Neutrino Theory and Phenomenology

PART 1

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NBIA PhD School
Neutrinos underground & in the heavens

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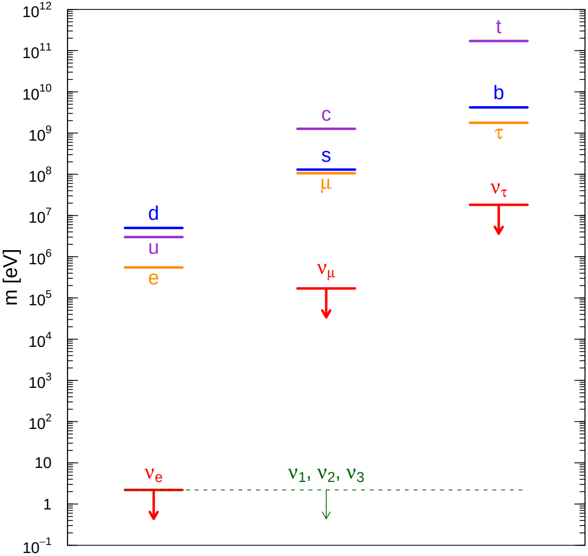
Neutrino Theory and Phenomenology

- Neutrino Masses and Mixing
- Neutrino Oscillations in Vacuum
- Neutrino Oscillations in Matter

Neutrino Masses and Mixing

Fermion Mass Spectrum



Dirac Mass

- Dirac Equation: $(i\partial\!\!\!/ - m)\,\nu(x) = 0 \qquad (\partial\!\!\!/ \equiv \gamma^\mu \partial_\mu)$
- Dirac Lagrangian: $\mathscr{L}(x) = \overline{\nu}(x)\,(i\partial\!\!\!/ - m)\,\nu(x)$
- Chiral decomposition: $\nu_L \equiv P_L \nu\,, \qquad \nu_R \equiv P_R \nu\,, \qquad \nu = \nu_L + \nu_R$

  Left and Right-handed Projectors: $\quad P_L \equiv \dfrac{1-\gamma^5}{2}\,, \quad P_R \equiv \dfrac{1+\gamma^5}{2}$

$$P_L^2 = P_L\,, \quad P_R^2 = P_R\,, \quad P_L + P_R = 1\,, \quad P_L P_R = P_R P_L = 0$$

$$\mathscr{L} = \overline{\nu_L} i\partial\!\!\!/ \nu_L + \overline{\nu_R} i\partial\!\!\!/ \nu_R - m\,(\overline{\nu_L}\nu_R + \overline{\nu_R}\nu_L)$$

- In SM only ν_L by assumption $\Longrightarrow$ no neutrino mass
  Note that all the other elementary fermion fields (charged leptons and quarks) have both left and right-handed components
- Oscillation experiments have shown that neutrinos are massive
- Simplest and natural extension of the SM: consider also ν_R
  as for all the other elementary fermion fields

Higgs Mechanism in SM

- Higgs Doublet: $\Phi(x) = \begin{pmatrix} \phi_+(x) \\ \phi_0(x) \end{pmatrix}$ $\qquad |\Phi|^2 = \Phi^\dagger \Phi = \phi_+^\dagger \phi_+ + \phi_0^\dagger \phi_0$

- Higgs Lagrangian: $\mathscr{L}_{\text{Higgs}} = (D_\mu \Phi)^\dagger (D^\mu \Phi) - V(|\Phi|^2)$

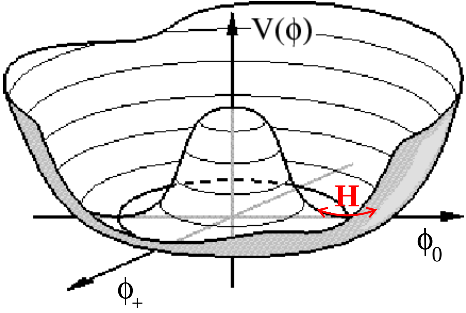
- Higgs Potential: $V(|\Phi|^2) = \mu^2 |\Phi|^2 + \lambda |\Phi|^4$

- $\mu^2 < 0$ and $\lambda > 0 \quad \Longrightarrow \quad V(|\Phi|^2) = \lambda \left(|\Phi|^2 - \frac{v^2}{2} \right)^2$

$$v \equiv \sqrt{-\frac{\mu^2}{\lambda}} = \left(\sqrt{2} G_{\text{F}}\right) \simeq 246\,\text{GeV}$$

