Nonlinear evolution of the vertical shear instability in accretion discs

Richard Nelson

Queen Mary, University of London

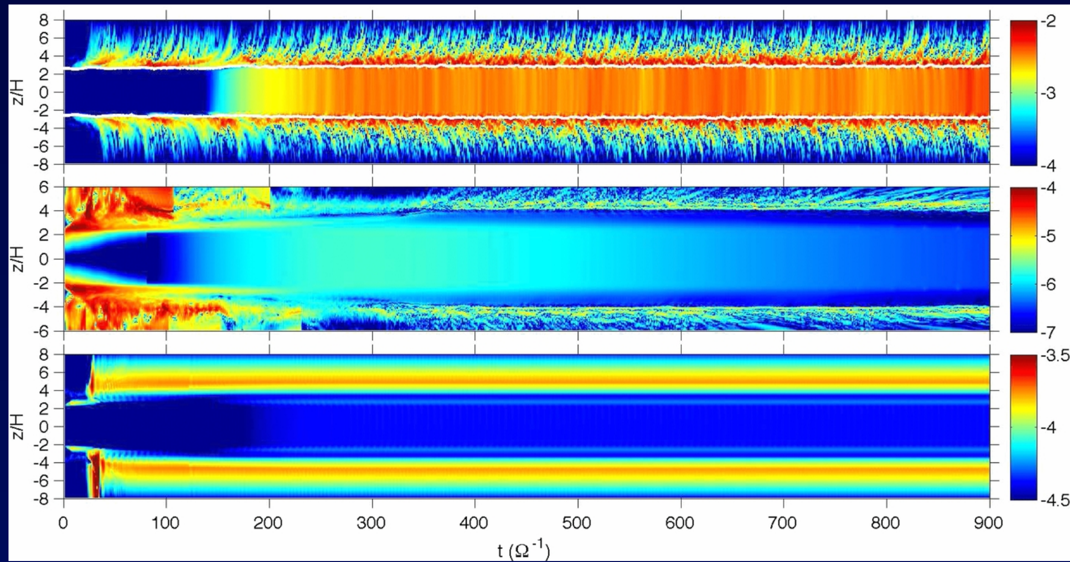
Collaborators: Oliver Gressel (Copenhagen), Orkan Umurhan (NASA-AMES),

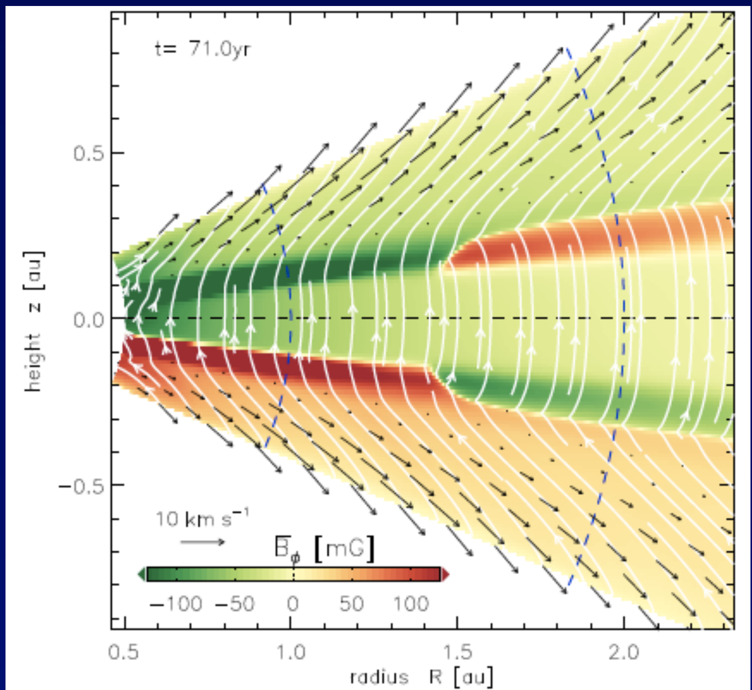
Samuel Richard (QMUL)

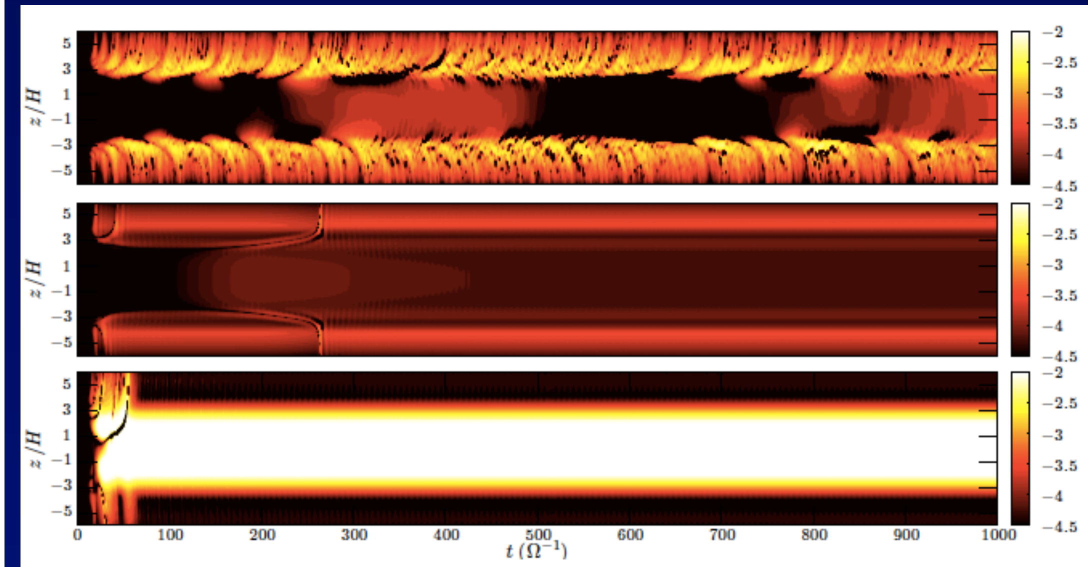
Queen Mary
University of London

Ohmic resistivity + ambipolar diffusion + Hall EMFs with vertical B-field

➔ laminar disc with angular momentum transport occurring through magnetised wind and large-scale coherent magnetic stresses (Bai & Stone 2013; Kunz & Lesur 2013; Lesur, Kunz & Fromang 2014)



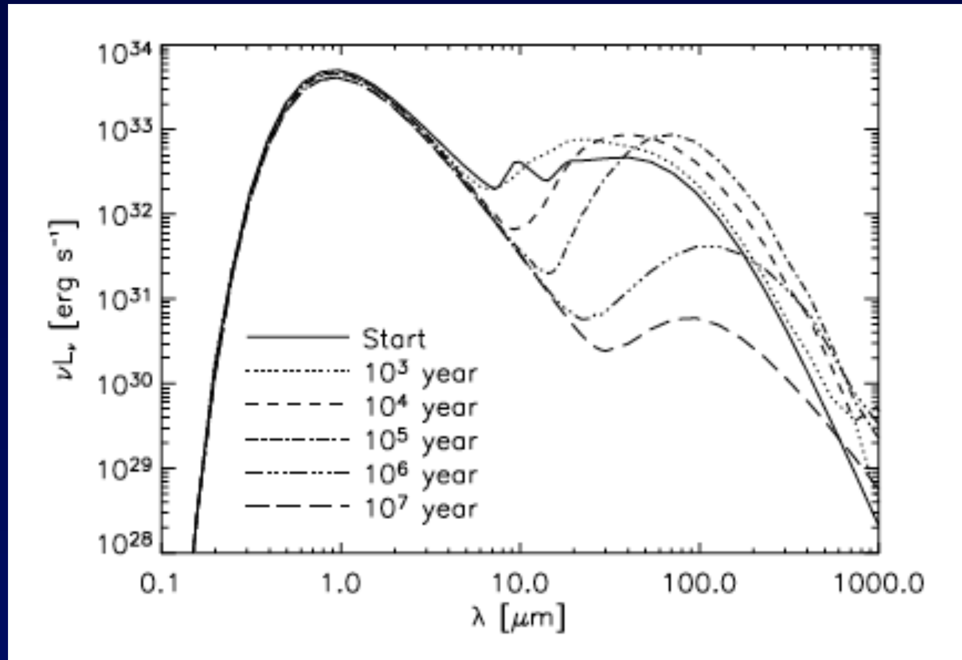


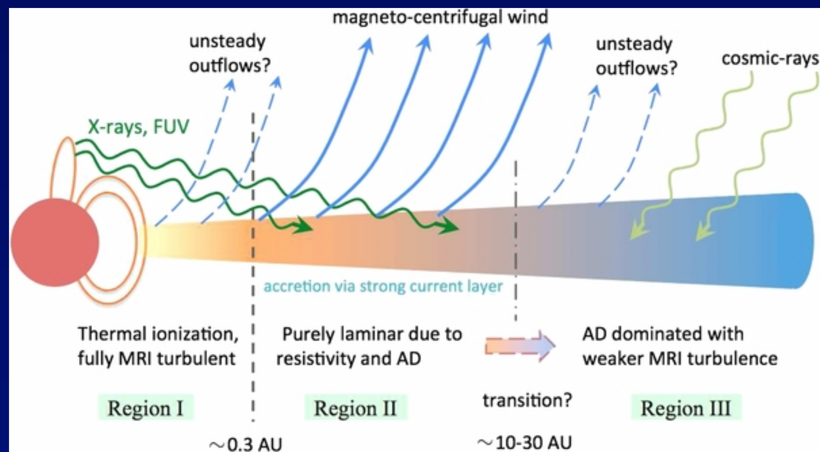


Direct evidence of turbulence in outer disc:
CO and CS linewidths of ~300 m/s (HD163296), ~40 m/s (TW Hydra) and ~ 100 m/s (DM Tau) observed (Hughes et al 2011; Guilloteau et al 2012)

Indirect evidence for turbulence in discs:
Models of dust growth in laminar discs fail to reproduce SEDs of observed T Tauri stars without small particle replenishment and turbulent mixing (Dullemond & Dominik 2005)

Protoplanetary discs appear to be turbulent – hydrodynamic origin?





Hydrodynamic stability in a disc

Quasi-keplerian discs with Ω=Ω(R) stable according to Rayleigh criterion

$$\frac{\mathrm{d}j^2}{\mathrm{d}r} > 0$$

Compressible adiabatic fluid with Ω=Ω(R,Z) stable when Solberg-Hoiland criteria are satisfied

$$\frac{1}{R^3}\frac{\partial j^2}{\partial R} + \frac{1}{\rho C_p}\left(\left|\frac{\partial P}{\partial R}\right|\frac{\partial S}{\partial R} + \left|\frac{\partial P}{\partial Z}\right|\frac{\partial S}{\partial Z}\right) > 0$$

$$\frac{\partial j^2}{\partial R}\frac{\partial S}{\partial Z} - \frac{\partial j^2}{\partial Z}\frac{\partial S}{\partial R} > 0.$$

In protoplanetary discs expect: radial entropy gradients to provide destabilizing influence
vertical entropy gradients to provide stabilizing influence

Rotating fluid in which thermal diffusion occurs much more rapidly than viscous diffusion is unstable to Goldreich-Schubert-Fricke instability for perturbation modes that satisfy

$$\frac{\partial j^2}{\partial R} - \frac{k_R}{k_Z}\frac{\partial j^2}{\partial Z} < 0.$$

Goldreich & Schubert (1967, Fricke (1968),
Urpin & Brandenburg (1998), Urpin (2003),
Arlt & Urpin (2004),
Nelson, Gressel & Umurhan (2013)

$$\frac{\partial j}{\partial Z} \simeq q\left(\frac{H}{R}\right)\frac{\partial j}{\partial R}$$

➔ thin disc with radial power-law T profile

Expect unstable modes in thin accretion discs to have k_R / k_Z ≈ R / H
i.e. radial wavelengths smaller than vertical wavelengths by factor H / R

Disc models and simulations

2D axisymmetric and 3D disc models run with NIRVANA and NIRVANA-III

Density, temperature and angular velocity profiles

$$T(R) = T_0 \left(\frac{R}{R_0}\right)^q$$

$$\rho_{\rm mid}(R) = \rho_0 \left(\frac{R}{R_0}\right)^p$$

$$\rho(R,Z) = \rho_0 \left(\frac{R}{R_0}\right)^p \exp\left(\frac{GM}{c_s^2}\left[\frac{1}{\sqrt{R^2+Z^2}} - \frac{1}{R}\right]\right),$$

$$\Omega(R,Z) = \Omega_K \left[(p+q)\left(\frac{H}{R}\right)^2 + (1+q) - \frac{qR}{\sqrt{R^2+Z^2}}\right]^{1/2}$$

$$P = K_s \rho^\gamma$$

$$K_s(R) = K_0 \left(\frac{R}{R_0}\right)^s$$

$$K_s = c_{\rm v}(\gamma - 1)\exp\left(\frac{S}{c_{\rm v}}\right)$$

$$\rho(R,Z) = \left(\rho_{\rm mid}^{(\gamma-1)} + \frac{(\gamma-1)}{\gamma}\frac{GM}{K_s}\left[\frac{1}{\sqrt{R^2+Z^2}} - \frac{1}{R}\right]\right)^{1/(\gamma-1)}$$

$$\Omega(R,Z) = \Omega_K \left[\frac{p}{\mathcal{M}_{\rm mid}^2} + \frac{s}{\gamma \mathcal{M}^2} + (1+s) - \frac{sR}{\sqrt{R^2+Z^2}}\right]^{1/2}$$

Thermal evolution: Isothermal, adiabatic or Newtonian cooling:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{(T - T_0)}{\tau_{\rm relax}}$$

See Nelson, Gressel & Umurhan (2013) for details

Disc model stability according to Solberg-Hoiland criteria

1 $$\frac{1}{R^3}\frac{\partial j^2}{\partial R} + \frac{1}{\rho C_p}\left(\left|\frac{\partial P}{\partial R}\right|\frac{\partial S}{\partial R} + \left|\frac{\partial P}{\partial Z}\right|\frac{\partial S}{\partial Z}\right) > 0$$

2 $$\frac{\partial j^2}{\partial R}\frac{\partial S}{\partial Z} - \frac{\partial j^2}{\partial Z}\frac{\partial S}{\partial R} > 0.$$

$$\frac{\partial j}{\partial Z} \simeq q\left(\frac{H}{R}\right)\frac{\partial j}{\partial R}$$

$$q\left(\frac{H}{R}\right)\frac{\partial S}{\partial R} > \frac{\partial S}{\partial Z}$$

➔ Instability condition for thin disc (H/R<<1)

$$T(R) = T_0\left(\frac{R}{R_0}\right)^q$$

$$\rho_{\rm mid}(R) = \rho_0\left(\frac{R}{R_0}\right)^p$$

$$S = C_v \log(T\rho^{1-\gamma})$$

Models with $T \sim R^q$ (p=-1.5, q=-1, -1/2, -1/4):
dS/dZ > 0 – stabilizing
q=-1 disc has dS/dR < 0 – destabilizing but weak compared to dj^2/dR term

