+\frac{1}{e^2}\psi B)$$

- Carrying out the change to H for the gauge fields, the supercharge becomes

$$Q=i\int_x\underbrace{\psi^{a\dagger}(x)M^{ab}(x)}_{\chi^{\dagger b}}\left[\int_y\mathcal{G}(x,y)p^b(y)+\frac{k}{4c_A}\bar{J}^b(x)\right]-\frac{1}{e^2}\frac{2\pi}{c_A}\int\underbrace{\psi^a(M^{\dagger-1})^{ab}}_{\chi^b}\;\bar{\partial}J^b$$

- This identifies Choice II as the proper change to gauge invariant variables for the fermions.

- The wave functions are of the form

$$\Xi = \exp\left(\frac{k}{2}\left[S_{wzw}(M^\dagger) - S_{wzw}(M) + S_{wzw}(H)\right]\right)\ \Phi(H, \chi, \chi^\dagger)$$

- Absorbing the exponential factor into the measure, the inner product for Φ's involves

$$d\mu = d\mu(H)\ \exp\left[(k + (2-n)c_A)\, S_{wzw}(H)\right]$$

where we know $n = 1$ from previous arguments.

- We obtain the supercharges (in terms of the gauge-invariant variables) and the Hamiltonian is given by $\mathcal{H} = \frac{1}{2}\{Q, Q^\dagger\}$. The terms relevant for the mass are

$$\mathcal{H} = \frac{e^2}{4\pi}(k + 2\,c_A - n\,c_A)\int\left[J^a\frac{\delta}{\delta J^a} + \chi^{a\dagger}(H^{-1})^{ab}\chi^b + \cdots\right]$$

We see equality of boson and fermion masses.

- The action is

$$
\begin{aligned}
S_{YM} &= -\frac{1}{4\,e^2}\int F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2\,e^2}\int D_\mu\phi^a_A D^\mu\phi^a_A + \frac{1}{2\,e^2}\int F^a_A F^a_A \\
&\quad - \frac{i}{2e^2}\int \bar{\psi}^a_I\,\gamma^\mu D_\mu\,\psi^a_I - \frac{i}{2\,e^2}\int \bar{\omega}^a\,\gamma^\mu D_\mu\,\omega^a - \frac{i}{2\,e^2}\int \epsilon_{ABC}\,\bar{\psi}^a_A\,\psi^b_B\,\phi^c_C f^{abc} \\
&\quad + \frac{i}{e^2}\int \bar{\psi}^a_A\,\omega^b\,\phi^a_A f^{abc} - \frac{1}{4\,e^2}\int f^{abc}f^{amn}\,\phi^b_B\,\phi^c_C\,\phi^m_B\,\phi^n_C \\
S_{CS} &= -\frac{k}{4\pi}\epsilon^{\mu\nu\rho}\int \mathrm{Tr}\,(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho) \\
&\quad + \frac{k}{8\,\pi}\int \left(-i\,\bar{\psi}^a_I\,\psi^a_I + i\,\bar{\omega}^a\,\omega^a + 2\,F^a_A\,\Phi^a_A - \frac{1}{3}f^{abc}\epsilon_{ABC}\,\phi^a_A\,\phi^b_B\,\phi^c_C\right)
\end{aligned}
$$

- Fermions are ω^a and ψ^a_A, $A = 1, 2, 3$, corresponding to an $SO(3)$ R-symmetry.
- Setting $\omega,\ \phi_1,\ \phi_2,\ \psi_3\ =\ 0 \Longrightarrow \mathcal{N} = 2$ theory

- The supercharge contains, among other terms,

$$Q = \int \left[i\,\psi^{a\dagger} \frac{\delta}{\delta A^a} + \epsilon_{IJK} \psi^a_J \left(\Pi^a_{\phi_K} + i\,\frac{k}{4\pi} \phi^a_K \right) - \omega^a \left(\Pi^a_{\phi_I} - i\,\frac{k}{4\pi} \phi^a_I \right) + \cdots \right]$$

- The choice of gauge-invariant fermion variables is Choice II for ψ^a; this follows from the term connecting it to A^a.
- Given this, ω^a must go with Choice I.
- So we get $-2\,c_A$ from two of the three ϕ's; the contributions from the third ϕ and from ω cancel out. i.e., $n = 2$,

$$d\mu = d\mu(H)\ \exp\left[(k + (2-n)c_A)\, S_{wzw}(H)\right] = d\mu(H)\ \exp\left[k\, S_{wzw}(H)\right] \to d\mu(H)$$

- Independently, we can check that the SUSY algebra

$$\{Q, Q^\dagger\} = 2\,\mathcal{H}, \qquad [Q, \mathcal{H}] = 0$$

requires $n = 2$.

