Vertically Global, Radially Local Models for Astrophysical Disks

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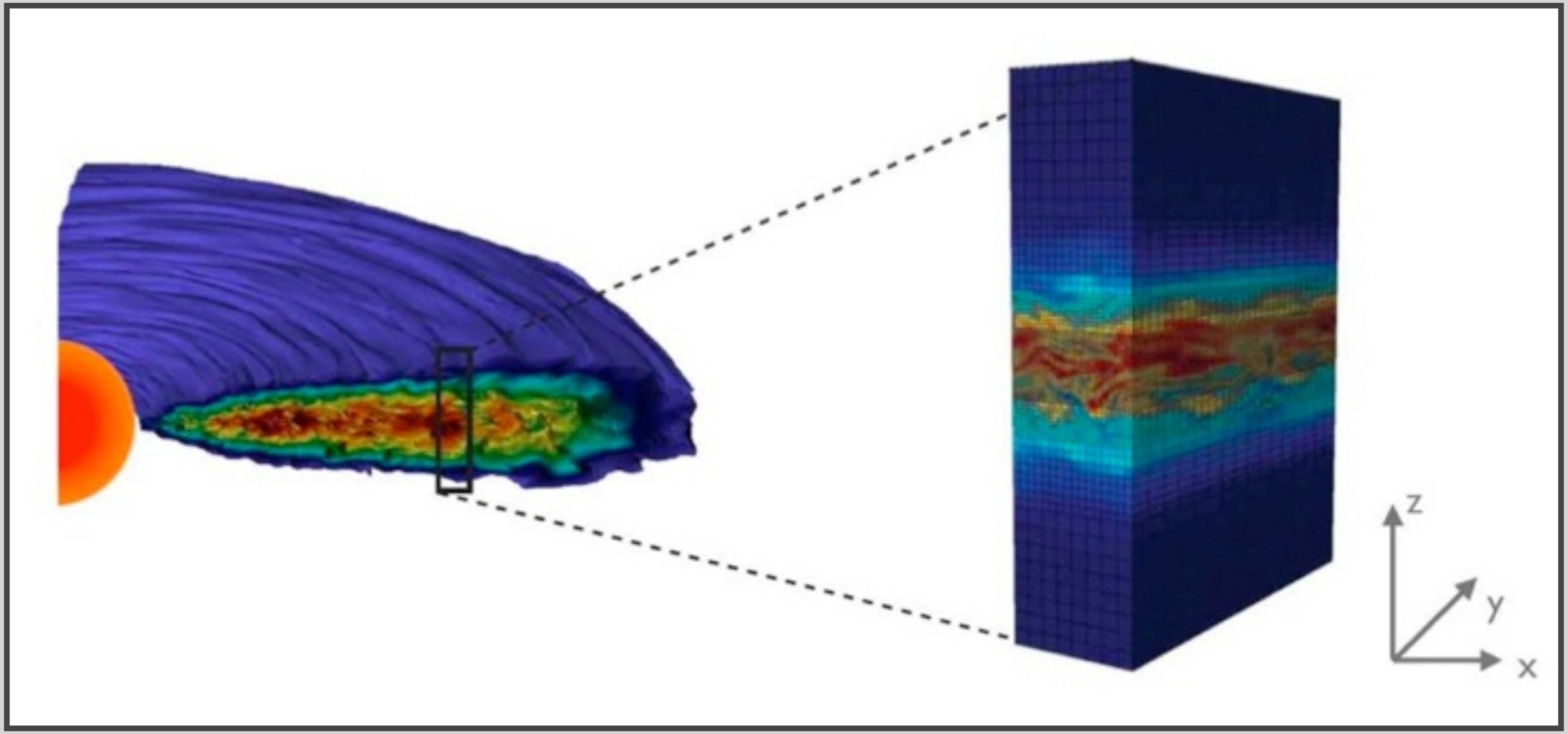
Local Models of Astrophysical Disks



Beckwith, Armitage and Simon, 2011

Essence of the local approximation: expand the equations of motion around a point corotating with the disk

Deriving Local Disks Models

- Write down equations of motion
- Find an equilibrium solution
- Expand the equations around fiducial point
- Define boundary conditions
- Solve problems!

Equations for an ideal MHD Fluid

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

continuity

momentum

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{v} = \Omega_{\mathrm{F}}^2 \boldsymbol{r} - 2\boldsymbol{\Omega}_{\mathrm{F}} \times \boldsymbol{v} - \nabla \Phi - \frac{\nabla P}{\rho} + \frac{1}{\rho} \boldsymbol{J} \times \boldsymbol{B}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{B} = (\boldsymbol{B} \cdot \nabla)\, \boldsymbol{v} - \boldsymbol{B}\, (\nabla \cdot \boldsymbol{v})$$

induction

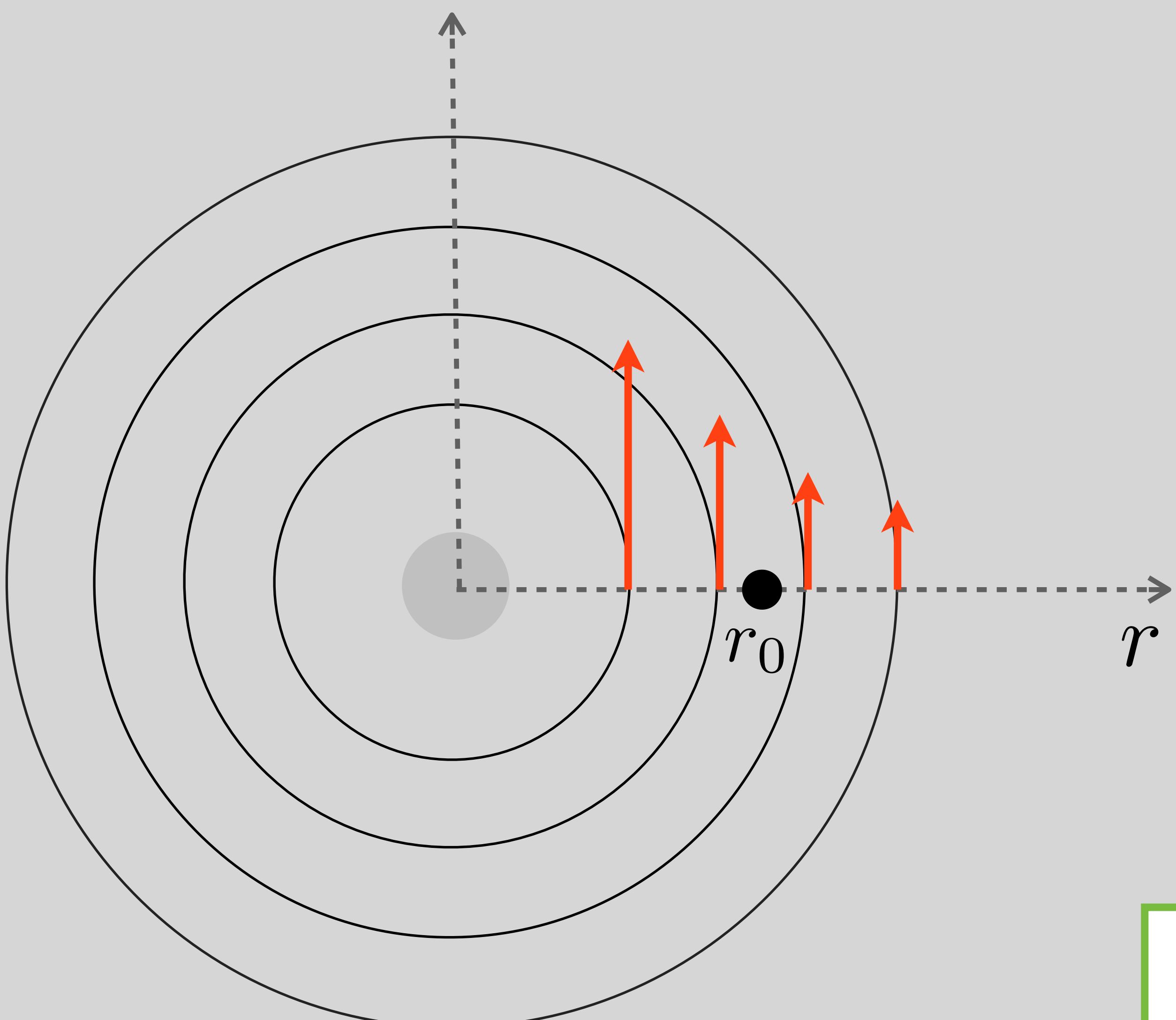
$$\frac{\partial e}{\partial t} + \nabla \cdot (e \boldsymbol{v}) = -P\, (\nabla \cdot \boldsymbol{v})$$

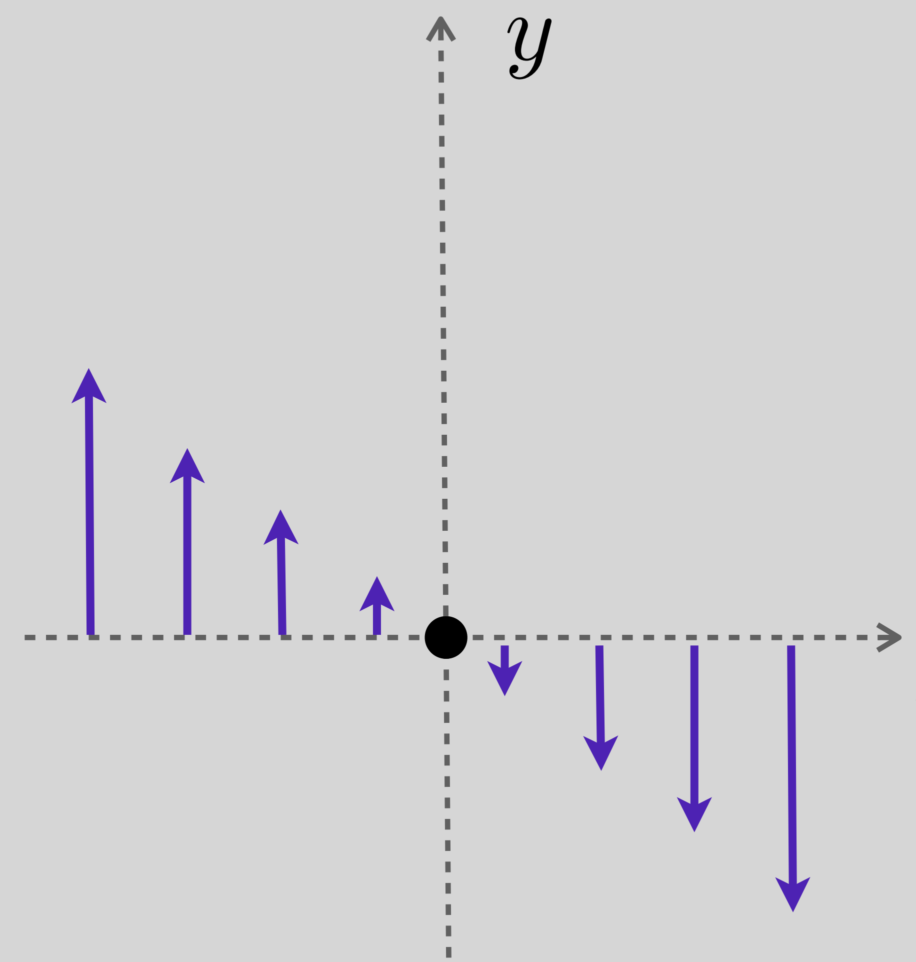
energy

The Local Approximation

$$V(r) = r\,\Omega(r)$$

$$V(x) \simeq V_0 + S_0\,x$$





$$V(x) \simeq r_0 \left[\Omega_0 + \left.\frac{d\Omega}{dr}\right|_0 (r - r_0) \right]$$