Stable according to eqn 2
All $T \sim R^q$ runs stable according to S-H criteria

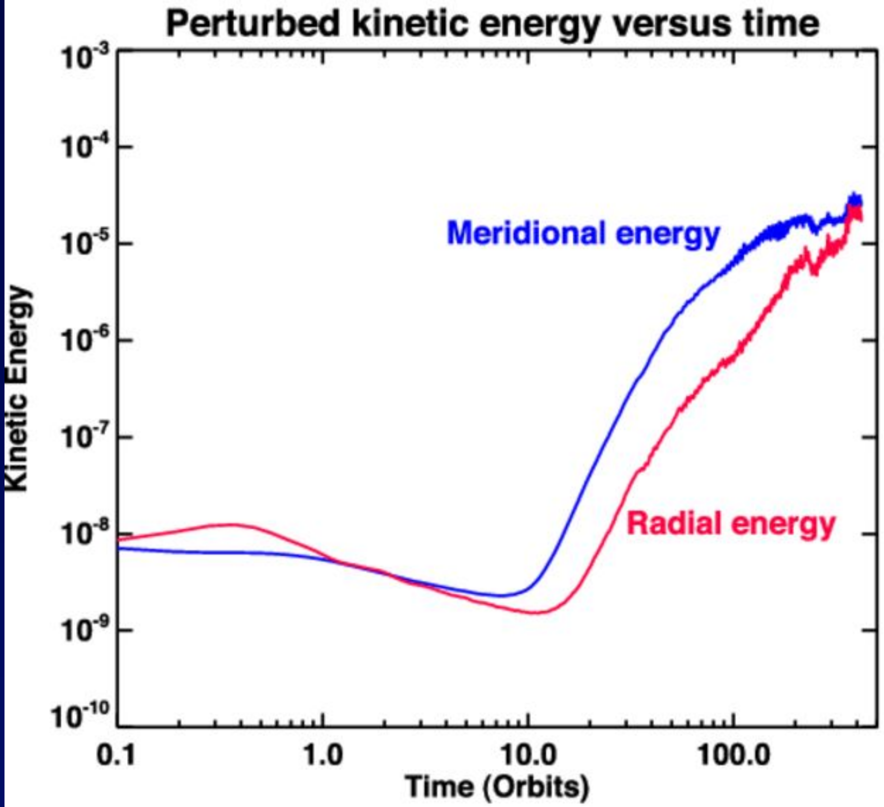
$$K_s(R) = K_0\left(\frac{R}{R_0}\right)^s$$

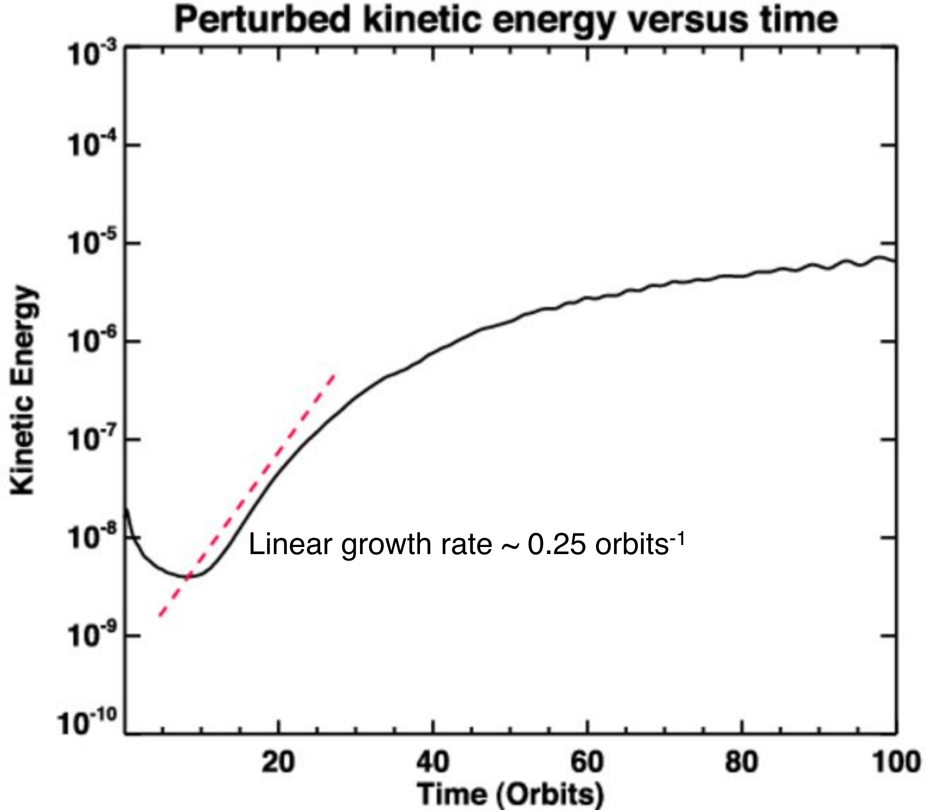
$$K_s = c_v(\gamma - 1)\exp\left(\frac{S}{c_v}\right)$$

Models with $K_s \sim R^s$ (p=0, s=-1):
dS/dZ = 0
dS/dR < 0 - unstable according to eqn 2

p=-3/2, s=0
dS/dZ = dS/dR = 0 – marginally stable

Fiducial locally-isothermal model





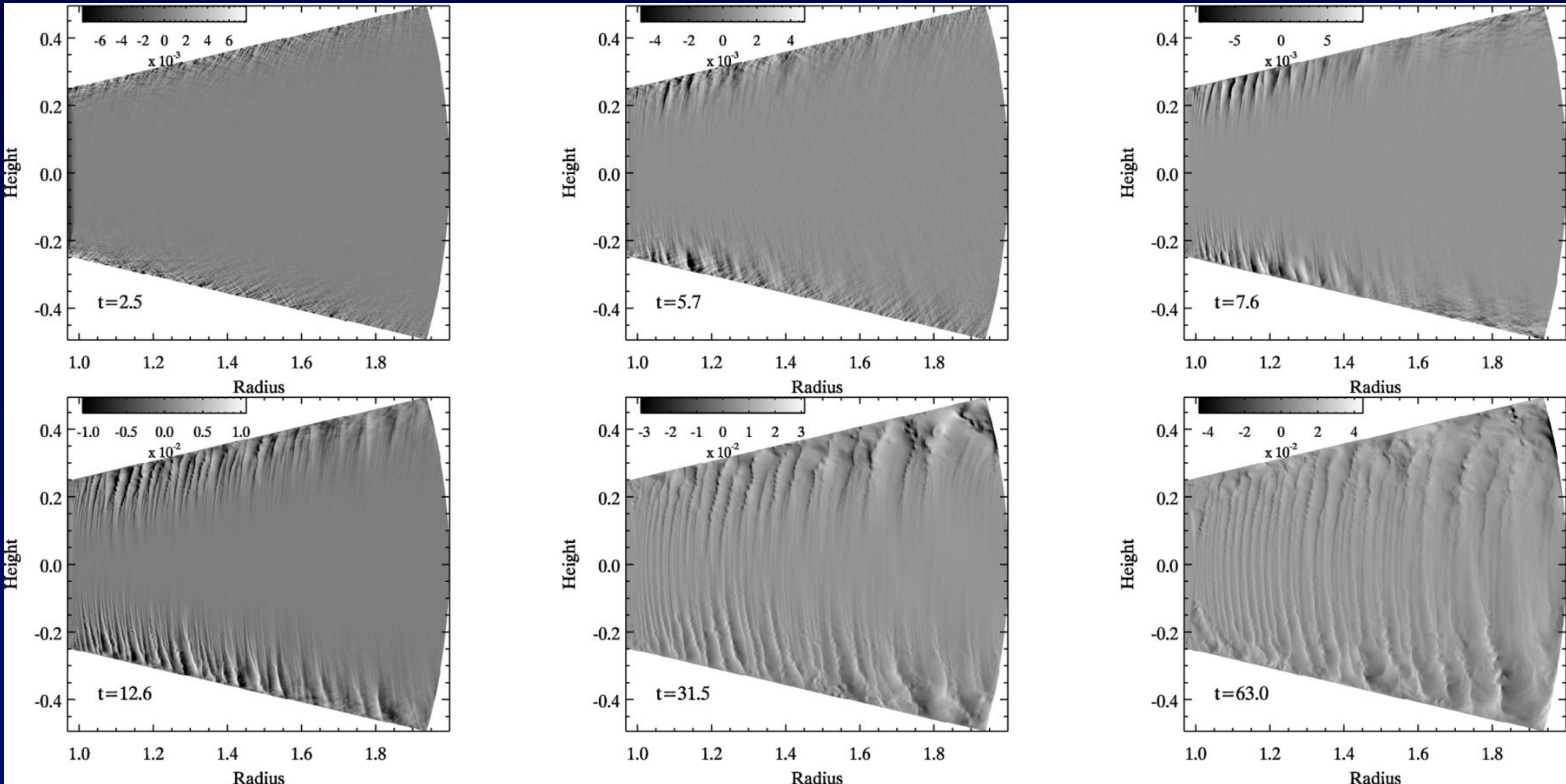
$$T(R) = T_0 \left(\frac{R}{R_0}\right)^q$$

q=-1.5

$$\rho_{\rm mid}(R) = \rho_0 \left(\frac{R}{R_0}\right)^p$$

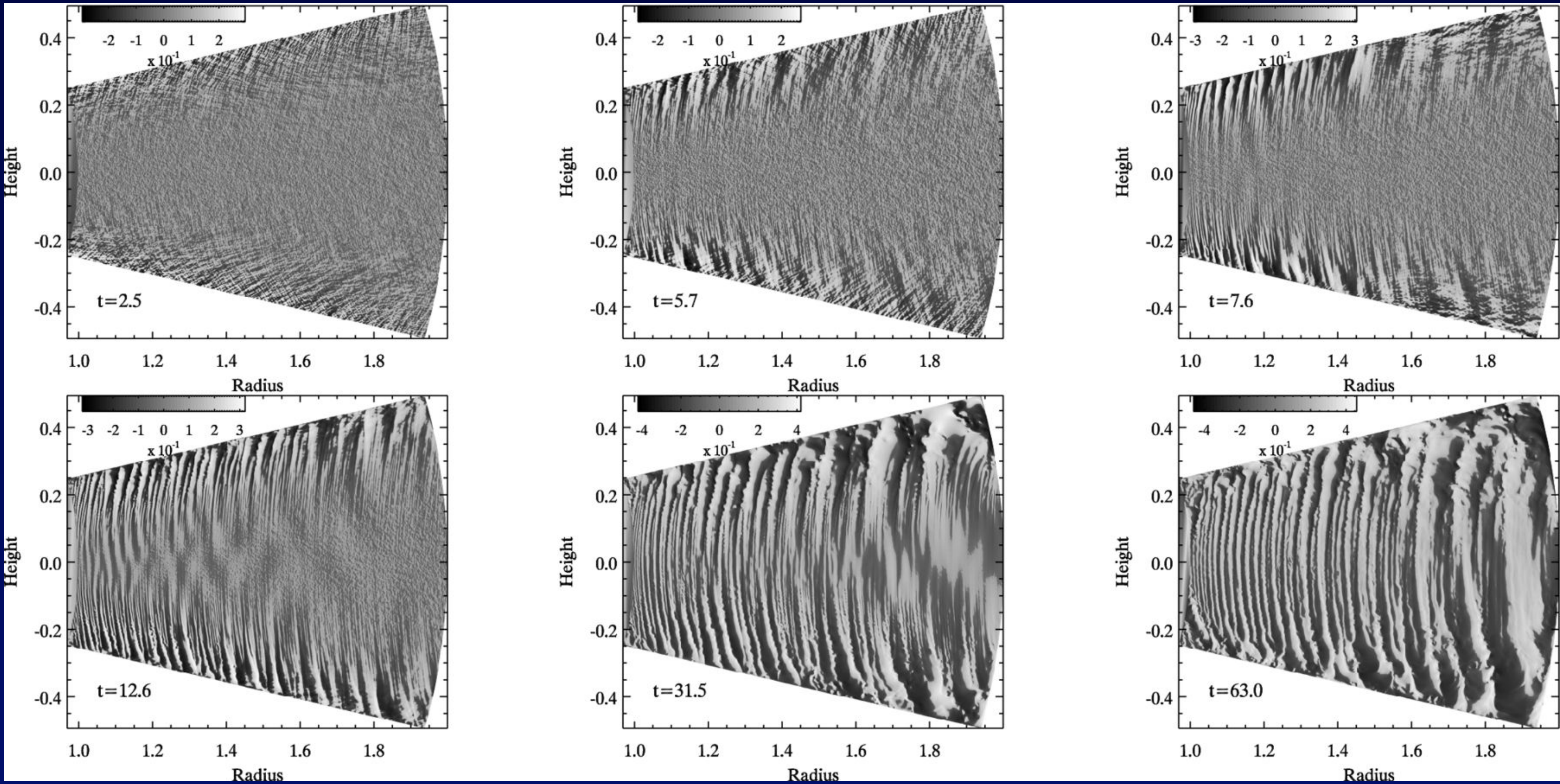
p=-1

$$e_\theta = \frac{1}{2}\int_V \rho v_\theta^2 dV, \quad e_r = \frac{1}{2}\int_V \rho v_r^2 dV.$$



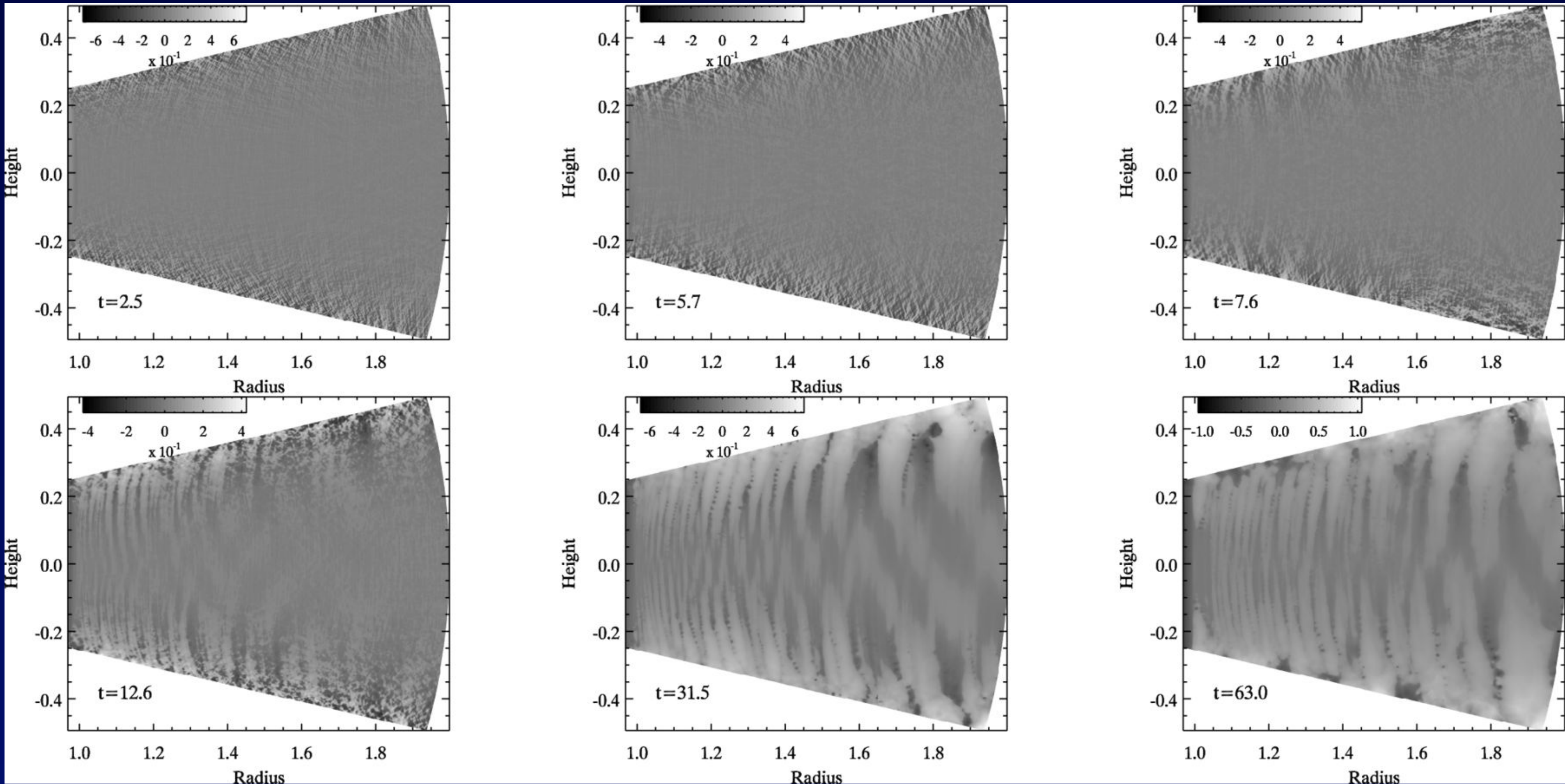
Numerical set-up
$N_R \times N_\vartheta = (1328 \times 1000)$
±5 vertical scale heights
Reflecting B.C.s
Seed noise $\delta v = 0.01\ c_s$

