# The $D^6\mathcal{R}^4$ term from three loop maximal supergravity

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# Outline of the talk

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- Introduction and motivation
- The one loop four graviton amplitude in supergravity
- The two loop four graviton amplitude in supergravity
- The three loop four graviton amplitude in supergravity
- The symmetries of the Mercedes skeleton and an auxiliary $T^3$
- The renormalised three loop amplitude

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- It is important to understand the effective action of string theory not only from the point of view of phenomenology, but also from the point of view of understanding various duality symmetries of string theory.
- These perturbative and non-perturbative (U-duality) symmetries are useful in defining the theory non–perturbatively and give us an insight into the quantitative properties of the theory.
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- However, this has proved to be difficult to obtain in generic circumstances.

- The effective action encodes the S matrices of the theory and contains non–trivial moduli dependence of the interactions for various compactifications.
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- Among the various interactions in these BPS multiplets, we shall consider the $\mathcal{R}^4$, $D^4\mathcal{R}^4$ and $D^6\mathcal{R}^4$ interactions.
- These interactions are $1/2$, $1/4$ and $1/8$ BPS respectively.
- The moduli dependent coefficient functions of these interactions can be calculated using various methods using spacetime and world sheet techniques, supersymmetry and duality (Green, Gutperle '97; Green, Gutperle, Vanhove '97; Kiritsis, Pioline '97; Green, Sethi '98; Obers, Pioline '98; Green, Kwon, Vanhove '99; Green, Vanhove '05; Basu '07; Basu '07; Green, Miller, Russo, Vanhove '10; Green, Miller, Russo, Vanhove '10).

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- We shall analyse the role played by maximal supergravity in determining these interactions.
- This will lead to an interplay of how these interactions arise from supergravity and are UV divergent, and how string theory regulates these divergences to yield finite moduli dependent answers.
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- The calculations can be done in arbitrary dimensions, but we shall focus on $T^2$ compacitification of $N = 1, d = 11$ supergravity.
- This is because for $T^d$ compactifications for $d \geq 3$, from the M theory point of view, there are contributions from wrapped (Euclidean) membranes and five branes whose effects are not captured by supergravity.
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- The supergravity analysis is helpful because multi–loop amplitudes in maximal supergravity have been calculated (Green, Schwarz, Brink '82; Green, Gutperle, Vanhove '97; Bern, Dixon, Dunbar, Perelstein, Rozowsky '98; Bern, Carrasco, Dixon, Johansson, Kosower '07; Bern, Carrasco, Dixon, Johannson, Roiban '08; Green, Russo, Vanhove '08; Bern, Carrasco, Dixon, Johannson, Roiban '09; Bern, Carrasco, Johansson '10).

- The $\mathcal{R}^4$ term receives contributions only upto 1 loop.
- The $D^4\mathcal{R}^4$ term receives contributions only upto 2 loops.
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- We now briefly discuss the contributions from 1 and 2 loops in supergravity.

- The one loop four graviton amplitude in 11 dimensions is given by

$$\mathcal{A}_4^{(1)} = \frac{4\pi^2\kappa_{11}^4}{(2\pi)^{11}}\Big[I(S,T)+I(S,U)+I(T,U)\Big]\mathcal{K}.$$

- $\mathcal{K}$ is $\mathcal{R}^4$ at the linearised level, and $2\kappa_{11}^2 = (2\pi)^8 l_{11}^9$.
- $S = -G^{MN}(k_1+k_2)_M(k_1+k_2)_N$, $T = -G^{MN}(k_1+k_4)_M(k_1+k_4)_N$ and $U = -G^{MN}(k_1+k_3)_M(k_1+k_3)_N$, where $G_{MN}$ is the M theory metric.
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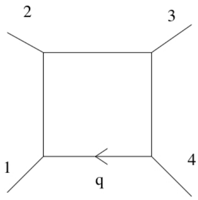
- The one loop box diagram $I(S,T)$ is




$$= \int d^{11}q \frac{1}{q^2(q+k_1)^2(q+k_1+k_2)^2(q-k_4)^2}$$

- For the background $\mathbb{R}^{8,1} \times T^2$, where $\mathcal{V}_2$ and $\Omega$ are the dimensionless volume and complex structure of $T^2$ , the loop momenta are split into non–compact parts and KK modes, the 4 propagators are written introducing 4 Schwinger parameters, so that $I(S,T)$ equals

$$\frac{\pi^{1/2}}{4l_{11}^2\mathcal{V}_2}\int_0^\infty \frac{d\sigma}{\sigma^{3/2}}\int d\omega_3 d\omega_2 d\omega_1 \sum_{l_I} e^{-\sigma G^{IJ} l_I l_J / l_{11}^2 - \sigma Q(S,T;\omega_i)}$$

where

$$Q(S,T;\omega_i) = -S\omega_1(\omega_3-\omega_2) - T(\omega_2-\omega_1)(1-\omega_3),$$

and

$$0 \le \omega_1 \le \omega_2 \le \omega_3 \le 1.$$

- When $S = T = 0$, the integral has an UV divergence which is evaluated using Poison resummation to yield