- For Yang-Mills theory in 4 dimensions, we will consider the covariant functional integral
- The use of complex components $A, \bar{A}$ was crucial for YM(2+1), so we choose a complex 4-manifold $\mathbb{M}^4$
- The simplest choice is

$$\mathbb{CP}^2 = SU(3)/U(2)$$

- This is not particularly restrictive. The standard (Fubini-Study) metric for $\mathbb{CP}^2$ is

$$ds^2 = \frac{dz^a \, d\bar{z}^{\bar{a}}}{(1 + z \cdot \bar{z}/R^2)} - \frac{\bar{z} \cdot dz \, z \cdot d\bar{z}}{R^2(1 + z \cdot \bar{z}/R^2)}$$

By taking $R \to \infty$ at the end, one can (almost) recover $\mathbb{R}^4$.

- The space $\mathcal{G}_*$ is defined by

$$\mathcal{G}_* = \{g(\vec{x}) : \mathbb{CP}^2 \to SU(N), \; g \to 1 \text{ at one chosen point}\}$$

- Once again, $\mathcal{C} = \mathcal{A}/\mathcal{G}_*$ has nontrivial topology. In addition to instanton sectors, we have things like $\Pi_1(\mathcal{C}) = \mathbb{Z}$, for $SU(N \geq 3)$ theories.
- We can use a group element $g \in SU(3)$ as coordinates for $\mathbb{CP}^2$. Functions on $\mathbb{CP}^2$ are of the form

$$f = \sum_{R,\alpha} C_\alpha^{(R)} \langle R, \alpha | \, g \, | R, \beta \rangle$$

where the state $|R, \beta\rangle$ is invariant under $U(2) \subset SU(3)$

- Derivatives are given by right translations on g, $\nabla_a \sim i R_a$,

$$R_a \, g = g \, t_a$$

$\{t_a\}$ = basis for the Lie algebra of $SU(3)$

- For a vector field, we need $|R, \beta\rangle$ to be an $SU(2)$ doublet with $Y = \pm 1$.
- There are 3 choices for $|R, \beta\rangle$:
  - R_i, $R_{\bar{i}}$ acting on a state which is $SU(2)$ singlet, with $Y = 0$
  - $R_{\bar{i}}$ acting on a state which is $SU(2)$ singlet, with $Y = 2$
  - R_i acting on a state which is $SU(2)$ singlet, with $Y = -2$
- This means that an Abelian vector field can be parametrized as

$$
\begin{aligned}
A_i &= -\nabla_i \theta + \epsilon_{ij} \, \nabla_{\bar{j}} \chi \\
A_{\bar{j}} &= \quad \nabla_{\bar{i}} \bar{\theta} - \epsilon_{ij} \, \nabla_j \bar{\chi}
\end{aligned}
$$

  - θ has $Y = 0$, it is a complex function
  - χ has $Y = 2$; so it cannot be considered as a function on $\mathbb{CP}^2$; rather $\epsilon_{ij}\chi$ is a 2-form

- The nonabelian generalization is

$$A_i = -\nabla_i M M^{-1} + \epsilon_{ij} \bar{D}_{\bar{j}} \phi$$

$$A_{\bar{i}} = \ M^{\dagger -1} \nabla_{\bar{i}} M^{\dagger} - \epsilon_{ij} D_j \phi^{\dagger}$$

$$D_i \phi^{\dagger} = \nabla_i \phi^{\dagger} + [-\nabla_i M M^{-1}, \phi^{\dagger}], \qquad \bar{D}_{\bar{i}} \phi = \nabla_{\bar{i}} \phi + [M^{\dagger -1} \nabla_{\bar{i}} M^{\dagger}, \phi]$$