Vertical velocity perturbations
(linear grey-scale)
Note $k_R >> k_Z$



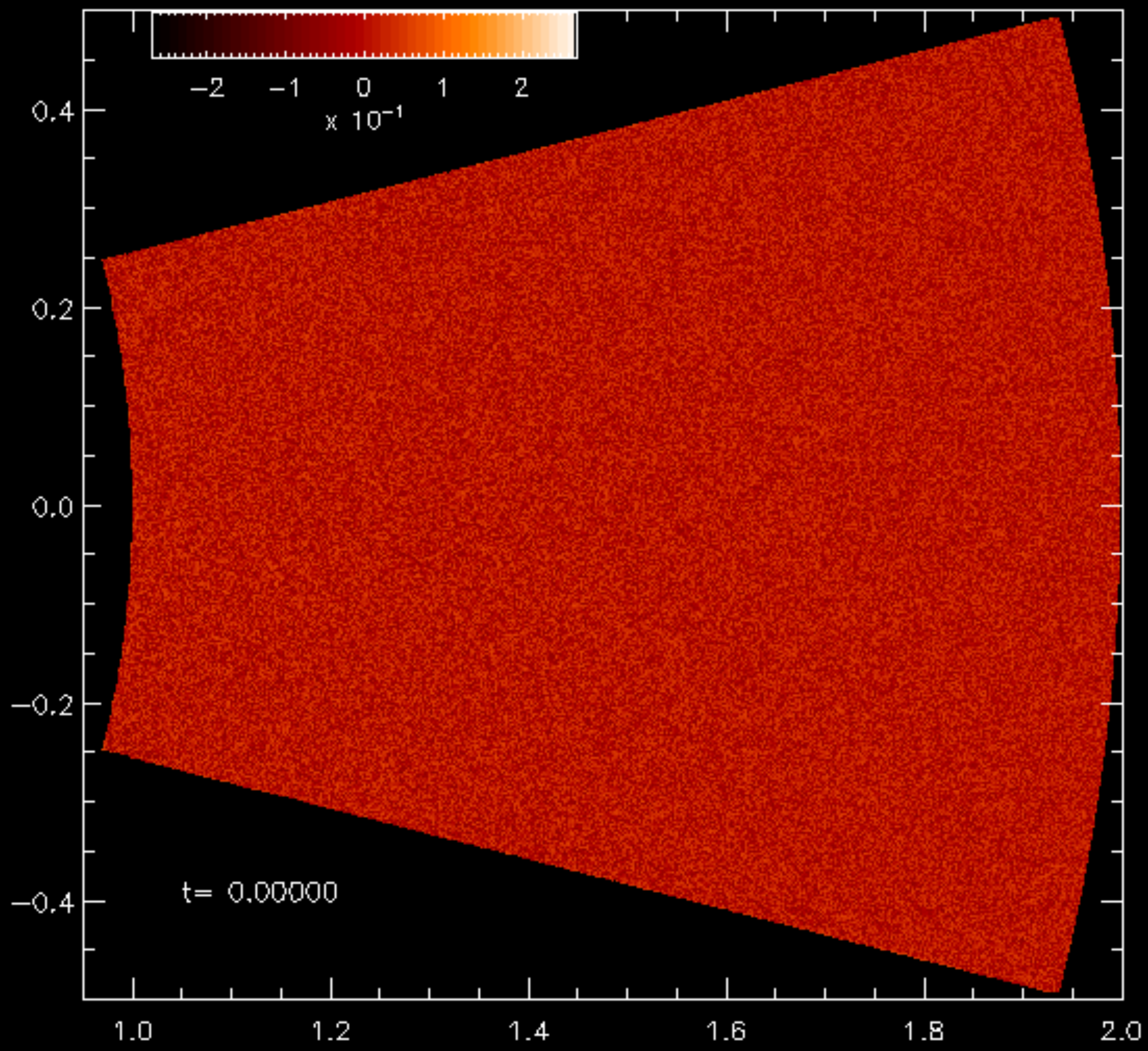
Vertical velocity perturbations (stretched grey-scale)

Note $k_R >> k_Z$



Density perturbations
(linear grey-scale)

Vertical velocity perturbations



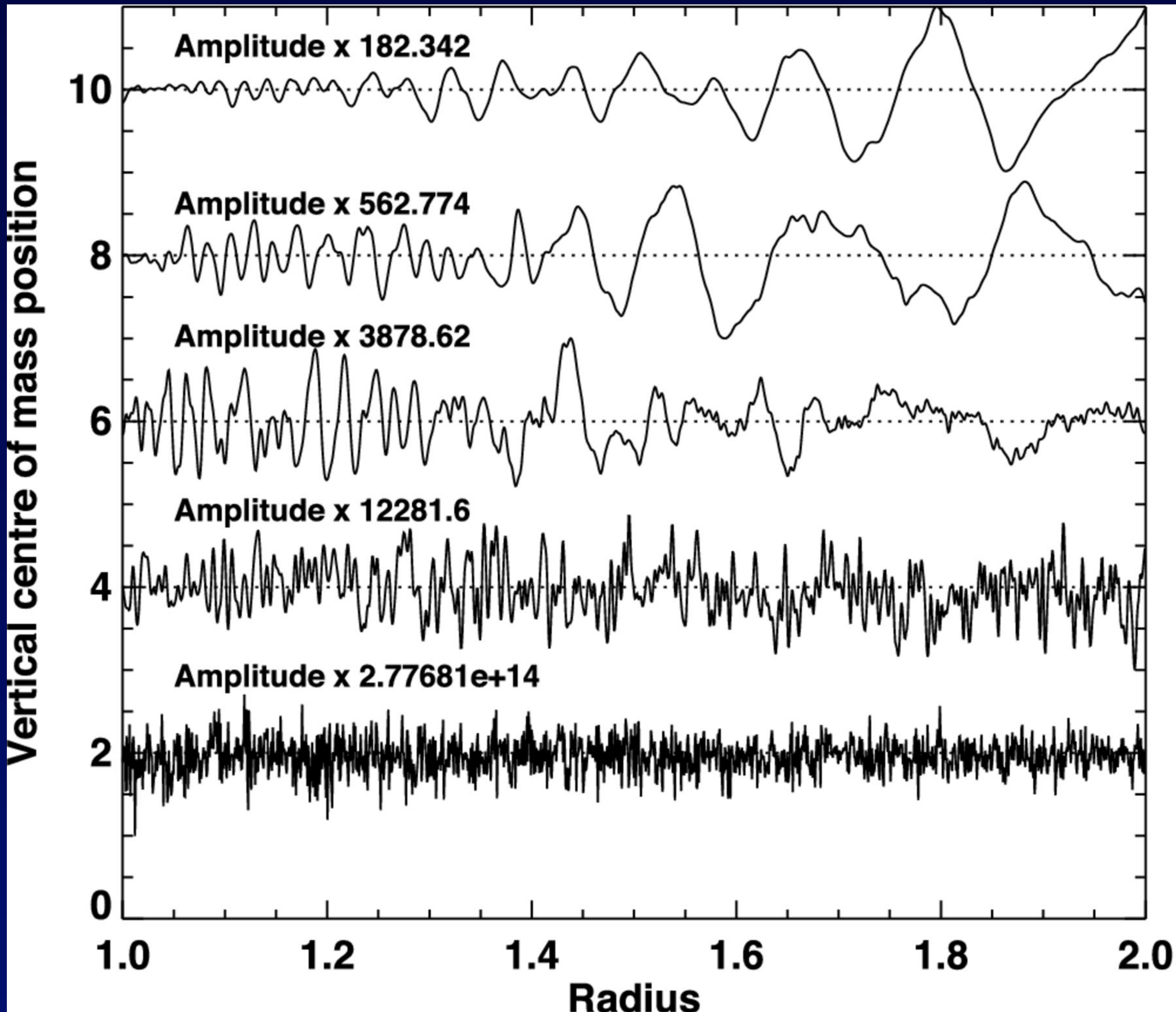
t=93

t=65

t=28

t=9

t=0.1

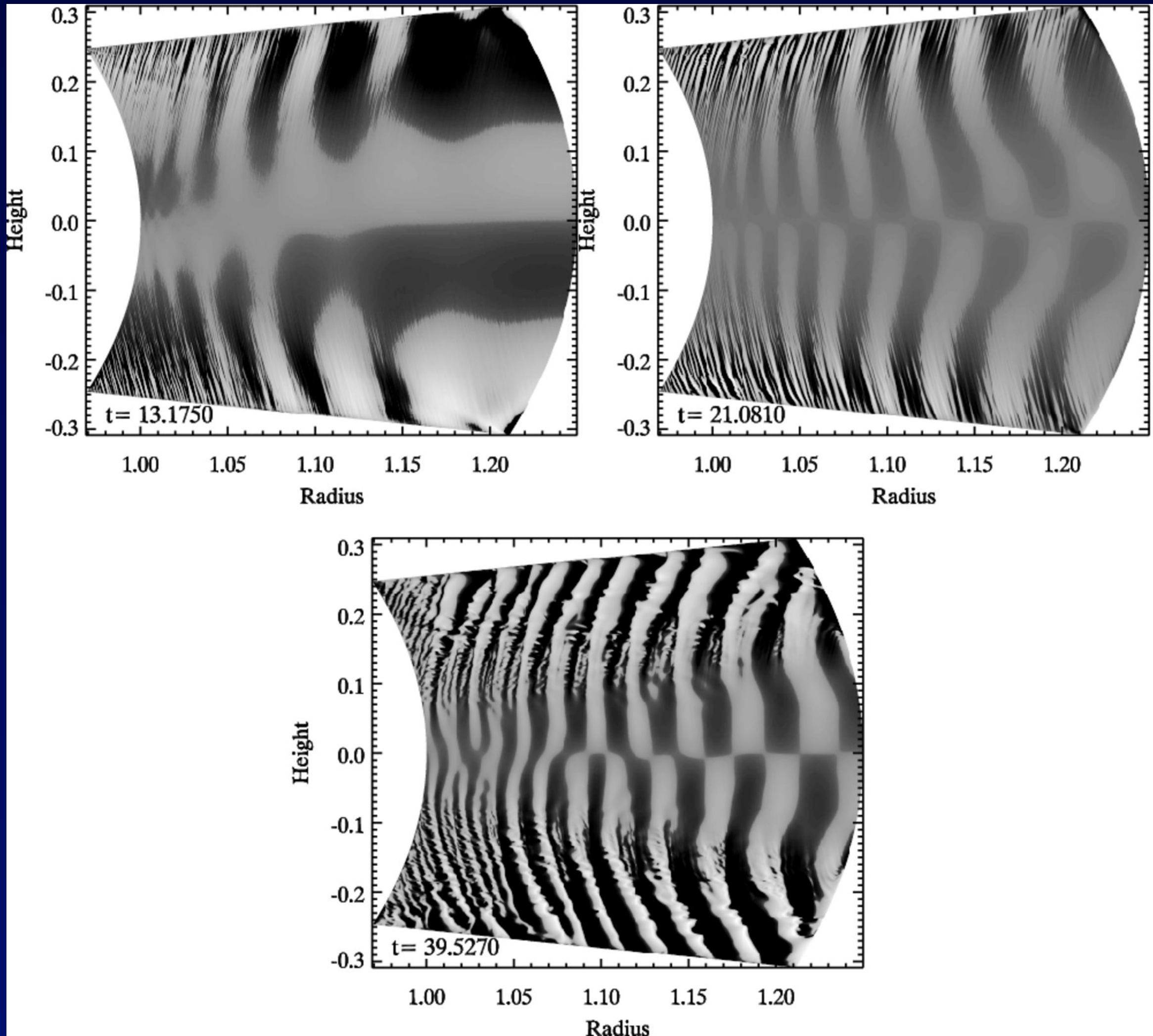


Coherent corrugation of disc

Noise

Vertical centre of mass position of disc annuli versus radius and time

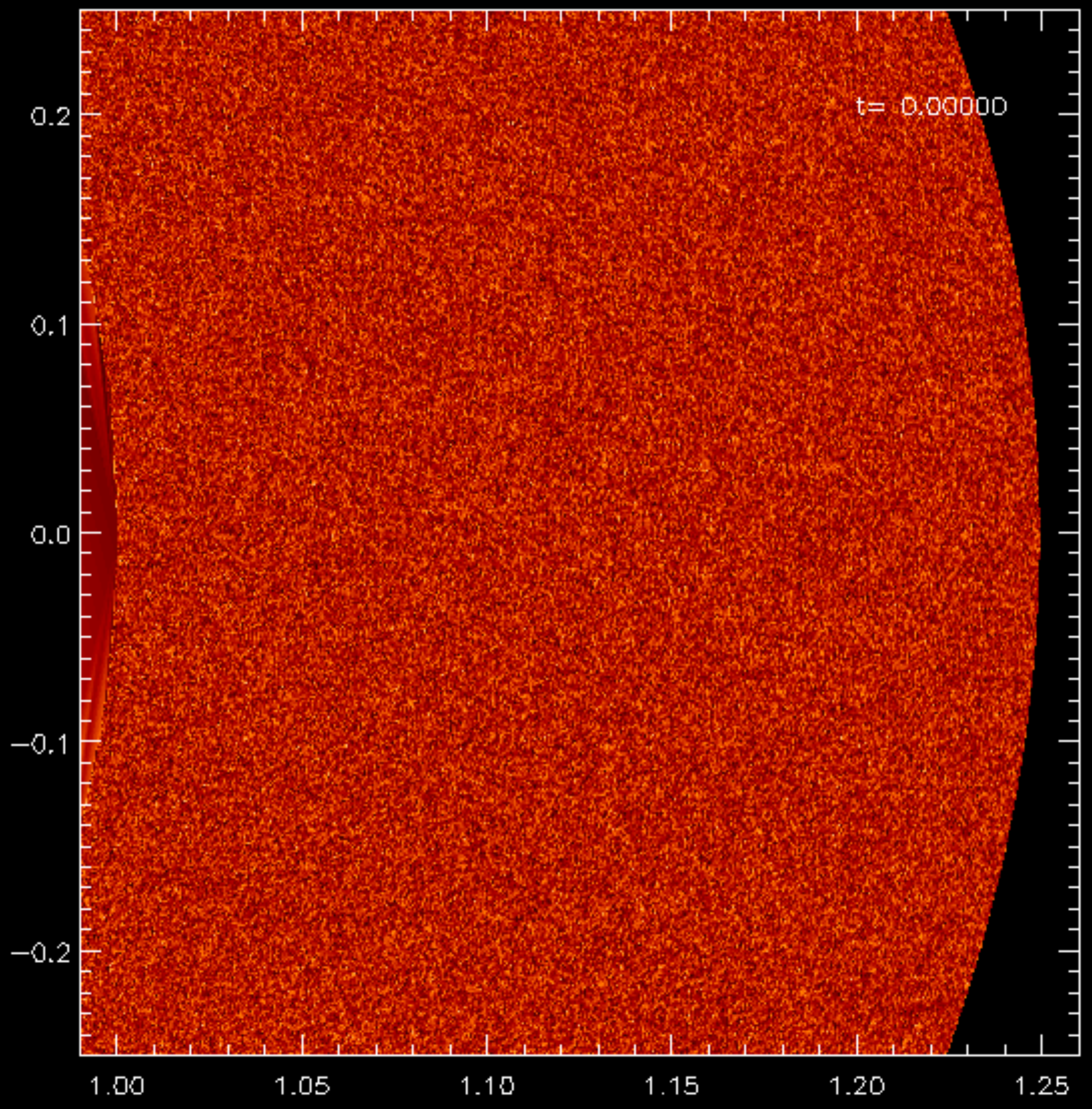
Higher resolution. Smaller seed noise.