Vertically Local & Radially Local

- 1. Bulk disk flow

$$\boldsymbol{V}(r) = r\Omega(r)\hat{\boldsymbol{\phi}}$$

$$\Omega^2(r) \equiv \frac{1}{r}\frac{\partial \Phi}{\partial r} = \frac{GM}{r^3}$$

$$\frac{1}{\rho_h}\frac{\partial P(\rho_h)}{\partial z} = -\frac{\partial \Phi}{\partial z}$$

- 2. Taylor expand in r

$$V(x) \equiv V_0 + S_0 x$$

$$\Omega(x) \equiv \Omega_0 + \left.\frac{\partial \Omega(r)}{\partial r}\right|_{r=r_0} x$$

$$S_0 \equiv r_0 \left.\frac{\partial \Omega(r)}{\partial r}\right|_{r=r_0}$$

- 3. Define departures

$$\boldsymbol{w} \equiv \boldsymbol{v} - V(x)\hat{\boldsymbol{y}}$$

Standard Shearing Box - Model

$$\boldsymbol{w} \equiv \boldsymbol{v} - S_0 x \hat{\boldsymbol{y}}$$

$$\mathcal{D}_0 \equiv \partial_t + S_0 x \partial_y$$

$$\begin{aligned}(\mathcal{D}_0 + \boldsymbol{w} \cdot \nabla)\, \boldsymbol{w} = &-2\Omega_0 \hat{\boldsymbol{z}} \times \boldsymbol{w} - S_0 w_x \hat{\boldsymbol{y}} \\ &- \frac{\nabla P}{\rho} - \frac{\partial \Phi_0(z)}{\partial z} \hat{\boldsymbol{z}} + \frac{1}{\rho} \boldsymbol{J} \times \boldsymbol{B}\end{aligned}$$

$$(\mathcal{D}_0 + \boldsymbol{w} \cdot \nabla)\, \boldsymbol{B} = S_0 B_x \hat{\boldsymbol{y}} + (\boldsymbol{B} \cdot \nabla)\, \boldsymbol{w} - \boldsymbol{B}\, (\nabla \cdot \boldsymbol{w})$$

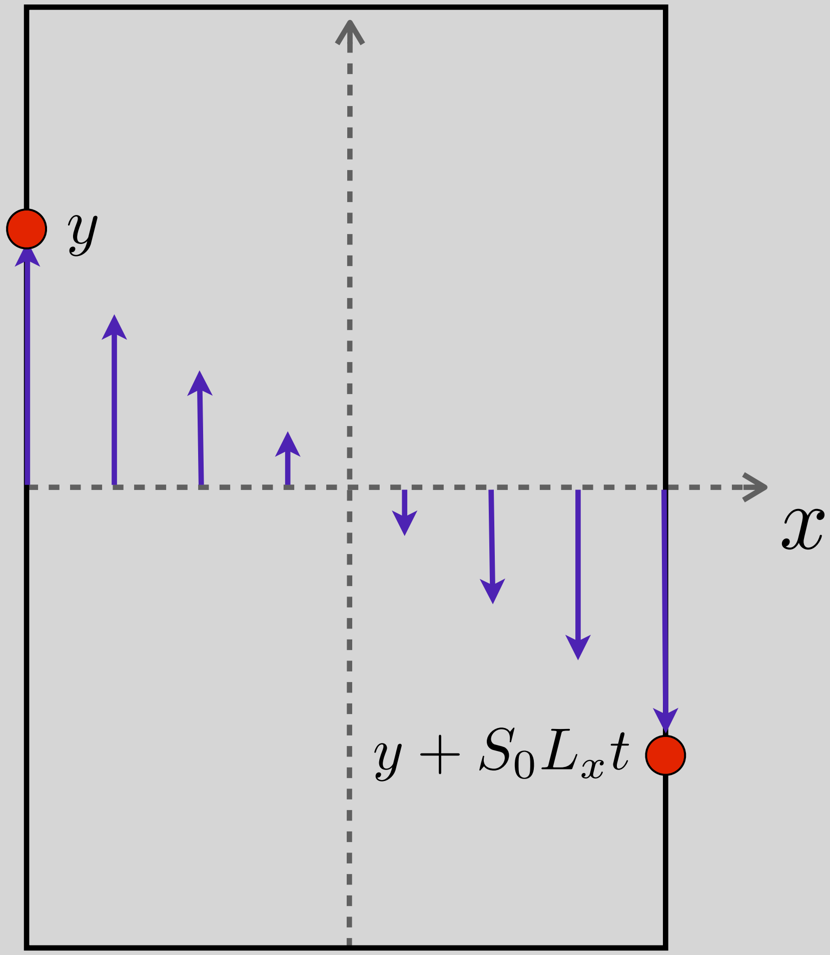
Isothermal, thin disks pressure support is negligible and angular frequency is height-independent

Shearing Periodic Boundaries

$$\mathcal{D}_0^{\mathrm{SSB}} \equiv \partial_t + x S_0 \partial_y$$

$$y' = y - x\, S_0\, t$$

$$\mathcal{D}_0^{\mathrm{SSB}} = \partial_t'$$

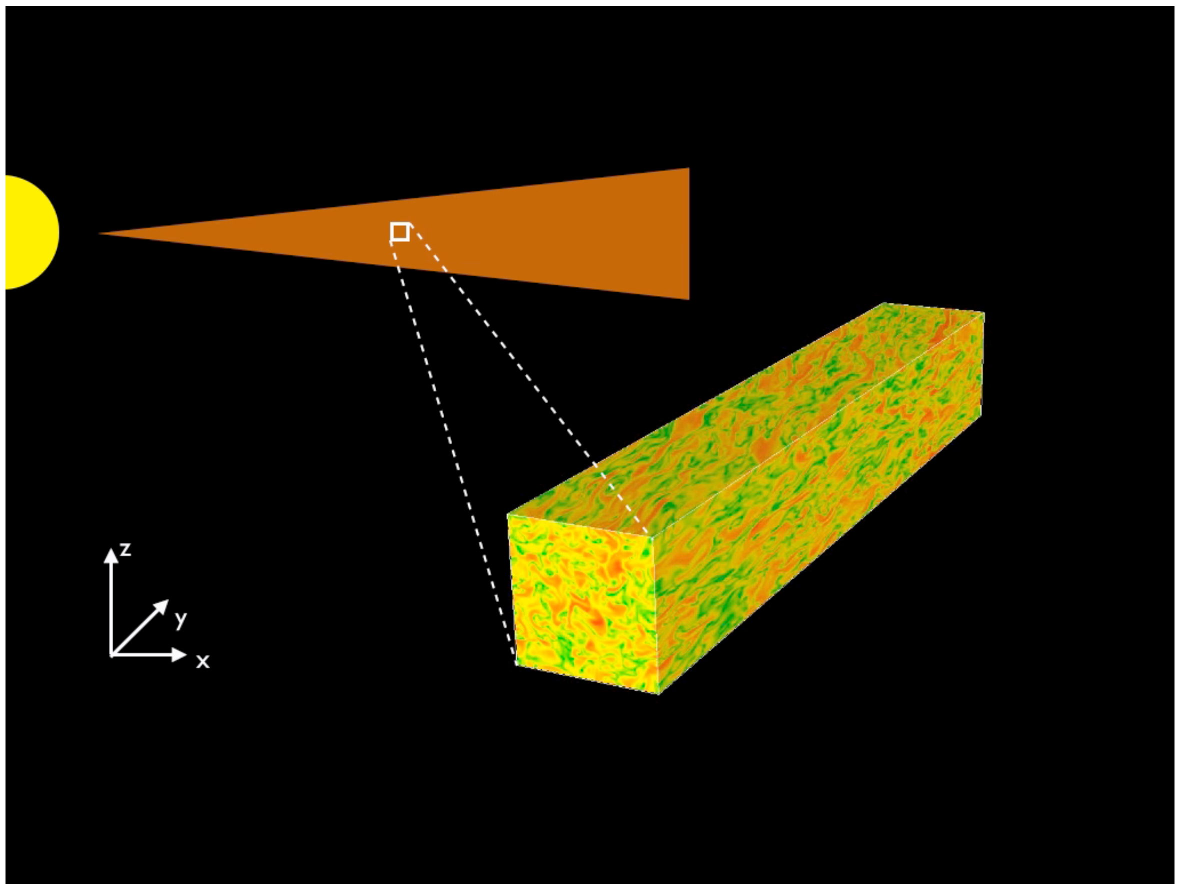


SSB

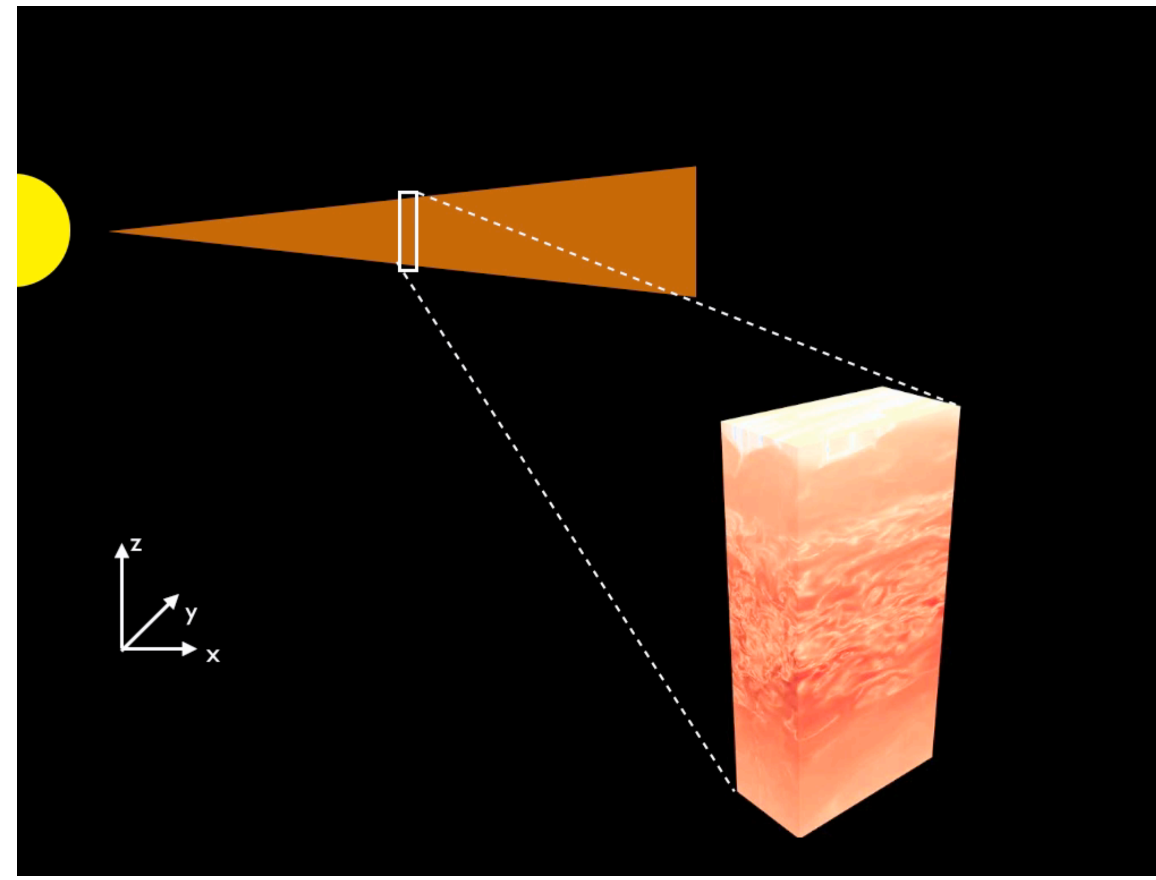
$$f(x', y', z', t') = f(x' + L_x, y', z', t')$$

$$f(x, y, z, t) = f(x + L_x, y + S_0 L_x t, z, t)$$

The Standard Shearing Box (SSB)



$$\frac{\partial \Phi}{\partial z} \simeq 0$$



$$\frac{\partial \Phi}{\partial z} \simeq z\Omega_0^2$$

from J. Simon's website

The Standard Shearing Box (SSB)

