Neutrino Theory and Phenomenology

PART 4

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Neutrino Unbound: http://www.nu.to.infn.it

NBIA PhD School
Neutrinos underground & in the heavens

The Niels Bohr International Academy, Copenhagen, Denmark

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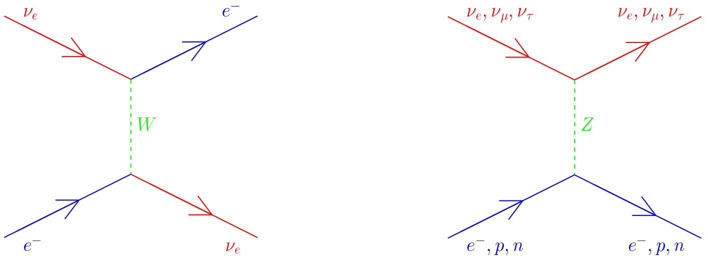
Neutrino Oscillations in Matter

Effective Potentials in Matter

coherent interactions with medium: forward elastic CC and NC scattering



$$\boxed{V_{\text{CC}} = \sqrt{2} G_{\text{F}} N_e} \qquad V_{\text{NC}}^{(e^-)} = -V_{\text{NC}}^{(p)} \quad \Rightarrow \quad \boxed{V_{\text{NC}} = V_{\text{NC}}^{(n)} = -\frac{\sqrt{2}}{2} G_{\text{F}} N_n}$$

$$V_e = V_{\text{CC}} + V_{\text{NC}} \qquad V_\mu = V_\tau = V_{\text{NC}}$$

only $V_{\text{CC}} = V_e - V_\mu = V_e - V_\tau$ is important for flavor transitions

antineutrinos: $\overline{V}_{\text{CC}} = -V_{\text{CC}}$ $\quad \overline{V}_{\text{NC}} = -V_{\text{NC}}$

Evolution of Neutrino Flavors in Matter

- Flavor neutrino ν_α with momentum p: $|\nu_\alpha(p)\rangle = \sum_k U^*_{\alpha k} |\nu_k(p)\rangle$
- Evolution is determined by Hamiltonian
- Hamiltonian in vacuum: $\mathcal{H} = \mathcal{H}_0$

$$\mathcal{H}_0 |\nu_k(p)\rangle = E_k |\nu_k(p)\rangle \qquad E_k = \sqrt{p^2 + m_k^2}$$

- Hamiltonian in matter: $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$ $\qquad \mathcal{H}_I |\nu_\alpha(p)\rangle = V_\alpha |\nu_\alpha(p)\rangle$
- Schrödinger evolution equation: $i \frac{\mathrm{d}}{\mathrm{d}t} |\nu(p,t)\rangle = \mathcal{H} |\nu(p,t)\rangle$
- Initial condition: $|\nu(p,0)\rangle = |\nu_\alpha(p)\rangle$
- For $t > 0$ the state $|\nu(p,t)\rangle$ is a superposition of all flavors:

$$|\nu(p,t)\rangle = \sum_\beta \varphi_\beta(p,t) |\nu_\beta(p)\rangle$$

- Transition probability: $P_{\nu_\alpha \to \nu_\beta} = |\varphi_\beta|^2$

evolution equation of states

$$i \, \frac{\mathrm{d}}{\mathrm{d}t} \, |\nu(p,t)\rangle = \mathcal{H} |\nu(p,t)\rangle \,, \qquad |\nu(p,0)\rangle = |\nu_\alpha(p)\rangle$$

flavor transition amplitudes

$$\varphi_\beta(p,t) = \langle \nu_\beta(p) | \nu(p,t) \rangle \,, \qquad \varphi_\beta(p,0) = \delta_{\alpha\beta}$$

evolution of flavor transition amplitudes

$$i \, \frac{\mathrm{d}}{\mathrm{d}t} \, \varphi_\beta(p,t) = \langle \nu_\beta(p) | \mathcal{H} | \nu(p,t) \rangle$$

$$i \, \frac{\mathrm{d}}{\mathrm{d}t} \, \varphi_\beta(p,t) = \langle \nu_\beta(p) | \mathcal{H}_0 | \nu(p,t) \rangle + \langle \nu_\beta(p) | \mathcal{H}_I | \nu(p,t) \rangle$$

$$i\,\frac{\mathrm{d}}{\mathrm{d}t}\,\varphi_\beta(p,t) = \langle\nu_\beta(p)|\mathcal{H}_0|\nu(p,t)\rangle + \langle\nu_\beta(p)|\mathcal{H}_I|\nu(p,t)\rangle$$

$$\langle\nu_\beta(p)|\mathcal{H}_0|\nu(p,t)\rangle =$$

$$\sum_\rho \sum_{k,j} \underbrace{\langle\nu_\beta(p)|\nu_k(p)\rangle}_{U_{\beta k}} \underbrace{\langle\nu_k(p)|\mathcal{H}_0|\nu_j(p)\rangle}_{\delta_{kj}E_k} \underbrace{\langle\nu_j(p)|\nu_\rho(p)\rangle}_{U^*_{\rho j}} \underbrace{\langle\nu_\rho(p)|\nu(p,t)\rangle}_{\varphi_\rho(p,t)}$$

$$= \sum_\rho \sum_k U_{\beta k} E_k U^*_{\rho k}\,\varphi_\rho(p,t)$$

$$\langle\nu_\beta(p)|\mathcal{H}_I|\nu(p,t)\rangle = \sum_\rho \underbrace{\langle\nu_\beta(p)|\mathcal{H}_I|\nu_\rho(p)\rangle}_{\delta_{\beta\rho}V_\beta} \underbrace{\langle\nu_\rho(p)|\nu(p,t)\rangle}_{\varphi_\rho(p,t)}$$

$$= \sum_\rho \delta_{\beta\rho} V_\beta\,\varphi_\rho(p,t)$$

$$\boxed{i\,\frac{\mathrm{d}}{\mathrm{d}t}\,\varphi_\beta = \sum_\rho \left(\sum_k U_{\beta k}\,E_k\,U^*_{\rho k} + \delta_{\beta\rho}\,V_\beta \right) \varphi_\rho}$$

ultrarelativistic neutrinos: $E_k = p + \frac{m_k^2}{2E}$ $\quad E = p \quad t = x$

$$V_e = V_{CC} + V_{NC} \qquad V_\mu = V_\tau = V_{NC}$$

$$i\,\frac{d}{dx}\,\varphi_\beta(p,x) = (p + V_{NC})\,\varphi_\beta(p,x) + \sum_\rho \left(\sum_k U_{\beta k}\,\frac{m_k^2}{2E}\,U_{\rho k}^* + \delta_{\beta e}\,\delta_{\rho e}\,V_{CC} \right) \varphi_\rho(p,x)$$

$$\psi_\beta(p,x) = \varphi_\beta(p,x)\,e^{ipx + i\int_0^x V_{NC}(x')\,dx'}$$

$$\Downarrow$$

$$i\,\frac{d}{dx}\,\psi_\beta = e^{ipx + i\int_0^x V_{NC}(x')\,dx'} \left(-p - V_{NC} + i\,\frac{d}{dx} \right) \varphi_\beta$$