- Vacuum: V_{min} for $|\Phi|^2 = \frac{v^2}{2} \Longrightarrow \langle \Phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$

- Spontaneous Symmetry Breaking: $\text{SU}(2)_L \times \text{U}(1)_Y \to \text{U}(1)_Q$



- Unitary Gauge: $\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix} \Longrightarrow |\Phi|^2 = \frac{v^2}{2} + vH + \frac{1}{2} H^2$

- $V = \lambda \left(|\Phi|^2 - \frac{v^2}{2} \right)^2 = \lambda v^2 H^2 + \lambda v H^3 + \frac{\lambda}{4} H^4$

$$m_H = \sqrt{2\lambda v^2} = \sqrt{-2\mu^2} \simeq 126\,\text{GeV}$$

$$-\mu^2 \simeq (89\,\text{GeV})^2 \qquad \lambda = -\frac{\mu^2}{v^2} \simeq 0.13$$

SM Extension: Dirac ν Masses

$$L_L \equiv \begin{pmatrix} \nu_L \\ \ell_L \end{pmatrix} \qquad \ell_R \qquad \nu_R$$

Lepton-Higgs Yukawa Lagrangian

$$\mathscr{L}_{H,\mathrm{L}} = -y^{\ell}\,\overline{L_L}\,\Phi\,\ell_R - y^{\nu}\,\overline{L_L}\,\widetilde{\Phi}\,\nu_R + \text{H.c.}$$

Spontaneous Symmetry Breaking

$$\Phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + H(x) \end{pmatrix} \qquad \widetilde{\Phi} = i\sigma_2\,\Phi^* = \frac{1}{\sqrt{2}}\begin{pmatrix} v + H(x) \\ 0 \end{pmatrix}$$

$$\begin{aligned} \mathscr{L}_{H,\mathrm{L}} = & -\frac{y^{\ell}}{\sqrt{2}}\begin{pmatrix} \overline{\nu_L} & \overline{\ell_L} \end{pmatrix}\begin{pmatrix} 0 \\ v + H(x) \end{pmatrix}\ell_R \\ & -\frac{y^{\nu}}{\sqrt{2}}\begin{pmatrix} \overline{\nu_L} & \overline{\ell_L} \end{pmatrix}\begin{pmatrix} v + H(x) \\ 0 \end{pmatrix}\nu_R + \text{H.c.} \end{aligned}$$

$$\mathscr{L}_{H,\mathrm{L}} = -\, y^{\ell}\, \frac{v}{\sqrt{2}}\, \overline{\ell_L}\, \ell_R - y^{\nu}\, \frac{v}{\sqrt{2}}\, \overline{\nu_L}\, \nu_R$$

$$- \frac{y^{\ell}}{\sqrt{2}}\, \overline{\ell_L}\, \ell_R\, H - \frac{y^{\nu}}{\sqrt{2}}\, \overline{\nu_L}\, \nu_R\, H + \mathrm{H.c.}$$

$$m_{\ell} = y^{\ell}\, \frac{v}{\sqrt{2}} \qquad\qquad m_{\nu} = y^{\nu}\, \frac{v}{\sqrt{2}}$$

$$g_{\ell H} = \frac{y^{\ell}}{\sqrt{2}} = \frac{m_{\ell}}{v} \qquad\qquad g_{\nu H} = \frac{y^{\nu}}{\sqrt{2}} = \frac{m_{\nu}}{v}$$

$$v = \left(\sqrt{2}\, G_{\mathrm{F}}\right)^{1/2} = 246\,\mathrm{GeV}$$

PROBLEM: $y^{\nu} \lesssim 10^{-11} \ll y^{e} \sim 10^{-6}$

Three-Generations Dirac Neutrino Masses

$L'_{eL} \equiv \begin{pmatrix} \nu'_{eL} \\ \ell'_{eL} \equiv e'_L \end{pmatrix}$	$L'_{\mu L} \equiv \begin{pmatrix} \nu'_{\mu L} \\ \ell'_{\mu L} \equiv \mu'_L \end{pmatrix}$	$L'_{\tau L} \equiv \begin{pmatrix} \nu'_{\tau L} \\ \ell'_{\tau L} \equiv \tau'_L \end{pmatrix}$
$\ell'_{eR} \equiv e'_R$	$\ell'_{\mu R} \equiv \mu'_R$	$\ell'_{\tau R} \equiv \tau'_R$
ν'_{eR}	$\nu'_{\mu R}$	$\nu'_{\tau R}$