Pros:

- Offers a controlled environment

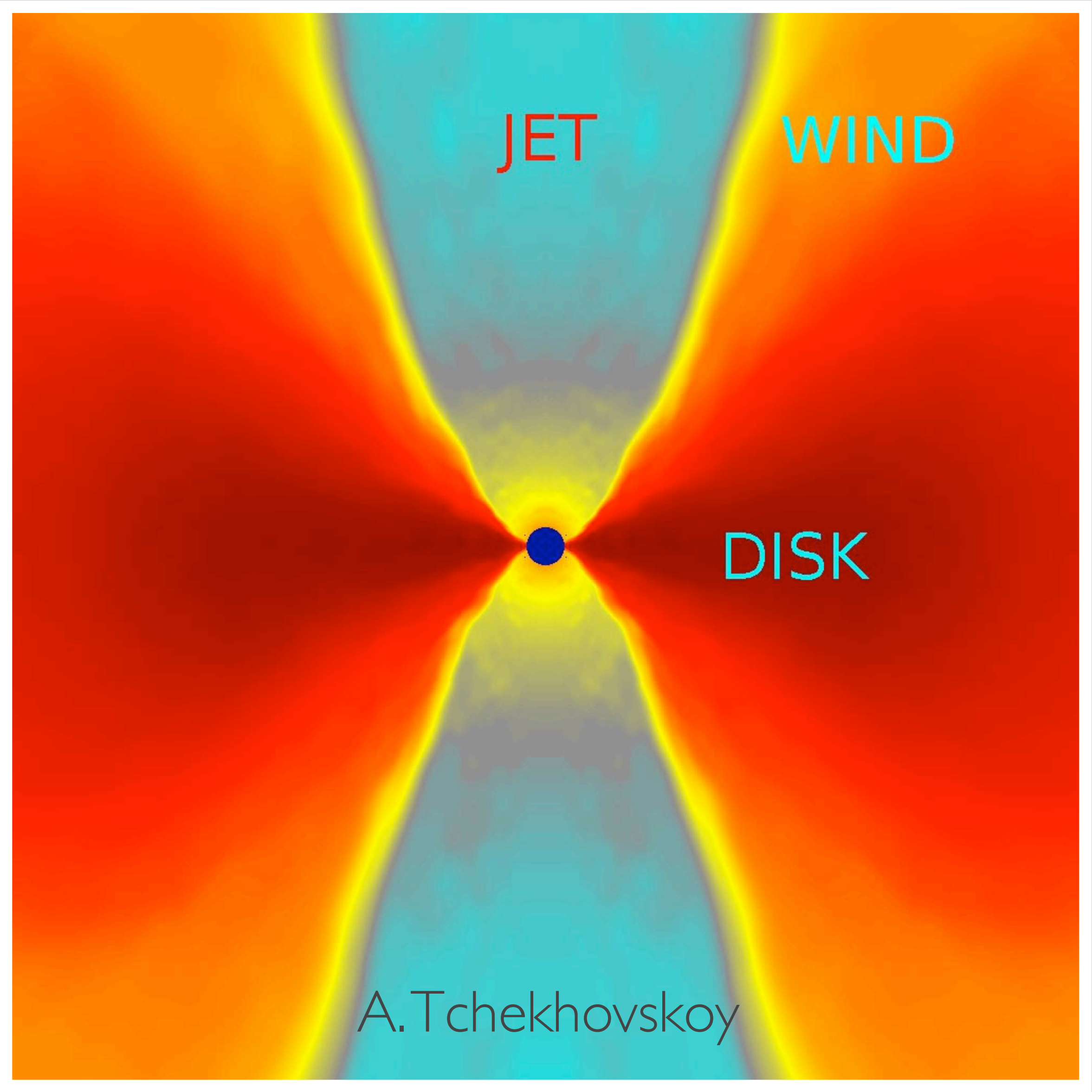
- Can study small-scale turbulent dynamics

Cons:

- Unable to address any global disc dynamics

- Framework valid for isothermal, thin disks

What if we care about z ~ R?



$$\Phi(r,z) = \frac{-GM}{\sqrt{r^2+z^2}}$$

$$\frac{\partial \Phi}{\partial z} = z\,\Omega_0^2 \left(1 + \frac{z^2}{r_0^2}\right)^{-3/2}$$

Beyond Vertically Local Models

- Isothermal disks are barotropic $P = P(\rho)$
- Barotropic fluids rotate on cylinders $\frac{d\Omega}{dz} = 0$

- Non-trivial thermal structure implies $\frac{d\Omega}{dz} \neq 0$
- Thick disks have pressure support $\nabla P \neq 0$

Vertically Global & Radially Local

- 1. Bulk disk flow

$$\boldsymbol{V}(r,z) = r\left[\Omega(r,z) - \Omega_{\mathrm{F}}\right]\hat{\boldsymbol{\phi}}$$

$$\Omega^2(r,z) \equiv \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r\rho_h}\frac{\partial P(\rho_h, e_h)}{\partial r}$$

$$\frac{1}{\rho_h}\frac{\partial P(\rho_h, e_h)}{\partial z} = -\frac{\partial \Phi}{\partial z}$$

- 2. Taylor expand in r

$$V(x,z) \equiv V_0(z) + S_0(z)x$$

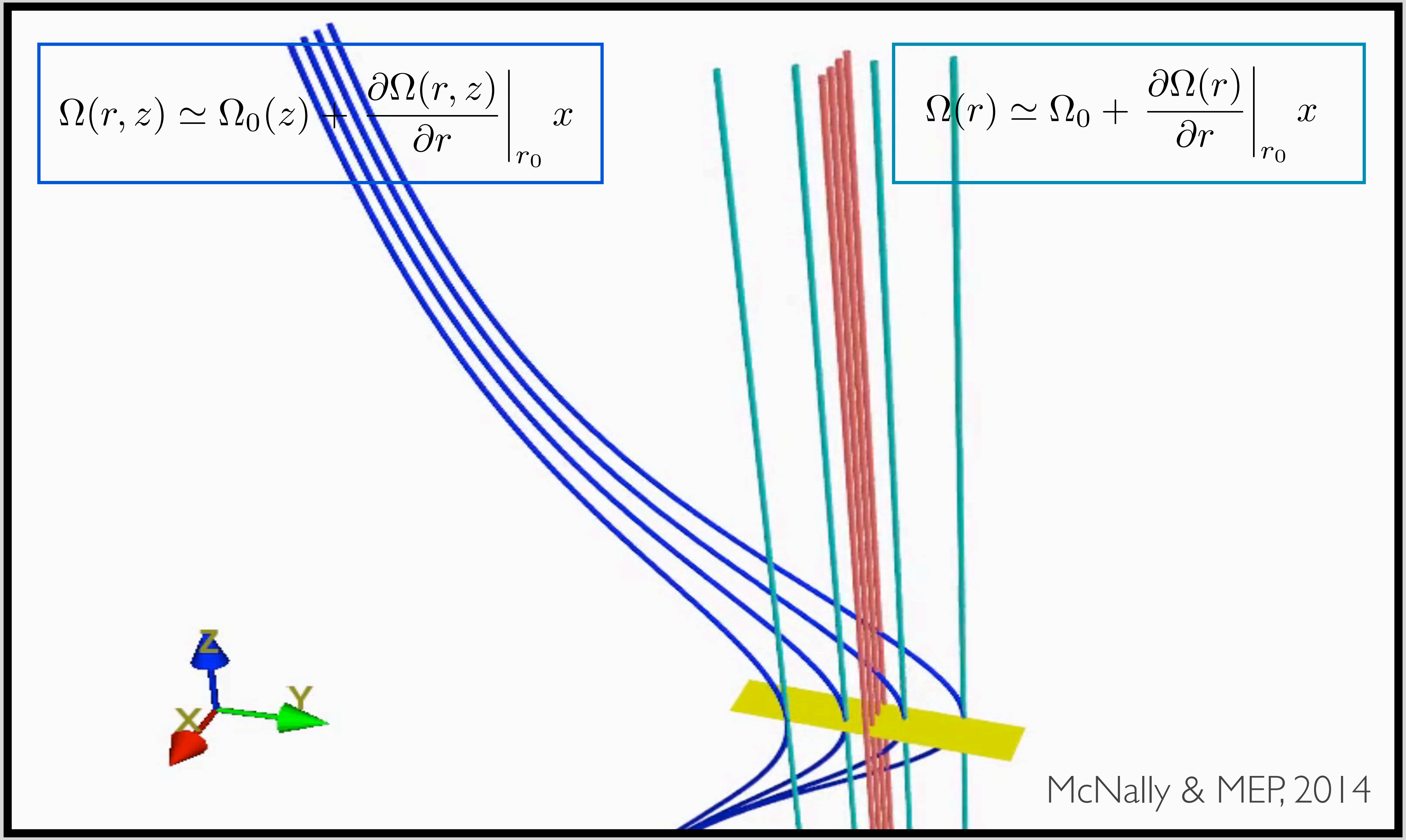
$$\Omega(x,z) \equiv \Omega_0(z) + \left.\frac{\partial \Omega(r,z)}{\partial r}\right|_{r=r_0} x$$

$$S_0(z) \equiv r_0 \left.\frac{\partial \Omega(r,z)}{\partial r}\right|_{r=r_0}$$

- 3. Define departures

$$\boldsymbol{w} \equiv \boldsymbol{v} - \left[V_0(z) + S_0(z)x\right]\hat{\boldsymbol{y}}$$

What is New?