$$\boxed{i\,\frac{d}{dx}\,\psi_\beta = \sum_\rho \left(\sum_k U_{\beta k}\,\frac{m_k^2}{2E}\,U_{\rho k}^* + \delta_{\beta e}\,\delta_{\rho e}\,V_{CC} \right) \psi_\rho}$$

$$\boxed{P_{\nu_\alpha \to \nu_\beta} = |\varphi_\beta|^2 = |\psi_\beta|^2}$$

evolution of flavor transition amplitudes in matrix form

$$\boxed{i \frac{\mathrm{d}}{\mathrm{d}x} \Psi_\alpha = \frac{1}{2E} \left(U \, \mathbb{M}^2 \, U^\dagger + \mathbb{A} \right) \Psi_\alpha}$$

$$\Psi_\alpha = \begin{pmatrix} \psi_e \\ \psi_\mu \\ \psi_\tau \end{pmatrix} \qquad \mathbb{M}^2 = \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} \qquad \mathbb{A} = \begin{pmatrix} A_{CC} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_{CC} = 2EV_{CC} = 2\sqrt{2}EG_F N_e$$

effective mass-squared matrix in vacuum

$$\mathbb{M}^2_{VAC} = U \, \mathbb{M}^2 \, U^\dagger \xrightarrow{\text{matter}} U \, \mathbb{M}^2 \, U^\dagger + 2 \, E \, \underset{\uparrow}{\mathbb{V}} = \mathbb{M}^2_{MAT}$$

effective mass-squared matrix in matter

potential due to coherent forward elastic scattering

Two-Neutrino Mixing

$\nu_e \to \nu_\mu$ transitions with $U = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}$

$$U\,\mathbb{M}^2\,U^\dagger = \begin{pmatrix} \cos^2\vartheta m_1^2 + \sin^2\vartheta m_2^2 & \cos\vartheta\sin\vartheta\left(m_2^2 - m_1^2\right) \\ \cos\vartheta\sin\vartheta\left(m_2^2 - m_1^2\right) & \sin^2\vartheta m_1^2 + \cos^2\vartheta m_2^2 \end{pmatrix}$$

$$= \frac{1}{2}\,\underset{\substack{\uparrow \\ \text{irrelevant common phase}}}{\Sigma m^2} + \frac{1}{2}\begin{pmatrix} -\Delta m^2\cos 2\vartheta & \Delta m^2\sin 2\vartheta \\ \Delta m^2\sin 2\vartheta & \Delta m^2\cos 2\vartheta \end{pmatrix}$$

$$\Sigma m^2 \equiv m_1^2 + m_2^2 \qquad\qquad \Delta m^2 \equiv m_2^2 - m_1^2$$

$$\boxed{i\,\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}\psi_e\\ \psi_\mu\end{pmatrix} = \frac{1}{4E}\begin{pmatrix}-\Delta m^2\cos 2\vartheta + 2A_{CC} & \Delta m^2 \sin 2\vartheta\\ \Delta m^2 \sin 2\vartheta & \Delta m^2 \cos 2\vartheta\end{pmatrix}\begin{pmatrix}\psi_e\\ \psi_\mu\end{pmatrix}}$$

initial ν_e $\Longrightarrow$ $\begin{pmatrix}\psi_e(0)\\ \psi_\mu(0)\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}$

$$\boxed{\begin{array}{l} P_{\nu_e\to\nu_\mu}(x) = |\psi_\mu(x)|^2 \\ P_{\nu_e\to\nu_e}(x) = |\psi_e(x)|^2 = 1 - P_{\nu_e\to\nu_\mu}(x) \end{array}}$$

Constant Matter Density

$$\boxed{i \frac{\mathrm{d}}{\mathrm{d}x} \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \frac{1}{4E} \begin{pmatrix} -\Delta m^2 \cos 2\vartheta + 2A_{\mathrm{CC}} & \Delta m^2 \sin 2\vartheta \\ \Delta m^2 \sin 2\vartheta & \Delta m^2 \cos 2\vartheta \end{pmatrix} \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix}}$$

$$\frac{\mathrm{d}A_{\mathrm{CC}}}{\mathrm{d}x} = 0$$

diagonalization of effective hamiltonian: $\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos\vartheta_{\mathrm{M}} & \sin\vartheta_{\mathrm{M}} \\ -\sin\vartheta_{\mathrm{M}} & \cos\vartheta_{\mathrm{M}} \end{pmatrix} \begin{pmatrix} \psi_1^{\mathrm{M}} \\ \psi_2^{\mathrm{M}} \end{pmatrix}$

$$\begin{pmatrix} \cos\vartheta_{\mathrm{M}} & -\sin\vartheta_{\mathrm{M}} \\ \sin\vartheta_{\mathrm{M}} & \cos\vartheta_{\mathrm{M}} \end{pmatrix} \begin{pmatrix} -\Delta m^2 \cos 2\vartheta + 2A_{\mathrm{CC}} & \Delta m^2 \sin 2\vartheta \\ \Delta m^2 \sin 2\vartheta & \Delta m^2 \cos 2\vartheta \end{pmatrix} \begin{pmatrix} \cos\vartheta_{\mathrm{M}} & \sin\vartheta_{\mathrm{M}} \\ -\sin\vartheta_{\mathrm{M}} & \cos\vartheta_{\mathrm{M}} \end{pmatrix} =$$

$$= \begin{pmatrix} A_{\mathrm{CC}} - \Delta m^2_{\mathrm{M}} & 0 \\ 0 & A_{\mathrm{CC}} + \Delta m^2_{\mathrm{M}} \end{pmatrix}$$

$$i \frac{\mathrm{d}}{\mathrm{d}x} \begin{pmatrix} \psi_1^{\mathrm{M}} \\ \psi_2^{\mathrm{M}} \end{pmatrix} = \left[\underset{\substack{\uparrow \\ \text{irrelevant common phase}}}{\frac{A_{\mathrm{CC}}}{4E}} + \frac{1}{4E} \begin{pmatrix} -\Delta m^2_{\mathrm{M}} & 0 \\ 0 & \Delta m^2_{\mathrm{M}} \end{pmatrix} \right] \begin{pmatrix} \psi_1^{\mathrm{M}} \\ \psi_2^{\mathrm{M}} \end{pmatrix}$$

Effective Mixing Angle in Matter

$$\tan 2\vartheta_{\mathrm{M}} = \frac{\tan 2\vartheta}{1 - \dfrac{A_{\mathrm{CC}}}{\Delta m^2 \cos 2\vartheta}}$$

Effective Squared-Mass Difference

$$\Delta m^2_{\mathrm{M}} = \sqrt{\left(\Delta m^2 \cos 2\vartheta - A_{\mathrm{CC}}\right)^2 + \left(\Delta m^2 \sin 2\vartheta\right)^2}$$

