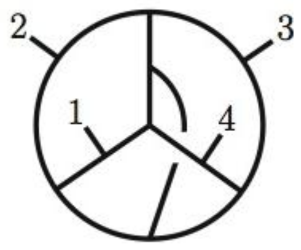
A Duality Between Color and Kinematics and Applications to Gravity

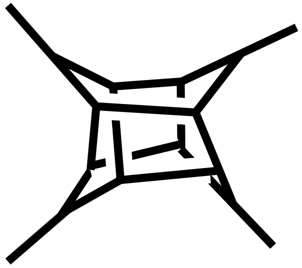
August 12, 2013, Copenhagen

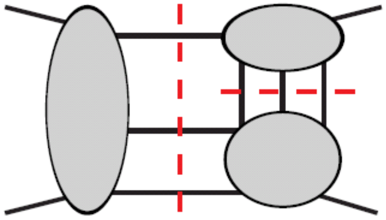
Current Themes in Higher Energy Physics and Cosmology

Zvi Bern, UCLA

Based on papers with John Joseph Carrasco, Scott Davies, Tristan Dennen, Lance Dixon, Yu-tin Huang, Henrik Johansson Josh Nohle and Radu Roiban.







Outline

1) **Remarkable progress in scattering amplitudes.**

2) **A hidden structure in gauge and gravity amplitudes.**
   - **a duality between color and kinematics.**
   - **gravity as a double copy of gauge theory.**

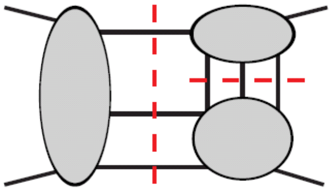
3) **Application: Surprises in UV properties of point-like theories of quantum gravity.**

Some new results from past year:

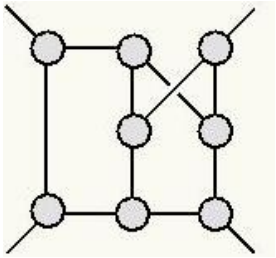
1) **Clear examples where standard symmetry arguments fail to predict (and postdict) UV finiteness in supergravity.**
2) **Uncovered precise reason behind surprising UV cancellations in supergravity, at least in a 2 loop $D = 5$ case.**

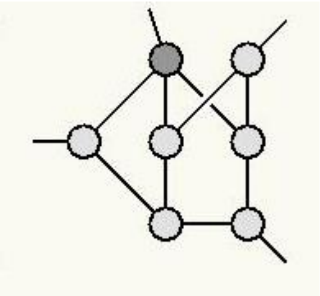
Constructing Multiloop Amplitudes

We do have powerful tools for complete calculations including nonplanar contributions and for discovering new structures:



- **Unitarity Method.**
  ZB, Dixon, Dunbar, Kosower
  ZB, Carrasco, Johansson , Kosower
- **On-shell recursion.**
  Britto, Cachazo Feng and Witten; Arkani-Hamed et al
- **Duality between color and kinematics.**
  ZB, Carrasco and Johansson
- **Advanced loop integration technology.**
  Chetyrkin, Kataev and Tkachov; A.V. Smirnov; V. A. Smirnov, Vladimirov; Marcus, Sagnotti; Cazkon; etc



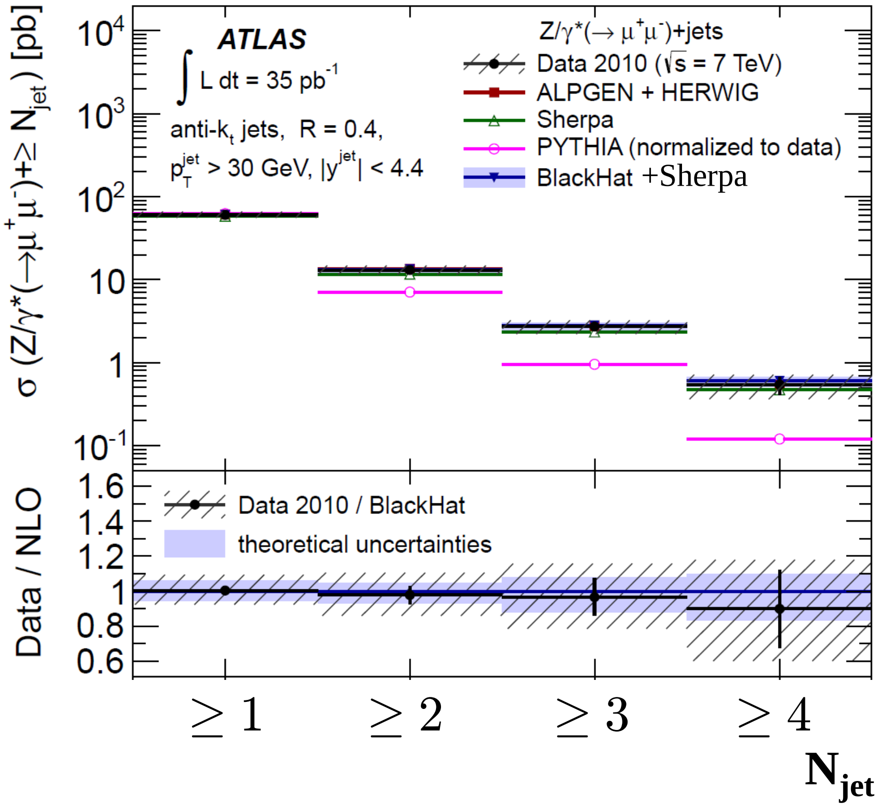


In this talk we will explain how the duality between color and kinematics allows us to probe deeply the UV properties of supergravity theories.

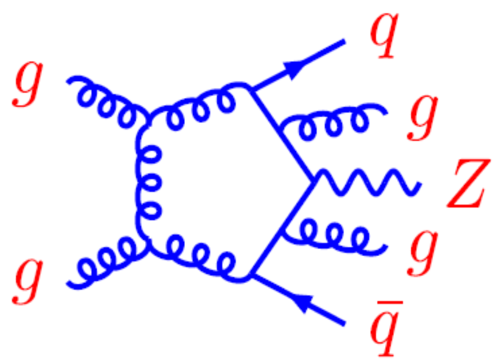
ATLAS Comparison Against NLO QCD

ZB, Dixon, Febres Cordero, Ita, Kosower, Maitre, Ozeren [BlackHat collaboration]

Z +1, 2, 3, 4 jets inclusive



Unitarity approach is used to carry out state of the art NLO QCD computations of multijet processes at the LHC.



very good agreement with no tuning

A triumph for unitarity method. Beyond Feynman diagrams.

- **Even W + 5 jets in NLO QCD has also been done.** BlackHat (2013)
- **Serious advance in our ability to do NLO QCD calculations.**

Review: The structure of gravity scattering amplitudes

Gravity vs Gauge Theory

Consider the gravity Lagrangian

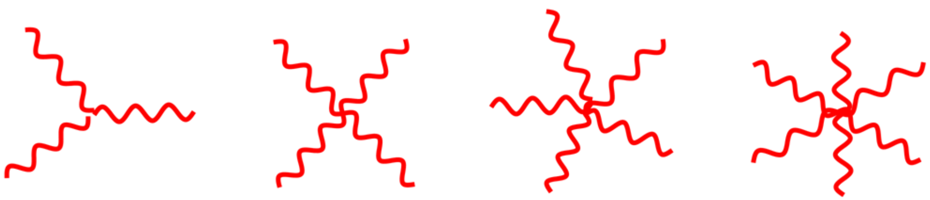
$$L_{\text{gravity}} = \frac{2}{\kappa^2}\sqrt{-g}\,R$$

$$\kappa^2 = 32\pi G_{\text{Newton}}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

metric ($g_{\mu\nu}$), flat metric ($\eta_{\mu\nu}$), graviton field ($h_{\mu\nu}$)

Infinite number of complicated interactions



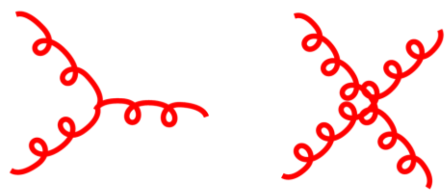
$+ \cdots$

terrible mess

Non-renormalizable

Compare to Yang-Mills Lagrangian on which QCD is based

$$L_{\text{YM}} = \frac{1}{g^2}F^2$$

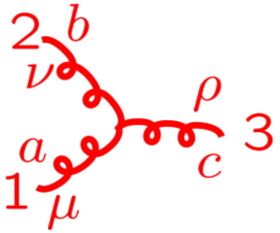


Only three and four point interactions

Gravity seems so much more complicated than gauge theory.