Two distinct modes seen to grow when initial seed noise is small ($\delta v = 10^{-6}\ c_s$)

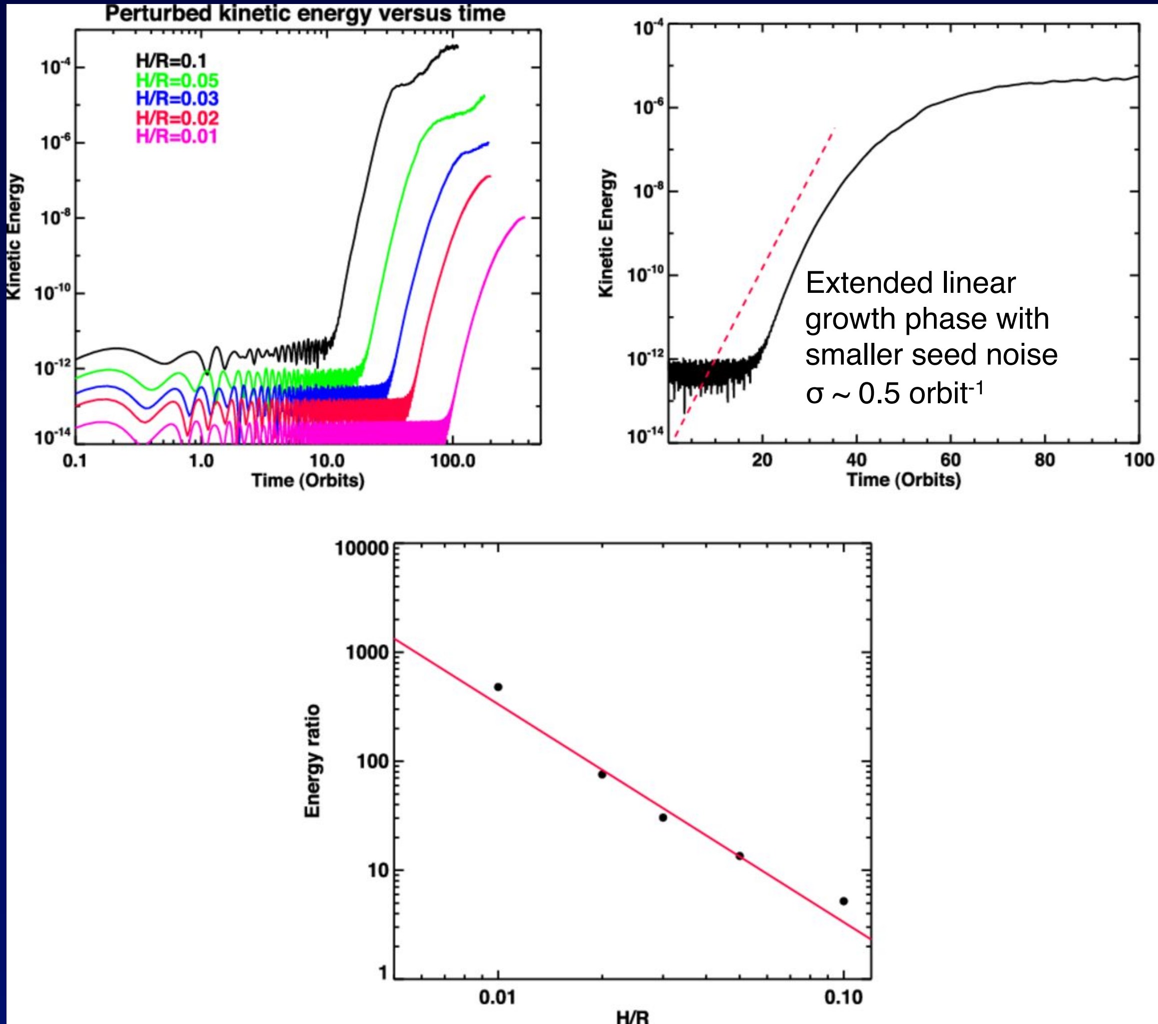
High altitude "finger modes" – grow fastest and descend toward midplane from surface

Lower altitude "body modes" – start as breathing modes (antisymmetric oscillation about midplane) and develop into fundamental corrugation modes



t= 0.00000
0.2
0.1
0.0
−0.1
−0.2
1.00
1.05
1.10
1.15
1.20
1.25

Evolution as function of H/R



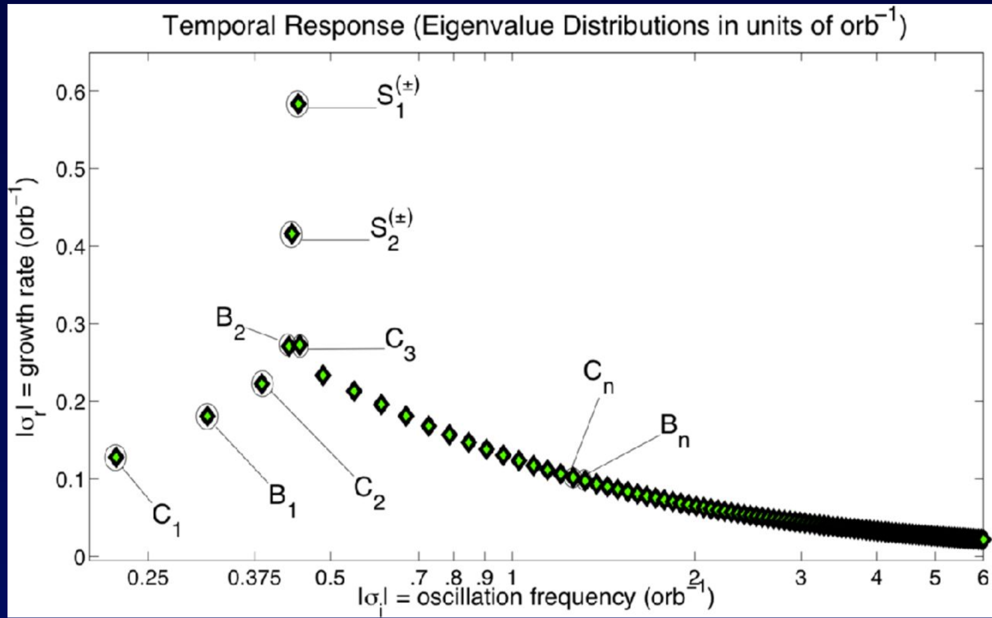
Linear analysis predicts growth rate σ ~ H/R and predicts $e_\vartheta / e_r \sim (H/R)^{-2}$

Asymptotic analysis using scalings suggested by simulations and GS67 results in simplified equation that describes eigenmodes of system:

$$\frac{\partial^2}{\partial \tau^2}\frac{\partial^2 \tilde{\Pi}}{\partial x^2} = -\frac{\partial^2 \tilde{\Pi}}{\partial z^2} + \left(1 + q\frac{\partial}{\partial x}\right) z \frac{\partial \tilde{\Pi}}{\partial z}$$

$$\tilde{\Pi} = \Pi(z) e^{ikx+\sigma\tau} + \text{c.c.}$$

$$\frac{d^2\Pi}{dz^2} - (1 + ikq) z \frac{d\Pi}{dz} + \sigma^2 k^2 \Pi = 0$$

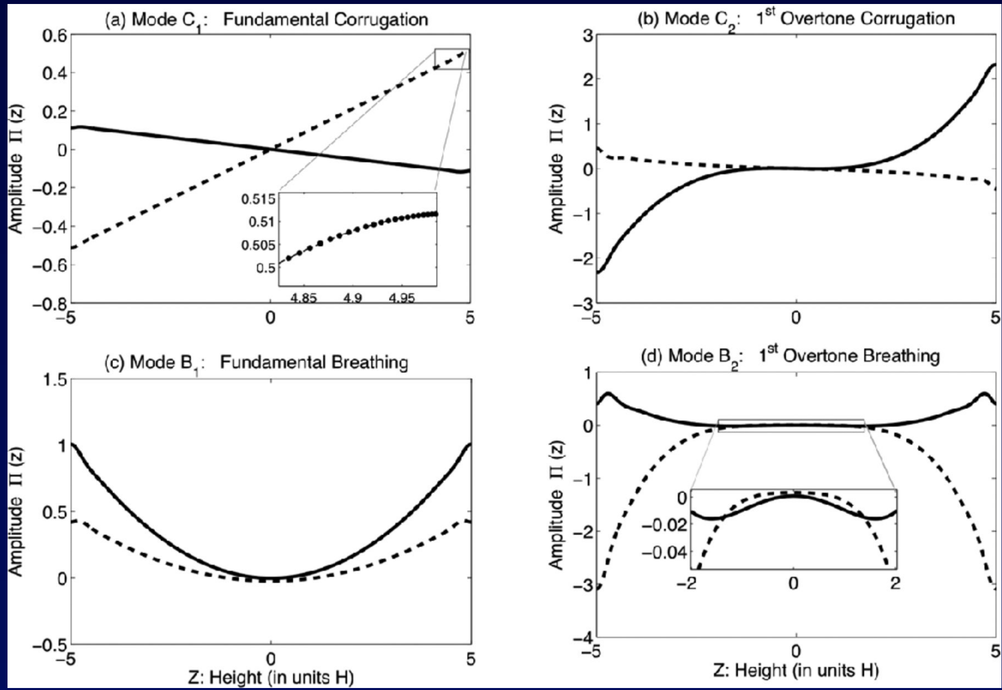


➔ Predicts existence of two mode types:

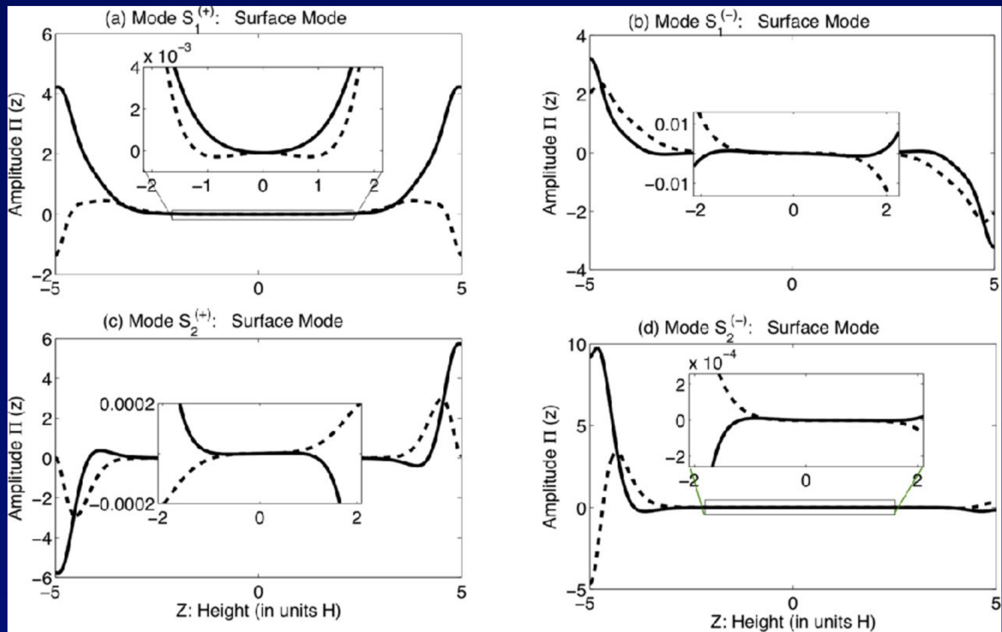
Surface modes: rapidly growing with $k_R/k_Z << 1$

