1 **Coin tossing**

You flip a coin $n = 10$ times and you obtain 8 heads.

(a) What is the likelihood function for this measurement? Identify explicitly what are the data and what is the free parameter you are trying to estimate.

(b) What is the Maximum Likelihood Estimate for the probability of obtaining heads in one flip, p?

(c) Approximate the likelihood function as a Gaussian around its peak and derive the 1σ confidence interval for p. How would you report your result for p?

(d) With how many σ confidence can you exclude the hypothesis that the coin is fair? (Hint: compute the distance between the MLE for p and $p = 1/2$ and express the result in number of σ).

(e) You now flip the coin 1000 times and obtain 800 heads. What is the MLE for p now and what is the 1σ confidence interval for p? With how many σ confidence can you exclude the hypothesis that the coin is fair now?

2 **Counting experiment**

(a) An experiment counting particles emitted by a radioactive decay measures r particles per unit time interval. The counts are Poisson distributed. If λ is the average number of counts per per unit time interval, write down the appropriate probability distribution function for r.

(b) Now we seek to determine λ by repeatedly measuring for M times the number of counts per unit time interval. This series of measurements yields a sequence of counts $\hat{\mathbf{r}} = \{\hat{r}_1, \hat{r}_2, \hat{r}_3, ..., \hat{r}_M\}$. Each measurement is assumed to be independent. Derive the combined likelihood function for λ, $\mathcal{L}(\lambda) = P(\hat{\mathbf{r}}|\lambda)$, given the measured sequence of counts $\hat{\mathbf{r}}$.

(c) Use the Maximum Likelihood Principle applied to the the log likelihood $\ln \mathcal{L}(\lambda)$ to show that the Maximum Likelihood estimator for the average rate λ is just the average of the measured counts, $\hat{\mathbf{r}}$, i.e.

$$\lambda_{\rm ML} = \frac{1}{M}\sum_{i=1}^{M}\hat{r}_i \, .$$

(d) By considering the Taylor expansion of $\ln \mathcal{L}(\lambda)$ to second order around $\lambda_{\rm ML}$, derive the Gaussian approximation for the likelihood $\mathcal{L}(\lambda)$ around the Maximum Likelihood point (see Eq. (63) in the handout), and show that it can be written as

$$\mathcal{L}(\lambda) \approx L_0 \exp\left(-\frac{1}{2}\frac{M}{\lambda_{\rm ML}}(\lambda - \lambda_{\rm ML})^2\right) ,$$

where L_0 is a normalization constant.

(e) Compare with the equivalent expression for M Gaussian-distributed measurements to show that the variance σ^2 of the Poisson distribution is given by $\sigma^2 = \lambda$.

3 **Gaussian measurements with different variance**

This problem generalizes Examples 22 and 27 in the handout to the case where the measurements have different uncertainties among them.

You measure the flux F of photons from a laser source using 4 different instruments and you obtain the following results (units of 10^4 photons/cm^2):

$$34.7 \pm 5.0, \quad 28.9 \pm 2.0, \quad 27.1 \pm 3.0, \quad 30.6 \pm 4.0. \tag{1}$$

(a) Write down the likelihood for each measurement, and explain why a Gaussian approximation is justified in this case.

(b) Write down the total likelihood for the combination of the 4 measurements.

(c) Find the MLE of the photon flux, $F_{\rm ML}$, and show that it is given by:

$$F_{\rm ML} = \sum_i \frac{\hat{n}_i}{\hat{\sigma}_i^2/\bar{\sigma}^2}, \tag{2}$$

where

$$\frac{1}{\bar{\sigma}^2} \equiv \sum_i \frac{1}{\hat{\sigma}_i^2}. \tag{3}$$

(d) Compute $F_{\rm ML}$ from the data above and compare it with the sample mean.

(e) Find the 1σ confidence interval for your MLE for the mean, and show that it is given by:

$$\left(\sum_i \frac{1}{\hat{\sigma}_i^2}\right)^{-1/2}. \tag{4}$$

Evaluate the confidence interval for the above data. How would you summarize your measurement of the flux F?

4 Photon counts and source strength

An astronomer measures the photon flux from a distant star using a very sensitive instrument that counts single photons. After one minute of observation, the instrument has collected $\hat{r}$ photons. One can assume that the photon counts, $\hat{r}$, are distributed according to the Poisson distribution. The astronomer wishes to determine λ, the emission rate of the source.