Three Vertices

Standard Feynman diagram approach.



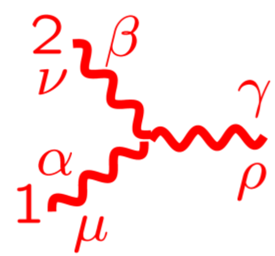
Three-gluon vertex:

$$V_{3\,\mu\nu\sigma}^{abc} = -gf^{abc}(\eta_{\mu\nu}(k_1-k_2)_\rho+\eta_{\nu\rho}(k_1-k_2)_\mu+\eta_{\rho\mu}(k_1-k_2)_\nu)$$

$$k_i^2 = E_i^2 - \vec{k}_i^2 \neq 0$$

Three-graviton vertex:

$$\begin{aligned}
&G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1, k_2, k_3) = \\
&\quad \mathrm{sym}[-\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) \\
&\qquad + P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) \\
&\qquad + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) \\
&\qquad + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})]
\end{aligned}$$



About 100 terms in three vertex

Naïve conclusion: Gravity is a nasty mess.

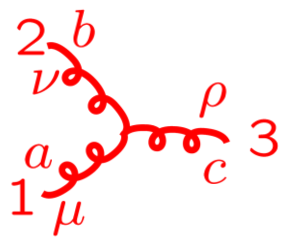
Simplicity of Gravity Amplitudes

People were looking at perturbative gravity the wrong way. On-shell viewpoint much more powerful.

On-shell **three vertices contains all information:**

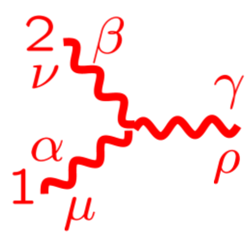
$$k_i^2 = 0$$

gauge theory:



$$-gf^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$

gravity:



$$i\kappa(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic}) \times (\eta_{\alpha\beta}(k_1 - k_2)_\gamma + \text{cyclic})$$

double copy of Yang-Mills vertex.

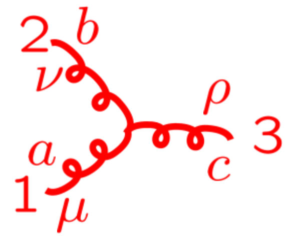
- **Using modern on-shell methods, any gravity scattering amplitude constructible solely from *on-shell* 3 vertex.**
- **Higher-point vertices irrelevant! On-shell recursion for trees, and unitarity method for loops.**

Duality Between Color and Kinematics

ZB, Carrasco, Johansson (BCJ)

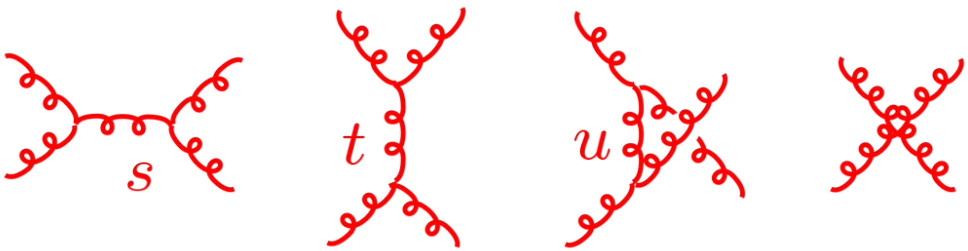
coupling constant → color factor → momentum dependent kinematic factor

$$-gf^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$



Color factors based on a Lie algebra: $[T^a, T^b] = if^{abc}T^c$

Jacobi Identity $f^{a_1a_2b}f^{ba_4a_3} + f^{a_4a_2b}f^{ba_3a_1} + f^{a_4a_1b}f^{ba_2a_3} = 0$



Use 1 = *s/s* = *t/t* = *u/u* to assign 4-point diagram to others.

$$\mathcal{A}_4^{\text{tree}} = g^2\left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right)$$

$s = (k_1 + k_2)^2$
$t = (k_1 + k_4)^2$
$u = (k_1 + k_3)^2$

Color factors satisfy Jacobi identity: $c_u = c_s - c_t$

Numerator factors satisfy similar identity: $n_u = n_s - n_t$

Color and kinematics satisfy the same identity

Duality Between Color and Kinematics

Consider five-point tree amplitude:

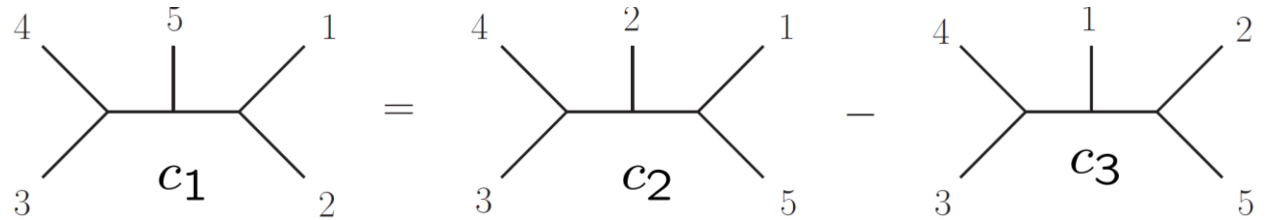
ZB, Carrasco, Johansson (BCJ)

$$\mathcal{A}_5^{\text{tree}} = \sum_{i=1}^{15} \frac{c_i\, n_i}{\prod_{\alpha_i} p^2_{\alpha_i}}$$

color factor

kinematic numerator factor

Feynman propagators



$$c_1 \equiv f^{a_3 a_4 b} f^{b a_5 c} f^{c a_1 a_2}, \quad c_2 \equiv f^{a_3 a_4 b} f^{b a_2 c} f^{c a_1 a_5}, \quad c_3 \equiv f^{a_3 a_4 b} f^{b a_1 c} f^{c a_2 a_5}$$

$$n_i \sim k_4 \cdot k_5\, k_2 \cdot \varepsilon_1\, \varepsilon_2 \cdot \varepsilon_3\, \varepsilon_4 \cdot \varepsilon_5 + \cdots$$

$$\boxed{c_1 - c_2 + c_3 = 0 \Leftrightarrow n_1 - n_2 + n_3 = 0}$$

Claim: At n-points we can always find a rearrangement so color and kinematics satisfy the same algebraic constraint equations.