- The 3 independent gauge-invariant variables are

$$H = M^{\dagger} M, \quad \chi = M^{-1} \phi M, \quad \bar{\chi} = M^{\dagger} \phi^{\dagger} M^{\dagger -1}$$

- The metric on the space of potentials $\mathcal{A}$ is

$$ds^2_{\mathcal{A}} = \int d\mu \, \delta A^a_{\bar{i}} \, \delta A^a_i = \frac{1}{2} \int d\mu \ \xi^{\dagger}_A \, \mathcal{M}_{AB} \, \xi_B$$

$$(\xi_1, \xi_2, \xi_3, \xi_4) = (\delta M M^{-1}, \ \delta\phi, \ M^{\dagger -1} \delta M^{\dagger}, \ \delta\phi^{\dagger})$$

- $\mathcal{M}_{AB}$ is a 4×4 matrix of operators; the nonzero elements are

$$\mathcal{M}_{11} = \mathcal{M}_{33} = (-\bar{D}_{\bar{i}} D_i \; + \bar{D}_{\bar{k}}\, \Phi\, \Phi^\dagger\, D_k)$$

$$\mathcal{M}_{22} = \mathcal{M}_{44} = (-D_i \bar{D}_{\bar{i}})$$

$$\mathcal{M}_{13} = 2\, (\epsilon_{ik} \bar{D}_{\bar{i}}\, \Phi\, \bar{D}_{\bar{k}}), \qquad \mathcal{M}_{23} = 2\, (-D_k \Phi \bar{D}_{\bar{k}})$$

$$\mathcal{M}_{31} = 2\, (\epsilon_{ik} D_i\, \Phi^\dagger\, D_k), \qquad \mathcal{M}_{32} = 2\, (D_k \Phi^\dagger\, \bar{D}_{\bar{k}})$$

where $(\Phi)^{ab} = \phi^c f^{abc}$

- The volume element is thus given by

$$d\mu_{\mathcal{A}} = \sqrt{\det \mathcal{M}}\;\, d\mu_{SL(N,\mathbb{C})}\;\, [d\phi\, d\phi^\dagger]$$

$$\Longrightarrow \quad d\mu(\mathcal{C}) = \sqrt{\det \mathcal{M}}\;\, d\mu(H)\;\, [d\chi\, d\bar{\chi}]$$

- Our strategy will be to calculate the Jacobian in powers of Φ, $\Phi^\dagger$, starting with the terms $(-\bar{D}_{\bar{i}} D_i)$ and $(-D_i \bar{D}_{\bar{i}})$ in $\mathcal{M}_{11} = \mathcal{M}_{33}$ and $\mathcal{M}_{22} = \mathcal{M}_{44}$
- The degree of divergence is higher than in 2 or 2+1 dimensions, because we are in 4 dimensions
- We calculate the leading divergence as

$$d\mu(\mathcal{C}) = d\mu(H)\,[d\chi\, d\bar{\chi}]\ \exp\left[2\Lambda^2\, S_{4d}(H, \chi, \bar{\chi}) + \text{subleading} + \text{finite}\right]$$

where $S_{4d}(H, \chi, \bar{\chi})$ is given as follows

- On any 4-dimensional Kähler manifold, we can define the generalized WZW action

$$\begin{aligned} S_{4d}(N) &= \frac{1}{2\pi}\int d\mu\, \mathrm{Tr}(\nabla_k N\, \nabla_{\bar{k}} N^{-1}) + \frac{i}{12\pi}\int \omega \wedge \mathrm{Tr}(N^{-1} dN)^3 \\ &= \frac{1}{2\pi}\int d\mu\, g^{a\bar{a}}\, \mathrm{Tr}(\partial_a N\, \bar{\partial}_{\bar{a}} N^{-1}) + \frac{i}{12\pi}\int \omega \wedge \mathrm{Tr}(N^{-1} dN)^3 \end{aligned}$$

where ω is the Kähler 2-form, $\omega_{a\bar{a}} = (i/2) g_{a\bar{a}}$.