(a) What is the likelihood function for the measurement? Identify explicitly what is the unknown parameter and what are the data in the problem.

(b) If the true rate is $\lambda = 10$ photons/minute, what is the probability of observing $\hat{r} = 15$ photons in one minute?

(c) Find the Maximum Likelihood Estimate for the rate λ (i.e., the number of photons per minute). What is the maximum likelihood estimate if the observed number of photons is $\hat{r} = 10$?

(d) Upon reflection, the astronomer realizes that the photon flux is the superposition of photons coming from the star plus "background" photons coming from other faint sources within the field of view of the instrument. The background rate is supposed to be known, and it is given by λ_b photons per minute (this can be estimated e.g. by pointing the telescope away from the source and measuring the photon counts there, when the telescope is only picking up background photos). She then points to the star again, measuring $\hat{r}_t$ photons in a time t_t. What is her maximum likelihood estimate of the rate λ_s from the star in this case?

Hint: The total number of photons $\hat{r}_t$ is Poisson distributed with rate $\lambda = \lambda_s + \lambda_b$, where λ_s is the rate for the star.

(e) What is the source rate (i.e., the rate for the star) if $\hat{r}_t = 30$, $t_t = 2$ mins, and $\lambda_b = 12$ photons per minute? Is it possible that the measured average rate from the source (i.e., $\hat{r}_t/t_t$) is less than λ_b? Discuss what happens in this case and comment on the physicality of this result.

5 Dice throwing

Two dice, each with the shape of a regular tetrahedron (i.e., a polyhedron composed of 4 equilateral, identical triangles), are thrown together. The 4 sides of each dice are numbered from 1 to 4.

(a) Sketch the probability distribution of the sum of their values.

(b) Now 1000 dice of the same sort are thrown. What is the probability distribution of the sum of their values? *Hint: for this part, use the Central Limit Theorem.*

6 **Bayesian estimation of the flux**

An astronomer wishes to know the (mono-chromatic) flux of a particular source and makes a photometric measurement which registers $N_{\rm src}$ photons.

Assume that all the photons have come from the source itself (*i.e.*, there is no background or or source confusion) and that the known calibration constant, C, is such that a source of true flux $F_{\rm src}$ would, on average, yield $F_{\rm src}/C$ photons in such a measurement (*i.e.*, a generic estimate of the source's flux would be $\hat{F}_{\rm src} \simeq CN_{\rm src}$).

(a) What is the model parameter that the astronomer is trying to infer?

(b) What are data?

(c) What is the likelihood [*i.e.*, the probability $\mathrm{P}(N_{\rm src}|F_{\rm src})$]?

(d) What prior information might the astronomer have *before* making (or at least making use of) the measurement?

(e) If the astronomer had access to a catalogue of sources of similar fluxes from a different part of the sky, how might this catalogue be used to generate an appropriate, if approximate, prior distribution for the source's true flux, $F_{\rm src}$?

(f) If the distribution of source fluxes was known to increase as $\mathrm{P}(F_{\rm src}) \propto F_{\rm src}^{-5/2}$, what would the resultant posterior information on the source's flux be upon combining this knowledge about the source population and the data on the particular source of interest? Is this prior normaliseable (*i.e.*, proper)?

(g) Assuming, for simplicity, that $C = 1$, plot (with the help of the computer) both the likelihood, $\mathrm{P}(N_{\rm src}|F_{\rm src})$, and the posterior distribution, $\mathrm{P}(F_{\rm src}|N_{\rm src})$, as a function of $F_{\rm src}$ in i) the case that $N_{\rm src} = 5$ (plausible for an X-ray observation) and ii) the case that $N_{\rm src} = 10^4$ (plausible for an optical observation). Are any of these functions approximately Gaussian? What is the probability that the source has $F_{\rm src} = 0$? What is the probability that the source has $F_{\rm src} < 0$? How did utilising the photometric measurement of the source affect these probabilities?

(h) What would be a reasonable "best estimate" of the source's flux? (There are several plausible answers.) How do these best estimates relate to the naive estimate $\hat{F}_{\rm src} = CN_{\rm src}$? Does this make sense?