Lepton-Higgs Yukawa Lagrangian

$$\mathscr{L}^{\mathrm{D}} = - \sum_{\alpha,\beta=e,\mu,\tau} \left[Y'^{\ell}_{\alpha\beta} \, \overline{L'_{\alpha L}} \, \Phi \, \ell'_{\beta R} + Y'^{\nu}_{\alpha\beta} \, \overline{L'_{\alpha L}} \, \widetilde{\Phi} \, \nu'_{\beta R} \right] + \text{H.c.}$$

Spontaneous Symmetry Breaking

$$\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \qquad \widetilde{\Phi} = i\sigma_2 \, \Phi^* = \frac{1}{\sqrt{2}} \begin{pmatrix} v \\ 0 \end{pmatrix}$$

$$\mathscr{L}^{\mathrm{D}} = - \sum_{\alpha,\beta=e,\mu,\tau} \left[\frac{v}{\sqrt{2}} Y'^{\ell}_{\alpha\beta} \, \overline{\ell'_{\alpha L}} \, \ell'_{\beta R} + \frac{v}{\sqrt{2}} Y'^{\nu}_{\alpha\beta} \, \overline{\nu'_{\alpha L}} \, \nu'_{\beta R} \right] + \mathrm{H.c.}$$

$$\mathscr{L}^{\mathrm{D}} = - \left[\overline{\boldsymbol{\ell}'_L} \, M'^{\ell} \, \boldsymbol{\ell}'_R + \overline{\boldsymbol{\nu}'_L} \, M'^{\nu} \, \boldsymbol{\nu}'_R \right] + \mathrm{H.c.}$$

$$\boldsymbol{\ell}'_L \equiv \begin{pmatrix} e'_L \\ \mu'_L \\ \tau'_L \end{pmatrix} \qquad \boldsymbol{\ell}'_R \equiv \begin{pmatrix} e'_R \\ \mu'_R \\ \tau'_R \end{pmatrix} \qquad \boldsymbol{\nu}'_L \equiv \begin{pmatrix} \nu'_{eL} \\ \nu'_{\mu L} \\ \nu'_{\tau L} \end{pmatrix} \qquad \boldsymbol{\nu}'_R \equiv \begin{pmatrix} \nu'_{eR} \\ \nu'_{\mu R} \\ \nu'_{\tau R} \end{pmatrix}$$

$$M'^{\ell} = \frac{v}{\sqrt{2}} \, Y'^{\ell} \qquad\qquad M'^{\nu} = \frac{v}{\sqrt{2}} \, Y'^{\nu}$$

$$M'^{\ell} \equiv \begin{pmatrix} M'^{\ell}_{ee} & M'^{\ell}_{e\mu} & M'^{\ell}_{e\tau} \\ M'^{\ell}_{\mu e} & M'^{\ell}_{\mu\mu} & M'^{\ell}_{\mu\tau} \\ M'^{\ell}_{\tau e} & M'^{\ell}_{\tau\mu} & M'^{\ell}_{\tau\tau} \end{pmatrix} \qquad M'^{\nu} \equiv \begin{pmatrix} M'^{\nu}_{ee} & M'^{\nu}_{e\mu} & M'^{\nu}_{e\tau} \\ M'^{\nu}_{\mu e} & M'^{\nu}_{\mu\mu} & M'^{\nu}_{\mu\tau} \\ M'^{\nu}_{\tau e} & M'^{\nu}_{\tau\mu} & M'^{\nu}_{\tau\tau} \end{pmatrix}$$

$$\mathscr{L}^{\mathrm{D}} = -\overline{\boldsymbol{\ell}'_L}\, M'^{\ell}\, \boldsymbol{\ell}'_R - \overline{\boldsymbol{\nu}'_L}\, M'^{\nu}\, \boldsymbol{\nu}'_R + \mathrm{H.c.}$$

Diagonalization of M'^{ℓ} and M'^{ν} with unitary V^{ℓ}_L, V^{ℓ}_R, V^{ν}_L, V^{ν}_R

$$\boldsymbol{\ell}'_L = V^{\ell}_L\, \boldsymbol{\ell}_L \qquad \boldsymbol{\ell}'_R = V^{\ell}_R\, \boldsymbol{\ell}_R \qquad \boldsymbol{\nu}'_L = V^{\nu}_L\, \mathbf{n}_L \qquad \boldsymbol{\nu}'_R = V^{\nu}_R\, \mathbf{n}_R$$