- $S_{4d}(H, \chi, \bar{\chi})$ is then given by

$$S_{4d}(H, \chi, \bar{\chi}) = S_{4d}(H) - \frac{1}{\pi} \int d\mu \, \mathrm{Tr} \Big[H^{-1} (\mathcal{D}_a \bar{\chi}) \, H \, \mathcal{D}_a \chi - (\mathcal{D}_a \bar{\chi}) \, \mathcal{D}_a \chi$$
$$- \epsilon^{\bar{a}\bar{b}} H^{-1} \nabla_{\bar{a}} H \, \mathcal{D}_{\bar{b}} \chi - \epsilon^{ab} \mathcal{D}_b \bar{\chi} \, \nabla_a H \, H^{-1} \Big]$$

$$\mathcal{D}_a = \nabla_a + [-\nabla_a H \, H^{-1}, \ \ ], \quad \mathcal{D}_{\bar{a}} = \nabla_{\bar{a}} + [H^{-1} \nabla_{\bar{a}} H, \ \ ]$$

- Implications:
  - To have a well-defined measure, we must have a similar counterterm with a bare parameter m_0^2, defining

$$d\mu(\mathcal{C}) = d\mu(H) \, [d\chi d\bar{\chi}] \, \sqrt{\det \mathcal{M}} \, \exp \Big[m_0^2 \, S_{4d}(H, \chi, \bar{\chi}) \Big]$$
$$= d\mu(H) \, [d\chi d\bar{\chi}] \, \exp \Big[m_R^2 \, S_{4d}(H, \chi, \bar{\chi}) + \ldots \Big]$$

  - The renormalized value m_R^2 defines a mass scale for the theory
  - This is the dimensional transmutation of YM(4)
  - The recalculation of asymptotic freedom in these variables will relate m_R^2 to Λ_{QCD}.

- Viewed as an action in its own right, the equations of motion for $S_{4d}(H)$ (with $\chi = \bar{\chi} = 0$) are the instanton equations
- It was first used by DONALDSON in his work on holomorphic vector bundles
- It was also obtained as the boundary action for Kähler-Chern-Simons theories (NAIR & SCHIFF; *later work by* LOSEV *et al + others*)
- It has a holomorphically factorized current algebra (similar to 2d) and a Polyakov-Wiegmann identity

$$S_{4d}(NM) = S_{4d}(N) + S_{4d}(M) - \frac{1}{\pi} \int d\mu \, \mathrm{Tr}(N^{-1} \nabla_{\bar{a}} N \, \nabla_a M \, M^{-1})$$

- It has the properties of a mass term, similar to YM(2), YM(2+1) and screening masses in YM(4) at finite temperature and/or density
- It describes the target space dynamics of (world-sheet) $\mathcal{N} = 2$ heterotic string theory (OOGURI & VAFA)

- The geometry of the gauge orbit space seems to be at the heart of the mass gap question for Yang-Mills theories
- There are direct and indirect ways to get some understanding of the gauge orbit space
- The expectations for the mass gap based on the geometry of $\mathcal{C}$ are consistent with other arguments.
- There is a lot of work for YM(4), including supersymmetry
- Thanks to my collaborators:

  DIMITRA KARABALI; CHANJU KIM; ABHISHEK AGARWAL; ALEXANDR YELNIKOV
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Thank You

Additional Slides

- There may be a useful generalization to 3+1 dimensions. Define

$$d\mu(\mathcal{C})_{3d} = \frac{[dA]}{vol(\mathcal{G}_*)}\ \exp\left(-\frac{1}{4\mu}\int F^2\right)\Bigg]_{\mu\to\infty}$$

- This leads to

$$\begin{aligned}
\int d\mu(\mathcal{C})_{3d} &= \int \frac{[dA]}{vol(\mathcal{G}_*)}\ e^{\left(-\frac{1}{4\mu}\int F^2\right)}\Bigg]_{\mu\to\infty}\\
&= \langle 0|\ e^{-\beta\mathcal{H}}\ |0\rangle\Big]_{\beta,\mu\to\infty} = \int d\mu(\mathcal{C})_{2d}\ \Psi_0^*\Psi_0\Big]_{\mu\to\infty}\\
&= \int d\mu(\mathcal{C})_{2d}\ \exp\left(-\frac{\pi}{2\mu^2 c_A}\int F^2\right) < \infty
\end{aligned}$$

- On the torus, the Hamiltonian has a part which is Laplacian for the zero modes,

$$A_z = M \left[\frac{i\pi\, a}{\mathrm{Im}\, \tau} \right] M^{-1} - \partial_z M\, M^{-1}$$

- As one of the torus directions becomes small, there is an accumulation of the eigenvalues of the Laplacian for the zero modes.
- This could be the signal for deconfinement.

