$\nabla \cdot B = 0$ Condition in numerical codes

$\nabla \cdot B$ Condition

- Numerically, the solenoidal condition is fulfilled only at the truncation level and non-solenoidal components may be generated during the evolution;

- Magnetic monopoles causes unphysical accelerations of the plasma in the direction parallel to the field lines1;

- $\nabla \cdot B = 0$ cannot be satisfied for any type of discretization;

- Robustness of a method can be assessed on practical basis by extensive numerical testing.

[1] BrackBill & Barnes (1980)

Cell Centered vs Staggered

- *Cell Centered* Methods: magnetic field treated as volume average over the zone:

  - Projection method (BrackBill & Barnes, 1980)
  - Powell's 8-wave formulation (Powell 1994, Powell et al. 1999)
  - Field CD (Toth 2000)
  - Divergence cleaning (Dedner 2002)

- *Staggered* (*face-centered*):

  - magnetic field has a staggered representation where field components live on the face they are normal to (Evans & Hawley 1988).

Projection Method

- Correct the magnetic field after the time step is completed;
- Starting from $\boldsymbol{B}^n$ we obtain $\boldsymbol{B}^*$ which is not divergence-free.

- Then, using Hodge-projection: $\boldsymbol{B}^* = \nabla \times \boldsymbol{A} + \nabla\phi$
- Taking the divergence of both sides gives

$$\nabla^2\phi = \nabla \cdot \boldsymbol{B}^*$$

  which can be solved for the scalar function ϕ.
- The magnetic field is then corrected as $\boldsymbol{B}^{n+1} = \boldsymbol{B}^* - \nabla\phi$
- Cons: requires the solution of a Poisson equation.

Powell's Method (8 wave)

- Start from the primitive form of the MHD equation without discarding the $\nabla\cdot\mathbf{B}$ term → quasi-conservative form

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0$$

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla\cdot\left(\rho\mathbf{u}\mathbf{u} + \left(p + \frac{\mathbf{B}\cdot\mathbf{B}}{2\mu_0}\right)\mathbf{I} - \frac{\mathbf{B}\mathbf{B}}{\mu_0}\right) = -\frac{1}{\mu_0}\mathbf{B}\nabla\cdot\mathbf{B}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla\cdot(\mathbf{u}\mathbf{B} - \mathbf{B}\mathbf{u}) = -\mathbf{u}\nabla\cdot\mathbf{B}$$

$$\frac{\partial E}{\partial t} + \nabla\cdot\left[\left(E + p + \frac{\mathbf{B}\cdot\mathbf{B}}{2\mu_0}\right)\mathbf{u} - \frac{1}{\mu_0}(\mathbf{u}\cdot\mathbf{B})\mathbf{B}\right] = -\frac{1}{\mu_0}(\mathbf{u}\cdot\mathbf{B})\nabla\cdot\mathbf{B}$$

- Just use vector identities:

$(\nabla\times\mathbf{B})\times\mathbf{B} = (\mathbf{B}\cdot\nabla)\mathbf{B} - (\nabla\mathbf{B})\cdot\mathbf{B} = \nabla\cdot(\mathbf{B}\mathbf{B}) - (\nabla\cdot\mathbf{B})\mathbf{B} - (\nabla\mathbf{B})\mathbf{B}$

Powell's Method (8 wave)

- The non-conservative form is discretized by introducing an 8^{th} wave in the Riemann solver associated with jumps in the normal component of magnetic field.

- With the non-conservative formulation $\nabla\cdot\mathbf{B}$ errors generated by the numerical solution do not accumulate at a fixed grid point but, rather, propagate together with the flow.

- For many problems the 8-wave formulation works.

- However, in problems containing strong shocks, the non-conservative source terms can produce incorrect jump conditions and consequently the scheme can produce incorrect results

Hyperbolic Divergence Cleaning

- The divergence constraint is coupled to Faraday's law by introducing a new scalar field function ψ (generalized Lagrangian multiplier).
- The second and third Maxwell's equations are thus replaced by

$$\begin{cases} \nabla \cdot \mathbf{B} = 0, \\ \dfrac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \end{cases} \quad \Rightarrow \quad \begin{cases} \mathcal{D}(\psi) + \nabla \cdot \mathbf{B} = 0, \\ \dfrac{\partial \mathbf{B}}{\partial t} + \nabla \psi = \nabla \times (\mathbf{v} \times \mathbf{B}), \end{cases}$$

where $\mathcal{D}$ is a linear differential operator.