- *Rotation rate depends on z*
- *Shear rate depends on z*

Vertically Global & Radially Local

- 4. Taylor-expand background equilibrium in r

$$\rho_h(r,z) = \rho_{h0}(z) + \mathrm{O}\left(\frac{x}{r_0}\right)$$

$$e_h(r,z) = e_{h0}(z) + \mathrm{O}\left(\frac{x}{r_0}\right)$$

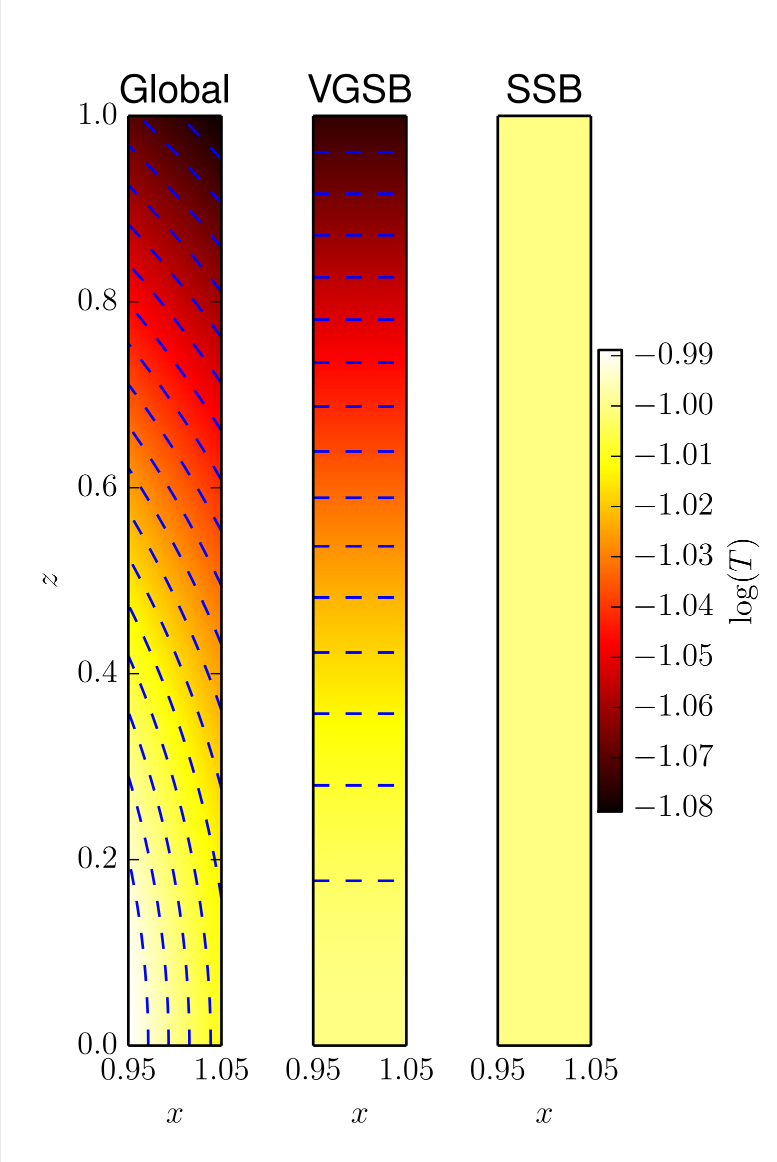
$$\frac{\nabla P(\rho_h, e_h)}{\rho_h} = \frac{\nabla P(\rho_{h0}, e_{h0})}{\rho_{h0}} + \mathrm{O}\left(\frac{x}{r_0}\right)$$

$$\frac{1}{\rho_{h0}} \frac{\partial P(\rho_{h0}, e_{h0})}{\partial z} = -\frac{\partial \Phi_0(z)}{\partial z}$$

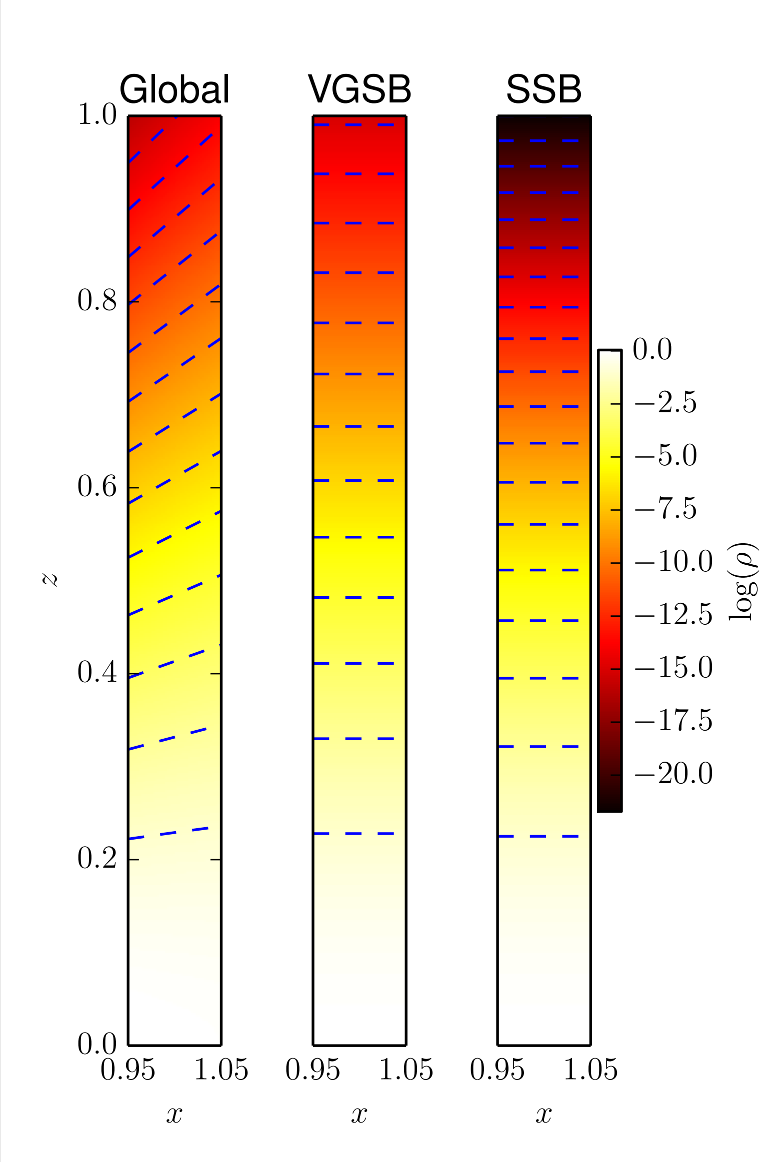
This eliminates radial gradients, which is necessary for domains with shearing-periodic boundaries

Vertically Global & Radially Local

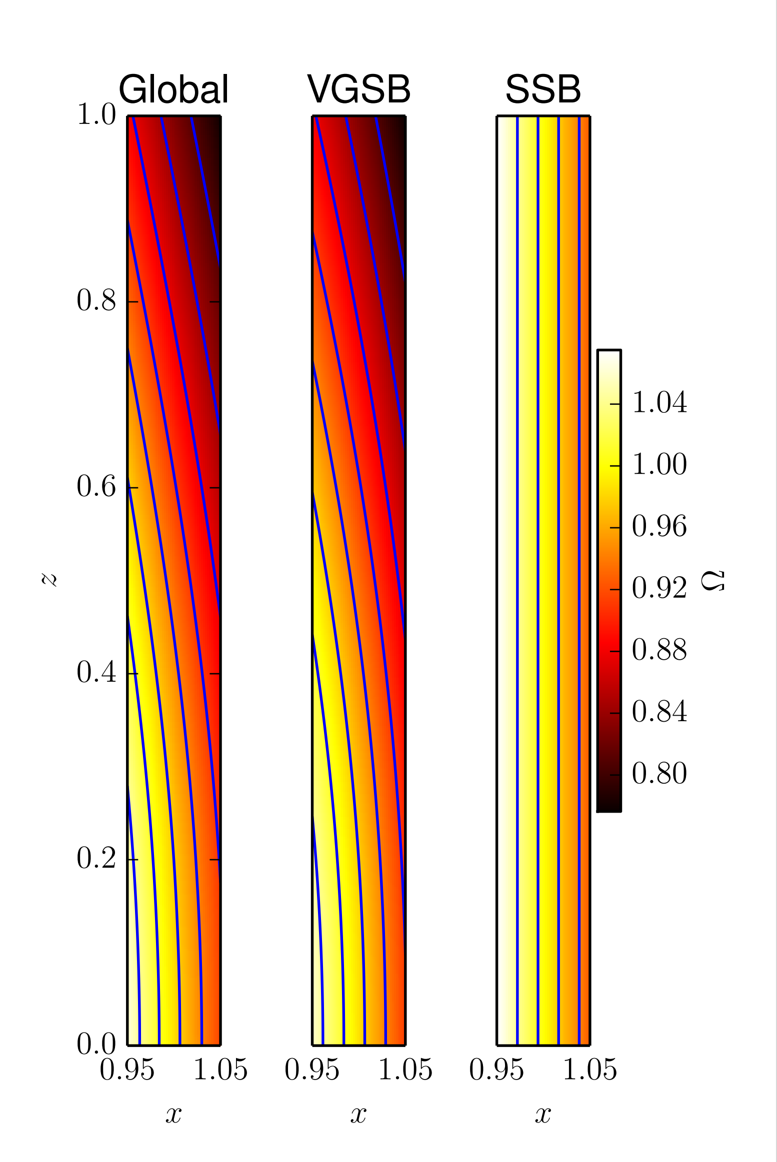
Temperature



Density



Angular Frequency



McNally & MEP, 2014

Vertically Global, Radially Local

$$\boldsymbol{w} \equiv \boldsymbol{v} - [V_0(z) + S_0(z)x]\,\hat{\boldsymbol{y}}$$

$$\mathcal{D}_0 \equiv \partial_t + [V_0(z) + S_0(z)x]\,\partial_y$$

momentum

$$(\mathcal{D}_0 + \boldsymbol{w}\cdot\nabla)\,\boldsymbol{w} + w_z \frac{\partial V(x,z)}{\partial z}\hat{\boldsymbol{y}} =$$

$$- 2\Omega_0(z)\hat{\boldsymbol{z}} \times \boldsymbol{w} - S_0(z) w_x \hat{\boldsymbol{y}} - \frac{\nabla P}{\rho} - \frac{\partial \Phi_0(z)}{\partial z}\hat{\boldsymbol{z}} + \frac{1}{\rho}\boldsymbol{J}\times\boldsymbol{B}$$

induction

$$(\mathcal{D}_0 + \boldsymbol{w}\cdot\nabla)\,\boldsymbol{B} - B_z \frac{\partial V(x,z)}{\partial z}\hat{\boldsymbol{y}} =$$

$$S_0(z) B_x \hat{\boldsymbol{y}} + (\boldsymbol{B}\cdot\nabla)\,\boldsymbol{w} - \boldsymbol{B}\,(\nabla\cdot\boldsymbol{w})$$

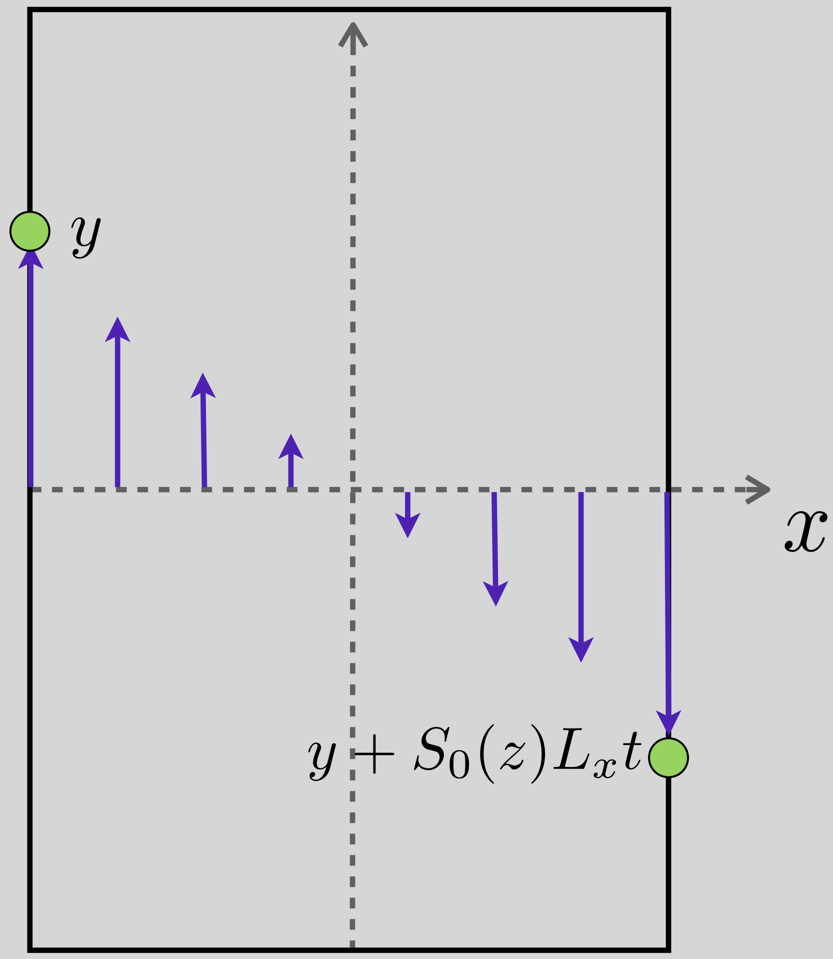
Not-consistent with shearing-periodic boundaries!