- The space of gauge potentials has the bundle structure

$$\begin{array}{ccc} \mathcal{G}_* \rightarrow & \mathcal{A} & \\ & \downarrow & \\ & \mathcal{A}/\mathcal{G}_* & \end{array}$$

- This bundle is nontrivial. In particular, $\Pi_2(\mathcal{A}/\mathcal{G}_*) = \mathbb{Z}$ and $\Pi_n(\mathcal{A}) = 0$. There are noncontractible 2-spheres in $\mathcal{A}/\mathcal{G}_*$
- An example of such a configuration is

$$H = \cosh 2f + \mathcal{J} \sinh 2f$$
$$f = \frac{1}{2} \log \left(\frac{z\,\bar{z} + w\,\bar{w} + \mu^2}{z\,\bar{z} + w\,\bar{w}} \right)$$

$w,\ \bar{w}$ are coordinates of the 2-sphere in $\mathcal{A}/\mathcal{G}_*$.

- The matrix $\mathcal{J}$ is given by

$$\mathcal{J} = \begin{pmatrix} z\,\bar{z} - w\,\bar{w} & 2\,\bar{w}\,z \\ 2\,w\,\bar{z} & w\,\bar{w} - z\,\bar{z} \end{pmatrix}$$

- $w = 0, z = 0$ is a singular point. Move singularity to another point by

$$H \to V\,H\,\bar{V}, \qquad \bar{V} = \exp\left[\sigma_3 \left(\log \frac{\bar{z}}{\bar{z} - \bar{a}}\right)\right]$$

- $S_{wzw}(H)$ unchanged, finite. The volume element is insensitive to this problem.

- Consider, as an example, in $SU(2)$ gauge theory

$$
\begin{aligned}
A_n &= (-it^3)\, i\, n\, (z\,\bar{z})^{n-1} \frac{(z\, d\bar{z} - \bar{z}\, dz)}{[1 + (z\,\bar{z})^n]} \\
F_n &= (-it^3)\,(-4\, n^2) \frac{(z\,\bar{z})^{n-1}}{[1 + (z\,\bar{z})^n]^2}\, dx_1 \wedge dx_2 \qquad \text{(Nothing pathological)}
\end{aligned}
$$

where $z = x_1 - i\, x_2,\ \bar{z} = x_1 + i\, x_2$.

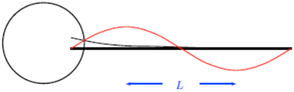
- In this case,

$$s^2_{\mathcal{C}}(A, 0) = 8\pi n$$

For any value L^2, we can find an A, namely, A_n, with $n \geq (L^2/8\pi)$, for which $s^2(A, 0) > L^2$.

- These are the so-called "spikes" on $\mathcal{C}$.

- A long wavelength standing wave on such a spike can have arbitrarily low energy, seemingly vitiating Feynman's argument.



- May be not ! It could be similar to the 2-dim Schrödinger problem

$$\mathcal{H} = -\frac{\nabla^2}{2M} + \lambda(x^2 + x^2y^2)$$

- The potential is zero along the y-axis, $(x = 0)$, and one can, *a priori*, think of long wavelength wave functions along this direction.
- The valley along the y-axis gets narrower as x becomes large. The zero-point energy of transverse directions (roughly, $\omega \sim \sqrt{1 + y^2}$) lifts the potential.
- Something similar could happen for YM, but we need a measure for the transverse directions.
- This can be done as follows

All calculations have to be done with proper regularization.

