Non-Abelian strings in Yang-Mills theories and beyond: 2D-4D correspondence, spin-orbit interaction generating unexpected Goldstone modes, and all that
☆ Hanany-Tong, 2003

☆ ☆ Auzzi et al., 2003

☆ ☆ ☆ Shifman-Yung, 2003 – ...

❖ Gaiotto, 2012 & Gaiotto, Gukov, Seiberg, 2013 “surface defects”...

Outline: a) Non-Abelian strings in FT (N=2, N=1),
b) World sheet models,
c) 2D-4D correspondence,
d) Applications (e.g. in He-3B)
Abrikosov strings (1950s);

Cosmic strings (Kibble 1970s, Witten 1985);

Non-Abelian strings in susy Yang-Mills:

Bulk $G \times G \rightarrow \text{CF locking} \rightarrow \left( G_{\text{diag}} \rightarrow H \right) \rightarrow G/H$ coset model on the world sheet $\rightarrow$ (susy in bulk $\rightarrow$ susy on ws)

$N=2 \rightarrow N=(2,2)$

$N=1 \rightarrow N=(2,0)$ nonminimal

$N=0 \rightarrow N=0$
Superconductor of the 2\textsuperscript{nd} kind

\[ \text{Cooper pair condensate} \]

\[ \text{Abrikosov (ANO) vortex (flux tube)} \]

DUAL MEISSNER EFFECT (Nambu-‘t Hooft-Mandelstam, \sim 1975)
Superconductor of the 2\textsuperscript{nd} kind

Cooper pair condensate

\[ \mathbf{B} \]

Abrikosov (ANO) vortex (flux tube)

DUAL MEISSNER EFFECT (Nambu-'t Hooft-Mandelstam, ∼1975)
ANO strings are there because of \( U(1) \)!

New strings:

\[ \pi_1(\text{U}(1) \times \text{SU}(2)) \text{ nontrivial due to } \mathbb{Z}_2 \text{ center of SU}(2) \]

\[ \sqrt{\xi} e^{i\alpha} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ T = 4\pi\xi \]

Non-Abelian

\[ \sqrt{\xi} \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & 1 \end{pmatrix} \]

\[ T_U(1) \pm T^3_{\text{SU}(2)} \]

\[ T = 2\pi\xi \]

\( \text{SU}(2)/\text{U}(1) \leftrightarrow \) orientational moduli; \( \text{O}(3) \) \( \sigma \) model

\( \text{string center in perp. plane} \)
★ ANO strings are there because of U(1)!
★ New strings:
\( \pi_1(SU(2) \times U(1)) = \mathbb{Z}_2: \) rotate by \( \pi \) around 3-d axis in \( SU(2) \)
→ -1; another -1 rotate by \( \pi \) in \( U(1) \)

\( \pi_1(U(1) \times SU(2)) \) nontrivial due to \( \mathbb{Z}_2 \) center of \( SU(2) \)

\[ \begin{align*}
\sqrt{\xi} \ e^{i\alpha} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \quad ANO \\
\sqrt{\xi} \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & 1 \end{pmatrix} & \quad Non-Abelian
\end{align*} \]

\( T = 4\pi\xi \) for ANO
\( T = 2\pi\xi \) for Non-Abelian

\( SU(2)/U(1) \) \( \leftrightarrow \) orientational moduli; \( O(3) \) \( \sigma \) model

\( T = 4\pi\xi \) for ANO
\( T = 2\pi\xi \) for Non-Abelian
“Non-Abelian” string is formed if all non-Abelian degrees of freedom participate in dynamics at the scale of string formation.

Classically gapless excitation

\[ SU(2)/U(1) = \mathbb{CP}(1) \sim O(3) \] sigma model

\[
S = \int d^2 x \left\{ \frac{2}{g^2} \frac{\partial_{\mu} \bar{\phi} \partial^{\mu} \phi - (\Delta m)^2 \bar{\phi} \phi}{(1 + \bar{\phi} \phi)^2} + \text{fermions} \right\}
\]
$Z_2$ string junction = kink

Evolution in dimensionless parameter $m^2/\xi$
\[ \xi = 0 \]
\[ \Delta m \neq 0 \]

The 't Hooft–Polyakov monopole

\[ \left| \Delta m \right| \ll \xi^{1/2} \]

Confined monopole, quasiclassical regime

\[ \Lambda_{CP(1)} \ll \left| \Delta m \right| \ll \xi^{1/2} \]

Almost free monopole

\[ \left| \Delta m \right| \gg \xi^{1/2} \]

Confined monopole, highly quantum regime

\[ \xi^{-1/2} \]

\[ \Lambda_{CP(1)}^{-1} \]

\[ \Delta m \to 0 \]
Kinks are confined in 4D (attached to strings).