- An efficient method may be obtained by choosing $\mathcal{D}(\psi) = c_h^{-2}\partial_t \psi + c_p^{-2}\psi$ yielding a mixed hyperbolic/parabolic correction.
- Direct manipulation leads to the telegraph equation:

$$\frac{\partial^2 \psi}{\partial t^2} + \frac{c_h^2}{c_p^2}\frac{\partial \psi}{\partial t} = c_h^2 \Delta \psi$$

→ errors are propagated to the domain at finite speed c_h and damped at the same time.

Hyperbolic Cleaning: GLM-MHD Equations

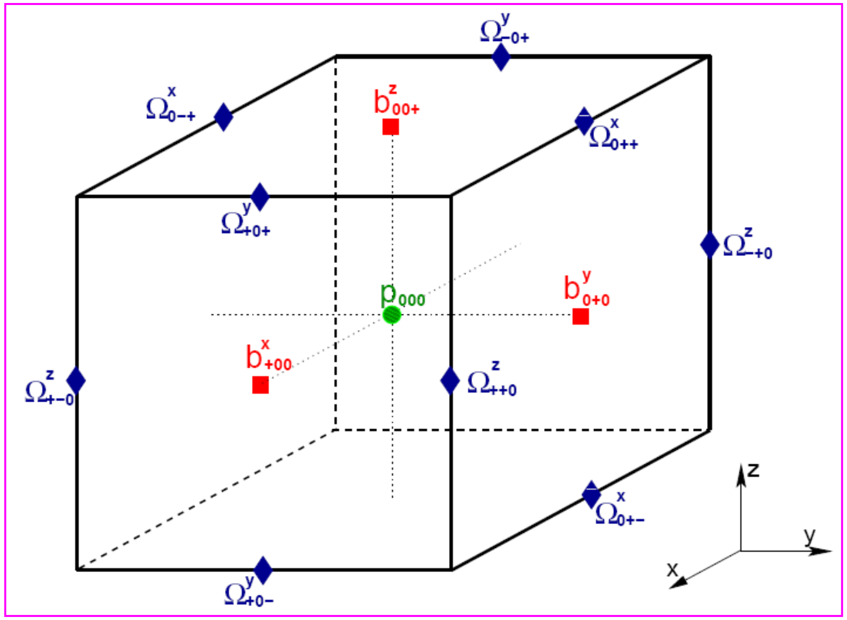
- The resulting system is called the generalized Lagrange multiplier (GLM-MHD) and includes 9 evolution equation:

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \\
&\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot \left[\rho \mathbf{v}\mathbf{v}^T - \mathbf{B}\mathbf{B}^T + \mathsf{I} \left(p + \frac{\mathbf{B}^2}{2} \right) \right] = 0, \\
&\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{v}\mathbf{B}^T - \mathbf{B}\mathbf{v}^T) + \nabla \psi = 0, \\
&\frac{\partial E}{\partial t} + \nabla \cdot \left[\left(E + p + \frac{\mathbf{B}^2}{2} \right) \mathbf{v} - (\mathbf{v} \cdot \mathbf{B})\mathbf{B} \right] = 0, \\
&\frac{\partial \psi}{\partial t} + c_h^2 \nabla \cdot \mathbf{B} = -\frac{c_h^2}{c_p^2} \psi,
\end{aligned}
$$

- Divergence errors propagate with speed c_h even at stagnation points where $v = 0$.

Constrained Transport

- Staggered magnetic field treated as an area-weighted average on the zone face.