Resonance ($\vartheta_{\mathrm{M}} = \pi/4$)

$$A^{\mathrm{R}}_{\mathrm{CC}} = \Delta m^2 \cos 2\vartheta \quad \Longrightarrow \quad N^{\mathrm{R}}_e = \frac{\Delta m^2 \cos 2\vartheta}{2\sqrt{2} E G_{\mathrm{F}}}$$

$$i \frac{\mathrm{d}}{\mathrm{d}x} \begin{pmatrix} \psi_1^{\mathrm{M}} \\ \psi_2^{\mathrm{M}} \end{pmatrix} = \frac{1}{4E} \begin{pmatrix} -\Delta m^2_{\mathrm{M}} & 0 \\ 0 & \Delta m^2_{\mathrm{M}} \end{pmatrix} \begin{pmatrix} \psi_1^{\mathrm{M}} \\ \psi_2^{\mathrm{M}} \end{pmatrix}$$

$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos\vartheta_{\mathrm{M}} & \sin\vartheta_{\mathrm{M}} \\ -\sin\vartheta_{\mathrm{M}} & \cos\vartheta_{\mathrm{M}} \end{pmatrix} \begin{pmatrix} \psi_1^{\mathrm{M}} \\ \psi_2^{\mathrm{M}} \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \psi_1^{\mathrm{M}} \\ \psi_2^{\mathrm{M}} \end{pmatrix} = \begin{pmatrix} \cos\vartheta_{\mathrm{M}} & -\sin\vartheta_{\mathrm{M}} \\ \sin\vartheta_{\mathrm{M}} & \cos\vartheta_{\mathrm{M}} \end{pmatrix} \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix}$$

$$\nu_e \to \nu_\mu \quad \Longrightarrow \quad \begin{pmatrix} \psi_e(0) \\ \psi_\mu(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} \psi_1^{\mathrm{M}}(0) \\ \psi_2^{\mathrm{M}}(0) \end{pmatrix} = \begin{pmatrix} \cos\vartheta_{\mathrm{M}} \\ \sin\vartheta_{\mathrm{M}} \end{pmatrix}$$

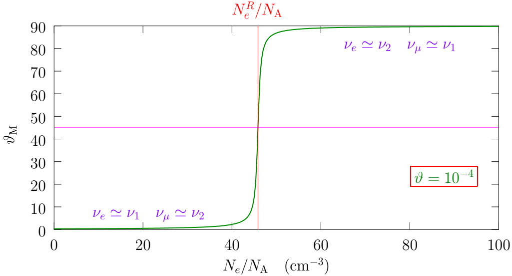
$$\psi_1^{\mathrm{M}}(x) = \cos\vartheta_{\mathrm{M}} \exp\left(i \frac{\Delta m^2_{\mathrm{M}} x}{4E} \right)$$

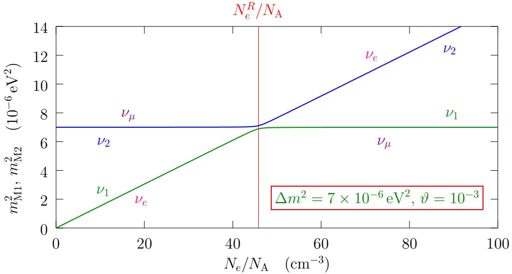
$$\psi_2^{\mathrm{M}}(x) = \sin\vartheta_{\mathrm{M}} \exp\left(-i \frac{\Delta m^2_{\mathrm{M}} x}{4E} \right)$$

$$P_{\nu_e \to \nu_\mu}(x) = |\psi_\mu(x)|^2 = \left| -\sin\vartheta_{\mathrm{M}} \psi_1^{\mathrm{M}}(x) + \cos\vartheta_{\mathrm{M}} \psi_2^{\mathrm{M}}(x) \right|^2$$

$$\boxed{P_{\nu_e \to \nu_\mu}(x) = \sin^2 2\vartheta_{\mathrm{M}} \ \sin^2\left(\frac{\Delta m^2_{\mathrm{M}} x}{4E} \right)}$$

MSW Effect (Resonant Transitions in Matter)





$$\boxed{\begin{aligned}\nu_e &= \cos\vartheta_M\,\nu_1 + \sin\vartheta_M\,\nu_2\\ \nu_\mu &= -\sin\vartheta_M\,\nu_1 + \cos\vartheta_M\,\nu_2\end{aligned}}$$

$$\boxed{\tan 2\vartheta_M = \frac{\tan 2\vartheta}{1 - \dfrac{A_{CC}}{\Delta m^2 \cos 2\vartheta}}}$$

$$\boxed{\begin{aligned}\Delta m^2_M = \Big[&\left(\Delta m^2 \cos 2\vartheta - A_{CC}\right)^2\\ &+ \left(\Delta m^2 \sin 2\vartheta\right)^2\Big]^{1/2}\end{aligned}}$$

$$\boxed{i\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}\psi_e\\ \psi_\mu\end{pmatrix} = \frac{1}{4E}\begin{pmatrix}-\Delta m^2\cos 2\vartheta + 2A_{\mathrm{CC}} & \Delta m^2 \sin 2\vartheta\\ \Delta m^2 \sin 2\vartheta & \Delta m^2\cos 2\vartheta\end{pmatrix}\begin{pmatrix}\psi_e\\ \psi_\mu\end{pmatrix}}$$

tentative diagonalization: $\begin{pmatrix}\psi_e\\ \psi_\mu\end{pmatrix} = \begin{pmatrix}\cos\vartheta_{\mathrm{M}} & \sin\vartheta_{\mathrm{M}}\\ -\sin\vartheta_{\mathrm{M}} & \cos\vartheta_{\mathrm{M}}\end{pmatrix}\begin{pmatrix}\psi_1^{\mathrm{M}}\\ \psi_2^{\mathrm{M}}\end{pmatrix}$

$$i\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}\cos\vartheta_{\mathrm{M}} & \sin\vartheta_{\mathrm{M}}\\ -\sin\vartheta_{\mathrm{M}} & \cos\vartheta_{\mathrm{M}}\end{pmatrix}\begin{pmatrix}\psi_1^{\mathrm{M}}\\ \psi_2^{\mathrm{M}}\end{pmatrix} =$$

$$= \frac{1}{4E}\begin{pmatrix}-\Delta m^2\cos 2\vartheta + 2A_{\mathrm{CC}} & \Delta m^2 \sin 2\vartheta\\ \Delta m^2 \sin 2\vartheta & \Delta m^2\cos 2\vartheta\end{pmatrix}\begin{pmatrix}\cos\vartheta_{\mathrm{M}} & \sin\vartheta_{\mathrm{M}}\\ -\sin\vartheta_{\mathrm{M}} & \cos\vartheta_{\mathrm{M}}\end{pmatrix}\begin{pmatrix}\psi_1^{\mathrm{M}}\\ \psi_2^{\mathrm{M}}\end{pmatrix}$$

if matter densitity is not constant $\mathrm{d}\vartheta_{\mathrm{M}}/\mathrm{d}x \neq 0$

$$i\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}\psi_1^{\mathrm{M}}\\ \psi_2^{\mathrm{M}}\end{pmatrix} = \left[\underset{\substack{\uparrow\\ \text{irrelevant common phase}}}{\frac{A_{\mathrm{CC}}}{4E}} + \frac{1}{4E}\begin{pmatrix}-\Delta m^2_{\mathrm{M}} & 0\\ 0 & \Delta m^2_{\mathrm{M}}\end{pmatrix} + \begin{pmatrix}0 & -i\frac{\mathrm{d}\vartheta_{\mathrm{M}}}{\mathrm{d}x}\\ i\frac{\mathrm{d}\vartheta_{\mathrm{M}}}{\mathrm{d}x} & 0\end{pmatrix}\right]\begin{pmatrix}\psi_1^{\mathrm{M}}\\ \psi_2^{\mathrm{M}}\end{pmatrix}$$