Kinks are confined in 2D:

Kink = Confined Monopole

4D ↔ 2D Correspondence

World-sheet theory ↔ strongly coupled bulk theory inside

Dewar flask
1. \( \mathbf{N} = 2 \) SUSY bulk \( \rightarrow \) \( \mathbf{N} = (2,2) \) CP(N-1) model

\[
\mathcal{L} = \frac{1}{e^2} \left( \frac{1}{4} F_{\mu \nu}^2 + |\partial_\mu \sigma|^2 + \frac{1}{2} D^2 \right) + i D \left( \bar{n}_i n^i - 2\beta \right) \\
+ \left| \nabla_\mu n^i \right|^2 + 2 \sum_i \left| \sigma - \frac{m_i}{\sqrt{2}} \right|^2 |n^i|^2 \\
+ \text{fermions}
\]
$E_{\text{vac}}=0$ always, SUSY unbroken, $Z_{2N}$ always broken, (N degenerate vacua)

Crossover instead of phase transition

Strong-coupling $\leftrightarrow$ Higgs regime
II. $\mathbf{N} = 1$ SUSY bulk

$\mathbf{N} = (0,2)$ CP($N$-1) model

Supersymmetry is broken, generally speaking !!!
Phase transitions possible

All phase transitions are of the second kind!
Break $N = 2$ down to $N = 1$ in the bulk

Deformation of the bulk: $W = \mu (A^a)^2 + \mu' A^2$

Heterotic deformation of the world-sheet theory:

(2,2) supersymmetry is broken down to (0,2)

$$L_{\text{heterotic}} = \zeta_R^\dagger i \partial_L \zeta_R + \left[ \gamma \zeta_R R (i \partial_L \phi^\dagger) \psi_R + H.c. \right] - g_0^2 |\gamma|^2 \left( \zeta_R^\dagger \zeta_R \right) \left( R \psi_L^\dagger \psi_L \right)$$

at small $\gamma$

$\zeta_R$ is Goldstino

$$E_{\text{vac}} = |\gamma|^2 \left| \langle R \psi_R^\dagger \psi_L \rangle \right|^2$$

(0,2) supersymmetry is spontaneously broken!
$\gamma \gg 1$ (u \gg 1)

Strong coupling. Chiral $Z_N$ spont. broken.

No confinement

Coulomb/confining. Chiral $Z_N$ unbroken

Higgs phase
Weak coupling
Chiral $Z_N$ broken

SUSY restored here
No-supersymmetry applications
★ Internal symmetries $G \rightarrow H$,

\[ V^{"\text{rel}"} = V_G - V_H, \quad V^{"\text{non-rel}"} = (V_G - V_H)/2. \]

★ ★ This is not the case for geometric symmetries! (Ivanov-Ogievetsky, 1975; Low-Manohar, 2002)

★ ★ ★ E.g. (structureless) string breaks two translations and two rotations, but one should consider only translational zero modes!!

$M_{zx}$ & $M_{zy}$ broken, $T_x$ & $T_y$ broken.

ANO vortex string (flux tube)
Goldstone modes (gapless excitations) on strings

** Internal symmetries $G \rightarrow H$, 

\[ \nu_{\text{rel}} = \nu_G - \nu_H, \quad \nu_{\text{non-rel}} = (\nu_G - \nu_H)/2. \]

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** ** ** E.g. (structureless) string breaks two translations and two rotations, but one should consider only translational zero modes!!!

$M_{zx}$ & $M_{zy}$ broken, 
$T_x$ & $T_y$ broken.

ANO vortex string (flux tube)
What if order parameter carries Lorentz indices?

\[ M_{zx}(z) \sim T_x \]
Low-energy excitations (gapless modes) on vortices

\[ \Delta H_{GL} = \left( \frac{T}{2} \right) \left( \partial_z x_{\text{perp}} \partial_z x_{\text{perp}} \right) + \text{h.d.} \]

- Kelvin modes or Kelvons
  - 2 NG gapless modes in relat.
  - 1 NG gapless mode in non-rel.

- Nambu-Goto → String Theory
  - \( E_{\text{str}} = TL + C/L \)

- \( E_{\text{excit}} \ll m_Y \sim \text{ev} \)

- Time derivatives can be linear or quadratic.
In this paper we report a new phenomenon which occurs in superfluids with a tensorial order parameter. We are motivated by analogous developments explained in detail in Section 3. They can be applied in general to unconventional Goldstone modes in condensed matter physics. The arguments that lead us to this conclusion give rise to a novel spectrum of the gapless bulk excitations. Such a breaking of the axial symmetry leads to the presence of gapless Kelvin modes on the vortices. Both the non-Abelian bulk and the Kelvin excitations have been recently observed. Note that the breaking of the axial symmetry in the core of the condensate has a tensorial structure and has to be described by a $n$ by $n$ matrix. In the ground state, the number of the Nambu-Goldstone excitations in the bulk is limited to the $L=1, S=1$ case, while in the more symmetric B phase the ground state preserves a locked $U(1)_S$ symmetry. The expressions above imply that both phases admit a non-trivial set of non-Abelian $SO_L(3)$ and $SO_S(3)$ quasispinors. The unbroken $SO_L(3)$ which distinguishes the two phases is realized and studied experimentally. It is also one of the most well-studied systems from a practical point of view.

In this paper we point out that, in addition to the Nambu-Goldstone modes in the bulk, there exist novel Nambu-Goldstone modes – to be referred to as nong-Abelian – localized on the mass vortex. Therefore, contrived NG modes in the bulk arise independently from the breaking of spatial rotation symmetry.