Shearing Periodic Boundaries(z)

$$\mathcal{D}_0 \equiv \partial_t + [V_0(z) + S_0(z)x]\,\partial_y$$

$$y' = y - [V_0(z) + S_0(z)\,x]\,t$$

$$\mathcal{D}_0 = \partial_t'$$



VGSB

$$f(x', y', z', t') = f(x' + L_x, y', z', t')$$

$$f(x, y, z, t) = f(x + L_x, y + S_0(z)L_x t, z, t)$$

For this to work we only need the approx.

$$\partial_z V(x, z) = \partial_z V_0(z) + x\,\partial_z S_0(z)$$

$$\partial_z V(x, z) \simeq \partial_z V_0(z)$$

Vertically Global Shearing Box

$$\boldsymbol{w} \equiv \boldsymbol{v} - [V_0(z) + S_0(z)x]\,\hat{\boldsymbol{y}}$$

$$\mathcal{D}_0 \equiv \partial_t + [V_0(z) + S_0(z)x]\,\partial_y$$

momentum

$$(\mathcal{D}_0 + \boldsymbol{w}\cdot\nabla)\,\boldsymbol{w} + w_z \frac{\partial V_0(z)}{\partial z}\hat{\boldsymbol{y}} = - 2\Omega_0(z)\hat{\boldsymbol{z}} \times \boldsymbol{w} - S_0(z) w_x \hat{\boldsymbol{y}} - \frac{\nabla P}{\rho} - \frac{\partial \Phi_0(z)}{\partial z}\hat{\boldsymbol{z}} + \frac{1}{\rho}\boldsymbol{J} \times \boldsymbol{B}$$

induction

$$(\mathcal{D}_0 + \boldsymbol{w}\cdot\nabla)\,\boldsymbol{B} - B_z \frac{\partial V_0(z)}{\partial z}\hat{\boldsymbol{y}} = S_0(z) B_x \hat{\boldsymbol{y}} + (\boldsymbol{B}\cdot\nabla)\,\boldsymbol{w} - \boldsymbol{B}\,(\nabla\cdot\boldsymbol{w})$$

Amenable to shearing-periodic boundary conditions !!

Conserved Fluid Properties

- Kelvin's circulation theorem

$$\partial_t(\nabla \times \boldsymbol{v}) = \nabla \times [\boldsymbol{v} \times (\nabla \times \boldsymbol{v})]$$

$$\Gamma = \oint \boldsymbol{v} \cdot d\boldsymbol{l}$$

↓

$$[\partial_t + \boldsymbol{v} \cdot \nabla](\nabla \cdot (\nabla \times \boldsymbol{v})) = 0$$

→

$$[\partial_t + \boldsymbol{v} \cdot \nabla]\,\Gamma = 0$$

- Alfven's frozen-in theorem

$$\partial_t \boldsymbol{B} = \nabla \times [\boldsymbol{v} \times (\nabla \times \boldsymbol{B})]$$

$$\Phi_{\mathrm{B}} = \int \boldsymbol{B} \cdot d\boldsymbol{S}$$

↓

$$[\partial_t + \boldsymbol{v} \cdot \nabla](\nabla \cdot \boldsymbol{B}) = 0$$

→

$$[\partial_t + \boldsymbol{v} \cdot \nabla]\,\Phi_{\mathrm{B}} = 0$$

Conserved Fluid Properties in VGSB

- Kelvin's circulation theorem

$$(\mathcal{D}_0 + \boldsymbol{w}\cdot\nabla)\,\Gamma = \int_{\mathrm{S}} \left[\nabla \times \left(x w_z \frac{\partial S_0(z)}{\partial z}\hat{\boldsymbol{y}} \right)\right] + \dots$$

- Alfven's frozen-in theorem

$$(\mathcal{D}_0 + \boldsymbol{w}\cdot\nabla)\,(\nabla\cdot\boldsymbol{B}) = -x\frac{\partial S_0(z)}{\partial z}\frac{\partial B_z}{\partial y}$$

- If barotropic or axisymmetric

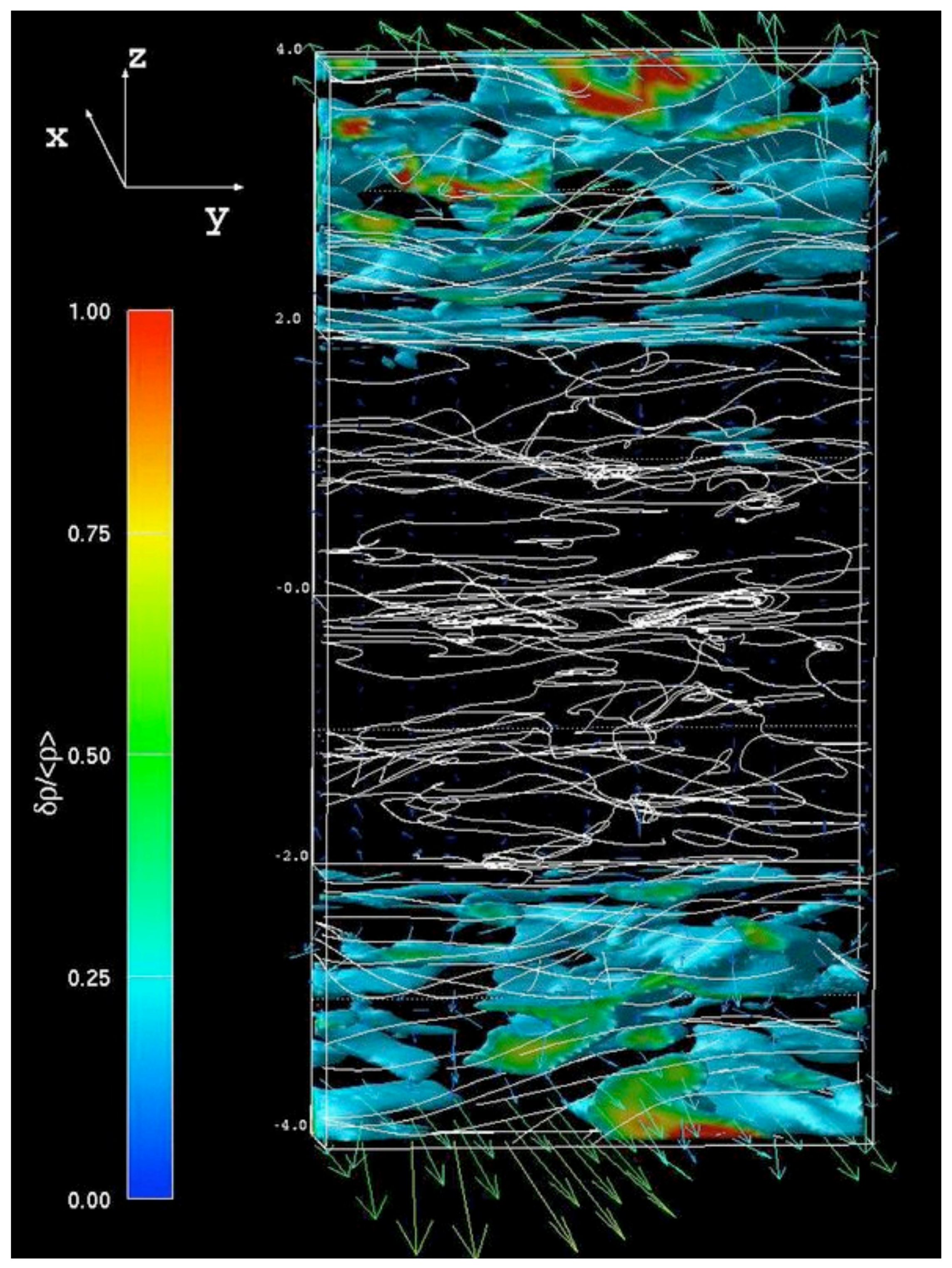
$$(\mathcal{D}_0 + \boldsymbol{w}\cdot\nabla)\,\Gamma = 0$$

$$(\mathcal{D}_0 + \boldsymbol{w}\cdot\nabla)\,\Phi_{\mathrm{B}} = 0$$

When Will It Matter?

- Hydrodynamic Disk Instabilities
- Disk Convection
- Disk Coronae and Thick Disks
- Disk Winds
- Interstellar Medium and Galactic Disks

Disk Winds



Suzuki & Inutsuka 2009

Usual assumption

$$\frac{\partial \Phi}{\partial z} \simeq z\Omega_0^2$$

makes it hard to launch winds

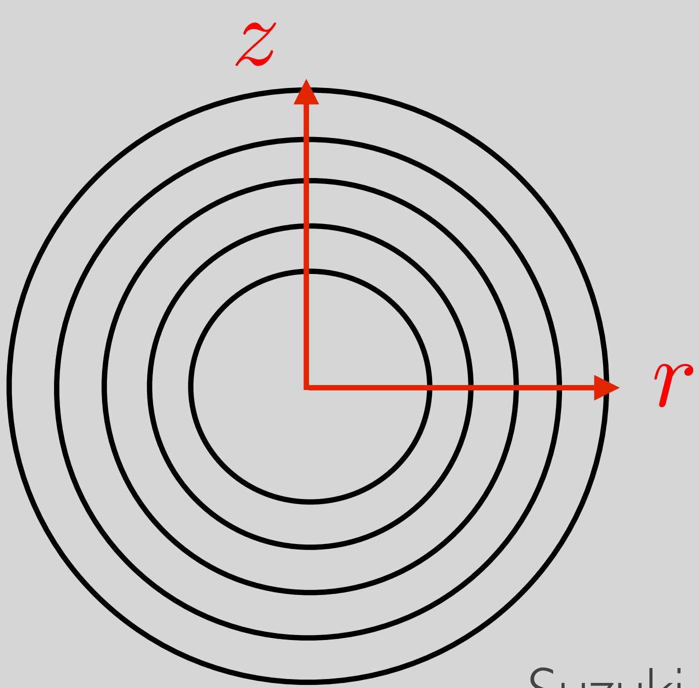
Some works have used

$$\frac{\partial \Phi}{\partial z} = z\,\Omega_0^2 \left(1 + \frac{z^2}{r_0^2}\right)^{-3/2}$$

without considering changes in rotation rate

Spherical Temperature Disk Structure

$$T(r,z) \equiv T_0 \frac{r_0}{\sqrt{r^2+z^2}}$$



Suzuki & Inutsuka, 2013

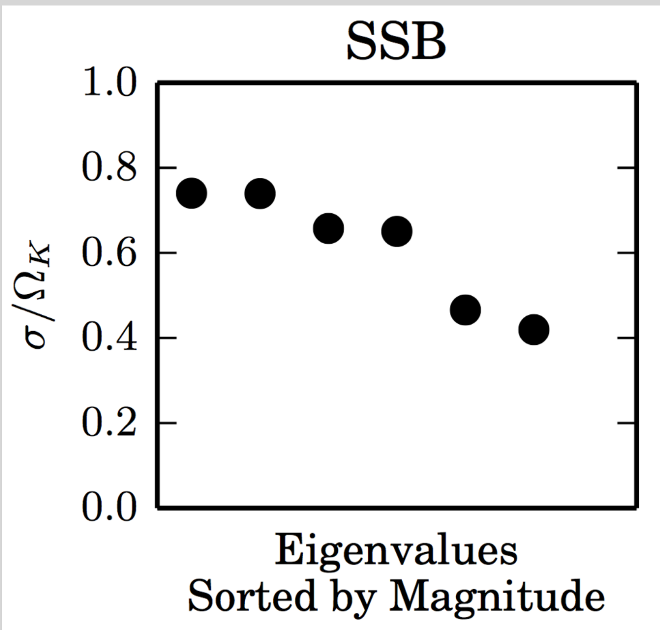
$$\rho(r,z) = \rho_0 \left(\frac{r}{\sqrt{r^2+z^2}}\right)^{\nu} \left(\frac{\sqrt{r^2+z^2}}{r_0}\right)^{1-\mu}$$