Important general remark: unitary transformations are allowed because they leave invariant the kinetic terms in the Lagrangian

$$\begin{aligned}
\mathscr{L}_{\mathrm{kin}} &= \overline{\boldsymbol{\ell}'_L} i\partial\!\!\!/ \boldsymbol{\ell}'_L + \overline{\boldsymbol{\ell}'_R} i\partial\!\!\!/ \boldsymbol{\ell}'_R + \overline{\boldsymbol{\nu}'_L} i\partial\!\!\!/ \boldsymbol{\nu}'_L + \overline{\boldsymbol{\nu}'_R} i\partial\!\!\!/ \boldsymbol{\nu}'_R \\
&= \overline{\boldsymbol{\ell}_L} V^{\ell\dagger}_L i\partial\!\!\!/ V^{\ell}_L \boldsymbol{\ell}_L + \ldots \\
&= \overline{\boldsymbol{\ell}_L} i\partial\!\!\!/ \boldsymbol{\ell}_L + \overline{\boldsymbol{\ell}_R} i\partial\!\!\!/ \boldsymbol{\ell}_R + \overline{\boldsymbol{\nu}_L} i\partial\!\!\!/ \boldsymbol{\nu}_L + \overline{\boldsymbol{\nu}_R} i\partial\!\!\!/ \boldsymbol{\nu}_R
\end{aligned}$$

$$\mathscr{L}^{\mathrm{D}} = -\overline{\boldsymbol{\ell}'_L}\, M'^{\ell}\, \boldsymbol{\ell}'_R - \overline{\boldsymbol{\nu}'_L}\, M'^{\nu}\, \boldsymbol{\nu}'_R + \text{H.c.}$$

$$\boldsymbol{\ell}'_L = V^{\ell}_L\, \boldsymbol{\ell}_L \qquad \boldsymbol{\ell}'_R = V^{\ell}_R\, \boldsymbol{\ell}_R \qquad \boldsymbol{\nu}'_L = V^{\nu}_L\, \mathbf{n}_L \qquad \boldsymbol{\nu}'_R = V^{\nu}_R\, \mathbf{n}_R$$

$$\mathscr{L}^{\mathrm{D}} = -\overline{\boldsymbol{\ell}_L}\, V^{\ell\dagger}_L M'^{\ell} V^{\ell}_R \boldsymbol{\ell}_R - \overline{\boldsymbol{\nu}_L}\, V^{\nu\dagger}_L M'^{\nu} V^{\nu}_R \boldsymbol{\nu}_R + \text{H.c.}$$

$$V^{\ell\dagger}_L \; M'^{\ell} \; V^{\ell}_R = M^{\ell} \qquad M^{\ell}_{\alpha\beta} = m^{\ell}_{\alpha}\, \delta_{\alpha\beta} \qquad (\alpha, \beta = e, \mu, \tau)$$

$$V^{\nu\dagger}_L \; M'^{\nu} \; V^{\nu}_R = M^{\nu} \qquad M^{\nu}_{kj} = m^{\nu}_{k}\, \delta_{kj} \qquad (k, j = 1, 2, 3)$$

Real and Positive m^{ℓ}_{α}, m^{ν}_{k}

Massive Chiral Lepton Fields

$\boldsymbol{\ell}_L = V_L^{\ell\dagger}\,\boldsymbol{\ell}'_L \equiv \begin{pmatrix} e_L \\ \mu_L \\ \tau_L \end{pmatrix}$	$\boldsymbol{\ell}_R = V_R^{\ell\dagger}\,\boldsymbol{\ell}'_R \equiv \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix}$
$\mathbf{n}_L = V_L^{\nu\dagger}\,\boldsymbol{\nu}'_L \equiv \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \nu_{3L} \end{pmatrix}$	$\mathbf{n}_R = V_R^{\nu\dagger}\,\boldsymbol{\nu}'_R \equiv \begin{pmatrix} \nu_{1R} \\ \nu_{2R} \\ \nu_{3R} \end{pmatrix}$

$$\begin{aligned} \mathscr{L}^{\text{D}} &= -\,\overline{\boldsymbol{\ell}_L}\,M^{\ell}\,\boldsymbol{\ell}_R - \overline{\mathbf{n}_L}\,M^{\nu}\,n_R + \text{H.c.} \\ &= -\sum_{\alpha=e,\mu,\tau} m_\alpha^{\ell}\,\overline{\ell_{\alpha L}}\,\ell_{\alpha R} - \sum_{k=1}^{3} m_k^{\nu}\,\overline{\nu_{kL}}\,\nu_{kR} + \text{H.c.} \end{aligned}$$