- We start with a regularization of the δ-function

$$\delta^{(2)}(u,w) \Longrightarrow \sigma(\vec{u},\vec{w},\epsilon) = \frac{1}{\pi\epsilon}\exp\left(-\frac{|u-w|^2}{\epsilon}\right)$$

- This is equivalent to

$$\begin{aligned} \bar{G}(\vec{x},\vec{y}) &= \frac{1}{\pi(x-y)} \\ \Longrightarrow \quad \bar{\mathcal{G}}(\vec{x},\vec{y}) &= \int_u \bar{G}(\vec{x},\vec{u})\,\sigma(\vec{u},\vec{y};\epsilon)H(u,\bar{y})H^{-1}(y,\bar{y}) \end{aligned}$$

- This simplifies as

$$\overline{\mathcal{G}}_{ma}(x,y) = \frac{1}{\pi(x-y)}\left[\delta_{ma} \ - \ e^{-\frac{(x-y)^2}{\epsilon}} \ [H(x,\bar{y})H^{-1}(y,\bar{y})]_{ma}\right]$$

All results checked using regularized expressions, with a single regulator from beginning to end.

- Absorb $\exp(2\, c_A S_{wzw})$ from the inner product into the wave function by $\Psi = e^{-c_A S_{wzw}(H)}\, \Phi$. The Hamiltonian acting on Φ is

$$\mathcal{H} \to e^{-c_A S_{wzw}(H)}\ \mathcal{H}\, e^{-c_A S_{wzw}(H)}$$

- Consider $H = e^{t^a \varphi^a} \approx 1 + t^a \varphi^a + \cdots$, a small φ limit appropriate for a (resummed) perturbation theory. The new Hamiltonian is

$$\mathcal{H} = \frac{1}{2} \int \left[-\frac{\delta^2}{\delta \phi^2} + \phi(-\nabla^2 + m^2)\phi + \cdots \right]$$

where $\phi_a(\vec{p}) = \sqrt{c_A\, p\bar{p}/(2\pi m)}\ \varphi_a(\vec{p})$.

- The vacuum wave function is

$$\Phi_0 \approx \exp\left[-\frac{1}{2} \int \phi^a \sqrt{m^2 - \nabla^2}\ \phi^a \right]$$

- Transforming back to Ψ,

$$\Psi_0 \approx \exp\left[-\frac{c_A}{\pi m}\int(\bar{\partial}\partial\varphi^a)\left[\frac{1}{m+\sqrt{m^2-\nabla^2}}\right](\bar{\partial}\partial\varphi^a)+\cdots\right]$$

- The full wave function must be a functional of J. The only form consistent with the above is

$$\Psi_0 = \exp\left[-\frac{2\pi^2}{e^2c_A^2}\int\bar{\partial}J^a(x)\left[\frac{1}{m+\sqrt{m^2-\nabla^2}}\right]_{x,y}\bar{\partial}J^a(y)\;+\cdots\right]$$

since $J \approx (c_A/\pi)\partial\varphi + \mathcal{O}(\varphi^2)$.

- This indicates the robustness of the Gaussian term in Ψ_0, since this argument only presumes
    1. Existence of a regulator, so that the transformation $\Psi \Longleftrightarrow \Phi$ can be carried out
    2. The two-dimensional anomaly calculation

- For modes of low momenta, this has the form

$$\Psi_0 \approx \exp\left[-\int \frac{B^2}{4m\, e^2}\right]$$

- For the expectation value of the Wilson loop, this gives

$$\langle W(C)\rangle = \langle \mathrm{Tr}_R\, \mathcal{P}e^{-\oint_C A}\rangle = \exp\left(-\sigma_R\, \mathrm{Area}(C)\right), \qquad \sigma_R = e^4\, \frac{c_A c_R}{4\pi}$$

- This agrees, within 2%, with numerical simulations for $SU(2)$ to $SU(6)$ (12 representations), G_2 (8 representations) and to within 1% of the extrapolated large N limit. (LUCINI & TEPER; BRINGOLTZ & TEPER; HARI DASS & MAJUMDAR; KISKIS & NARAYANAN; WELLEGHAUSEN *et al*)
- This is nice, but not the main point for today.
- We want to relate the measure calculation to three-dimensional covariant Feynman diagram calculations, motivated by supersymmetric theories.