Nontrivial constraints on amplitudes in field theory and string theory

BCJ, Bjerrum-Bohr, Feng,Damgaard, Vanhove, ; Mafra, Stieberger, Schlotterer; Cachazo; Tye and Zhang; Feng, Huang, Jia; Chen, Du, Feng; Du, Feng, Fu; Naculich, Nastase, Schnitzer

BCJ

Gravity and Gauge Theory

kinematic numerator → color factor

gauge theory:

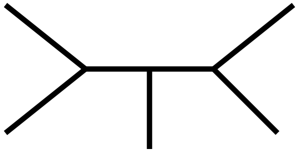
$$\frac{1}{g^{n-2}}\mathcal{A}_n^{\text{tree}}(1,2,3,\ldots,n) = \sum_i \frac{n_i\, c_i}{\prod_{\alpha_i} p_{\alpha_i}^2}$$

sum over diagrams with only 3 vertices

$$c_i \sim f^{a_1 a_2 b_1} f^{b_1 b_2 a_5} f^{b_2 a_4 a_5}$$

Assume we have:

$$c_1 + c_2 + c_3 = 0 \quad \Leftrightarrow \quad n_1 + n_2 + n_3 = 0$$



Then: $c_i \Rightarrow \tilde{n}_i$ **kinematic numerator of second gauge theory**

Proof: ZB, Dennen, Huang, Kiermaier

gravity:

$$-i\left(\frac{2}{\kappa}\right)^{(n-2)} \mathcal{M}_n^{\text{tree}}(1,2,\ldots,n) = \sum_i \frac{n_i\, \tilde{n}_i}{\prod_{\alpha_i} p_{\alpha_i}^2}$$

Encodes KLT tree relations

Gravity numerators are a double copy of gauge-theory ones.

This works for ordinary Einstein gravity and susy versions.

Cries out for a unified description of the sort given by string theory!

Gravity From Gauge Theory

BCJ

$$-i\left(\frac{2}{\kappa}\right)^{(n-2)}\mathcal{M}_n^{\text{tree}}(1,2,\ldots,n)=\sum_i\frac{n_i\,\tilde{n}_i}{\prod_{\alpha_i}p^2_{\alpha_i}}$$

n $\tilde{n}$

***N* = 8 sugra: (*N* = 4 sYM)× (*N* = 4 sYM)**
***N* = 4 sugra: (*N* = 4 sYM)× (*N* = 0 sYM)**
***N* = 0 sugra: (*N* = 0 sYM)× (*N* = 0 sYM)**

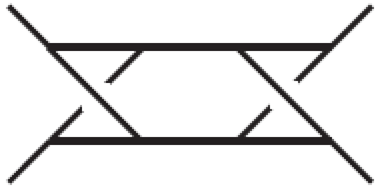
***N* = 0 sugra: graviton + antisym tensor + dilaton**

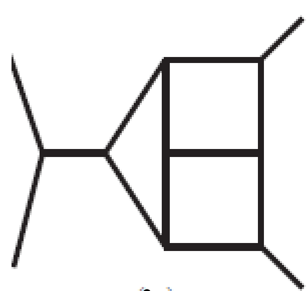
Spectrum controlled by simple tensor product of YM theories.
Recent papers show more sophisticated lower susy cases.

Carrasco, Chiodaroli, Günaydin and Roiban (2012); Borsten, Duff, Hughes and Nagy (2013)

Loop-Level Conjecture

ZB, Carrasco, Johansson (2010)





$$c_i + c_j + c_k = 0$$
$$n_i + n_j + n_k = 0$$

sum is over diagrams

kinematic numerator

color factor

$$\frac{(-i)^L}{g^{n-2+2L}} \mathcal{A}_n^{\rm loop} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

gauge theory

propagators

$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}} \mathcal{M}_n^{\rm loop} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

gravity

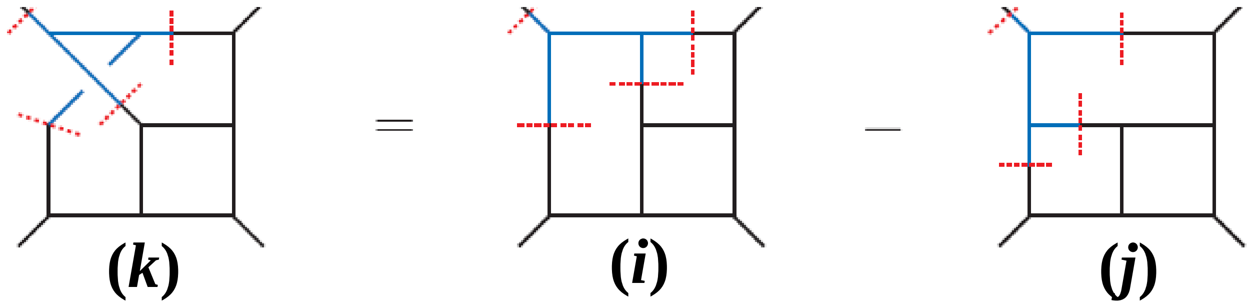
symmetry factor

Loop-level is identical to tree-level one except for symmetry factors and loop integration.
This works if numerator satisfies duality.

Gravity integrands are free!

BCJ

Ideas generalize to loops:



color factor

$$c_k = c_i - c_j$$

$$n_k = n_i - n_j$$

kinematic numerator

If you have a set of duality satisfying numerators.

To get:

gauge theory → gravity theory

simply take

color factor → kinematic numerator

$$c_k \longrightarrow n_k$$

Gravity loop integrands are trivial to obtain!

Generalized Gauge Invariance

BCJ
ZB, Dennen, Huang, Kiermaier
Tye and Zhang

gauge theory

$$\frac{(-i)^L}{g^{n-2+2L}}\mathcal{A}_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

sum is over diagrams

$$\sum_j \int \frac{d^{DL} p}{(2\pi)^{DL}} \frac{1}{S_j} \frac{\Delta_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2} = 0$$

$$n_i \to n_i + \Delta_i$$

$$(c_\alpha + c_\beta + c_\gamma) f(p_i) = 0$$

Above is just a definition of generalized gauge invariance

gravity

$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}}\mathcal{M}_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

$$n_i \to n_i + \Delta_i$$

$$\sum_j \int \frac{d^{DL} p}{(2\pi)^{DL}} \frac{1}{S_j} \frac{\Delta_j \tilde{n}_j}{\prod_{\alpha_j} p_{\alpha_j}^2} = 0$$

- **Gravity inherits generalized gauge invariance from gauge theory!**
- **Double copy works even if only one of the two copies has duality manifest!**
- **Very useful for *N* ≥4 supergravity amplitudes.**

ZB, Boucher-Veronneau and Johansson; Boucher-Veroneau and Dixon; ZB, Davies, Dennen, Huang

Symmetries

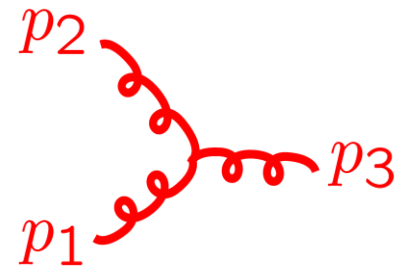
Monteiro and O'Connell

Consider self dual YM. Work in light-cone gauge.

$$u = t - z$$
$$w = x + iy$$

Generators of diffeomorphism invariance:

$$L_k = e^{-ik\cdot x}(-k_w\partial_u + k_u\partial_w)$$



The Lie Algebra:

YM vertex

$$[L_{p_1}, L_{p_2}] = iX(p_1, p_2)L_{p_1+p_2} = iF_{p_1 p_2}{}^k L_k$$

The $X(p_1, p_2)$ are YM vertices, valid for self-dual configurations.

Explains why numerators satisfy Jacobi Identity

YM inherits diffeomorphism symmetry of gravity.

We need to go beyond self dual.

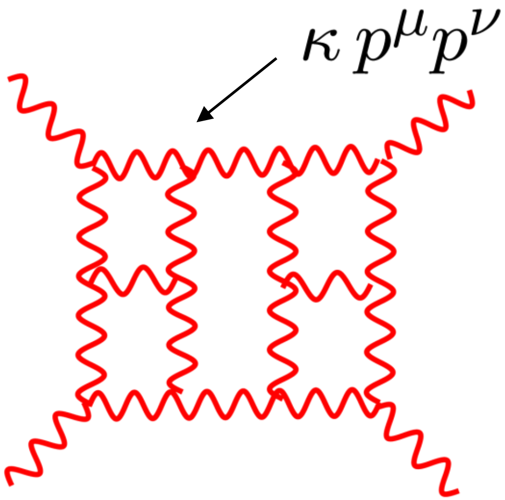
Various other attempts to understand Lagrangian and symmetries.