$$i \frac{\mathrm{d}}{\mathrm{d}x} \begin{pmatrix} \psi_1^{\mathrm{M}} \\ \psi_2^{\mathrm{M}} \end{pmatrix} = \left[\underbrace{\frac{1}{4E} \begin{pmatrix} -\Delta m^2_{\mathrm{M}} & 0 \\ 0 & \Delta m^2_{\mathrm{M}} \end{pmatrix}}_{\uparrow \text{ adiabatic}} + \underbrace{\begin{pmatrix} 0 & -i \frac{\mathrm{d}\vartheta_{\mathrm{M}}}{\mathrm{d}x} \\ i \frac{\mathrm{d}\vartheta_{\mathrm{M}}}{\mathrm{d}x} & 0 \end{pmatrix}}_{\uparrow \text{ non-adiabatic, maximum at resonance}} \right] \begin{pmatrix} \psi_1^{\mathrm{M}} \\ \psi_2^{\mathrm{M}} \end{pmatrix}$$

initial conditions:

$$\begin{pmatrix} \psi_1^{\mathrm{M}}(0) \\ \psi_2^{\mathrm{M}}(0) \end{pmatrix} = \begin{pmatrix} \cos\vartheta^0_{\mathrm{M}} & -\sin\vartheta^0_{\mathrm{M}} \\ \sin\vartheta^0_{\mathrm{M}} & \cos\vartheta^0_{\mathrm{M}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\vartheta^0_{\mathrm{M}} \\ \sin\vartheta^0_{\mathrm{M}} \end{pmatrix}$$

solution approximating all non-adiabatic $\nu_1^{\mathrm{M}} \leftrightarrows \nu_2^{\mathrm{M}}$ transitions in resonance

$$\begin{aligned}
\psi_1^{\mathrm{M}}(x) \simeq{}& \left[\cos\vartheta^0_{\mathrm{M}} \exp\left(i \int_0^{x_{\mathrm{R}}} \frac{\Delta m^2_{\mathrm{M}}(x')}{4E} \, \mathrm{d}x' \right) \mathcal{A}^{\mathrm{R}}_{11} + \sin\vartheta^0_{\mathrm{M}} \exp\left(-i \int_0^{x_{\mathrm{R}}} \frac{\Delta m^2_{\mathrm{M}}(x')}{4E} \, \mathrm{d}x' \right) \mathcal{A}^{\mathrm{R}}_{21} \right] \\
& \times \exp\left(i \int_{x_{\mathrm{R}}}^{x} \frac{\Delta m^2_{\mathrm{M}}(x')}{4E} \, \mathrm{d}x' \right) \\
\psi_2^{\mathrm{M}}(x) \simeq{}& \left[\cos\vartheta^0_{\mathrm{M}} \exp\left(i \int_0^{x_{\mathrm{R}}} \frac{\Delta m^2_{\mathrm{M}}(x')}{4E} \, \mathrm{d}x' \right) \mathcal{A}^{\mathrm{R}}_{12} + \sin\vartheta^0_{\mathrm{M}} \exp\left(-i \int_0^{x_{\mathrm{R}}} \frac{\Delta m^2_{\mathrm{M}}(x')}{4E} \, \mathrm{d}x' \right) \mathcal{A}^{\mathrm{R}}_{22} \right] \\
& \times \exp\left(-i \int_{x_{\mathrm{R}}}^{x} \frac{\Delta m^2_{\mathrm{M}}(x')}{4E} \, \mathrm{d}x' \right)
\end{aligned}$$

Averaged ν_e Survival Probability on Earth

$$\psi_e(x) = \cos\vartheta\,\psi_1^{\mathrm{M}}(x) + \sin\vartheta\,\psi_2^{\mathrm{M}}(x)$$

neglect interference (averaged over energy spectrum)

$$\begin{aligned}\overline{P}_{\nu_e\to\nu_e}(x) = |\langle\psi_e(x)\rangle|^2 = {} & \cos^2\vartheta\,\cos^2\vartheta_{\mathrm{M}}^0\,|\mathcal{A}_{11}^{\mathrm{R}}|^2 + \cos^2\vartheta\,\sin^2\vartheta_{\mathrm{M}}^0\,|\mathcal{A}_{21}^{\mathrm{R}}|^2 \\ & + \sin^2\vartheta\,\cos^2\vartheta_{\mathrm{M}}^0\,|\mathcal{A}_{12}^{\mathrm{R}}|^2 + \sin^2\vartheta\,\sin^2\vartheta_{\mathrm{M}}^0\,|\mathcal{A}_{22}^{\mathrm{R}}|^2\end{aligned}$$

conservation of probability (unitarity)

$$|\mathcal{A}_{12}^{\mathrm{R}}|^2 = |\mathcal{A}_{21}^{\mathrm{R}}|^2 = P_{\mathrm{c}} \qquad |\mathcal{A}_{11}^{\mathrm{R}}|^2 = |\mathcal{A}_{22}^{\mathrm{R}}|^2 = 1 - P_{\mathrm{c}}$$

$P_{\mathrm{c}} \equiv$ crossing probability

$$\boxed{\overline{P}_{\nu_e\to\nu_e}(x) = \frac{1}{2} + \left(\frac{1}{2} - P_{\mathrm{c}}\right)\cos 2\vartheta_{\mathrm{M}}^0\,\cos 2\vartheta}$$

[Parke, PRL 57 (1986) 1275]

Crossing Probability

$$P_c = \frac{\exp\left(-\frac{\pi}{2}\gamma F\right) - \exp\left(-\frac{\pi}{2}\gamma \frac{F}{\sin^2\vartheta}\right)}{1 - \exp\left(-\frac{\pi}{2}\gamma \frac{F}{\sin^2\vartheta}\right)}$$

[Kuo, Pantaleone, PRD 39 (1989) 1930]

adiabaticity parameter: $$\gamma = \left. \frac{\Delta m^2_{\mathrm{M}}/2E}{2|d\vartheta_{\mathrm{M}}/dx|} \right|_{\mathrm{R}} = \frac{\Delta m^2 \sin^2 2\vartheta}{2E\cos 2\vartheta \left| \frac{d\ln A_{\mathrm{CC}}}{dx} \right|_{\mathrm{R}}}$$

$A \propto x$ $\qquad F = 1$ (Landau-Zener approximation) [Parke, PRL 57 (1986) 1275]

$A \propto 1/x$ $\qquad F = \left(1 - \tan^2\vartheta\right)^2 / \left(1 + \tan^2\vartheta\right)$ [Kuo, Pantaleone, PRD 39 (1989) 1930]

$A \propto \exp(-x)$ $\qquad F = 1 - \tan^2\vartheta$

[Pizzochero, PRD 36 (1987) 2293]