- Thus, different magnetic field components live at different location;

- A discrete version of Stoke's theorem is used to update them:

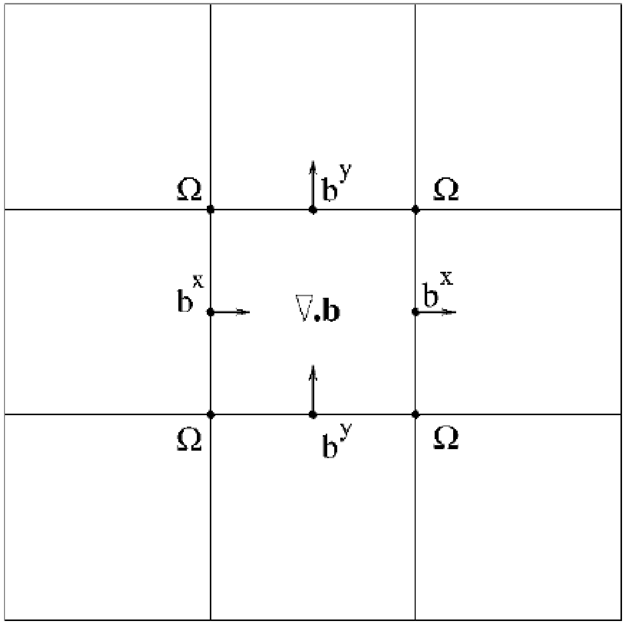
$$\int \left(\frac{\partial \boldsymbol{b}}{\partial t} + \nabla \times \boldsymbol{\mathcal{E}} \right) \cdot d\boldsymbol{S}_d = 0 \quad \Longrightarrow \quad \frac{db_{x_d}}{dt} + \frac{1}{S_d} \oint \boldsymbol{\mathcal{E}} \cdot d\boldsymbol{l} = 0$$

Constrained Transport in 2D

- In 2D, the emf is placed at cell corners.
- The discrete Stoke's theorem becomes

$$b^{x,n+1}_{j+1/2,k} = b^{x,n}_{j+1/2,k} - \Delta t \frac{\Omega_{j+1/2,k+1/2} - \Omega_{j+1/2,k-1/2}}{\Delta y}$$

$$b^{y,n+1}_{j,k+1/2} = b^{y,n}_{j,k+1/2} + \Delta t \frac{\Omega_{j+1/2,k+1/2} - \Omega_{j-1/2,k+1/2}}{\Delta x}$$