Body modes: breathing and corrugation disturbances

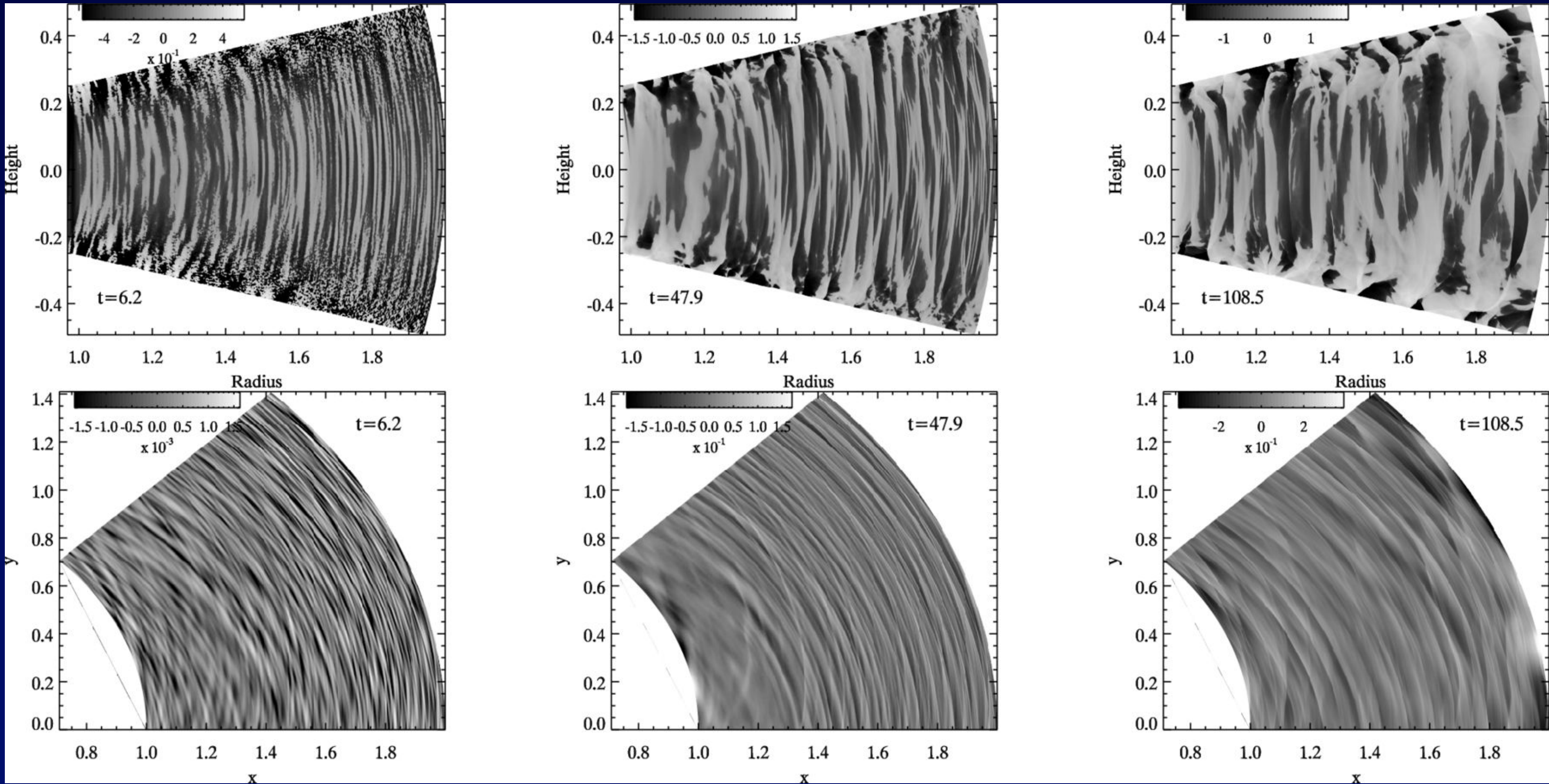
Corrugation and breathing modes



Surface modes

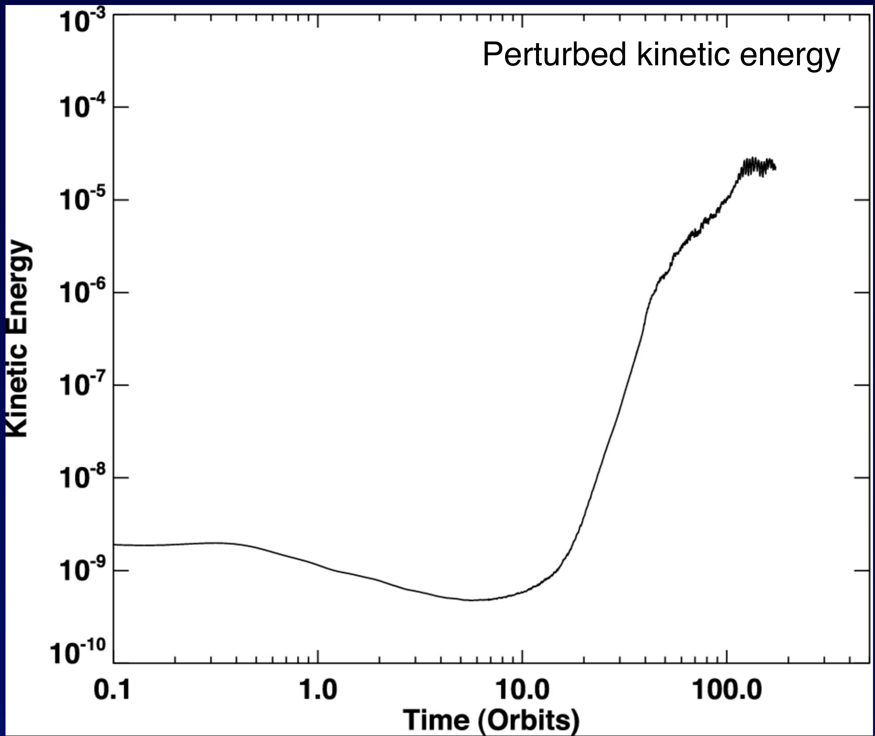


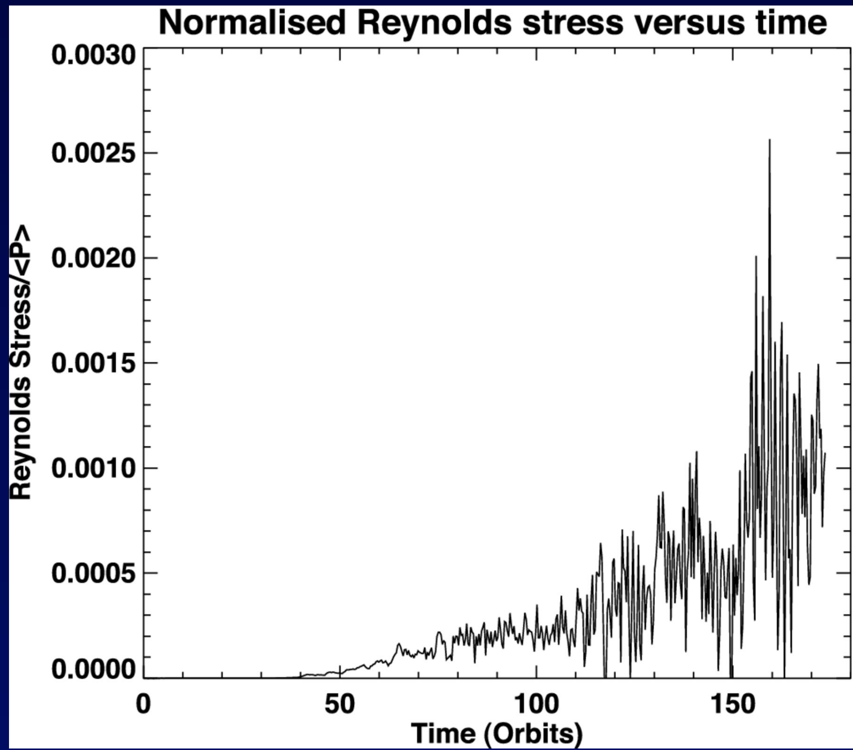
3D simulation



Upper panel: vertical velocity perturbations
Lower panel: midplane density perturbations

Locally isothermal equation of state
$N_r \times N_\theta \times N_\phi = 1328 \times 1000 \times 300$
in π/4



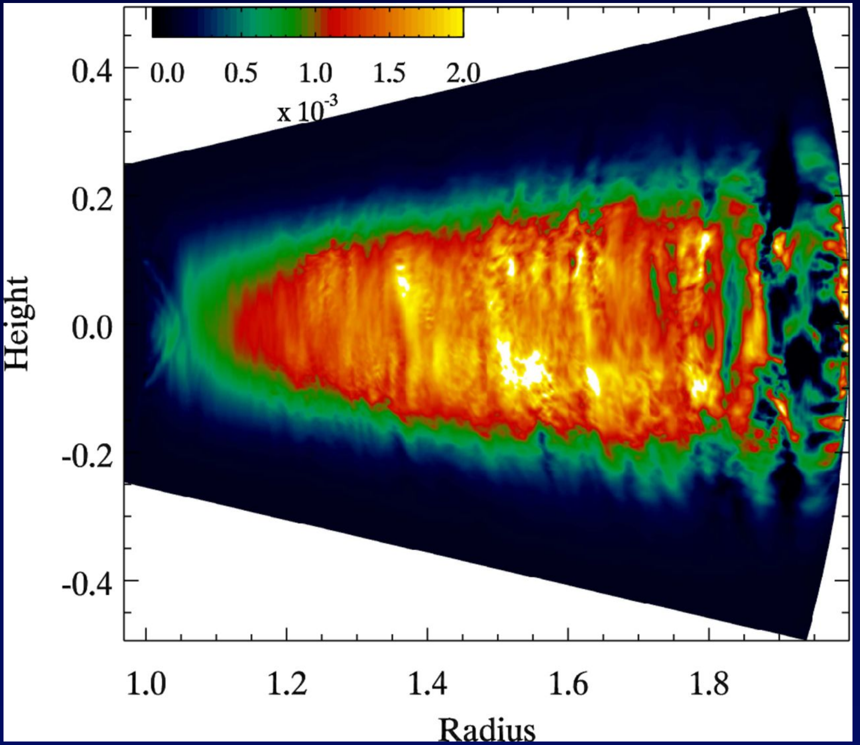


3D simulation show similar growth rate for perturbed energy as 2D runs

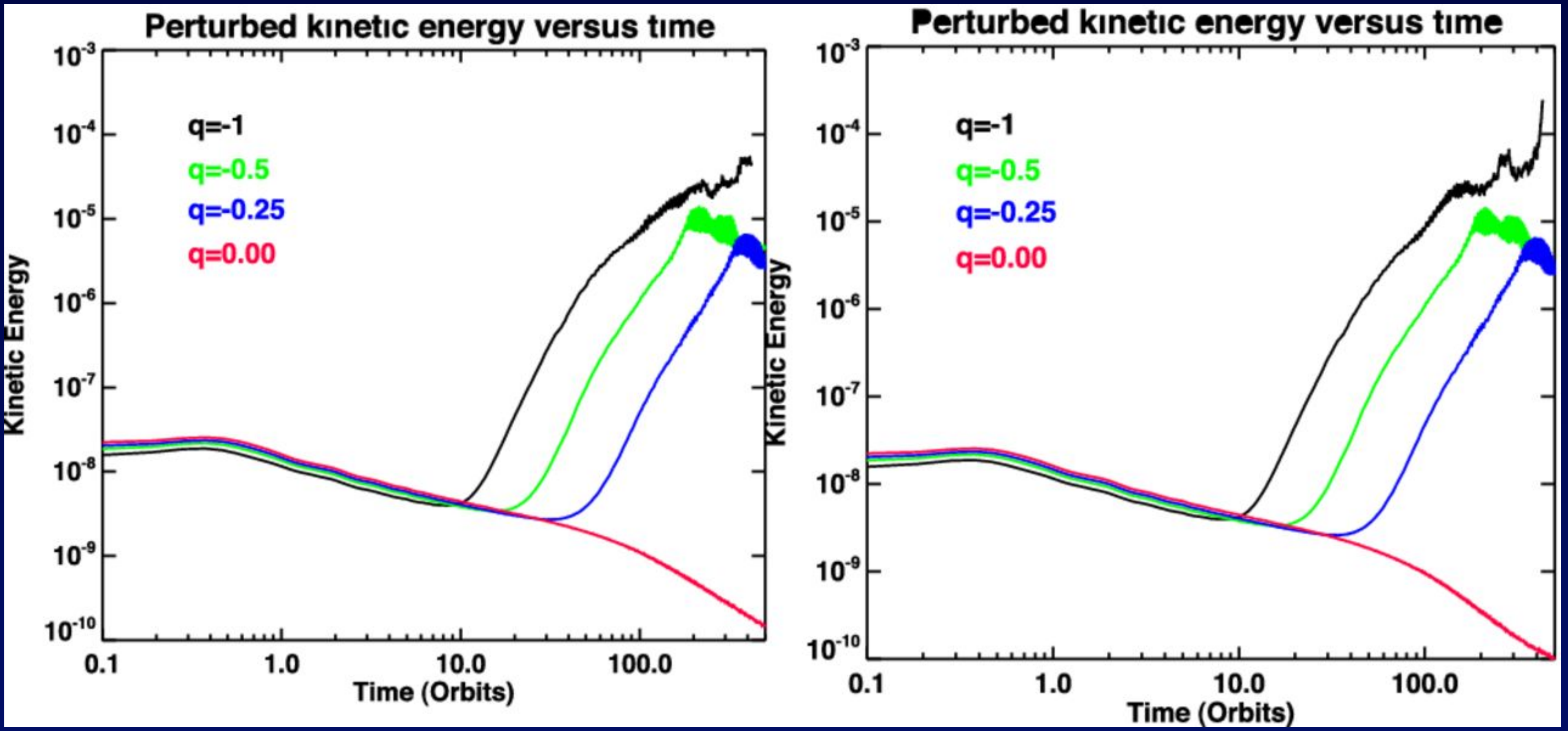
Quasi-turbulence in nonlinear saturated state gives rise to Reynolds stress with effective mean $\alpha \sim 10^{-3}$

Temporal and azimuthal average of
α = (Reynolds stress) / P

In the nonlinear phase the VSI leads to efficient transport of angular momentum outward



Evolution as function of radial temperature profile: T ~ R^q

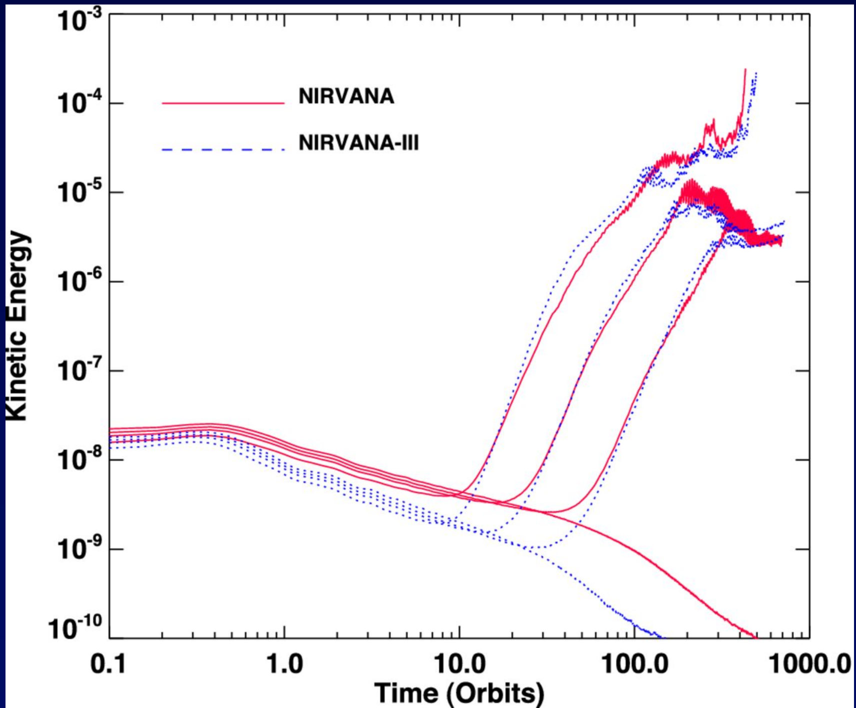


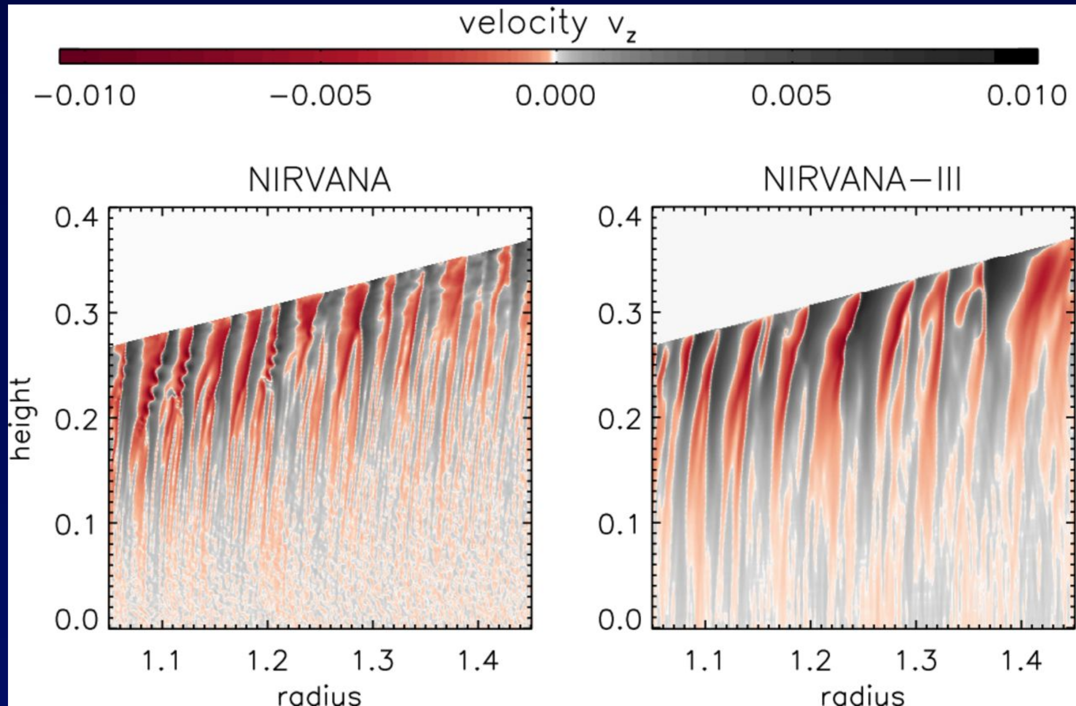
Reflecting boundary conditions

Outflow boundary conditions

Clearly require radial temperature gradient and associated vertical shear for instability