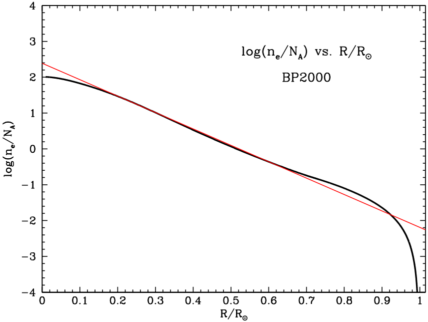
[Toshev, PLB 196 (1987) 170]

[Petcov, PLB 200 (1988) 373]

Review: [Kuo, Pantaleone, RMP 61 (1989) 937]

Solar Neutrinos

SUN: $N_e(x) \simeq N_e^{c} \exp\left(-\frac{x}{x_0}\right)$ $\quad N_e^{c} = 245\, N_A/\text{cm}^3 \quad x_0 = \frac{R_\odot}{10.54}$



$$\overline{P}^{\text{sun}}_{\nu_e\to\nu_e} = \frac{1}{2} + \left(\frac{1}{2} - P_c\right) \cos 2\vartheta^0_M \cos 2\vartheta$$

$$P_c = \frac{\exp\left(-\frac{\pi}{2}\gamma F\right) - \exp\left(-\frac{\pi}{2}\gamma \frac{F}{\sin^2\vartheta}\right)}{1 - \exp\left(-\frac{\pi}{2}\gamma \frac{F}{\sin^2\vartheta}\right)}$$

$$\gamma = \frac{\Delta m^2 \sin^2 2\vartheta}{2E \cos 2\vartheta \left|\frac{d \ln A_{CC}}{dx}\right|_R}$$

$$F = 1 - \tan^2\vartheta$$

$$A_{CC} = 2\sqrt{2} E G_F N_e$$

practical prescription:

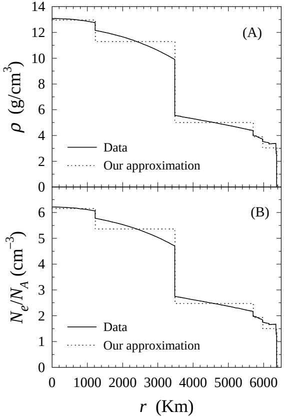
[Lisi et al., PRD 63 (2001) 093002]

$$\begin{cases} \text{numerical } |d \ln A_{CC}/dx|_R & \text{for } x \leq 0.904 R_\odot \\ |d \ln A_{CC}/dx|_R \to \dfrac{18.9}{R_\odot} & \text{for } x > 0.904 R_\odot \end{cases}$$

Electron Neutrino Regeneration in the Earth

$$P^{\text{sun+earth}}_{\nu_e\to\nu_e} = \overline{P}^{\text{sun}}_{\nu_e\to\nu_e} + \frac{\left(1 - 2\overline{P}^{\text{sun}}_{\nu_e\to\nu_e}\right)\left(P^{\text{earth}}_{\nu_2\to\nu_e} - \sin^2\vartheta\right)}{\cos 2\vartheta}$$

[Mikheev, Smirnov, Sov. Phys. Usp. 30 (1987) 759], [Baltz, Weneser, PRD 35 (1987) 528]



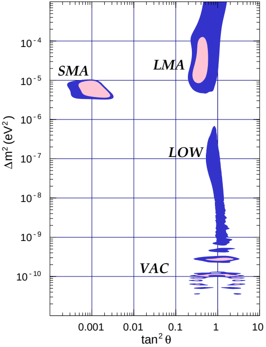
$P^{\text{earth}}_{\nu_2\to\nu_e}$ is usually calculated numerically approximating the Earth density profile with a step function.

Effective massive neutrinos propagate as plane waves in regions of constant density.

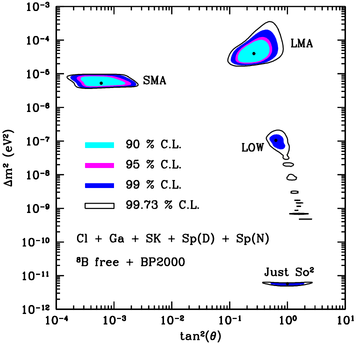
Wave functions of flavor neutrinos are joined at the boundaries of steps.

Solar Neutrino Oscillations

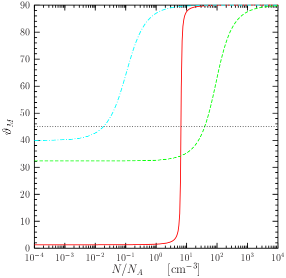
LMA (Large Mixing Angle):	$\Delta m^2 \sim 5 \times 10^{-5}\,\text{eV}^2$,	$\tan^2\vartheta \sim 0.8$
LOW (LOW Δm^2):	$\Delta m^2 \sim 7 \times 10^{-8}\,\text{eV}^2$,	$\tan^2\vartheta \sim 0.6$
SMA (Small Mixing Angle):	$\Delta m^2 \sim 5 \times 10^{-6}\,\text{eV}^2$,	$\tan^2\vartheta \sim 10^{-3}$
QVO (Quasi-Vacuum Oscillations):	$\Delta m^2 \sim 10^{-9}\,\text{eV}^2$,	$\tan^2\vartheta \sim 1$
VAC (VACuum oscillations):	$\Delta m^2 \lesssim 5 \times 10^{-10}\,\text{eV}^2$,	$\tan^2\vartheta \sim 1$



[de Gouvea, Friedland, Murayama, PLB 490 (2000) 125]



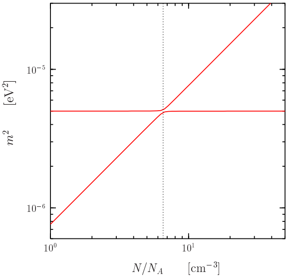
[Bahcall, Krastev, Smirnov, JHEP 05 (2001) 015]



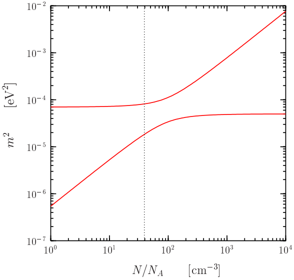
solid line: $\Delta m^2 = 5 \times 10^{-6}\,\text{eV}^2$
(typical SMA) $\tan^2\vartheta = 5 \times 10^{-4}$

dashed line: $\Delta m^2 = 7 \times 10^{-5}\,\text{eV}^2$
(typical LMA) $\tan^2\vartheta = 0.4$