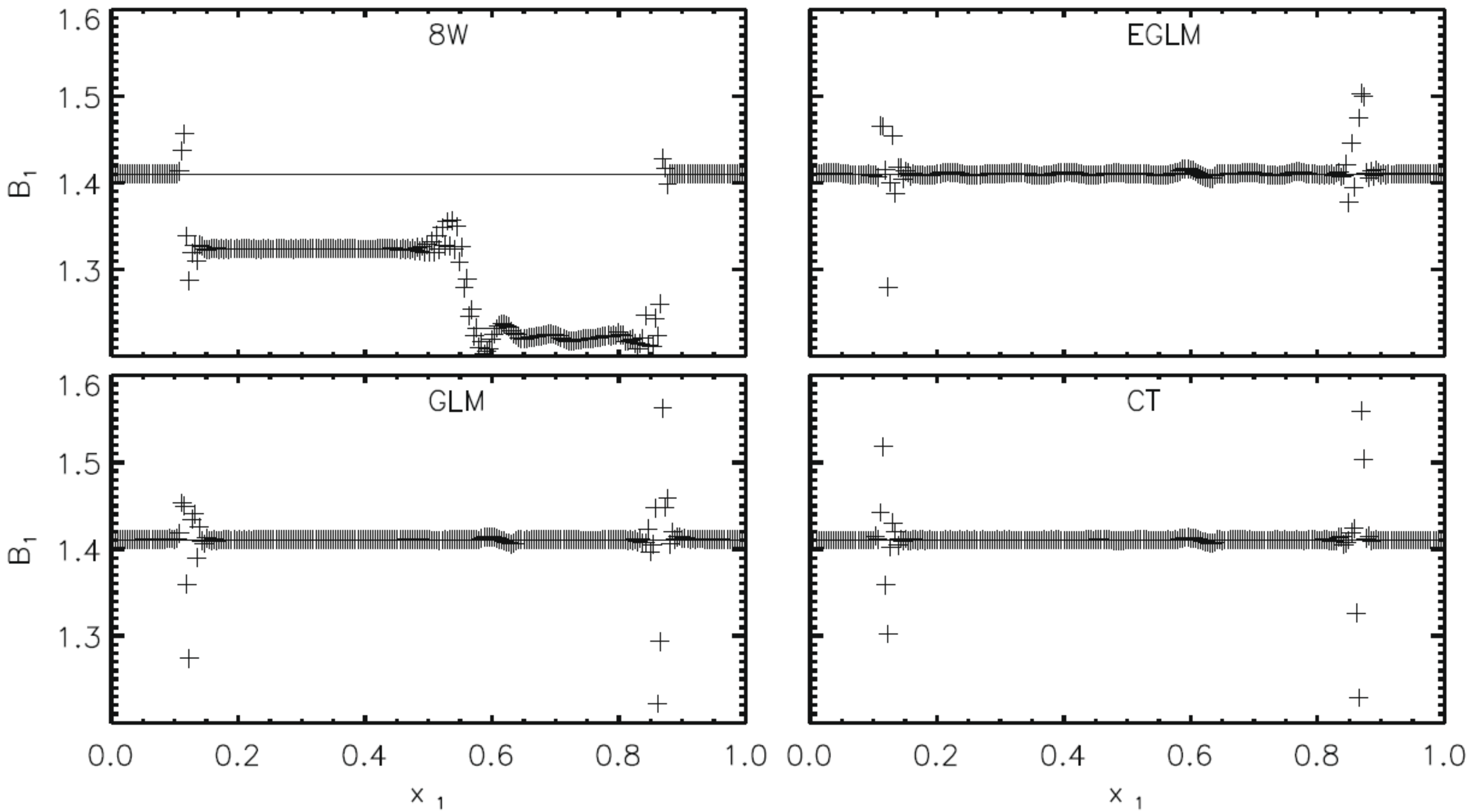


- It is easy to show that the numerical divergence of ***b*** defined by

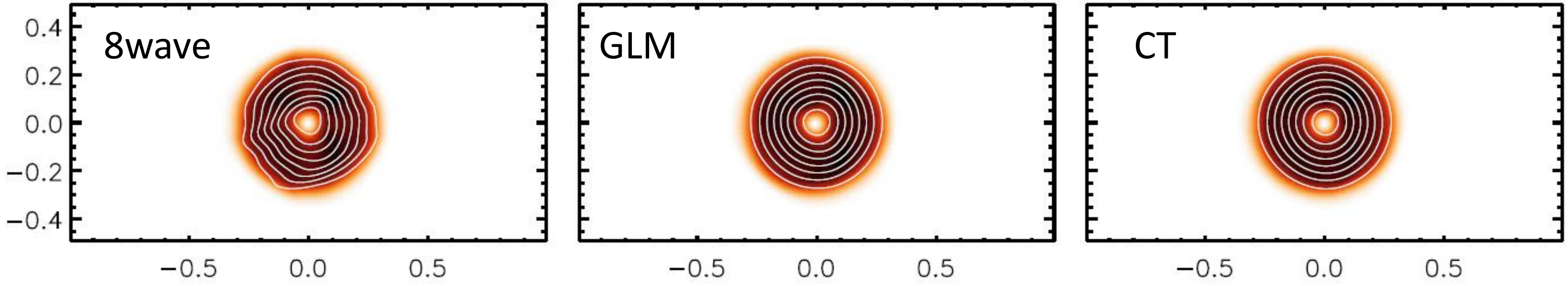
$$(\nabla \cdot \boldsymbol{b})_{j,k} = \frac{b^x_{j+1/2,k} - b^x_{j-1/2,k}}{\Delta x} + \frac{b^y_{j,k+1/2} - b^y_{j,k-1/2}}{\Delta y}$$

does not change due to perfect cancellation of term to machine accuracy.