ZB, Dennen, Huang, Kiermaier; Bjerrum-Bohr, Damgaard, Monteiro, O'Connell;
Boels, Isermann, Monteiro, O'Connell; Tolotti and Weinzierl

Application: UV Properties of Supergravity

Review: Power Counting at High Loop Orders

$$\kappa = \sqrt{32\pi G_N}$$ ⟵ Dimensionful coupling



Gravity: $$\int \prod_{i=1}^{L} \frac{d^D p_i}{(2\pi)^D} \frac{(\kappa\, p_j^\mu p_j^\nu)\cdots}{\text{propagators}}$$

Gauge theory: $$\int \prod_{i=1}^{L} \frac{d^D p_i}{(2\pi)^D} \frac{(g\, p_j^\nu)\cdots}{\text{propagators}}$$

Extra powers of loop momenta in numerator means integrals are badly behaved in the UV.

Non-renormalizable by power counting.

Reasons to focus on extended supegravity, especially *N* = 8.

- **With more susy expect better UV properties.**
- **High symmetry implies technical simplicity.**

Finiteness of *N* = 8 Supergravity?

If *N* = 8 supergravity is finite it would imply a new symmetry or non-trivial dynamical mechanism. No known symmetry can render a *D* = 4 gravity theory finite.
The discovery of such a mechanism would have a fundamental impact on our understanding of gravity.

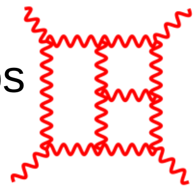
Note: Perturbative finiteness is not the only issue for consistent gravity: Nonperturbative completions? High energy behavior of theory? Realistic models?

Consensus opinion for the late 1970's and early 1980's: All supergravities would diverge by three loops and therefore not viable as fundamental theories.

Feynman Diagrams for Gravity

SUPPOSE WE WANT TO CHECK IF CONSENSUS OPINION IS TRUE

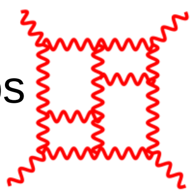
3 loops



$\sim 10^{20}$ TERMS

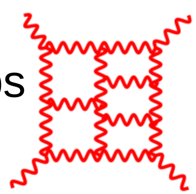
No surprise it has never been calculated via Feynman diagrams.

4 loops



$\sim 10^{26}$ TERMS

5 loops



$\sim 10^{31}$ TERMS

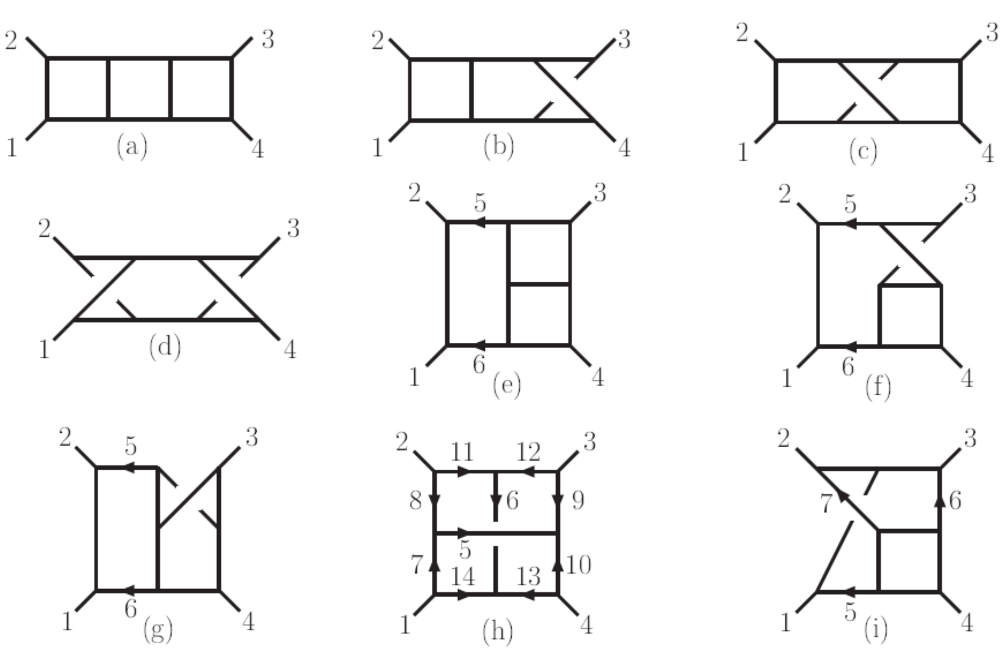
More terms than atoms in your brain!

- **Calculations to settle this seemed utterly hopeless!**
- **Seemed destined for dustbin of undecidable questions.**

Complete Three-Loop Result

Analysis of unitarity cuts shows highly nontrivial all-loop cancellations. ZB, Dixon and Roiban (2006); ZB, Carrasco, Forde, Ita, Johansson (2007)

To test completeness of cancellations, we decided to directly calculate potential three-loop divergence.



ZB, Carrasco, Dixon, Johansson, Kosower, Roiban (2007)

Three loops is not only ultraviolet finite it is "superfinite"— finite for $D < 6$.

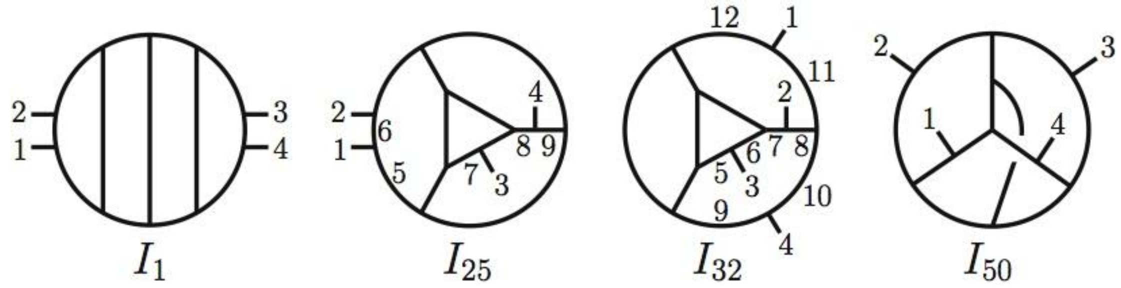
It is very finite!

Obtained via on-shell unitarity method.

Four-Loop Amplitude Construction

ZB, Carrasco, Dixon, Johansson, Roiban (2009)

Get 50 distinct diagrams or integrals (ones with two- or three-point subdiagrams not needed).



$$M_4^{\text{4-loop}} = \left(\frac{\kappa}{2}\right)^{10} stu M_4^{\text{tree}} \sum_{S_4} \sum_{i=1}^{50} c_i I_i$$

Integral

symmetry factor

leg perms

UV finite for *D* < 11/2
It's very finite!

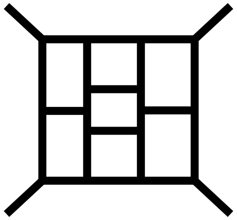
Duality between color and kinematic discovered by doing this calculation!

Today with duality we can reproduce this in a few days!

Recent Status of $N = 8$ Divergences

Consensus that in $N = 8$ supergravity trouble starts at 5 loops and by 7 loops we have valid UV counterterm in $D = 4$ under all known symmetries (suggesting divergences).