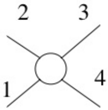
$$3I(0,0) = \frac{1}{4\pi^2} \cdot \frac{\pi^3}{4l_{11}^3}\Big[\frac{4\pi^{5/2}}{3}(\Lambda l_{11})^3 + \mathcal{V}_2^{-3/2} E_{3/2}(\Omega,\bar{\Omega})\Big],$$

where

$$E_s(\Omega,\bar{\Omega}) = \sum_{(m,n)\neq(0,0)} \frac{\Omega_2^s}{|m+n\Omega|^{2s}}$$

is an Eisenstein series of $SL(2,\mathbb{Z})$.

- The $\Lambda^3$ UV divergence is cancelled by the one loop counter term




$$= \delta\mathcal{A}_4^{(1)} = \frac{\kappa_{11}^4}{(2\pi)^{11}}\mathcal{K} \cdot \frac{\pi^3}{4l_{11}^3} c_1 .$$

where $c_1$ is moduli independent.

- Perturbative equality of the genus 1 four graviton type IIA and type IIB amplitudes in 9 dimensions sets

$$\frac{4\pi^{5/2}}{3}(\Lambda l_{11})^3 + c_1 = \frac{2\pi^2}{3}.$$

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- The other local contributions are given by

$$4\pi^2\Big[I(S,T)+I(S,U)+I(T,U)\Big]$$

$$=\pi^{9/2}\sum_{n=2}^{\infty}\frac{\mathcal{W}^n}{n!}(l_{11}^2\mathcal{V}_2)^{n-3/2}\Gamma(n-1/2)E_{n-1/2}(\Omega,\bar{\Omega}),$$

where

$$\mathcal{W}^n=G^n_{ST}+G^n_{SU}+G^n_{TU},$$

and

$$G^n_{ST}=\int_0^1 d\omega_3\int_0^{\omega_3}d\omega_2\int_0^{\omega_2}d\omega_1\Big(-Q(S,T;\omega_i)\Big)^n.$$

- We can now expand the Eisenstein series keeping terms upto exponentially suppressed contributions, and obtain results in the IIA and IIB string theories.
- To do so, we use the relations (Hull, Townsend '94; Witten '95; Aspinwall '95; Schwarz '95)

$$\mathcal{V}_2 = e^{\phi^B/3} r_B^{-4/3} = e^{\phi_A/3} r_A, \quad \Omega_1 = C, \quad \Omega_2 = e^{-\phi^B} = r_A e^{-\phi^A},$$

and

$$l_{11} = e^{\phi^A/3} l_s.$$

- Here $r_A$ ($r_B$) is the dimensionless radius of the circle in the IIA (B) theory, $\phi^A$ ($\phi^B$) is the IIA (B) dilaton, and $C$ is the R–R potential.

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- The contribution from the Eisenstein series is given by

$$E_s(\Omega,\bar{\Omega}) = 2\zeta(2s)\Omega_2^s + \frac{2\sqrt{\pi}\Gamma(s-1/2)}{\Gamma(s)}\zeta(2s-1)\Omega_2^{1-s} + \ldots.$$

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- We get that

$$\mathcal{A}_4^{(1)} = (2\pi^8 l_{11}^{15} r_B)\mathcal{K} r_B \Big[ 2\zeta(3) e^{-2\phi^B} + \frac{2\pi^2}{3}(1 + r_B^{-2}) + \frac{2\pi^2 l_s^4}{6! r_B^4} \sigma_2 \Big( \zeta(3) + 2\zeta(2) e^{2\phi^B} \Big) + \frac{l_s^6}{2 \cdot 4! r_B^6} \sigma_3 \Big( \frac{1}{21} \zeta(2)\zeta(5) + \frac{1}{9}\zeta(6) e^{4\phi^B} \Big) + O(k^8) \Big]$$

in the IIB theory, and

- in the IIA theory

$$\mathcal{A}_4^{(1)} = (2\pi^8 l_{11}^{15} r_A^{-1})\mathcal{K} r_A \Big[ 2\zeta(3) e^{-2\phi^A} + \frac{2\pi^2}{3}(1 + r_A^{-2}) + \frac{2\pi^2 l_s^4}{6!}\sigma_2\Big(\zeta(3) r_A^2 + 2\zeta(2) e^{2\phi^A}\Big) + \frac{l_s^6}{2\cdot 4!}\sigma_3\Big(\frac{1}{21}\zeta(2)\zeta(5) r_A^4 + \frac{1}{9}\zeta(6) e^{4\phi^A}\Big) + O(k^8)\Big].$$