Mixing

Charged-Current Weak Interaction Lagrangian

$$\mathscr{L}_{\mathrm{I}}^{(\mathrm{CC})} = -\frac{g}{2\sqrt{2}} j_W^\rho W_\rho + \mathrm{H.c.}$$

Weak Charged Current: $\quad j_W^\rho = j_{W,\mathrm{L}}^\rho + j_{W,\mathrm{Q}}^\rho$

Leptonic Weak Charged Current

$$j_{W,\mathrm{L}}^{\rho\dagger} = 2 \sum_{\alpha=e,\mu,\tau} \overline{\ell'_{\alpha L}} \gamma^\rho \nu'_{\alpha L} = 2 \overline{\boldsymbol{\ell}'_L} \gamma^\rho \boldsymbol{\nu}'_L$$

$$\boldsymbol{\ell}'_L = V_L^\ell \boldsymbol{\ell}_L \qquad \boldsymbol{\nu}'_L = V_L^\nu \mathbf{n}_L$$

$$j_{W,\mathrm{L}}^{\rho\dagger} = 2 \overline{\boldsymbol{\ell}_L} V_L^{\ell\dagger} \gamma^\rho V_L^\nu \mathbf{n}_L = 2 \overline{\boldsymbol{\ell}_L} \gamma^\rho V_L^{\ell\dagger} V_L^\nu \mathbf{n}_L = 2 \overline{\boldsymbol{\ell}_L} \gamma^\rho U \mathbf{n}_L$$

Mixing Matrix

$$\boxed{U = V_L^{\ell\dagger} V_L^\nu}$$

- Definition: Left-Handed Flavor Neutrino Fields

$$\boldsymbol{\nu}_L = U\,\mathbf{n}_L = V_L^{\ell\dagger}\,V_L^{\nu}\,\mathbf{n}_L = V_L^{\ell\dagger}\,\boldsymbol{\nu}'_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix}$$

- They allow us to write the Leptonic Weak Charged Current as in the SM:

$$j^{\rho\dagger}_{W,\mathrm{L}} = 2\,\overline{\boldsymbol{\ell}_L}\,\gamma^\rho\,\boldsymbol{\nu}_L = 2\sum_{\alpha=e,\mu,\tau}\overline{\ell_{\alpha L}}\,\gamma^\rho\,\nu_{\alpha L}$$

- Each left-handed flavor neutrino field is associated with the corresponding charged lepton field which describes a massive charged lepton:

$$j^{\rho\dagger}_{W,\mathrm{L}} = 2\left(\overline{e_L}\,\gamma^\rho\,\nu_{eL} + \overline{\mu_L}\,\gamma^\rho\,\nu_{\mu L} + \overline{\tau_L}\,\gamma^\rho\,\nu_{\tau L}\right)$$

- In practice left-handed flavor neutrino fields are useful for calculations in the SM approximation of massless neutrinos (interactions).

- If neutrino masses must be taken into account, it is necessary to use

$$j^{\rho\dagger}_{W,\mathrm{L}} = 2\,\overline{\boldsymbol{\ell}_L}\,\gamma^\rho\,U\,\mathbf{n}_L = 2\sum_{\alpha=e,\mu,\tau}\sum_{k=1}^{3}\overline{\ell_{\alpha L}}\,\gamma^\rho\,U_{\alpha k}\,\nu_{kL}$$

Flavor Lepton Numbers

Flavor Neutrino Fields are useful for defining
Flavor Lepton Numbers
as in the SM

	L_e	L_μ	L_τ
$(\nu_e\,,\,e^-)$	$+1$	0	0
$(\nu_\mu\,,\,\mu^-)$	0	$+1$	0
$(\nu_\tau\,,\,\tau^-)$	0	0	$+1$