Code comparison





ZEUS-like NIRVANA versus the NIRVANA-III Godunov code

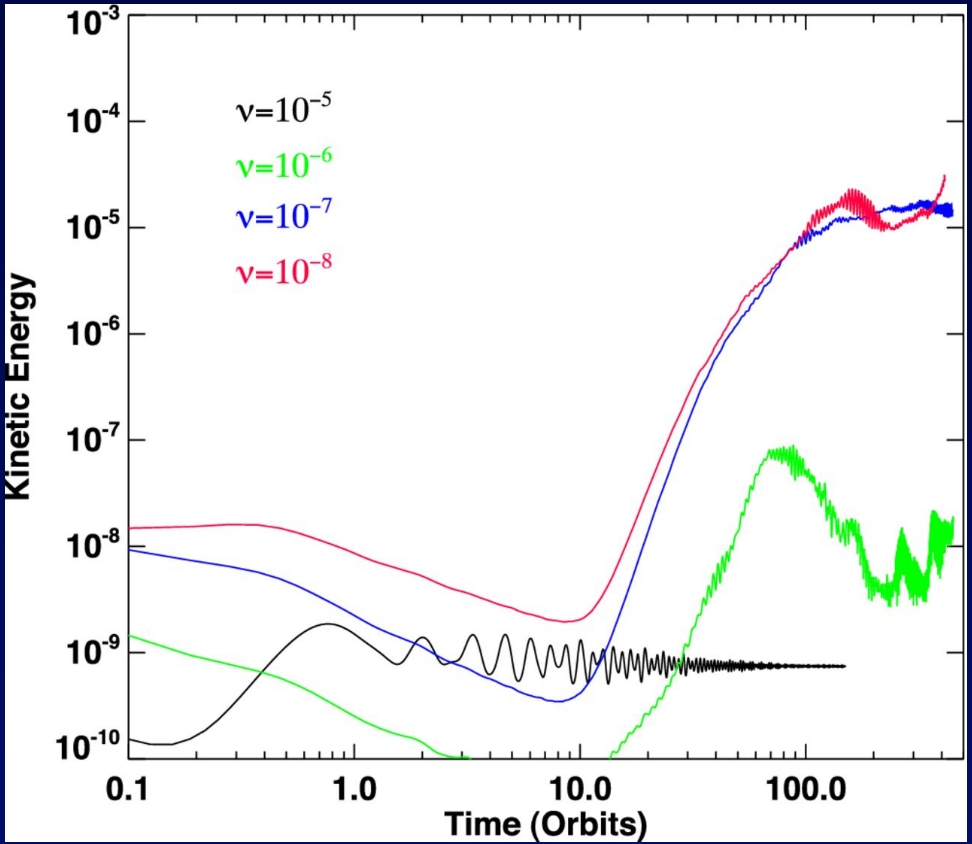
Evolution as function of viscosity

Simulations with constant kinematic viscosity show that instability is damped for $\nu \geq 10^{-6}$

➔ corresponds to $\alpha \geq 4 \times 10^{-4}$

Global simulations of discs with fully-developed MRI turbulence with $\alpha \sim 10^{-3}$ do not show evidence for VSI

(Fromang & Nelson 2006)



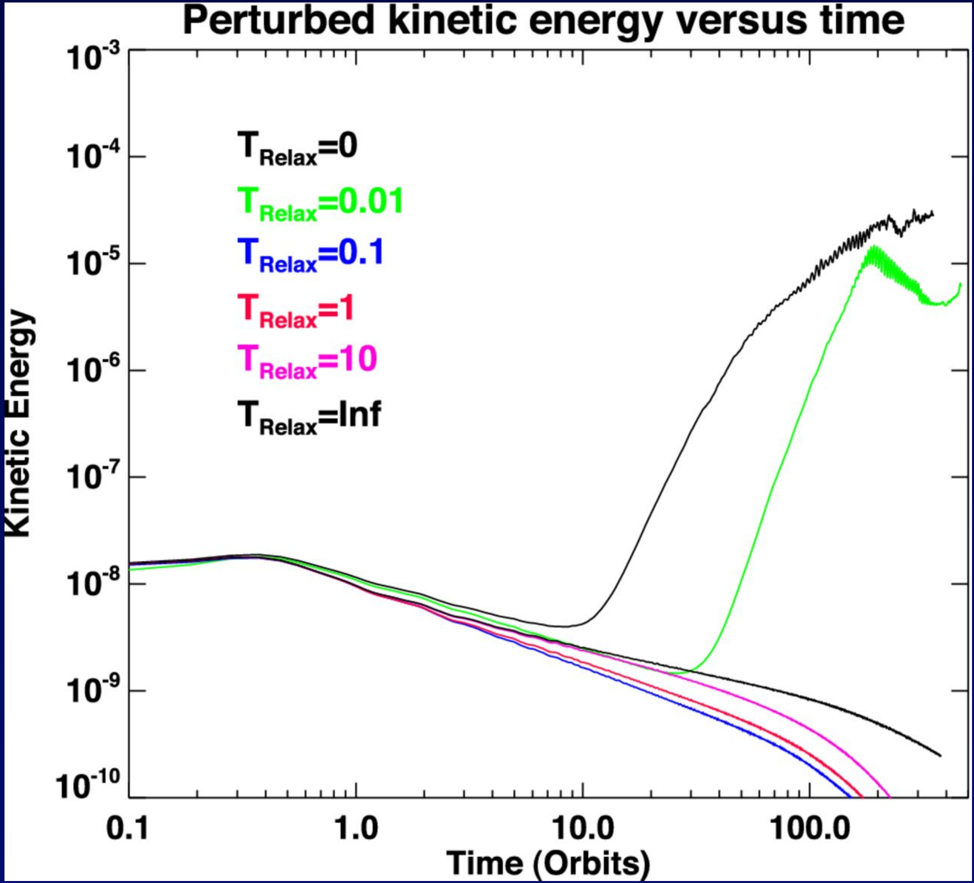
Evolution with thermal relaxation: $T \sim R^{-1}$

$$T(R) = T_0 \left(\frac{R}{R_0}\right)^q$$ q=-1

$$\rho_{\rm mid}(R) = \rho_0 \left(\frac{R}{R_0}\right)^p$$ p=-3/2

$$\frac{dT}{dt} = -\frac{(T - T_0)}{\tau_{\rm relax}}$$

Require short cooling times to overcome stabilizing influence of entropy gradients for these discs



Evolution with thermal relaxation: $K_s \sim R^{-1}$

$$K_s(R) = K_0 \left(\frac{R}{R_0}\right)^s \quad s=-1$$

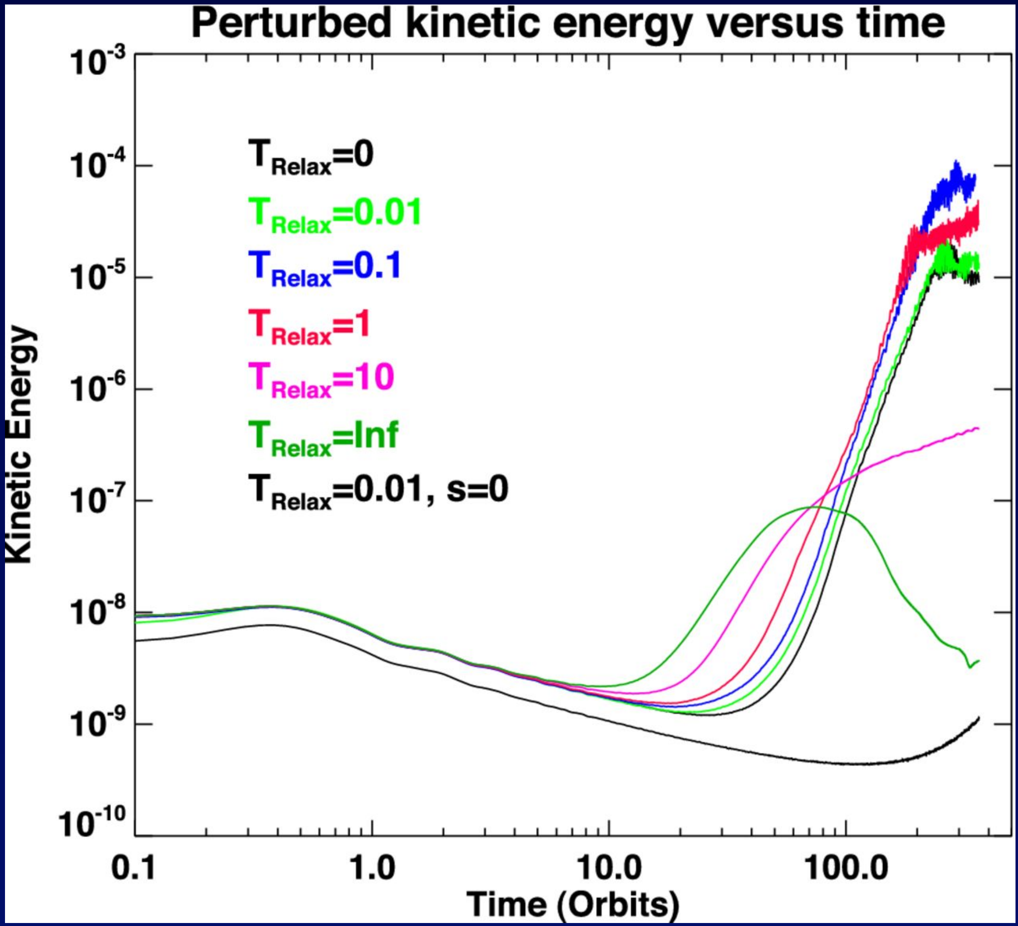
$$\rho_{\mathrm{mid}}(R) = \rho_0 \left(\frac{R}{R_0}\right)^p \quad p=0$$

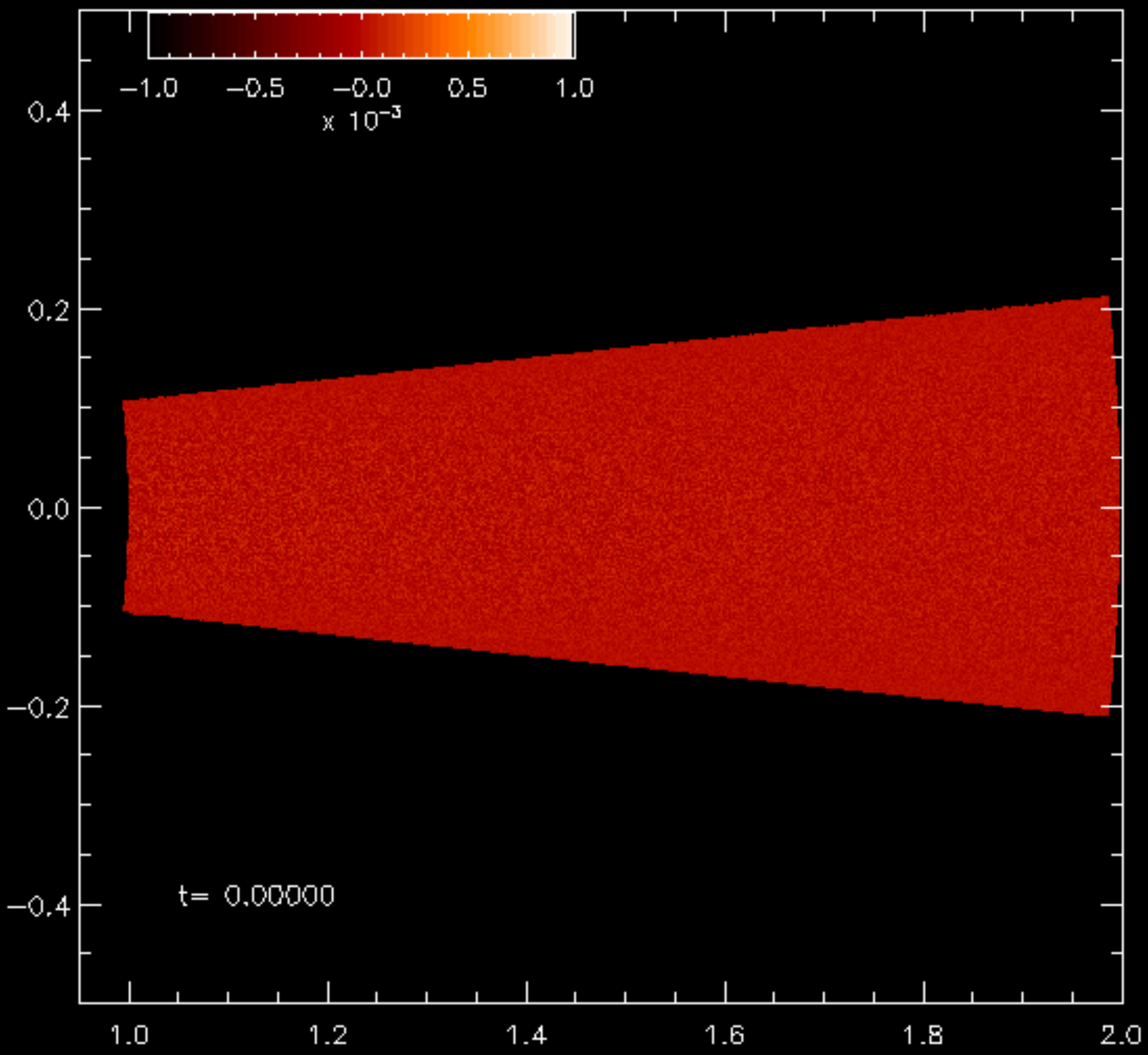
Model with s=-1, p=0 has no vertical entropy gradient to stabilize flow, and the adiabatic model is unstable according to the 2nd Solberg-Hoiland criterion