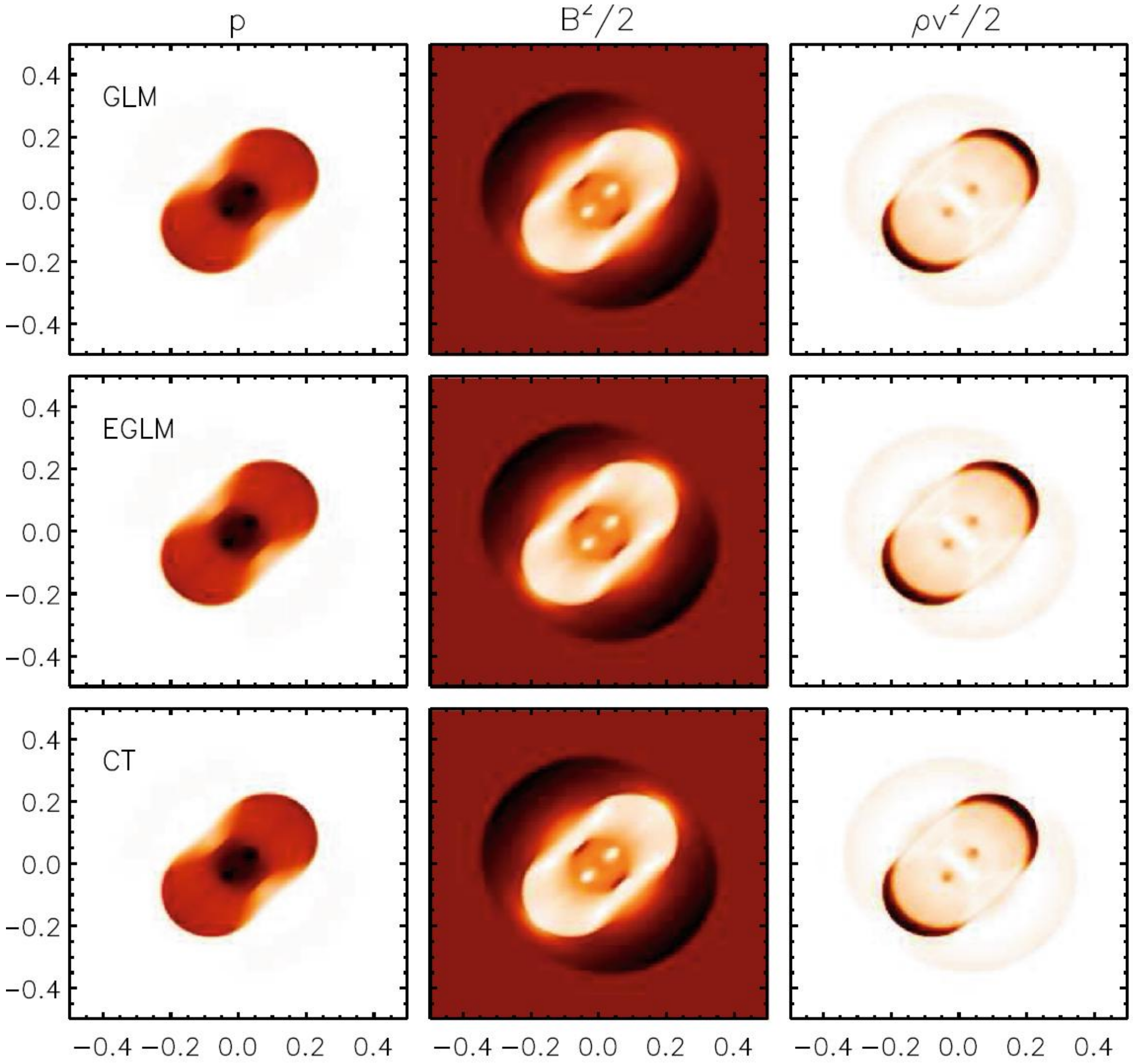
Comparison: rotated shock tube



Comparison: field loop advection



Comparison: blast wave



$\nabla\cdot B$ Condition

	Cell-Centered	*Staggered*
Pros	▪ keeps "native" code discretization ▪ better for I.C. and B.C. ▪ easier to extend to AMR grids ▪ Can be used in dimensionally split schemes	▪ keep $\nabla\cdot B = 0$ to machine accuracy ▪ elegant and consistent discretization ▪ lead to perfectly consistent, well posed Riemann problems
Cons	▪ require monopole control algorithm ▪ 8 wave / Projection: ➢ Jump of B at face → Riemann problem ➢ Break conservation (??)	▪ tricky extension to AMR ▪ more work on B.C. and I.C. ▪ Require solution of multi D Riemann problems (UCT, L. Del Zanna & Londrillo)