	L_e	L_μ	L_τ
$(\nu_e^c\,,\,e^+)$	-1	0	0
$(\nu_\mu^c\,,\,\mu^+)$	0	-1	0
$(\nu_\tau^c\,,\,\tau^+)$	0	0	-1

$$L = L_e + L_\mu + L_\tau$$

Standard Model: Lepton numbers are conserved

- L_e, L_μ, L_τ are conserved in the Standard Model with massless neutrinos

- Mass term:

$$\mathscr{L}^{\text{D}} = -\begin{pmatrix} \overline{\nu_{eL}} & \overline{\nu_{\mu L}} & \overline{\nu_{\tau L}} \end{pmatrix} \begin{pmatrix} m^{\text{D}}_{ee} & m^{\text{D}}_{e\mu} & m^{\text{D}}_{e\tau} \\ m^{\text{D}}_{\mu e} & m^{\text{D}}_{\mu\mu} & m^{\text{D}}_{\mu\tau} \\ m^{\text{D}}_{\tau e} & m^{\text{D}}_{\tau\mu} & m^{\text{D}}_{\tau\tau} \end{pmatrix} \begin{pmatrix} \nu_{eR} \\ \nu_{\mu R} \\ \nu_{\tau R} \end{pmatrix} + \text{H.c.}$$

L_e, L_μ, L_τ are not conserved

- L is conserved: $L(\nu_{\alpha R}) = L(\nu_{\beta L}) \implies |\Delta L| = 0$

Mixing Matrix

- $U = V_L^{\ell\dagger} \, V_L^{\nu} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}$

- Unitary $N \times N$ matrix depends on N^2 independent real parameters

$$N = 3 \quad \Longrightarrow \quad \begin{array}{ll} \dfrac{N(N-1)}{2} = 3 & \text{Mixing Angles} \\ \dfrac{N(N+1)}{2} = 6 & \text{Phases} \end{array}$$

- Not all phases are physical observables

- Only physical effect of mixing matrix occurs through its presence in the Leptonic Weak Charged Current

- Weak Charged Current: $j^{\rho\dagger}_{W,\mathrm{L}} = 2 \sum_{\alpha=e,\mu,\tau} \sum_{k=1}^{3} \overline{\ell_{\alpha L}} \, \gamma^{\rho} \, U_{\alpha k} \, \nu_{kL}$
- Apart from the Weak Charged Current, the Lagrangian is invariant under the global phase transformations (6 arbitrary phases)

$$\ell_{\alpha} \to e^{i\varphi_{\alpha}} \, \ell_{\alpha} \quad (\alpha = e, \mu, \tau) \, , \qquad \nu_{k} \to e^{i\varphi_{k}} \, \nu_{k} \quad (k = 1, 2, 3)$$

- Performing this transformation, the Charged Current becomes

$$j^{\rho\dagger}_{W,\mathrm{L}} = 2 \sum_{\alpha=e,\mu,\tau} \sum_{k=1}^{3} \overline{\ell_{\alpha L}} \, e^{-i\varphi_{\alpha}} \, \gamma^{\rho} \, U_{\alpha k} \, e^{i\varphi_{k}} \, \nu_{kL}$$

$$j^{\rho\dagger}_{W,\mathrm{L}} = 2 \, \underbrace{e^{-i(\varphi_{e}-\varphi_{1})}}_{1} \sum_{\alpha=e,\mu,\tau} \sum_{k=1}^{3} \overline{\ell_{\alpha L}} \, \underbrace{e^{-i(\varphi_{\alpha}-\varphi_{e})}}_{2} \, \gamma^{\rho} \, U_{\alpha k} \, \underbrace{e^{i(\varphi_{k}-\varphi_{1})}}_{2} \, \nu_{kL}$$

- There are 5 independent combinations of the phases of the fields that can be chosen to eliminate 5 of the 6 phases of the mixing matrix
- It is convenient to express the 3×3 unitary mixing matrix only in terms of the four physical parameters:

3 Mixing Angles and 1 Phase

Standard Parameterization of Mixing Matrix

$$\begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \nu_{3L} \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13} e^{i\delta_{13}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta_{13}} \\ -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta_{13}} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta_{13}} & s_{23} c_{13} \\ s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta_{13}} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta_{13}} & c_{23} c_{13} \end{pmatrix}$$

$c_{ab} \equiv \cos\vartheta_{ab}$ $\quad$ $s_{ab} \equiv \sin\vartheta_{ab}$ $\quad$ $0 \leq \vartheta_{ab} \leq \frac{\pi}{2}$ $\quad$ $0 \leq \delta_{13} < 2\pi$

3 Mixing Angles ϑ_{12}, ϑ_{23}, ϑ_{13} and 1 Phase δ_{13}

Standard Parameterization

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13} e^{i\delta_{13}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example of Different Phase Convention