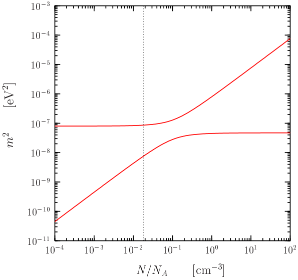
dash-dotted line: $\Delta m^2 = 8 \times 10^{-8}\,\text{eV}^2$
(typical LOW) $\tan^2\vartheta = 0.7$



typical SMA

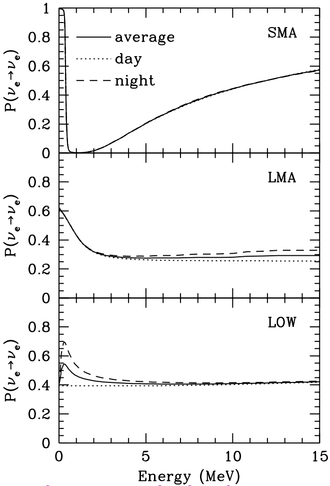


typical LMA



typical LOW

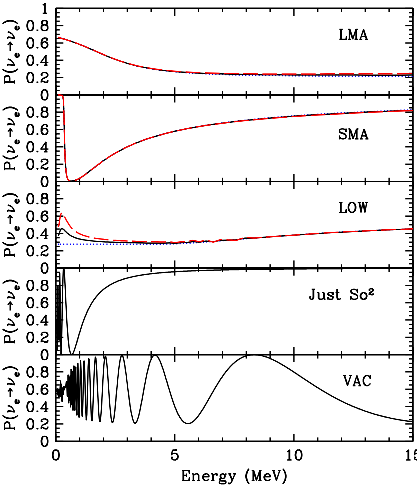
[Bahcall, Krastev, Smirnov, PRD 58 (1998) 096016]



SMA: $\Delta m^2 = 5.0 \times 10^{-6}\,\text{eV}^2$ $\sin^2 2\vartheta = 3.5 \times 10^{-3}$

LMA: $\Delta m^2 = 1.6 \times 10^{-5}\,\text{eV}^2$ $\sin^2 2\vartheta = 0.57$

LOW: $\Delta m^2 = 7.9 \times 10^{-8}\,\text{eV}^2$ $\sin^2 2\vartheta = 0.95$



[Bahcall, Krastev, Smirnov, JHEP 05 (2001) 015]

LMA: $\Delta m^2 = 4.2 \times 10^{-5}\,\text{eV}^2$ $\tan^2\vartheta = 0.26$

SMA: $\Delta m^2 = 5.2 \times 10^{-6}\,\text{eV}^2$ $\tan^2\vartheta = 5.5 \times 10^{-4}$

LOW: $\Delta m^2 = 7.6 \times 10^{-8}\,\text{eV}^2$ $\tan^2\vartheta = 0.72$

Just So^2: $\Delta m^2 = 5.5 \times 10^{-12}\,\text{eV}^2$ $\tan^2\vartheta = 1.0$

VAC: $\Delta m^2 = 1.4 \times 10^{-10}\,\text{eV}^2$ $\tan^2\vartheta = 0.38$

SNO: Sudbury Neutrino Observatory

water Cherenkov detector, Sudbury, Ontario, Canada

1 kton of D_2O, 9456 20-cm PMTs

2073 m underground, 6010 m.w.e.

$$
\begin{array}{ll}
\text{CC:} & \nu_e + d \to p + p + e^- \\
\text{NC:} & \nu + d \to p + n + \nu \\
\text{ES:} & \nu + e^- \to \nu + e^-
\end{array}
$$

$$
\left.
\begin{array}{l}
\text{CC threshold: } E_{\text{th}}^{\text{SNO}}(\text{CC}) \simeq 8.2\,\text{MeV} \\
\text{NC threshold: } E_{\text{th}}^{\text{SNO}}(\text{NC}) \simeq 2.2\,\text{MeV} \\
\text{ES threshold: } E_{\text{th}}^{\text{SNO}}(\text{ES}) \simeq 7.0\,\text{MeV}
\end{array}
\right\} \Longrightarrow {}^{8}\text{B},\ hep
$$

D_2O phase: 1999 – 2001

$$\frac{R_{\text{CC}}^{\text{SNO}}}{R_{\text{CC}}^{\text{SSM}}} = 0.35 \pm 0.02$$
$$\frac{R_{\text{NC}}^{\text{SNO}}}{R_{\text{NC}}^{\text{SSM}}} = 1.01 \pm 0.13$$
$$\frac{R_{\text{ES}}^{\text{SNO}}}{R_{\text{ES}}^{\text{SSM}}} = 0.47 \pm 0.05$$

[PRL 89 (2002) 011301]

NaCl phase: 2001 – 2002

$$\frac{R_{\text{CC}}^{\text{SNO}}}{R_{\text{CC}}^{\text{SSM}}} = 0.31 \pm 0.02$$
$$\frac{R_{\text{NC}}^{\text{SNO}}}{R_{\text{NC}}^{\text{SSM}}} = 1.03 \pm 0.09$$
$$\frac{R_{\text{ES}}^{\text{SNO}}}{R_{\text{ES}}^{\text{SSM}}} = 0.44 \pm 0.06$$

[nucl-ex/0309004]

$$\Phi_{\nu_e}^{\text{SNO}} = 1.76 \pm 0.11 \times 10^6 \, \text{cm}^{-2} \, \text{s}^{-1}$$

$$\Phi_{\nu_\mu,\nu_\tau}^{\text{SNO}} = 5.41 \pm 0.66 \times 10^6 \, \text{cm}^{-2} \, \text{s}^{-1}$$

SNO solved
solar neutrino problem

$\Downarrow$

Neutrino Physics
(April 2002)

[SNO, PRL 89 (2002) 011301, nucl-ex/0204008]

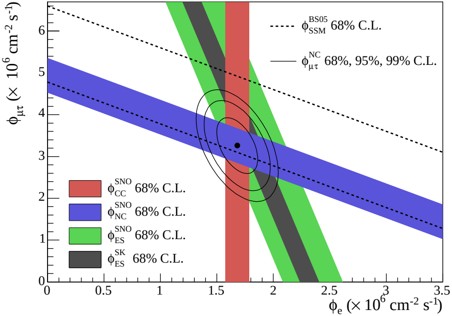
$\nu_e \to \nu_\mu, \nu_\tau$ oscillations

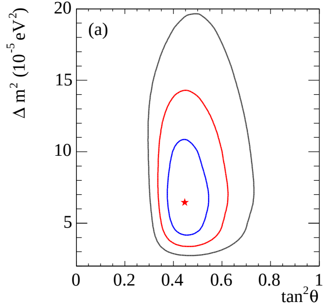
$\Downarrow$

Large Mixing Angle solution

$$\Delta m^2 \simeq 7 \times 10^{-5} \, \text{eV}^2$$

$$\tan^2 \vartheta \simeq 0.45$$





[SNO, PRC 72 (2005) 055502, nucl-ex/0502021]

KamLAND

Kamioka Liquid scintillator Anti-Neutrino Detector

long-baseline reactor $\bar{\nu}_e$ experiment

Kamioka mine (200 km west of Tokyo), 1000 m underground, 2700 m.w.e.