Bossard, Howe, Stelle; Elvang, Freedman, Kiermaier; Green, Russo, Vanhove ; Green and Bjornsson ; Bossard , Hillmann and Nicolai; Ramond and Kallosh; Broedel and Dixon; Elvang and Kiermaier; Beisert, Elvang, Freedman, Kiermaier, Morales, Stieberger



For $N = 8$ sugra in $D = 4$:

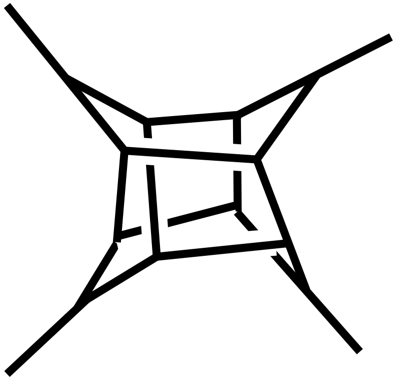
- **All counterterms ruled out until 7 loops!**
- **But D^8R^4 apparently available at 7 loops (1/8 BPS) under all known symmetries. (No known nonrenormalization theorem)**

Bossard, Howe, Stelle and Vanhove

Based on this a reasonable person would conclude that $N = 8$ supergravity almost certainly diverges at 7 loops in $D = 4$.

N = 8 Sugra 5 Loop Calculation

ZB, Carrasco, Dixon, Johannson, Roiban



~500 such diagrams with ~1000s terms each

Being reasonable and being right are not the same. Note duality between color and kinematics implies unaccounted for structures.

Place your bets:

- **At 5 loops in $D = 24/5$ does $N = 8$ supergravity diverge?**
- **At 7 loops in $D = 4$ does $N = 8$ supergravity diverge?**

$D^8 R^4$ **counterterms**

5 loops

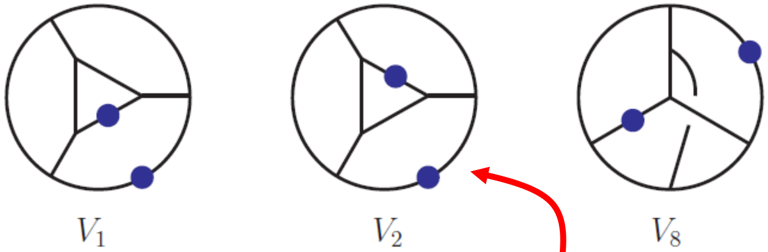
Kelly Stelle:
English wine
"It will diverge"

Zvi Bern:
California wine
"It won't diverge"

New Four-Loop $N = 8$ Surprise

ZB, Carrasco, Dixon, Johansson, Roiban (2012)

Critical dimension D =11/2.
Express UV divergences
in terms of vacuum like integrals.



doubled propagator

gauge theory

$$\mathcal{A}_4^{(4)}(1,2,3,4)\Big|_{\text{pole}}^{SU(N_c)} = -6\, g^{10}\, \mathcal{K}\, N_c^2 \Big(N_c^2\, V_1 + 12\, (V_1 + 2\, V_2 + V_8) \Big) \times \Big(s\,(\text{Tr}_{1324} + \text{Tr}_{1423}) + t\,(\text{Tr}_{1243} + \text{Tr}_{1342}) + u\,(\text{Tr}_{1234} + \text{Tr}_{1432}) \Big)$$

same divergence

gravity

$$\mathcal{M}_4^{(4)}\Big|_{\text{pole}} = -\frac{23}{8}\left(\frac{\kappa}{2}\right)^{10} stu(s^2+t^2+u^2)^2\, M_4^{\text{tree}}\, (V_1 + 2V_2 + V_8)$$

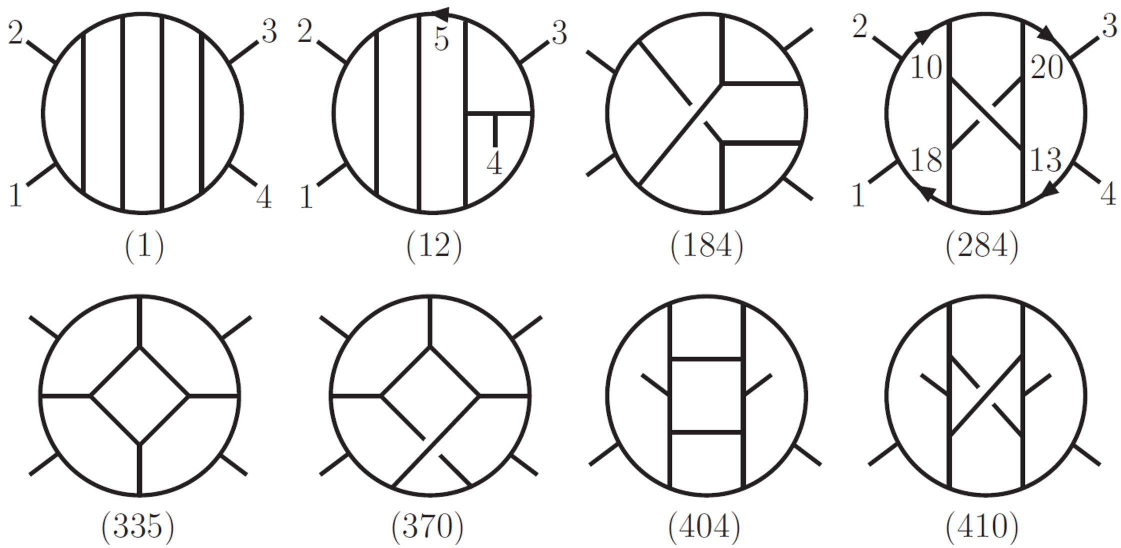
- **Gravity UV divergence is directly proportional to subleading-color single-trace divergence of $N = 4$ super-Yang-Mills theory.**
- **Same happens at 1-3 loops.**

Calculation of $N = 4$ sYM 5 Loop Amplitude

ZB, Carrasco, Johansson, Roiban (2012)

Key step for $N = 8$ supergravity is construction of complete 5 loop integrand of $N = 4$ sYM theory.

416 such diagrams with ~100s terms each



$$s = (k_1 + k_2)^2$$

diagram numerators

$$N_1 = s^4 , \qquad N_{12} = 2s^3 k_3 \cdot l_5 ,$$
$$N_{284} = 2s^2((l_{10} \cdot l_{20})^2 + (l_{13} \cdot l_{18})^2)$$

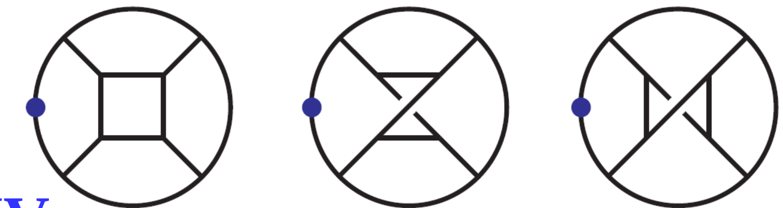
We are trying to figure out a BCJ form. If we can get it we should have five-loop $N = 8$ supergravity finished soon!
(Unclear how long it will take to get this new form.)

UV divergences $N = 4$ sYM 5 Loop Amplitude

ZB, Carrasco, Johansson, Roiban (2012)

Critical dimension where divergences first occur: $D = 26/5$

- **Expand in small external momenta.**
- **Apply integral consistency equations.**
- **Find all integral identities.**
- **Get astonishingly simple formula for UV**



(a) (b) (c)

$$A_4^{(5)}\Big|_{\rm div} = \frac{144}{5} g^{12} K \, N_c^3 \Big(N_c^2 V^{(a)} + 12 (V^{(\rm a)} + 2 V^{(\rm b)} + V^{(\rm c)}) \Big) \times \Big(s(\mathrm{Tr}_{1234} + \mathrm{Tr}_{1423}) + t(\mathrm{Tr}_{1243} + \mathrm{Tr}_{1342}) + u(\mathrm{Tr}_{1234} + \mathrm{Tr}_{1432}) \Big)$$

Amazing similarity to four loops

color trace

Use FIESTA: $V^{(\rm a)} = \frac{0.331K}{\epsilon}, \quad V^{(\rm b)} = \frac{0.310K}{\epsilon}, \quad V^{(\rm c)} = \frac{0.291K}{\epsilon}$

- **Proves known finiteness bound is saturated.**
- **Note amazing similarity to four loops and $N = 8$ sugra.**

Examples of Magical Cancellations?