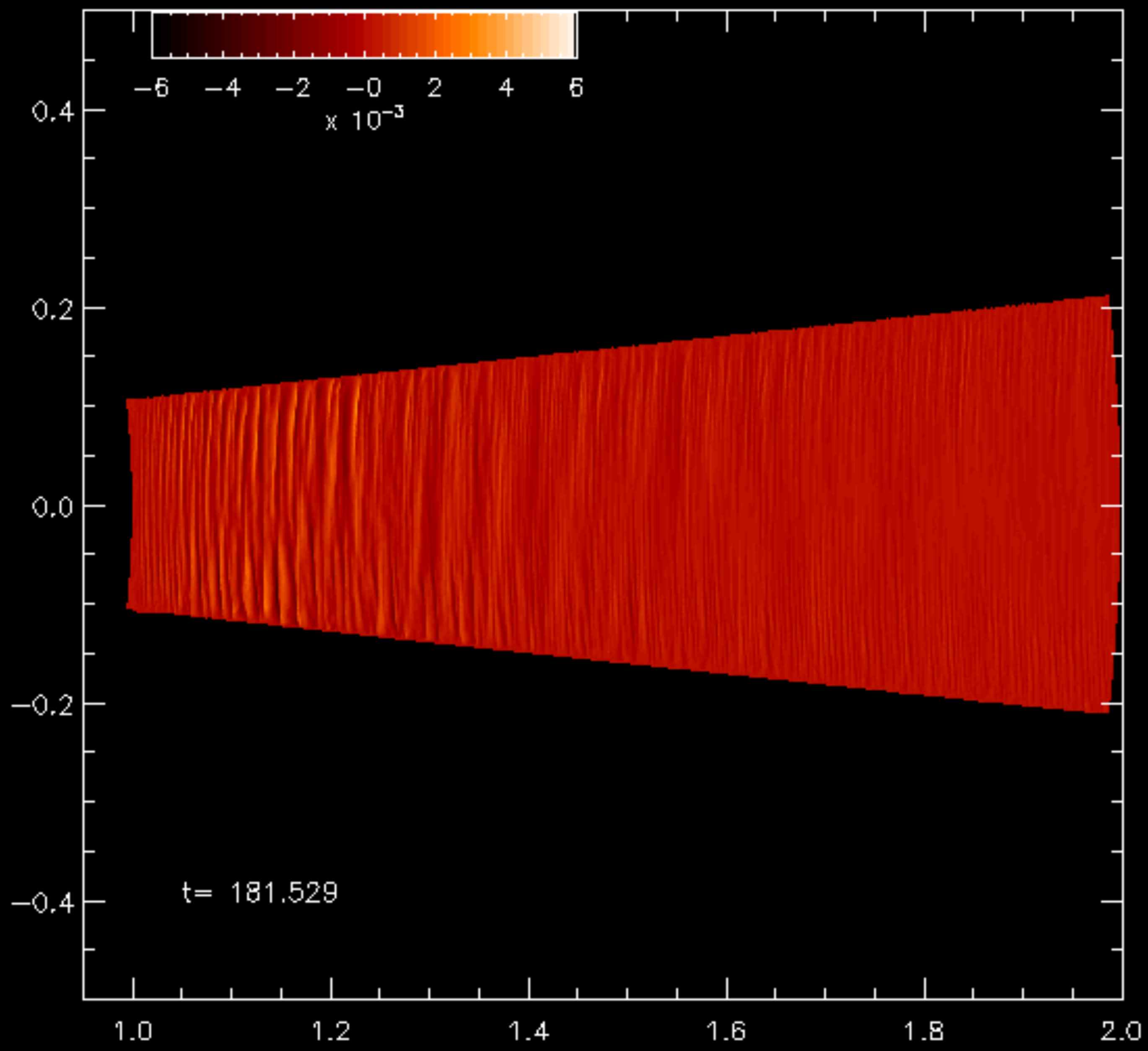
Instability is obtained with much longer cooling times – up to 10 local orbits

Adiabatic case: growth of velocity perturbations that die away
➔ this model evolves to a neighbouring stable solution??

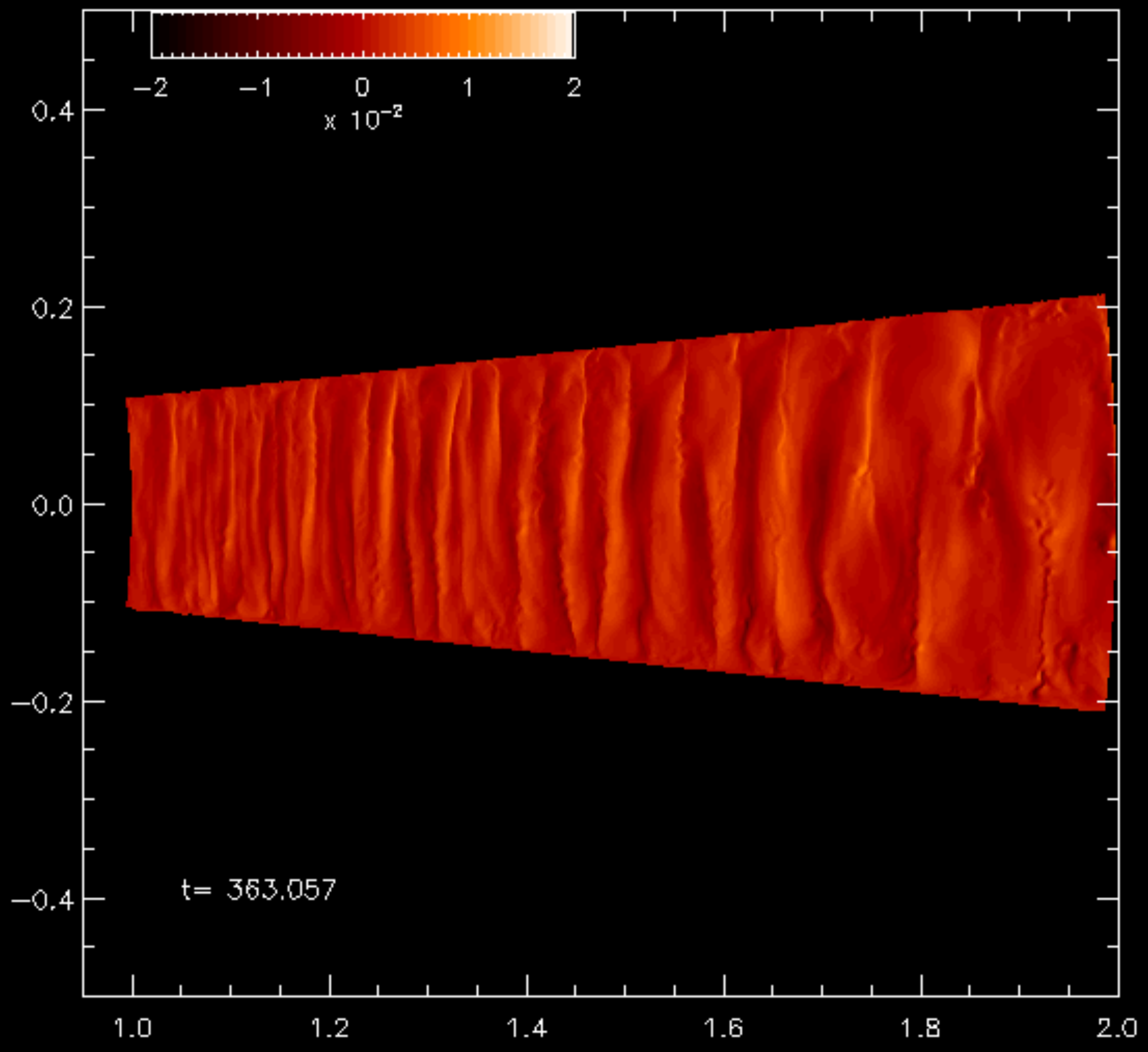
Model with s=0, p=-3/2 has no vertical shear
➔ stable against the VSI




−1.0 −0.5 −0.0 0.5 1.0
x 10^{-3}
0.4
0.2
0.0
−0.2
−0.4
t= 0.00000
1.0 1.2 1.4 1.6 1.8 2.0



−6 −4 −2 −0 2 4 6
x 10⁻³
0.4
0.2
0.0
−0.2
−0.4
t= 181.529
1.0
1.2
1.4
1.6
1.8
2.0



−2
−1
0
1
2
x 10⁻²
0.4
0.2
0.0
−0.2
−0.4
t= 363.057
1.0
1.2
1.4
1.6
1.8
2.0

Summary of runs

- Locally-isothermal disc models with $T \sim R^q$ display rapid growth of VSI
- nonlinear saturated state displays strong corrugation of the disc
- quasi-turbulence leads to outward angular momentum transport with $\alpha \sim 10^{-3}$

- Runs with thermal relaxation and $T \sim R^{-1}$ require $\tau_{relax} \leq 0.01$ orbits for VSI

- Runs with thermal relaxation and $K_s \sim R^{-1}$ require $\tau_{relax} \leq 10$ orbits for VSI
- absence of stabilising dS/dZ enhances instability

Preliminary results from a survey of the nonlinear phase in 3D

$$T(R) = T_0 \left(\frac{R}{R_0}\right)^q$$

$$\rho_{\rm mid}(R) = \rho_0 \left(\frac{R}{R_0}\right)^p$$

p=-3/2

q=-2

$$\frac{dT}{dt} = -\frac{(T - T_0)}{\tau_{\rm relax}}$$

Thermal relaxation used in all runs

$N_r \times N_\theta \times N_\phi = 500 \times 200 \times 300$
$1 \leq r \leq 1.5$
$-5h \leq \theta \leq 5h$
$0 \leq \phi \leq \pi/2$

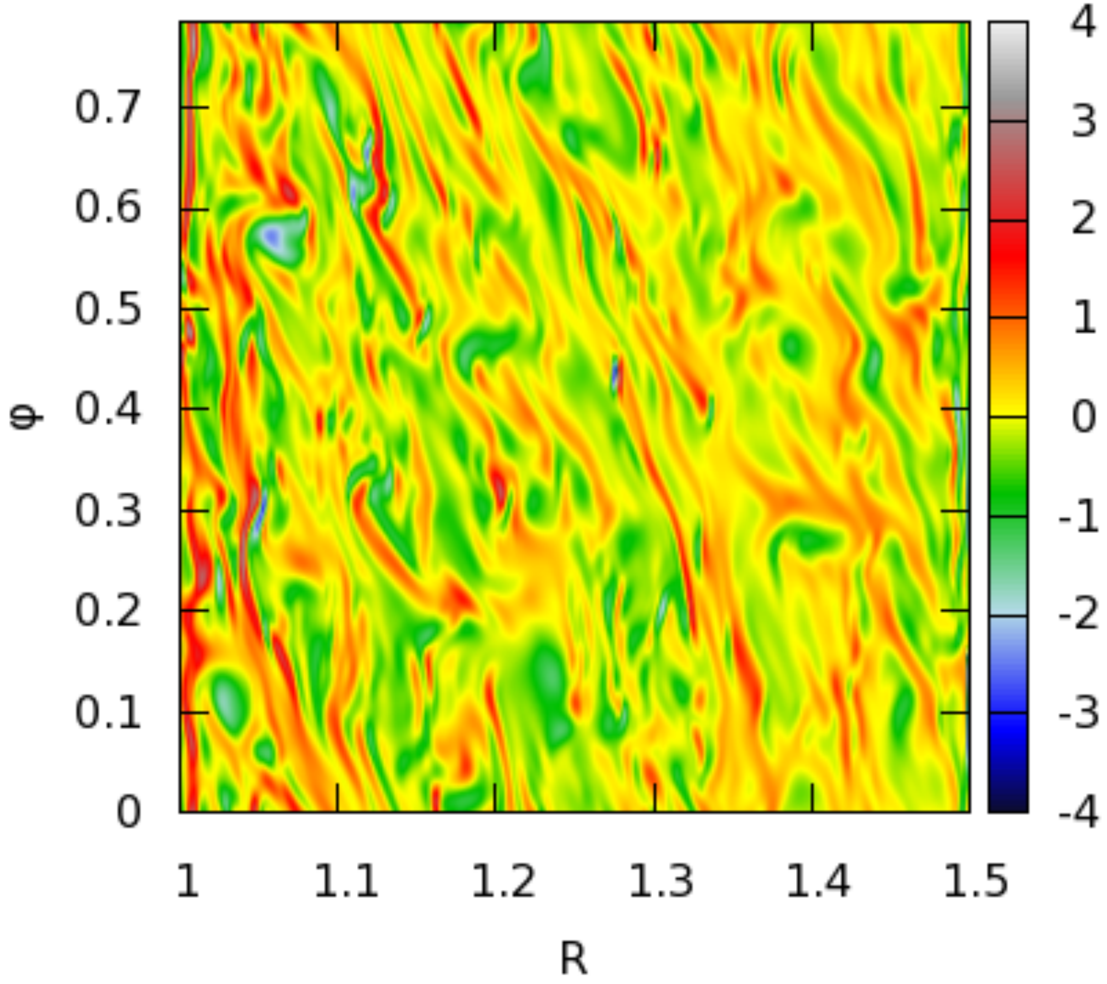
Key questions:

Can the VSI lead to the formation of long-lived vortices?
- source of finite amplitude vorticity perturbations that seed sub-critical baroclinic instability?

What levels of Reynolds stress can be generated by the VSI in saturated state?

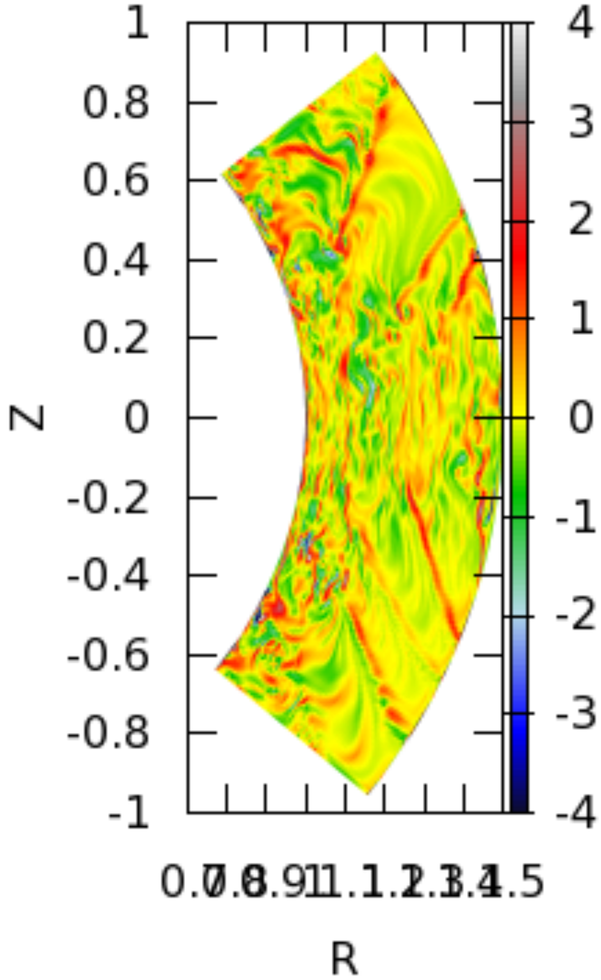
How does the nonlinear state vary as a function of disc parameters?