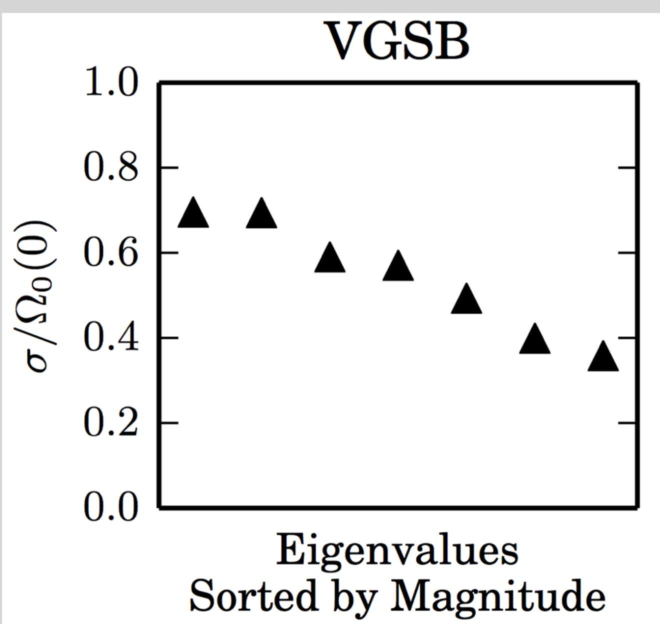
$$\Omega(r,z) = \sqrt{\nu}\,\frac{c_{\mathrm{s0}}}{r} \left(\frac{\sqrt{r^2+z^2}}{r_0}\right)^{-1/2}$$

$$\nu + \mu = \frac{v_{\mathrm{K0}}^2}{c_{\mathrm{s0}}^2}$$

Magnetorotational Instability (MRI)

Growth Rates

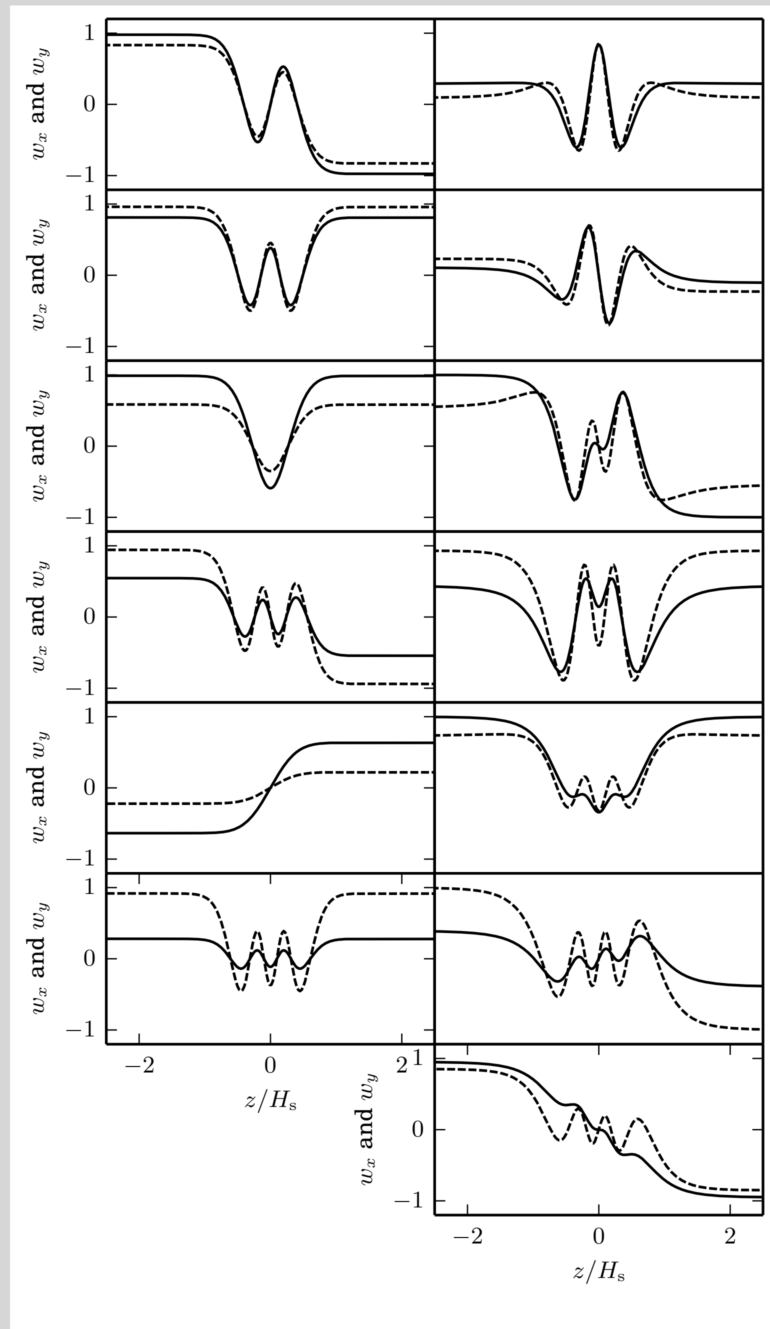




McNally & MEP, 2014

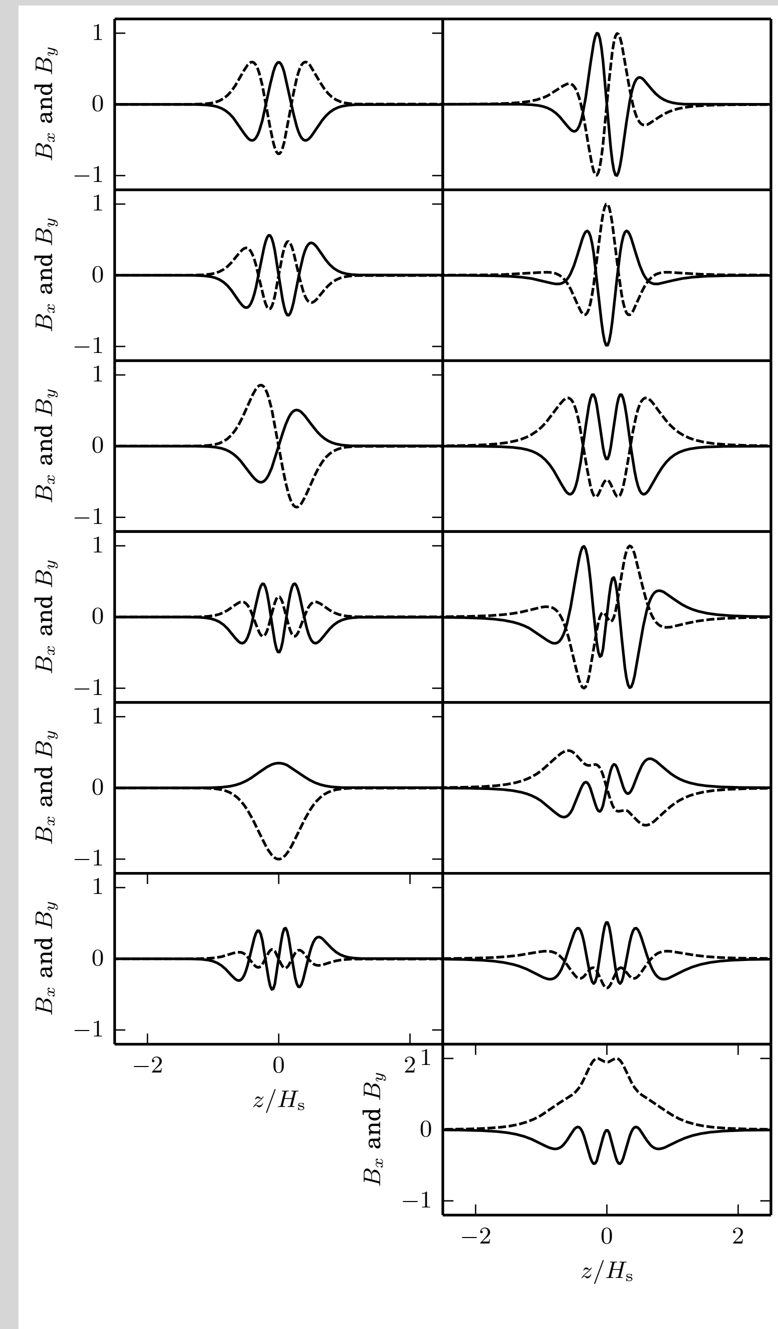
Velocity Field

SSB VGSB



Magnetic Field

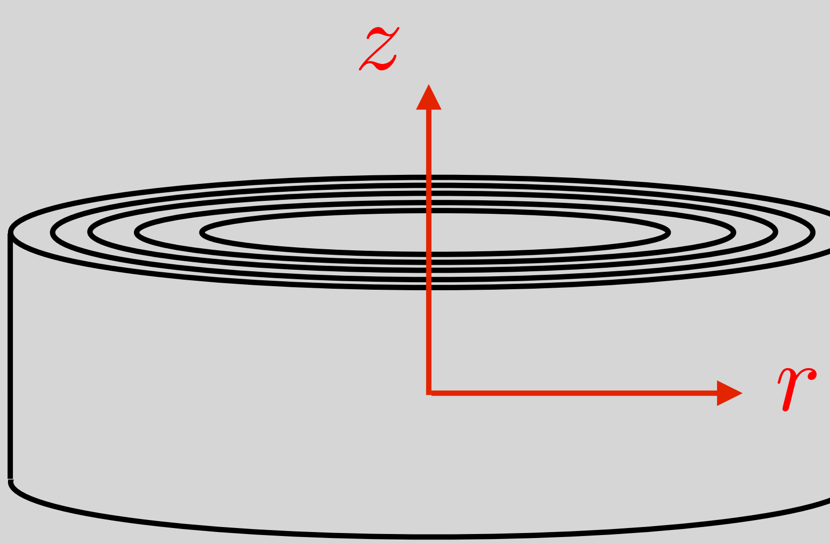
SSB VGSB



so far MRI only studied in isothermal disks!