53 nuclear power reactors in Japan and Korea

average distance from reactors: 180 km

- 6.7% of flux from one reactor at 88 km
- 79% of flux from 26 reactors at 138–214 km
- 14.3% of flux from other reactors at >295 km

1 kt liquid scintillator detector: $\bar{\nu}_e + p \to e^+ + n$, energy threshold: $E_{\text{th}}^{\bar{\nu}_e p} = 1.8\,\text{MeV}$

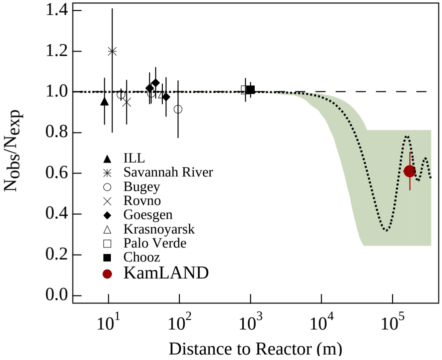
data taking: 4 March – 6 October 2002, 145.1 days (162 ton yr)

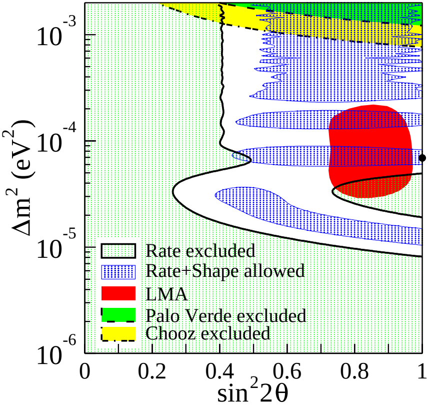
expected number of reactor neutrino events (no osc.): $N_{\text{expected}}^{\text{KamLAND}} = 86.8 \pm 5.6$
expected number of background events: $N_{\text{background}}^{\text{KamLAND}} = 0.95 \pm 0.99$
observed number of neutrino events: $N_{\text{observed}}^{\text{KamLAND}} = 54$

$$\frac{N_{\text{observed}}^{\text{KamLAND}} - N_{\text{background}}^{\text{KamLAND}}}{N_{\text{expected}}^{\text{KamLAND}}} = 0.611 \pm 0.085 \pm 0.041$$

99.95% C.L. evidence of $\bar{\nu}_e$ disappearance

confirmation of LMA (December 2002)



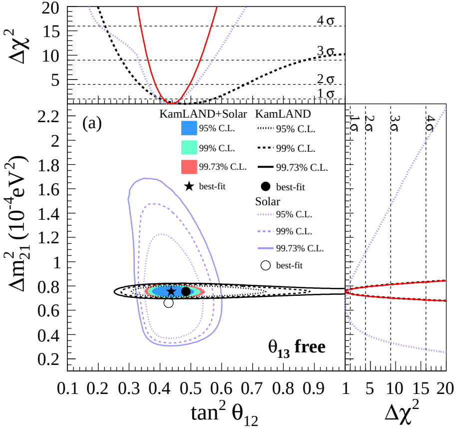


Shade: 95% C.L. LMA

Curve: $\begin{cases} \Delta m^2 = 5.5 \times 10^{-5}\,\text{eV}^2 \\ \sin^2 2\vartheta = 0.83 \end{cases}$

95% C.L.

[KamLAND, PRL 90 (2003) 021802, hep-ex/0212021]



$$\Delta m^2_{21} = 7.53^{+0.19}_{-0.18} \times 10^{-5}\ \text{eV}^2 \qquad \Delta m^2_{21} = 7.53 \pm 0.18 \times 10^{-5}\ \text{eV}^2$$

$$\tan^2\vartheta_{12} = 0.437^{+0.029}_{-0.026} \qquad \tan^2\vartheta_{12} = 0.436^{+0.029}_{-0.025}$$

$$\sin^2\vartheta_{13} = 0.023 \pm 0.015 \qquad \sin^2\vartheta_{13} = 0.023 \pm 0.002$$

[KamLAND, arXiv:1303.4667]

LMA Solar Neutrino Oscillations

best fit of reactor + solar neutrino data: $\Delta m^2 \simeq 7 \times 10^{-5}\,\text{eV}^2 \quad \tan^2\vartheta \simeq 0.4$

$$\boxed{\overline{P}^{\text{sun}}_{\nu_e\to\nu_e} = \frac{1}{2} + \left(\frac{1}{2} - P_{\text{c}}\right)\cos 2\vartheta^0_{\text{M}}\,\cos 2\vartheta}$$

$$P_{\text{c}} = \frac{\exp\left(-\frac{\pi}{2}\gamma F\right) - \exp\left(-\frac{\pi}{2}\gamma\frac{F}{\sin^2\vartheta}\right)}{1 - \exp\left(-\frac{\pi}{2}\gamma\frac{F}{\sin^2\vartheta}\right)} \qquad \gamma = \frac{\Delta m^2 \sin^2 2\vartheta}{2E\cos 2\vartheta \left|\frac{\text{d}\ln A}{\text{d}x}\right|_{\text{R}}} \qquad F = 1 - \tan^2\vartheta$$

$$A_{\text{CC}} \simeq 2\sqrt{2}EG_{\text{F}}N_e^{\text{c}}\exp\left(-\frac{x}{x_0}\right) \quad \Longrightarrow \quad \left|\frac{\text{d}\ln A}{\text{d}x}\right| \simeq \frac{1}{x_0} = \frac{10.54}{R_\odot} \simeq 3 \times 10^{-15}\,\text{eV}$$

$$\tan^2\vartheta \simeq 0.4 \quad \Longrightarrow \quad \sin^2 2\vartheta \simeq 0.82\,,\ \cos 2\vartheta \simeq 0.43 \qquad \gamma \simeq 2 \times 10^4 \left(\frac{E}{\text{MeV}}\right)^{-1}$$

$$\gamma \gg 1 \quad \Longrightarrow \quad P_{\text{c}} \ll 1 \quad \Longrightarrow \quad \boxed{\overline{P}^{\text{sun,LMA}}_{\nu_e\to\nu_e} \simeq \frac{1}{2} + \frac{1}{2}\cos 2\vartheta^0_{\text{M}}\,\cos 2\vartheta}$$

$$\boxed{\cos 2\vartheta_{\rm M}^0 = \frac{\Delta m^2 \cos 2\vartheta - A_{\rm CC}^0}{\sqrt{\left(\Delta m^2 \cos 2\vartheta - A_{\rm CC}^0\right)^2 + \left(\Delta m^2 \sin 2\vartheta\right)^2}}}$$

critical parameter [Bahcall, Peña-Garay, hep-ph/0305159]

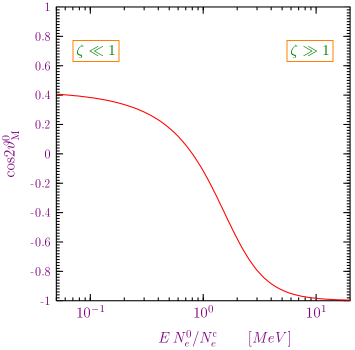
$$\zeta = \frac{A_{\rm CC}^0}{\Delta m^2 \cos 2\vartheta} = \frac{2\sqrt{2} E G_{\rm F} N_e^0}{\Delta m^2 \cos 2\vartheta} \simeq 1.2 \left(\frac{E}{\rm MeV}\right)\left(\frac{N_e^0}{N_e^{\rm c}}\right)$$