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} e^{i\delta_{23}} \\ 0 & -s_{23} e^{-i\delta_{23}} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example of Different Parameterization

$$U = \begin{pmatrix} c'_{12} & s'_{12} e^{-i\delta'_{12}} & 0 \\ -s'_{12} e^{i\delta'_{12}} & c'_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c'_{23} & s'_{23} \\ 0 & -s'_{23} & c'_{23} \end{pmatrix} \begin{pmatrix} c'_{13} & 0 & s'_{13} \\ 0 & 1 & 0 \\ -s'_{13} & 0 & c'_{13} \end{pmatrix}$$

Jarlskog Rephasing Invariant

- Simplest rephasing invariants: $|U_{\alpha k}|^2 = U_{\alpha k} U_{\alpha k}^*$, $U_{\alpha k} U_{\alpha j}^* U_{\beta k}^* U_{\beta j}$

$$\mathfrak{Im}\left[U_{\alpha k} U_{\alpha j}^* U_{\beta k}^* U_{\beta j}\right] = \pm J$$

$$J = \mathfrak{Im}\left[U_{e2} U_{e3}^* U_{\mu 2}^* U_{\mu 3}\right] = \mathfrak{Im} \begin{pmatrix} \cdot & \circ & \times \\ \cdot & \times & \circ \\ \cdot & \cdot & \cdot \end{pmatrix}$$

- In standard parameterization:

$$\begin{aligned} J &= c_{12} s_{12} c_{23} s_{23} c_{13}^2 s_{13} \sin\delta_{13} \\ &= \frac{1}{8} \sin 2\vartheta_{12} \sin 2\vartheta_{23} \cos\vartheta_{13} \sin 2\vartheta_{13} \sin\delta_{13} \end{aligned}$$

- Jarlskog invariant is useful for quantifying CP violation due to $U \neq U^*$ in a parameterization-independent way.
- All measurable CP-violation effects depend on J.

CP Violation

- $U \neq U^{*} \quad \Longrightarrow \quad$ CP Violation

- General conditions for CP violation (14 conditions):
  1. No two charged leptons or two neutrinos are degenerate in mass (6 conditions)
  2. No mixing angle is equal to 0 or $\pi/2$ (6 conditions)
  3. The physical phase is different from 0 or π (2 conditions)

- These 14 conditions are combined into the single condition $\det C \neq 0$

$$C = -i \, [M'^{\nu} \, M'^{\nu\dagger} \, , \, M'^{\ell} \, M'^{\ell\dagger}]$$

$$\begin{aligned} \det C = -2 \, J \, & \left(m_{\nu_2}^2 - m_{\nu_1}^2\right) \left(m_{\nu_3}^2 - m_{\nu_1}^2\right) \left(m_{\nu_3}^2 - m_{\nu_2}^2\right) \\ & \left(m_{\mu}^2 - m_{e}^2\right) \left(m_{\tau}^2 - m_{e}^2\right) \left(m_{\tau}^2 - m_{\mu}^2\right) \end{aligned}$$

[C. Jarlskog, Phys. Rev. Lett. 55 (1985) 1039, Z. Phys. C 29 (1985) 491]

[O. W. Greenberg, Phys. Rev. D 32 (1985) 1841]

[I. Dunietz, O. W. Greenberg, Dan-di Wu, Phys. Rev. Lett. 55 (1985) 2935]

Maximal CP Violation

- Maximal CP violation is defined as the case in which $|J|$ has its maximum possible value

$$|J|_{\text{max}} = \frac{1}{6\sqrt{3}}$$

- In the standard parameterization it is obtained for

$$\vartheta_{12} = \vartheta_{23} = \pi/4\,, \quad s_{13} = 1/\sqrt{3}\,, \quad \sin\delta_{13} = \pm 1$$

- This case is called Trimaximal Mixing. All the absolute values of the elements of the mixing matrix are equal to $1/\sqrt{3}$:

$$U = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \mp\frac{i}{\sqrt{3}} \\ -\frac{1}{2} \mp \frac{i}{2\sqrt{3}} & \frac{1}{2} \mp \frac{i}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{2} \mp \frac{i}{2\sqrt{3}} & -\frac{1}{2} \mp \frac{i}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & \mp i \\ -e^{\pm i\pi/6} & e^{\mp i\pi/6} & 1 \\ e^{\mp i\pi/6} & -e^{\pm i\pi/6} & 1 \end{pmatrix}$$