Cylindrical Temperature Disk Structure

$$T(r) \equiv T_0 \left(\frac{r}{r_0}\right)^q$$



Nelson, Gressel, & Umurhan 2013

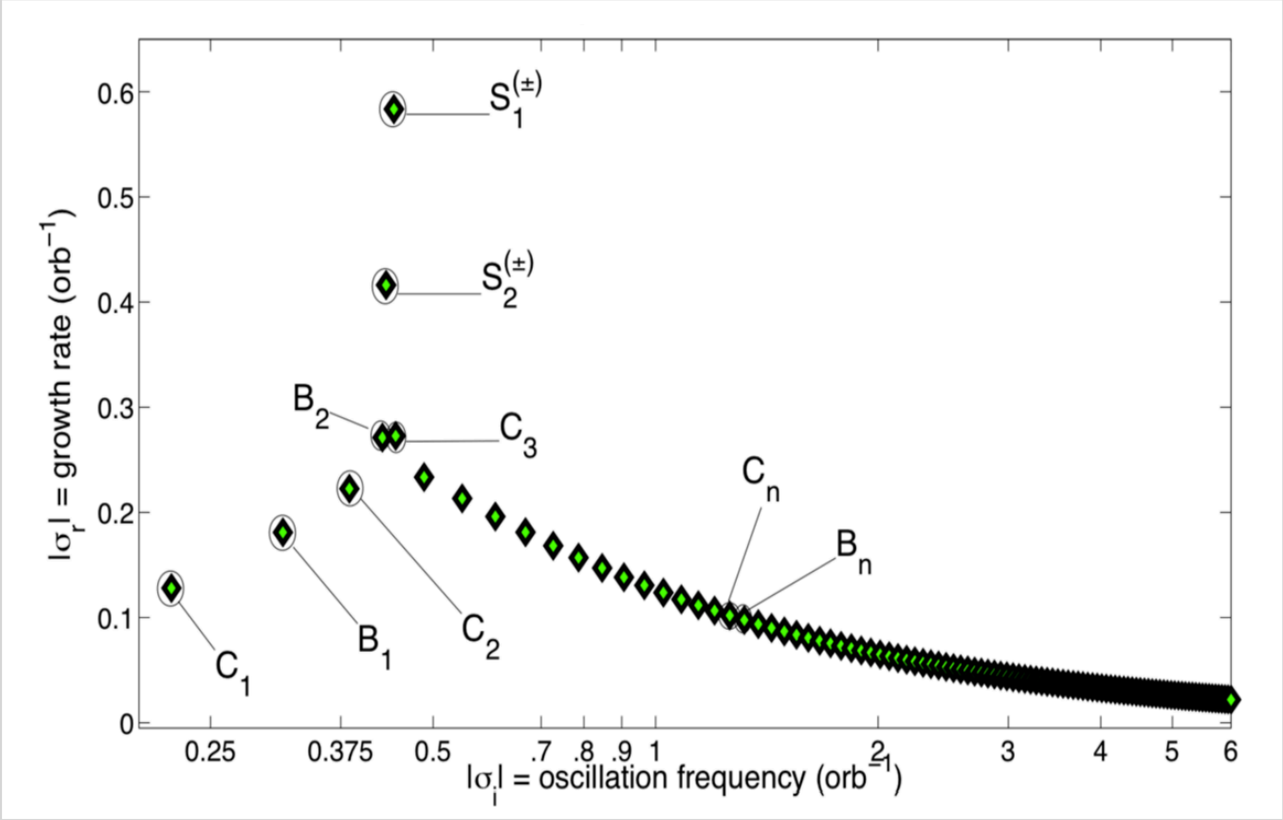
$$\rho(r,z) = \rho_0 \left(\frac{r}{r_0}\right)^p \exp\left[-\frac{v_{\mathrm{K}}^2}{c_{\mathrm{s}}^2}\left(1 - \frac{1}{\sqrt{1+(z/r)^2}}\right)\right]$$

$$\Omega(r,z) = \Omega_{\mathrm{K}} \sqrt{1 + (p+q)\frac{c_{\mathrm{s}}^2}{v_{\mathrm{K}}^2} + q\left(1 - \frac{1}{\sqrt{1+(z/r)^2}}\right)}$$

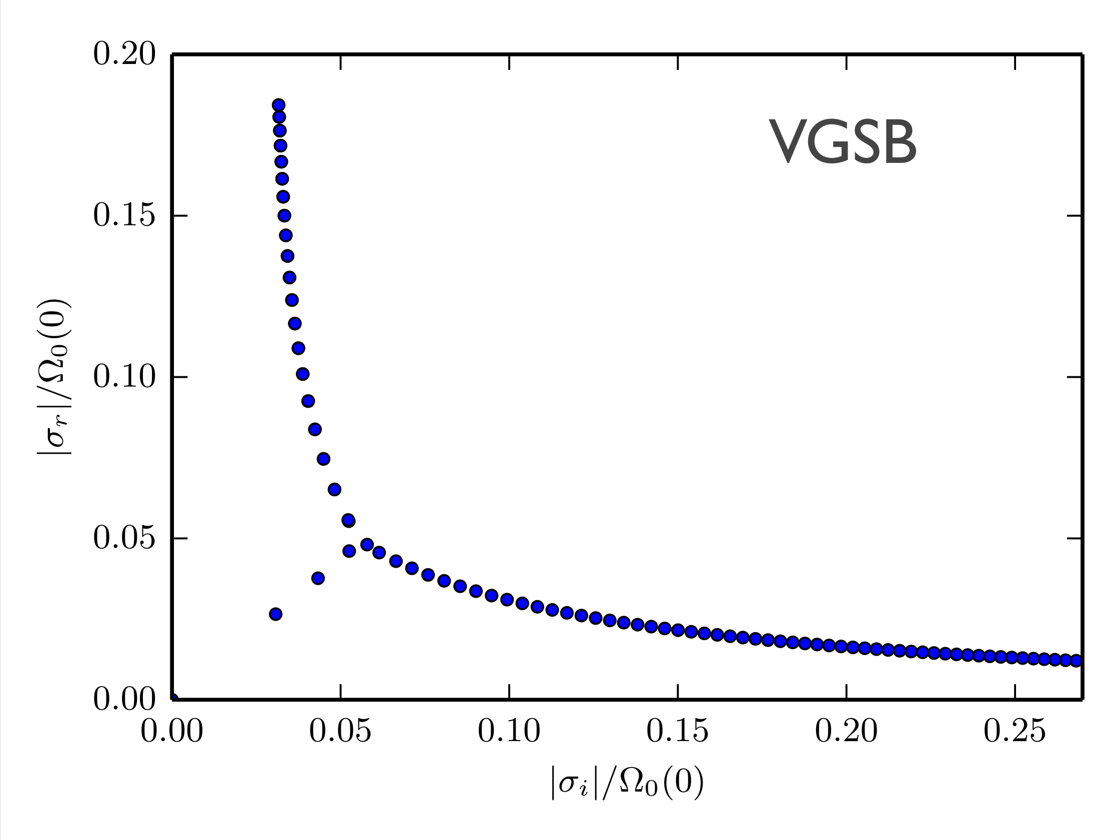
Vertical Shear Instability (VSI)

- Thin, isothermal hydro disks seem to be stable, but there are several instabilities that feed off vertical shear (GSF in 60's!)

- These could be important in low-ionization protoplanetary disks



Nelson, Gressel, & Umurhan 2013



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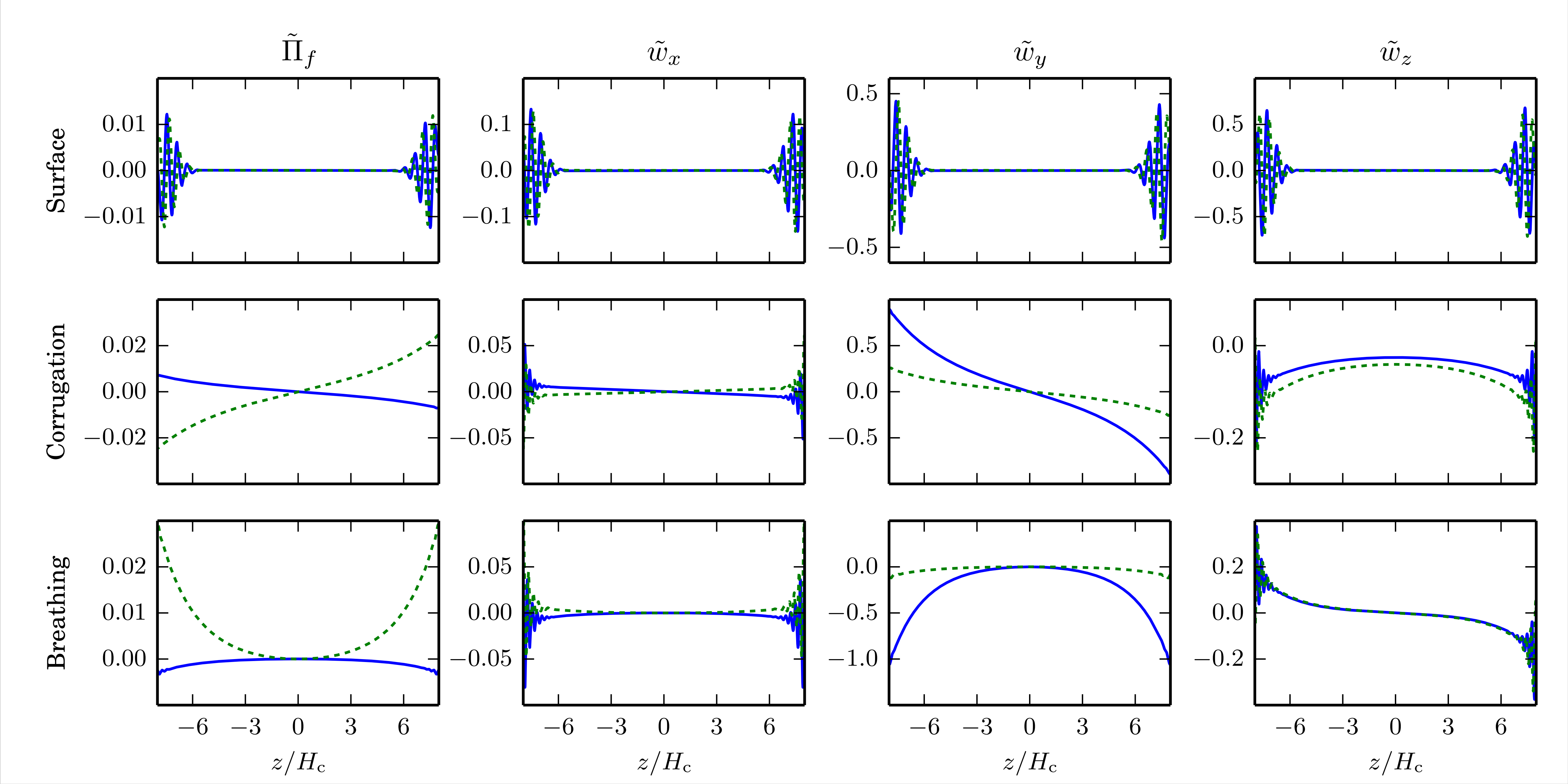
Unstable Modes In Hydro Disks