$3^\text{He}-B$ example

$3^\text{He}$ atoms

$^3\text{He}$ atom is a spin $\frac{1}{2}$ particle. The b antiquasymmetry of the wave function for a pair of $^3\text{He}$ atoms is realized by the Van der Walls potential. It is also one of the most well-studied systems from a practical point of view. It is definitely one of the most interesting states of matter which can be realized and studied experimentally. It is also one of the most well-studied systems from a practical point of view. It is also one of the most well-studied systems from a practical point of view. The He atom is a spin $\frac{1}{2}$ particle. The b antiquasymmetry of the wave function for a pair of $^3\text{He}$ atoms is realized by the Van der Walls potential. The He-B example

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\[ S_{\text{world sheet}} = \left( \mu^2 / 2 \beta \right) \int d^2 x \ (\partial_\mu S^i) \ (\partial_\mu S^i) \]

\[ S^i S^i = 1 \]

Clasically two "rotational" zero modes.

QMechanically may be lifted
Assume $\chi^i$ is spin field!

Add $\Delta L = \varepsilon (\partial_i \chi^i)(\partial_k \chi^k)$

If $\varepsilon \to 0$, geometric symmetry is enhanced

Poincaré $\times O(3)$

Two extra zero modes
What if $\varepsilon \neq 0$ but small?

$$\Delta_{\text{CP}(1)} S_{\text{world sheet}} = \varepsilon \int d^2 x \left\{ (\partial_z S^3)^2 - M^2 [1 - (S^3)^2] \right\}$$

$$\mathcal{L}_{x_{\perp}} = \frac{T}{2} (\partial_a \vec{x}_{\perp})^2 - \tilde{M}^2 (S^3)^2 (\partial_z \vec{x}_{\perp})^2,$$

★ ★ ★ EXTRA (quasi)gapless modes ★ ★ ★

★ ★ ★ Translational (Kelvon) and orientational (spin) modes mix with each other ★ ★ ★
\[ S_{\text{int}} = \frac{\alpha}{8\pi} \int d^2 \sigma \phi K^i_{\alpha \gamma} K^j_{\beta \gamma} \epsilon^{\alpha \beta} \epsilon_{ij}, \quad (11) \]

where \( K^i_{\alpha \gamma} \) is the extrinsic curvature of the worldsheet.
SQED with **two flavors**

\[
S_0 = \int d^4 x \left\{ -\frac{1}{4g^2} F_{\mu\nu}^2 + |D_\mu \varphi^A|^2 - \lambda (|\varphi^A|^2 - \xi)^2 \right\}.
\]

\[
\beta = \frac{2\lambda}{g^2} = 1 \rightarrow \text{SUSY} \rightarrow \beta \gg 1 \text{ SUSY}
\]

\[\xi \rightarrow \text{Fayet-Iliopoulos parameter}\]

\[
m_\gamma = \sqrt{2g\sqrt{\xi}}, \quad m_H = m_\gamma \sqrt{\beta}.
\]

\[m_H \gg m_\gamma \quad \leftarrow \text{Abrikosov/London limit}\]
Hopf solitons in two-component superconductors

Many field theories, in particular, supersymmetric Yang–Mills theories, support topologically stable solitons. Their stability is due to the existence of certain topological charges (in case of supersymmetry they are usually related to central charge of the relevant superalgebra [1]). In such cases one can perform the Bogomol’nyi completion [2] for the energy functional (in the instanton case, for the action) which selects the field configuration corresponding to the minimal energy in the sector with the given topological charge. Well-known examples are the Abrikosov–Nielsen–Olesen (ANO) strings [3], whose topological stability is due to $\pi_1(U(1)) = \mathbb{Z}$, instantons in the two-dimensional CP(1) model [4] whose topology is determined by $\pi_2(SU(2)/U(1)) = \mathbb{Z}$, and the Belavin–Polyakov–Schwartz–Tyupkina instantons [5] in four-dimensional Yang–Mills theory whose topological classification is based on $\pi_3(SU(2)) = \mathbb{Z}$.

Faddeev and Niemi discovered [6] a novel class of solitons, of the knot type, whose stability is due to the existence of the Hopf topological invariant.

The model with the solitonic knots considered by Faddeev and Niemi is a deformed O(3) nonlinear sigma model in four dimensions

$$L = \frac{F^2}{2} \partial_\mu \vec{S} \partial^\mu \vec{S} - \frac{\lambda}{4} \left( \partial_\mu \vec{S} \times \partial_\nu \vec{S} \right) \cdot \left( \partial_\mu \vec{S} \times \partial_\nu \vec{S} \right)$$

4D Skyrme-Faddeev model: nontriv. $S_3 \rightarrow S_2$
Conclusions

★ Non-Abelian strings in SUSY bulk $\rightarrow$

CP(N-1) models (heterotic & nonheterotic) on string; a wealth of phase transitions.

★ 2D $\leftrightarrow$ 4D Correspondence brings fruits and a treasure trove of novel 2D models with intriguing dynamics!

★ Unexpected applications in condensed matter (poorly explored).