GIM Mechanism

[S.L. Glashow, J. Iliopoulos, L. Maiani, Phys. Rev. D 2 (1970) 1285]

- The unitarity of V_L^ℓ, V_R^ℓ and V_L^ν implies that the expression of the neutral weak current in terms of the lepton fields with definite masses is the same as that in terms of the primed lepton fields:

$$
\begin{aligned}
j_{Z,\mathrm{L}}^\rho = & 2\, g_L^\nu \, \overline{\boldsymbol{\nu}_L'} \, \gamma^\rho \, \boldsymbol{\nu}_L' + 2\, g_L^l \, \overline{\boldsymbol{\ell}_L'} \, \gamma^\rho \boldsymbol{\ell}_L' + 2\, g_R^l \, \overline{\boldsymbol{\ell}_R'} \, \gamma^\rho \boldsymbol{\ell}_R' \\
= & 2\, g_L^\nu \, \overline{\mathbf{n}_L} \, V_L^{\nu\dagger} \, \gamma^\rho \, V_L^\nu \, \mathbf{n}_L + 2\, g_L^l \, \overline{\boldsymbol{\ell}_L} \, V_L^{\ell\dagger} \, \gamma^\rho \, V_L^\ell \, \boldsymbol{\ell}_L + 2\, g_R^l \, \overline{\boldsymbol{\ell}_R} \, V_R^{\ell\dagger} \, \gamma^\rho \, V_R^\ell \, \boldsymbol{\ell}_R \\
= & 2\, g_L^\nu \, \overline{\mathbf{n}_L} \, \gamma^\rho \, \mathbf{n}_L + 2\, g_L^l \, \overline{\boldsymbol{\ell}_L} \, \gamma^\rho \boldsymbol{\ell}_L + 2\, g_R^l \, \overline{\boldsymbol{\ell}_R} \, \gamma^\rho \boldsymbol{\ell}_R
\end{aligned}
$$

- The unitarity of U implies the same expression for the neutral weak current in terms of the flavor neutrino fields $\boldsymbol{\nu}_L = U \, \mathbf{n}_L$:

$$
\begin{aligned}
j_{Z,\mathrm{L}}^\rho &= 2\, g_L^\nu \, \overline{\boldsymbol{\nu}_L} \, U \, \gamma^\rho \, U^\dagger \, \boldsymbol{\nu}_L + 2\, g_L^l \, \overline{\boldsymbol{\ell}_L} \, \gamma^\rho \boldsymbol{\ell}_L + 2\, g_R^l \, \overline{\boldsymbol{\ell}_R} \, \gamma^\rho \boldsymbol{\ell}_R \\
&= 2\, g_L^\nu \, \overline{\boldsymbol{\nu}_L} \, \gamma^\rho \, \boldsymbol{\nu}_L + 2\, g_L^l \, \overline{\boldsymbol{\ell}_L} \, \gamma^\rho \boldsymbol{\ell}_L + 2\, g_R^l \, \overline{\boldsymbol{\ell}_R} \, \gamma^\rho \boldsymbol{\ell}_R
\end{aligned}
$$

Lepton Numbers Violating Processes

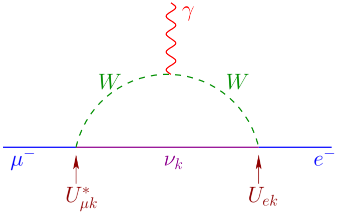
Dirac mass term allows L_e, L_μ, L_τ violating processes

Example: $\mu^\pm \to e^\pm + \gamma\,, \qquad \mu^\pm \to e^\pm + e^+ + e^-$

$$\boxed{\mu^- \to e^- + \gamma}$$

$\sum_k U_{\mu k}^* U_{ek} = 0 \Longrightarrow$ only part of ν_k propagator $\propto m_k$ contributes

$$\Gamma = \frac{G_F m_\mu^5}{192\pi^3} \underbrace{\frac{3\alpha}{32\pi} \left| \sum_k U_{\mu k}^* U_{ek} \frac{m_k^2}{m_W^2} \right|^2}_{\text{BR}}$$



Suppression factor: $\quad \frac{m_k}{m_W} \lesssim 10^{-11} \quad$ for $\quad m_k \lesssim 1\,\text{eV}$

$(\text{BR})_{\text{the}} \lesssim 10^{-47} \qquad (\text{BR})_{\text{exp}} \lesssim 10^{-11}$