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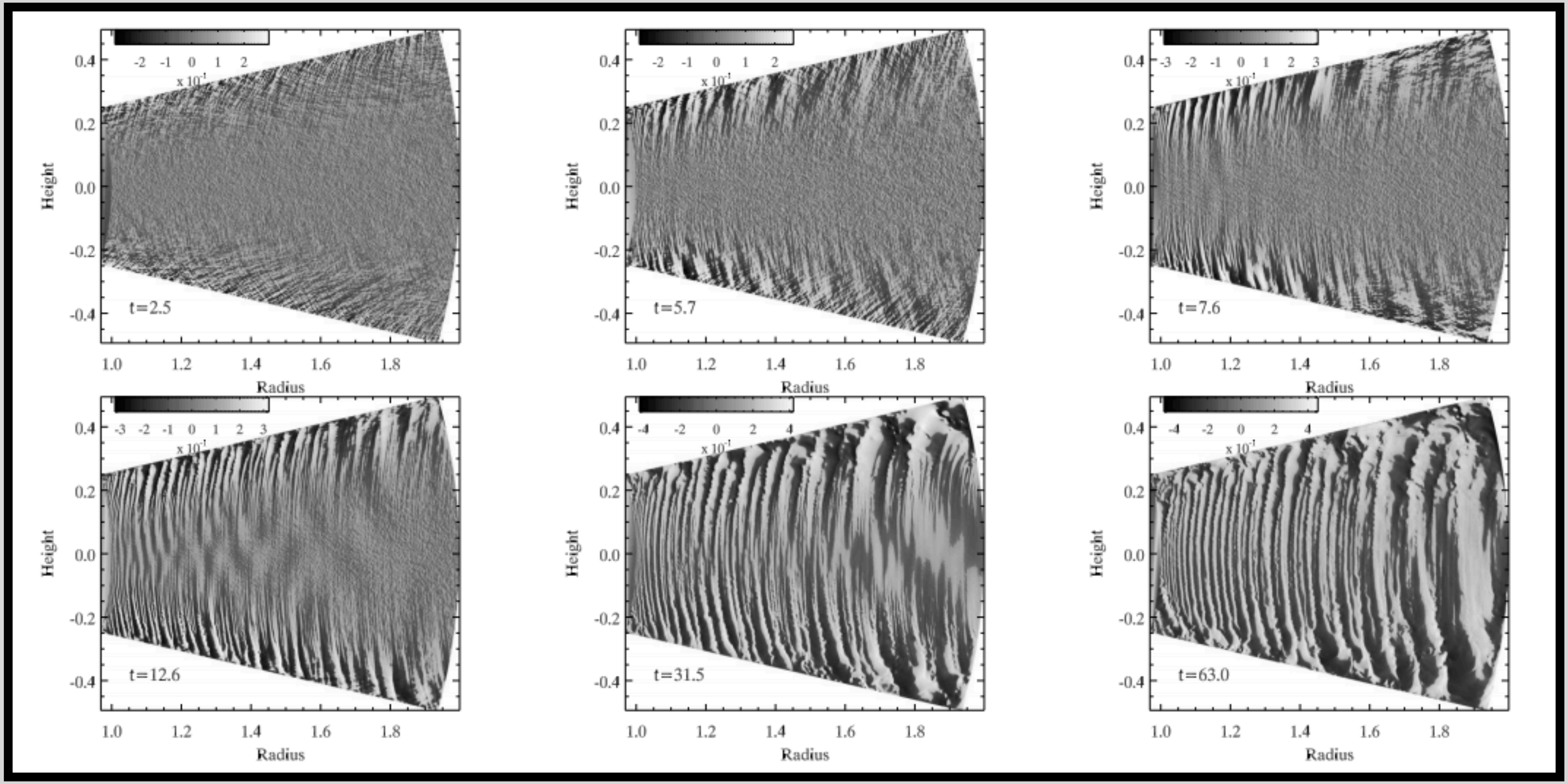
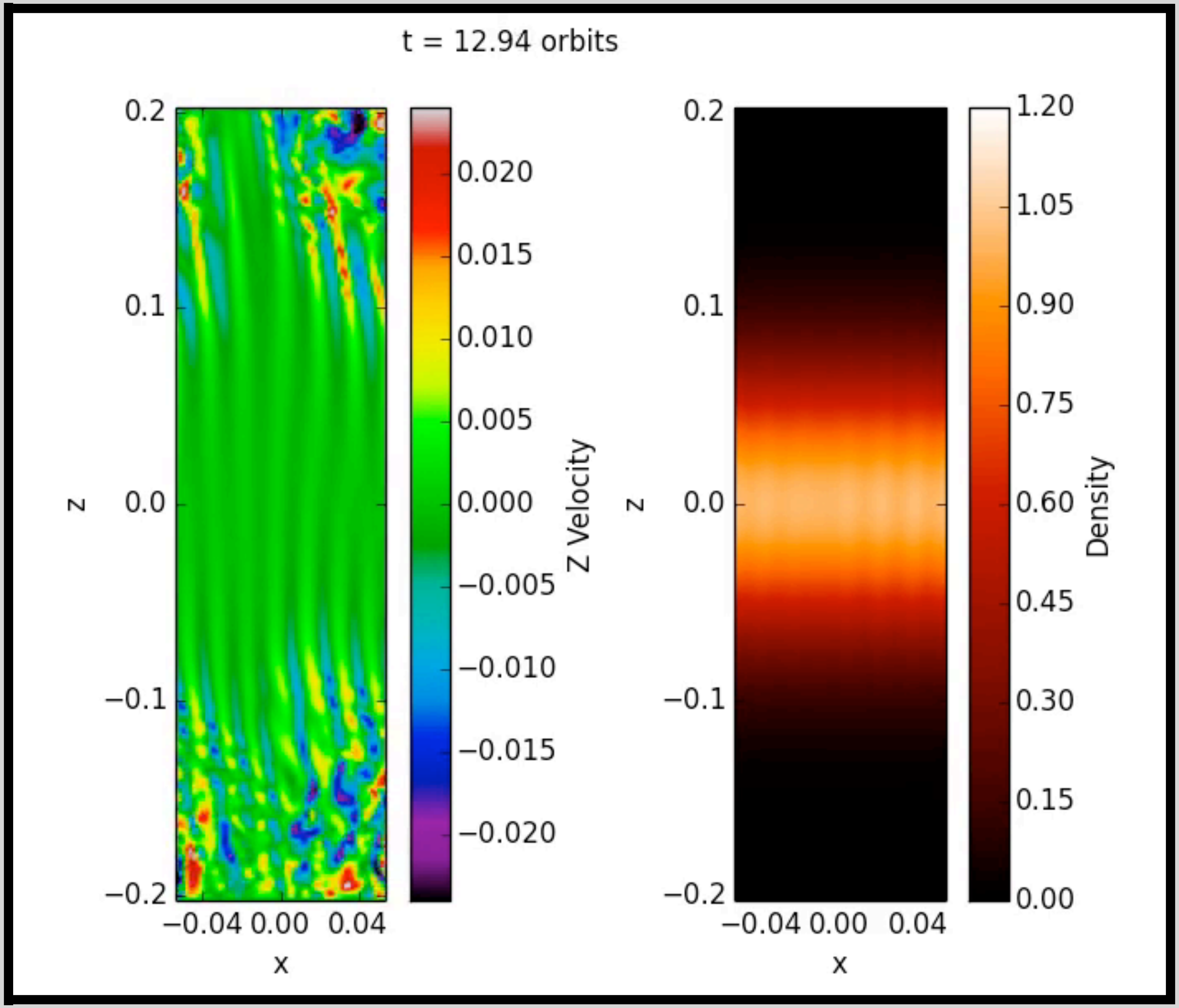
Hydrodynamic Disk Instabilities



Nelson, Gressel, & Umurhan 2013

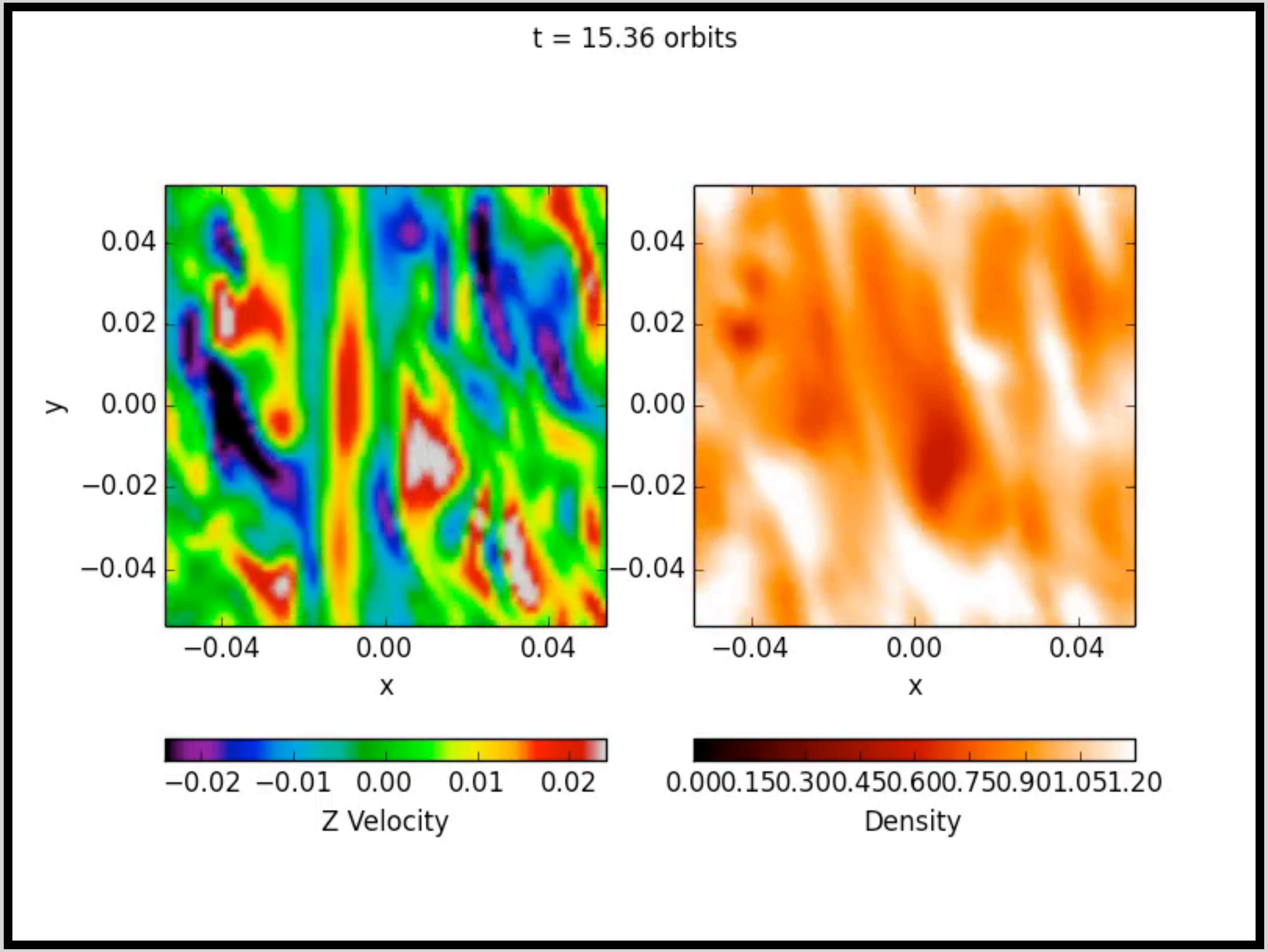
A First Implementation of the VGSB



$$q = -1.5 \qquad p = -1.0 \qquad c = 0.05\, v_{\rm K}$$

McNally & MEP, 2014

A First Implementation of the VGSB



$$q = -1.5 \qquad p = -1.0 \qquad c = 0.05\, v_{\mathrm{K}}$$

McNally & MEP, 2014

Thank you!

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