Fine, but do we have any examples where a divergence vanishes but the known symmetries imply valid counterterms?

Yes!

Two examples in half-maximal supergravity :

- **D = 5 at 2 loops.**
- **D = 4 at 3 loops.**

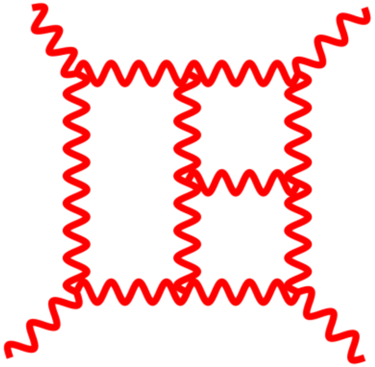
N = 4 Supergravity in D = 4

- $N = 4$ sugra at 3 loops ideal $D = 4$ test case to study.

  Cremmer, Scherk, Ferrara (1978)

- BCJ representation exists for $N = 4$ sYM 3-loop 4-pt amplitude

  ZB, Carrasco, Johansson (2010)



Consensus had it that a valid R^4 counterterm exists for this theory in $D = 4$.

Bossard, Howe, Stelle; Bossard, Howe, Stelle, Vanhove

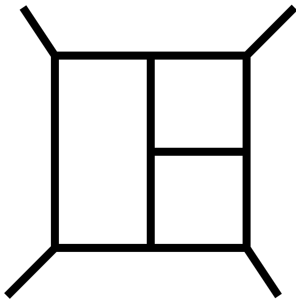
Is the consensus opinion true?

Three-Loop Construction

ZB, Davies, Dennen, Huang

N = 4 sugra : (N = 4 sYM) x (N = 0 YM)

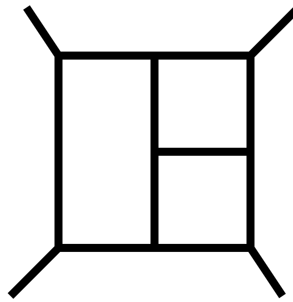
N = 4 sYM



$\sim l \cdot k \, s^2 t A_4^{\text{tree}}$

BCJ representation

pure YM



$\sim (\varepsilon_i \cdot l)^4 \, l^4$

Feynman representation

N = 4 sugra linear divergent

$$\int (d^D l)^3 \frac{k^7 l^9}{l^{20}}$$

simple to see finite for N=5,6 sugra

Ultraviolet divergences are obtained by series expanding small external momentum (or large loop momentum) and introducing a mass regulator for IR divergences.
In general subdivergences must be subtracted.

Dealing With Subdivergences

Vladymirov (1980); Marcus, Sagnotti (1984)

The problem was solve nearly 30 years ago.

Recursively subtract all subdivergences.

IR regulator dependent

reparametrize subintegral

$$\mathcal{S}\left[\int \prod_{i=1}^{L} dp_i\, I\right] \equiv \text{Div}\left[\int \prod_{i=1}^{L} dp_i\, I\right] - \sum_{l=1}^{L-1} \sum_{\substack{l-\text{loop}\\ \text{subloops}}} \text{Div}\left[\int \prod_{j=l+1}^{L} dp'_j\, \mathcal{S}\left[\int \prod_{i=1}^{l} dp'_i\, I\right]\right]$$

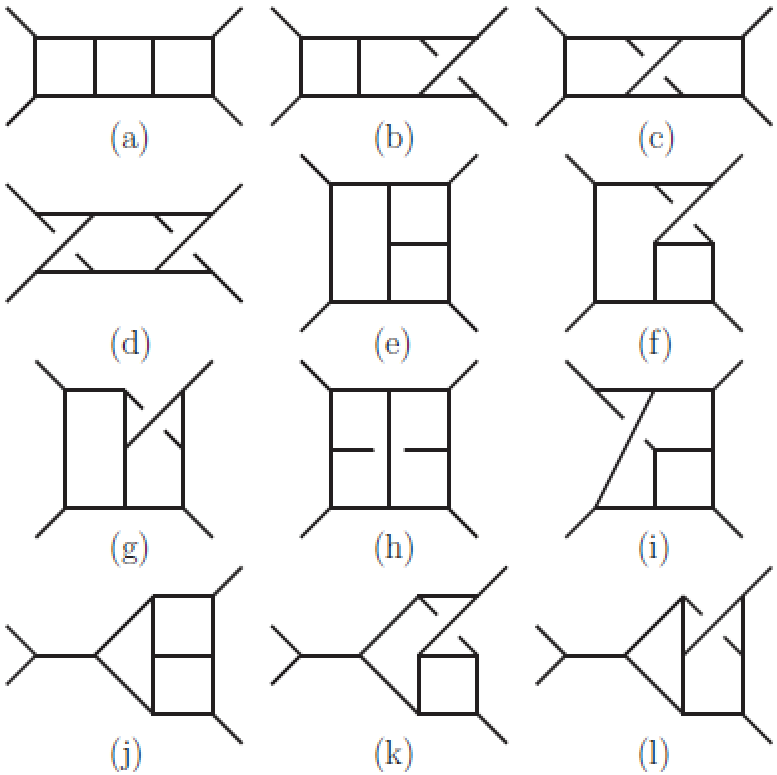
IR Regulator Independent

Nice consistency check: all log(*m*) terms must cancel

Extracting UV divergence in the presence of UV subdivergences and IR divergences is a well understood problem.

The N = 4 Supergravity UV Cancellation

ZB, Davies, Dennen, Huang



Graph	(divergence)$/(\langle 12\rangle^2[34]^2 st A^{\text{tree}}(\frac{\kappa}{2})^8)$
(a)-(d)	0
(e)	$\frac{263}{768}\frac{1}{\epsilon^3}+\frac{205}{27648}\frac{1}{\epsilon^2}+\left(-\frac{5551}{768}\zeta_3+\frac{326317}{110592}\right)\frac{1}{\epsilon}$
(f)	$-\frac{175}{2304}\frac{1}{\epsilon^3}-\frac{1}{4}\frac{1}{\epsilon^2}+\left(\frac{593}{288}\zeta_3-\frac{217571}{165888}\right)\frac{1}{\epsilon}$
(g)	$-\frac{11}{36}\frac{1}{\epsilon^3}+\frac{2057}{6912}\frac{1}{\epsilon^2}+\left(\frac{10769}{2304}\zeta_3-\frac{226201}{165888}\right)\frac{1}{\epsilon}$
(h)	$-\frac{3}{32}\frac{1}{\epsilon^3}-\frac{41}{1536}\frac{1}{\epsilon^2}+\left(\frac{3227}{2304}\zeta_3-\frac{3329}{18432}\right)\frac{1}{\epsilon}$
(i)	$\frac{17}{128}\frac{1}{\epsilon^3}-\frac{29}{1024}\frac{1}{\epsilon^2}+\left(-\frac{2087}{2304}\zeta_3-\frac{10495}{110592}\right)\frac{1}{\epsilon}$
(j)	$-\frac{15}{32}\frac{1}{\epsilon^3}+\frac{9}{64}\frac{1}{\epsilon^2}+\left(\frac{101}{12}\zeta_3-\frac{3227}{1152}\right)\frac{1}{\epsilon}$
(k)	$\frac{5}{64}\frac{1}{\epsilon^3}+\frac{89}{1152}\frac{1}{\epsilon^2}+\left(-\frac{377}{144}\zeta_3+\frac{287}{432}\right)\frac{1}{\epsilon}$
(l)	$\frac{25}{64}\frac{1}{\epsilon^3}-\frac{251}{1152}\frac{1}{\epsilon^2}+\left(-\frac{835}{144}\zeta_3+\frac{7385}{3456}\right)\frac{1}{\epsilon}$

Spinor helicity used to clean up table, but calculation for all states

***All* three-loop divergences cancel completely!**

All subdivergences cancel amongst themselves with uniform mass regulator.