Comparing schemes: Axisymmetric jet

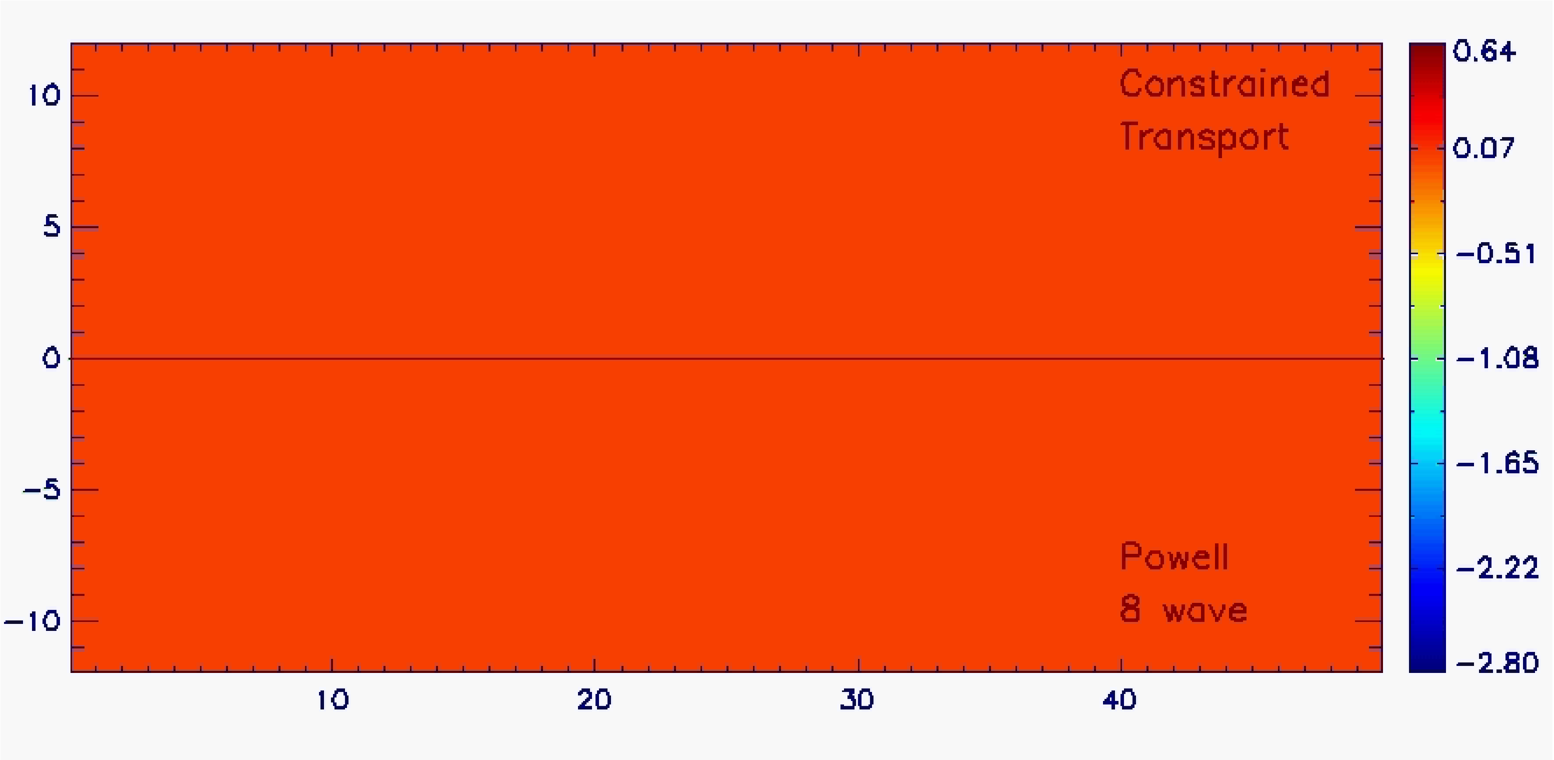
Jet Parameters:

$$\frac{\rho_b}{\rho_m} = 10^{-2}\,, \quad v_b = 10\,, \quad M_b \equiv \frac{v_b}{c_s} = 6\,, \quad \beta = \frac{B_z^2}{2p_b} = 1$$

Resolution: 240x1000

Fully Unsplit, staggered mag. field

Dim. Split; cell-centered (8wave)



Summary

- CT most consistent formulation for finite volume Godunov schemes;
- Projection method can be accurate but expensive;
- 8 wave prone to large errors in proximity of oblique shocks;
- GLM competitive alternative to CT;