H/R=0.2
τ_{relax} = 0.05

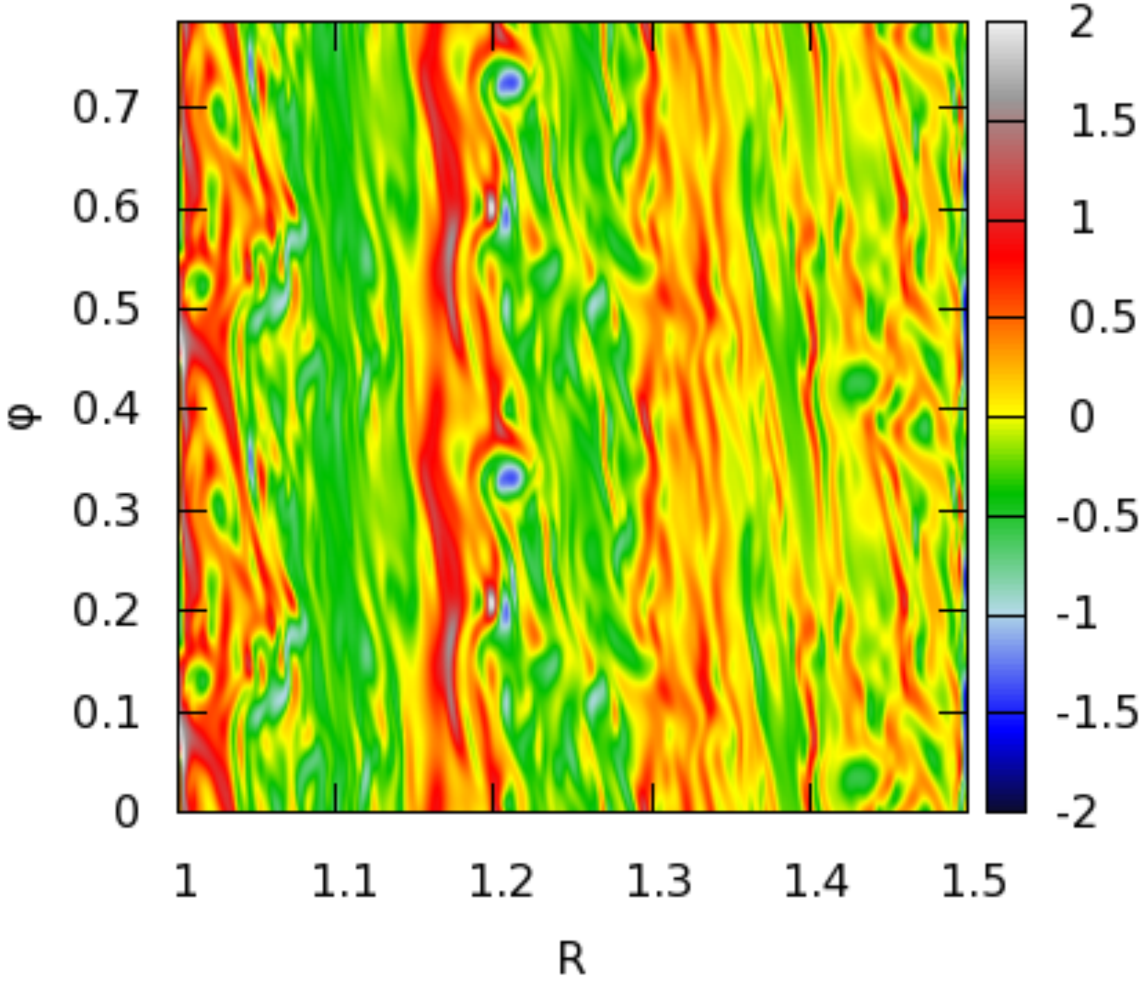


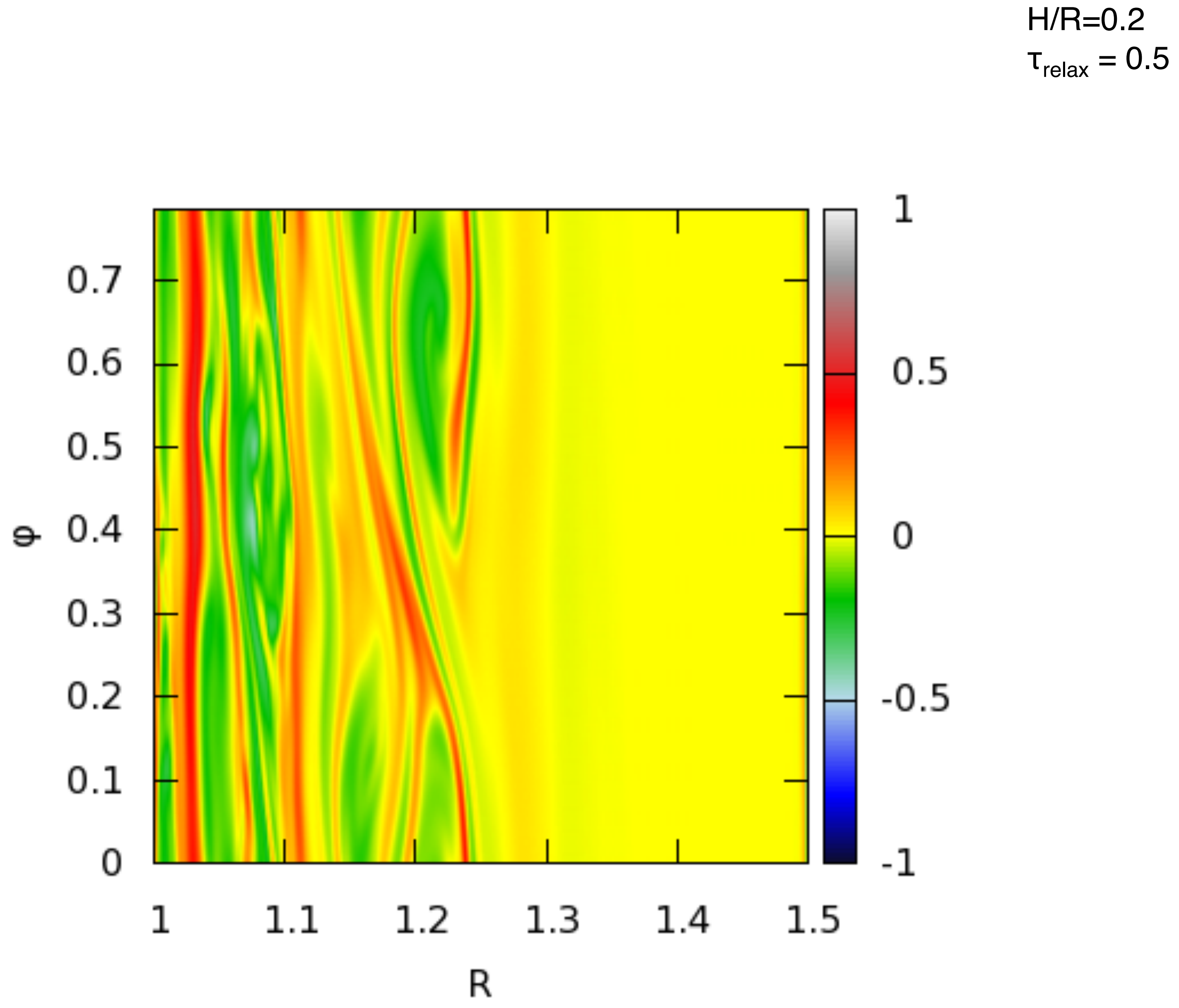
H/R=0.2

$\tau_{relax} = 0.05$



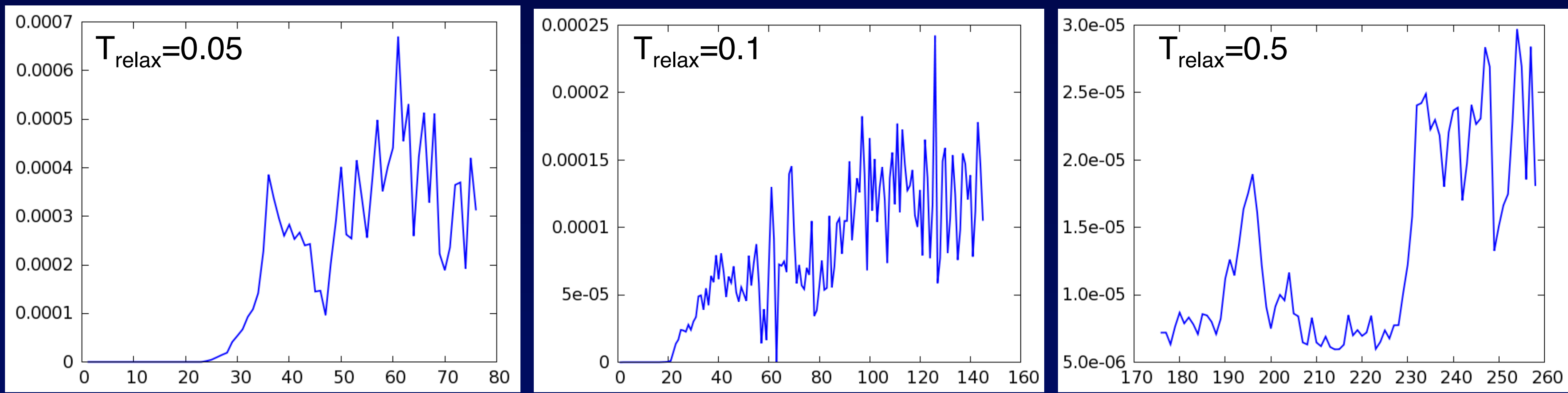
H/R=0.2
τ_{relax} = 0.1





H/R=0.2
τ_{relax} = 0.5
φ
R
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
1
1.1
1.2
1.3
1.4
1.5
1
0.5
0
-0.5
-1

“alpha” versus time



Reynolds stress scales inversely with thermal relaxation time

Summary of runs

Vorticity perturbations scale with inverse of thermal relaxation time

- short cooling times ➔ small aspect-ratio, short lived vortices
  + significant angular momentum transport ($\alpha \sim$ few x 10^{-4})
- longer cooling times ➔ large aspect-ratio, long-lived vortices
  + modest angular momentum transport ($\alpha \sim 10^{-5}$)

Hypothesis:

Small vortices unstable through rapid growth of elliptical instability

Cooling times may be sub-optimal for SBI to grow/maintain small vortices

Larger vortices survive longer because elliptical instability growth time is longer

Cooling time may be optimal for the SBI to grow/sustain large vortices

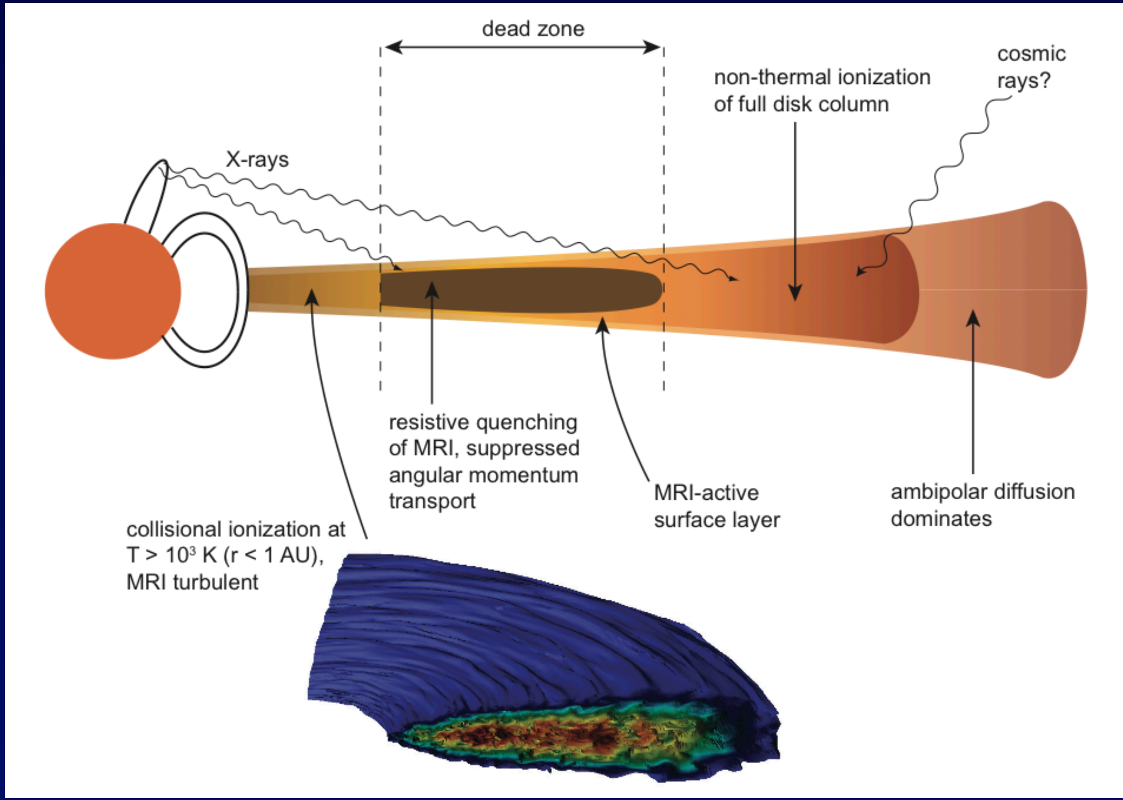
Application to protostellar/protoplanetary discs

VSI most likely to operate in dead zone where MRI is quenched

Radial temperature profiles in PPDs estimated from mm-observations to be power-laws with $-0.7 \leq q \leq -0.4$
(Andrews & Williams 2005)
Passively heated discs have $q \sim -3/7$
(Chiang & Goldreich 1997)
-in the required range for instability

Discs heated through stellar irradiation will have moderate $dT/dZ > 0$
(Dullemond, van Zadelhoff & Natta 2002)
- this is will provide a more stable entropy gradient than the locally-isothermal disc models

Require rapid thermal evolution across radial length scales associated with growing modes



Application to protostellar/protoplanetary discs

Thermal time scale due to radiative diffusion

$$\tau_{\mathrm{Rad}} = \frac{\Delta^2}{\mathcal{D}} \qquad \mathcal{D} = \frac{4acT^3}{3\kappa\rho^2 C_{\mathrm{v}}}$$

Use MMSN model and temperature structure from Chiang & Goldreich (1997)

$$\frac{\tau_{\mathrm{Rad}}}{P_{\mathrm{orb}}} = 168F^2\kappa\left(\frac{\Delta}{R}\right)^2\left(\frac{R}{20\,\mathrm{AU}}\right)^{-53/14}$$

$$\Delta = \lambda_R \simeq \frac{H^2}{R}$$

VSI likely to operate in outer regions (10 – 20 AU) of protoplanetary discs for MMSN disc

$$\frac{\tau_{\mathrm{Rad}}}{P_{\mathrm{orb}}} = 0.015$$ At 10 AU

$$\frac{\tau_{\mathrm{Rad}}}{P_{\mathrm{orb}}} = 0.001$$ At 20 AU

As grains grow and disc evolves, both opacity and disc density decrease, moving unstable regions inward

Conclusions and future work

- Vertical shear instability may act as a source of turbulence, angular momentum transport, vortices and mixing in protoplanetary discs – influencing both the evolution of the disc and of forming planets

- Require radiation-MHD simulations to examine the linear and nonlinear development of the vertical shear instability under realistic disc conditions