4-point 3-loop N = 4 sugra UV finite contrary to expectations

Tourkine and Vanhove have understood this result by extrapolating from two-loop heterotic string amplitudes.

Explanations?

Key Question:

Is there an ordinary symmetry explanation for this?

Or is something extraordinary happening?

Bossard, Howe and Stelle (2013) showed that 3 loop finiteness can be explained by ordinary superspace +duality symmetries, *assuming* a 16 supercharge off-shell superspace exists.

$$\int d^4x d^{16}\theta \, \frac{1}{\epsilon} \mathcal{L}$$

More θs implies more derivatives in operators

If true, then there is a perfectly good "ordinary" explanation.

Does this superspace exist in *D* = 5 or *D* = 4?

Not easy to construct: A harmonic superspace with infinite number of auxiliary fields and also it must be Lorentz non-covariant.

Explanations?

Prediction of superspace: If you add N = 4 vector multiplets, amplitude should develop no new 2, 3 loop divergences.

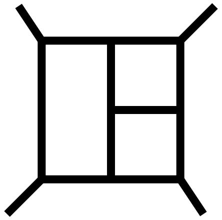
Bossard, Howe and Stelle (2013)

Highly nontrivial prediction because N = 4 supergravity with matter already diverges at one loop! Fischler (1979)

Prediction motivated us to check cases with vector multiplets.

ZB, Davies, Dennen (2013)

Four vector multiplet amplitude diverges at 2, 3 loops!



$$\mathcal{M}^{(3)}(1_{\rm H},2_{\rm H},3_{\rm H},4_{\rm H})\big|_{D=4\,\text{div.}} = 0\,,$$ ← **external graviton multiplet**

$$\mathcal{M}^{(3)}(1_{\rm H},2_{\rm H},3_{\rm V},4_{\rm V})\big|_{D=4\,\text{div.}} = 0\,,$$

$$\mathcal{M}^{(3)}(1_{\rm V},2_{\rm V},3_{\rm V},4_{\rm V})\big|_{D=4\,\text{div.}} = -\frac{1}{(4\pi)^6}\left(\frac{\kappa}{2}\right)^8 (s^2+t^2+u^2)stA^{(0)}_{Q=16} \times \frac{(D_s-2)^2}{4}\left(\frac{D_s-2}{2\epsilon^3}-\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\right)$$

UV divergence

matter multiplet

$n_{\rm V} = D_s - 4$

Similar story in D = 5

Adding vector multiplets causes diverges both at 2, 3 loops.
Desired superspaces hard to construct because they do not exist!
This leaves our supergravity friends puzzled.

What is the new magic?

Some recent papers:

- **Non-renormalization understanding from heterotic string.**

Tourkine and Vanhove (2012)

- **A hidden superconformal symmetry in $N = 4$ supergravity.**

Ferrara, Kallosh and van Proeyen (2012)

- **At 1,2 loops in $D = 4, 5$ a direct link between UV cancellations in nonsupersymmetry YM and finiteness of half-max supergravity**

ZB, Davies and Dennen (2013)

Half-maximal supergravity in $D = 5$ at 2 loop is excellent test case because story is very similar to $D = 4$, $N = 4$ sugra at 3 loops: Mysterious finiteness from standard symmetry considerations.

One-Loop Warmup in Half-Maximal Sugra

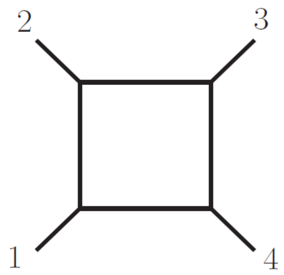
ZB, , Boucher-Veronneau ,Johansson
ZB, Davies, Dennen, Huang

Generic color decomposition:

color factor

$$\mathcal{A}_Q^{(1)} = ig^4 \Big[c_{1234}^{(1)} A_Q^{(1)}(1,2,3,4) + c_{1342}^{(1)} A_Q^{(1)}(1,3,4,2) + c_{1423}^{(1)} A_Q^{(1)}(1,4,2,3) \Big]$$

Q = # supercharges **Q = 0 is pure non-susy YM**



$c_{1234}^{(1)}$ **is color factor of this box diagram**

$$s = (k_1 + k_2)^2$$
$$t = (k_2 + k_3)^2$$

To get Q + 16 supergravity take 2[nd] copy *N* = 4 sYM

***N* = 4 sYM numerators very simple: independent of loop momentum**

$$n_{1234} = n_{1342} = n_{1423} = st A_{Q=16}^{\text{tree}}(1,2,3,4) \qquad c_{1234}^{(1)} \to n_{1234}$$

$$\mathcal{M}_{Q+16}^{(1)} = i \left(\frac{\kappa}{2}\right)^4 st A_{Q=16}^{\text{tree}}(1,2,3,4) \Big[A_Q^{(1)}(1,2,3,4) + A_Q^{(1)}(1,3,4,2) + A_Q^{(1)}(1,4,2,3) \Big]$$

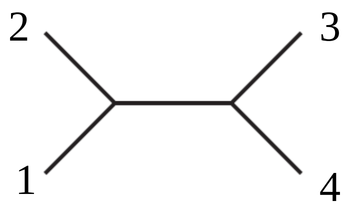
One-loop divergences in pure YM

ZB, Davies, Dennen, Huang

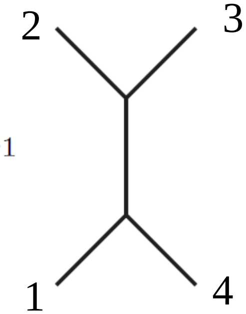
Go to a basis of color factors

Three independent one-loop color tensors

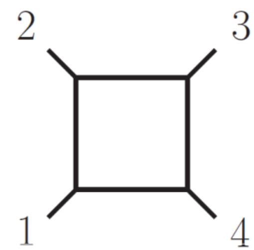
$$b_1^{(0)} = \tilde{f}^{a_1 a_2 b}\tilde{f}^{b a_3 a_4}$$



$$b_2^{(0)} = \tilde{f}^{a_2 a_3 b}\tilde{f}^{b a_4 a_1}$$



$$b_1^{(1)} \equiv c_{1234}^{(1)} = \tilde{f}^{a_1 b_2 b_1}\tilde{f}^{a_2 b_3 b_2}\tilde{f}^{a_3 b_4 b_3}\tilde{f}^{a_4 b_1 b_4}$$



All other color factors expressible in terms of these three:

one-loop color tensor

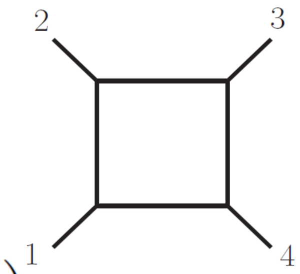
$$\mathcal{A}_Q^{(1)} = i g^4 \Big[b_1^{(1)} \Big(A_Q^{(1)}(1,2,3,4) + A_Q^{(1)}(1,3,4,2) + A_Q^{(1)}(1,4,2,3) \Big) - \frac{1}{2} C_A b_1^{(0)} A_Q^{(1)}(1,3,4,2) - \frac{1}{2} C_A b_2^{(0)} A_Q^{(1)}(1,4,2,3) \Big]$$

tree color tensor

$C_A = 2\, N_c$ for SU(N_c)

One-loop divergences in pure YM

ZB, Davies, Dennen, Huang



In a basis of color factors:

one-loop color tensor

$$\mathcal{A}_Q^{(1)} = ig^4\Big[b_1^{(1)}\Big(A_Q^{(1)}(1,2,3,4) + A_Q^{(1)}(1,3,4,2) + A_Q^{(1)}(1,4,2,3)\Big) - \frac{1}{2}C_A b_1^{(0)} A_Q^{(1)}(1,3,4,2) - \frac{1}{2}C_A b_2^{(0)} A_Q^{(1)}(1,4,2,3)\Big]$$

tree color tensor

Q **supercharges (mainly interested in *Q* = 0)**

$D = 4$: F^2 is only allowed counterterm by renormalizability
1-loop color tensor *not* allowed.