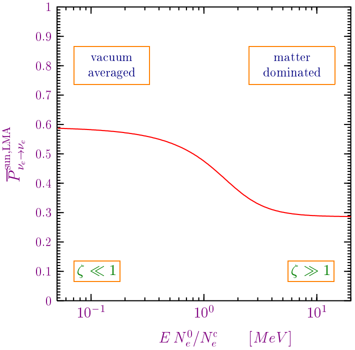
$$\zeta \ll 1 \implies \vartheta_{\rm M}^0 \simeq \vartheta \implies \overline{P}_{\nu_e\to\nu_e}^{\rm sun} \simeq 1 - \frac{1}{2}\sin^2 2\vartheta$$

vacuum averaged survival probability

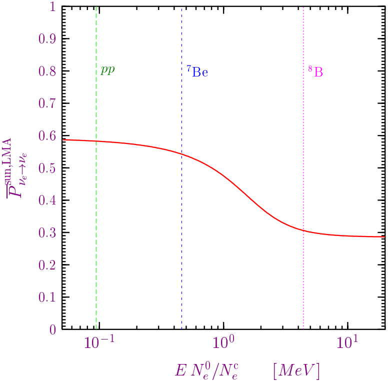
$$\zeta \gg 1 \implies \vartheta_{\rm M}^0 \simeq \pi/2 \implies \overline{P}_{\nu_e\to\nu_e}^{\rm sun} \simeq \sin^2 \vartheta$$

matter dominated survival probability





$$\langle E \rangle_{pp} \simeq 0.27\,\text{MeV}\,,\ \langle r_0 \rangle_{pp} \simeq 0.1\,R_\odot \implies \langle E\, N_e^0/N_e^c \rangle_{pp} \simeq 0.094\,\text{MeV}$$
$$E_{^7\text{Be}} \simeq 0.86\,\text{MeV}\,,\ \langle r_0 \rangle_{^7\text{Be}} \simeq 0.06\,R_\odot \implies \langle E\, N_e^0/N_e^c \rangle_{^7\text{Be}} \simeq 0.46\,\text{MeV}$$
$$\langle E \rangle_{^8\text{B}} \simeq 6.7\,\text{MeV}\,,\ \langle r_0 \rangle_{^8\text{B}} \simeq 0.04\,R_\odot \implies \langle E\, N_e^0/N_e^c \rangle_{^8\text{B}} \simeq 4.4\,\text{MeV}$$



each neutrino experiment is mainly sensitive to one flux

each neutrino experiment is mainly sensitive to ϑ

accurate *pp* experiment can improve determination of ϑ

[Bahcall, Peña-Garay, hep-ph/0305159]

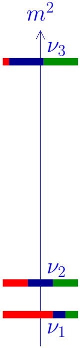
Mass Hierarchy

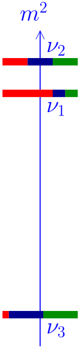
1. Matter Effect (Atmospheric, Long-Baseline, Supernova Experiments):
   - $\nu_e \leftrightarrows \nu_\mu$ MSW resonance: $\quad V = \dfrac{\Delta m^2_{13}\cos 2\vartheta_{13}}{2E} \quad \Leftrightarrow \quad \Delta m^2_{13} > 0 \quad$ NH
   - $\bar{\nu}_e \leftrightarrows \bar{\nu}_\mu$ MSW resonance: $\quad V = -\dfrac{\Delta m^2_{13}\cos 2\vartheta_{13}}{2E} \quad \Leftrightarrow \quad \Delta m^2_{13} < 0 \quad$ IH
2. Phase Difference (Reactor $\bar{\nu}_e \to \bar{\nu}_e$):

Normal Hierarchy

$$|\Delta m^2_{31}| = |\Delta m^2_{32}| + |\Delta m^2_{21}|$$

$$|\Delta m^2_{31}| > |\Delta m^2_{32}|$$

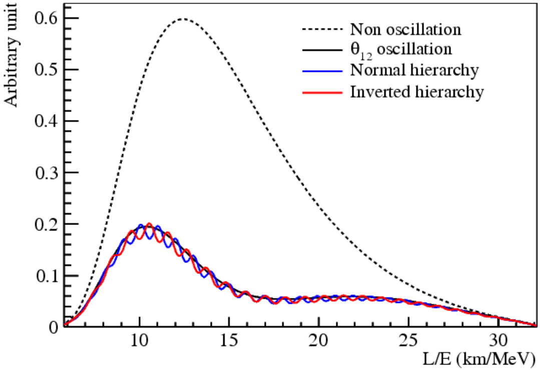




Inverted Hierarchy

$$|\Delta m^2_{31}| = |\Delta m^2_{32}| - |\Delta m^2_{21}|$$

$$|\Delta m^2_{31}| < |\Delta m^2_{32}|$$



S.T. Petcov et al., PLB533(2002)94
S.Choubey et al., PRD68(2003)113006
J. Learned et al., hep-ex/0612022

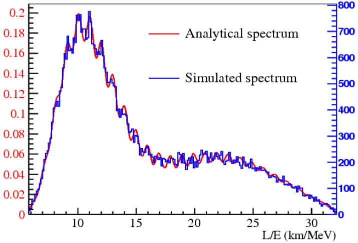
L. Zhan, Y. Wang, J. Cao, L. Wen,
PRD78:111103, 2008
PRD79:073007, 2009

Precision energy spectrum measurement: Looking for interference between P_{31} and P_{32}
➔ relative measurement

$$F(L/E) = \phi(E)\sigma(E)P_{ee}(L/E)$$

$$\begin{aligned}
P_{ee}(L/E) &= 1 - P_{21} - P_{31} - P_{32} \\
P_{21} &= \cos^4(\theta_{13})\sin^2(2\theta_{12})\sin^2(\Delta_{21}) \\
P_{31} &= \cos^2(\theta_{12})\sin^2(2\theta_{13})\sin^2(\Delta_{31}) \\
P_{32} &= \sin^2(\theta_{12})\sin^2(2\theta_{13})\sin^2(\Delta_{32})
\end{aligned}$$

$$\Delta_{21} << \Delta_{31} \approx \Delta_{32}$$



[Miao He, NuFact 2013]

In Neutrino Oscillations Dirac = Majorana

[Bilenky, Hosek, Petcov, PLB 94 (1980) 495; Doi, Kotani, Nishiura, Okuda, Takasugi, PLB 102 (1981) 323]

[Langacker, Petcov, Steigman, Toshev, NPB 282 (1987) 589]

Evolution of Amplitudes: $$i \frac{\mathrm{d}\psi_\alpha}{\mathrm{d}x} = \frac{1}{2E} \sum_\beta \left(U M^2 U^\dagger + 2EV \right)_{\alpha\beta} \psi_\beta$$

difference: $$\begin{cases} \text{Dirac:} & U^{(\mathrm{D})} \\ \text{Majorana:} & U^{(\mathrm{M})} = U^{(\mathrm{D})} D(\lambda) \end{cases}$$

$$D(\lambda) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & e^{i\lambda_{21}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i\lambda_{N1}} \end{pmatrix} \quad \Rightarrow \quad D^\dagger = D^{-1}$$

$$M^2 = \begin{pmatrix} m_1^2 & 0 & \cdots & 0 \\ 0 & m_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_N^2 \end{pmatrix} \quad \Longrightarrow \quad D M^2 = M^2 D \quad \Longrightarrow \quad D M^2 D^\dagger = M^2$$

$$U^{(\mathrm{M})} M^2 (U^{(\mathrm{M})})^\dagger = U^{(\mathrm{D})} D M^2 D^\dagger (U^{(\mathrm{D})})^\dagger = U^{(\mathrm{D})} M^2 (U^{(\mathrm{D})})^\dagger$$