- We have defined

$$\sigma_n \equiv S^n + T^n + U^n.$$

- The overall factor of $2\pi^8 l_{11}^{15} r_B = 2\pi^8 l_{11}^{15} r_A^{-1}$ is needed to correctly normalize the action and is common to multiloop amplitudes.
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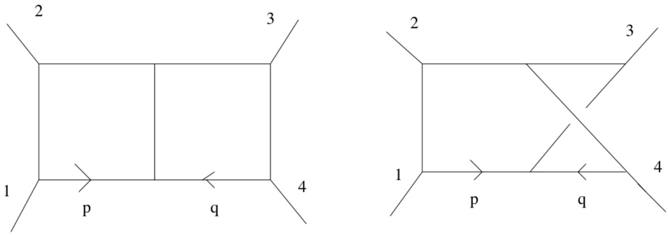
- The two loop four graviton amplitude in 11 uncompactified dimensions is given by

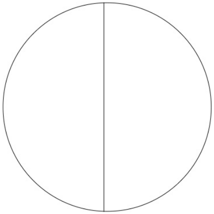
$$\mathcal{A}_4^{(2)} = \frac{(4\pi^2)^2\kappa_{11}^6}{(2\pi)^{22}}\Big[S^2\Big(I_P(S,T) + I_P(S,U) + I_{NP}(S,T) + I_{NP}(S,U)\Big) + 2 \ \ other \ \ permutations\Big]\mathcal{K}.$$

- The planar and non–planar contributions $I_P(S,T)$ and $I_{NP}(S,T)$ respectively are given by




and involve massless $\varphi^3$ diagrams.

- One proceeds as in the 1 loop case, and obtains an integral over 7 Schwinger parameters. Of them 3 are "radial" integrals, and 4 are "angular" integrals. This is evident from the unique two loop skeleton diagram



- The links of the skeleton diagram are the "radial" Schwinger parameters, while the "angular" ones arise from the insertion of the graviton vertex operators in the first quantised superparticle formalism.
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- After Poisson resumming, the leading $D^4\mathcal{R}^4$ term involves

$$(4\pi^2)^2[I_P(0,0)+I_{NP}(0,0)] \\ =\frac{\pi^{11}}{2l_{11}^8}\sum_{\hat{m}_I,\hat{n}_I}\int_0^\infty dV_2 V_2^3\int_{\mathcal{F}_2}\frac{d^2\tau}{\tau_2^2}e^{-\pi^2 G_{IJ}(\hat{m}+\hat{n}\tau)_I(\hat{m}+\hat{n}\bar{\tau})_J V_2/\tau_2},$$

where $d^2\tau = d\tau_1 d\tau_2$ and $\mathcal{F}_2$ is the fundamental domain of $SL(2,\mathbb{Z})$ defined by

$$\mathcal{F}_2=\{-\frac{1}{2}\le\tau_1\le\frac{1}{2},\tau_2\ge 0,|\tau|^2\ge 1\}.$$

- The UV divergences arise from the boundaries of moduli space, and lead to

$$(4\pi^2)^2[I_P(0,0) + I_{NP}(0,0)] = a\Lambda^8 + \frac{\pi^{13/2}\Lambda^3}{8 l_{11}^5 \mathcal{V}_2^{5/2}} E_{5/2}(\Omega, \bar{\Omega}) + \frac{\pi^4 \zeta(3)\zeta(4)}{2 l_{11}^8 \mathcal{V}_2^4},$$

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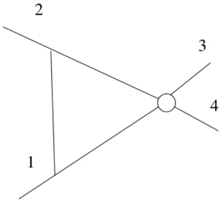
- The one loop subdivergence is cancelled by the counterterm diagram




which includes the one loop counterterm with coefficient $c_1$.