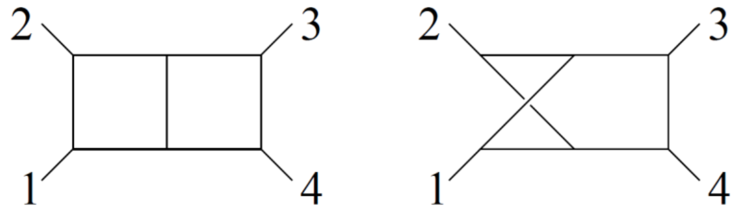
$D = 6$: F^3 counterterm: 1-loop color tensor again *not* allowed.

$$F^3 = f^{abc} F^{a\mu}_{\nu} F^{b\nu}_{\sigma} F^{c\sigma}_{\mu}$$

$$A_Q^{(1)}(1,2,3,4) + A_Q^{(2)}(1,3,4,2) + A_Q^{(1)}(1,4,2,3)\Big|_{D=4,6\ \text{div.}} = 0$$

$$\boxed{M_{Q+16}^{(1)}(1,2,3,4)\Big|_{D=4,6\ \text{div.}} = 0}$$

Two Loop Half Maximal Sugra in $D = 5$



ZB, Davies, Dennen, Huang

$$\mathcal{A}_Q^{(2)} = -g^6 \Big[c^{\rm P}_{1234} A^{\rm P}_Q(1,2,3,4) + c^{\rm P}_{3421} A^{\rm P}_Q(3,4,2,1) + c^{\rm NP}_{1234} A^{\rm NP}_Q(1,2,3,4) + c^{\rm NP}_{3421} A^{\rm NP}_Q(3,4,2,1) + \text{ cyclic} \Big]$$

$D = 5$ F^3 counterterm: 1,2-loop color tensors forbidden!

1) **Go to color basis.**
2) **Demand no forbidden color tensors in pure YM divegence.**
3) **Replace color factors with kinematic numerators.**

gravity $$\mathcal{M}^{(2)}_{16+Q}(1,2,3,4)\Big|_{D=5\,\text{div.}} = 0$$

Half-maximal supergravity four-point divergence vanishes because forbidden color tensor cancels in pure YM theory

Two Loop *D* = 5 UV Magic

ZB, Davies, Dennen

At least for 2 loops we have identified the source of unexpected UV cancellations in half-maximal supergravity:

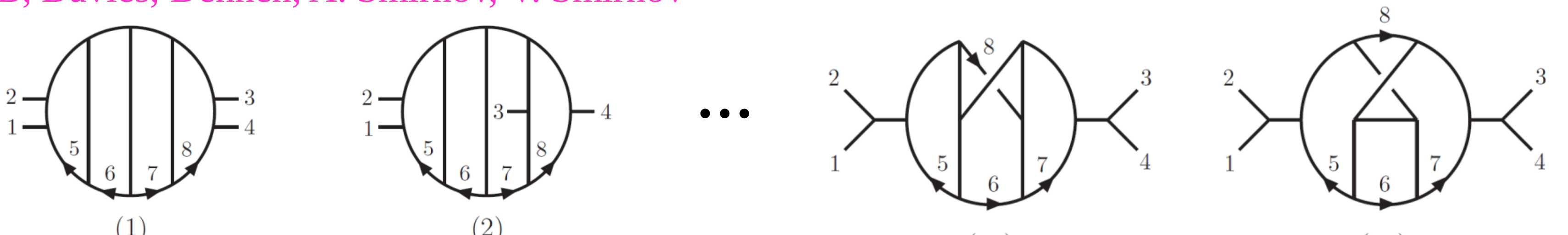
It is the *same* magic found by 't Hooft and Veltman 40 years ago preventing forbidden divergences appearing in ordinary non-susy gauge theory!

Completely explains the *D* = 5 two-loop half maximal sugra case, which still remains mysterious from standard supergravity viewpoint.

Still studying the more complicated higher-loop cases.

Four-loop $N = 4$ Sugra in Progress

ZB, Davies, Dennen, A. Smirnov, V. Smirnov


(1) (2) … (81) (82)

- **82 nonvanishing diagram types using $N = 4$ sYM BCJ form.**

ZB, Carrasco, Dixon, Johansson, Roiban,

- **Highly nontrivial $\sim 10^9$ terms. Complexity comes from $N = 0$ Yang-Mills side of double copy.**

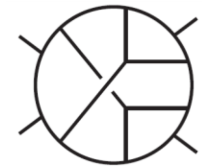
- **Mostly done, perhaps another month to finish.**

Does pure $N = 4$ supergravity diverge at four loops or is it finite?

- **If it is finite, we should be able to link it to YM cancellations.**
- **If it is divergent, what is the structure?**

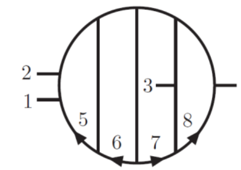
Important guidance for $N > 4$ (especially $N = 8$) supergravity theories which are the more likely theories to be UV finite.

Ongoing Work and Next Steps



* **Five loops: Compute the coefficient of the D^8R^4 five-loopcounterterm of N = 8 supergravity in D = 24/5.**

ZB, Carrasco, Johansson, Roiban



* **Four loops: find the coefficient of the D^2R^4 four loop counterterm in N=4 supergravity in D=4.**

ZB, Davies, Dennen, Smirnov[2]

* **Better means for finding BCJ representations of gauge theory.**

Bjerrum-Bohr, Dennen, Monteiro and O'Connell (2013)

* **Understanding duality between color and kinematics in string theory.** Mafra (2009); Tye and Zhang (2010); Bjerrum-Bohr, Damgaard, Sodergaard, Vanhove (2010); Mafra, Schlotterer and Steiberger (2011)

* **Finding the connection to Lagrangian description of gravity.**

ZB, Dennen, Huang, Kiermaier (2010); Tolotti and Weinzierl (2013)

* **Studying the duality in D = 3 for BLG and ABJM theories.**

Bargheer, He, and McLoughlin (2012); Huang and Johansson (2012, 2013)

* **Finding the underlying symmetry responsible for duality between color and kinematics.** Self dual: O'Connell and Monteiro (2011)

Summary

- A new duality conjectured between color and kinematics.
- When duality between color and kinematics manifest, gravity loop integrands follow *trivially* from gauge-theory ones.
- Surprisingly good UV behavior of supergravity uncovered.
- N = 4, D =4 sugra with no vector multiplets has no 3-loop 4-point divergence, contrary to standard symmetry considerations. Duality between color and kinematics needs to be taken into account.
- For D = 5, 2 loops we know precisely origin of the "magical UV cancellations": it is standard magic that restricts counterterms of nonsusy YM.

The duality between color and kinematics is revealing a remarkably close connection between perturbative gravity and gauge theories, including their UV properties. In the coming years we should expect many new developments.