- This yields

$$\delta\mathcal{A}_4^{(2)} = \frac{\pi^{11/2}\kappa_{11}^6}{(2\pi)^{22} l_{11}^5}\mathcal{K}\sigma_2 \cdot \frac{\pi^3}{4 l_{11}^3} c_1 \cdot \Big[\frac{2}{5}(\Lambda l_{11})^5 + \frac{3}{4\pi^{9/2}}\mathcal{V}_2^{-5/2} E_{5/2}(\Omega,\bar{\Omega})\Big].$$

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- The analysis for the $D^6\mathcal{R}^4$ term proceeds along the same lines. However the integrand is no more $SL(2,\mathbb{Z})_\tau$ invariant. Apart from the lattice factor, it involves a factor which satisfies Poisson equation on $\mathcal{F}_2$ with a source term which follows from the structure of supersymmetry (Basu, Sethi '08). Hence more care has to be taken.

- After regularising the one loop subdivergence the total amplitude is given by

$$\mathcal{A}_4^{(2)} = \frac{\kappa_{11}^6}{(2\pi)^{22} l_{11}^6} \mathcal{K}\sigma_3 \Big[ \frac{\pi^8}{144} \mathcal{V}_2^{-3/2} E_{3/2}(\Omega,\bar{\Omega}) + \frac{\pi^6}{96} \Big( \mathcal{V}_2^{-3} \mathcal{E}(\Omega,\bar{\Omega}) + \hat{c}(\Lambda l_{11})^6 \Big) \Big],$$

  where the last term involving the irrelevant constant $\hat{c}$ is the primitive two loop divergence.

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- Perturbative equality of the genus 2 IIA and IIB four graviton amplitudes implies that

$$\hat{e} + \hat{c}(\Lambda l_{11})^6 = 24\zeta(4)(1-\eta),$$

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- Dropping exponentially suppressed corrections, upto $O(D^6\mathcal{R}^4)$ we get that in the IIB theory

$$\mathcal{A}_4^{(2)} = (2\pi^8 l_{11}^{15} r_B)\mathcal{K} r_B \Big[\frac{l_s^4}{6!}\sigma_2\Big(45\zeta(5)e^{-2\phi^B} + 2\pi^2\zeta(3)r_B^2 + 4\pi^2\zeta(2)e^{2\phi^B}\Big) + \frac{l_s^6}{16\cdot 4!}\sigma_3\Big(4\zeta(3)^2 e^{-2\phi^B} + 8\zeta(2)\zeta(3)(1+r_B^{-2}) + 24\zeta(4)e^{2\phi^B}(1+(1-\eta)r_B^{-4}) + 16\zeta(2)^2 e^{2\phi_B} r_B^{-2} + \frac{8}{9}\zeta(6)e^{4\phi^B}\Big)\Big].$$

- In the IIA theory,

$$\mathcal{A}_4^{(2)} = (2\pi^8 l_{11}^{15} r_A^{-1})\mathcal{K} r_A \Big[\frac{l_s^4}{6!}\sigma_2\Big(45\zeta(5)e^{-2\phi^A} + \frac{2\pi^2}{r_A^4}\zeta(3) + \frac{4\pi^2}{r_A^4}\zeta(2)e^{2\phi^A}\Big) + \frac{l_s^6}{16\cdot 4!}\sigma_3\Big(4\zeta(3)^2 e^{-2\phi^A} + 8\zeta(2)\zeta(3)(1+r_A^{-2}) + 24\zeta(4)e^{2\phi^A}((1-\eta)+r_A^{-4}) + 16\zeta(2)^2 e^{2\phi_A} r_A^{-2} + \frac{8}{9r_A^6}\zeta(6)e^{4\phi^A}\Big)\Big].$$

- Let us set $\eta = 1$ and write the total renormalised contribution from 1 and 2 loops dropping overall numerical factors, for both the IIA and IIB theories.

- The $\mathcal{R}^4$ term has

$$2\zeta(3)e^{-2\phi_B} + 4\zeta(2)(1 + r_B^{-2})$$

and

$$2\zeta(3)e^{-2\phi_A} + 4\zeta(2)(1 + r_A^{-2})$$

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$$\frac{1}{6!}\Big(45\zeta(5)e^{-2\phi_B}+12\zeta(2)\zeta(3)(r_B^2+r_B^{-4})+24\zeta(2)^2e^{2\phi_B}(1+r_B^{-4})\Big)$$

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$$\frac{1}{2\cdot 4!}\Big[\frac{\zeta(3)^2}{2}e^{-2\phi_B}+\zeta(2)\zeta(3)(1+r_B^{-2})+\frac{\zeta(2)\zeta(5)}{21 r_B^6}$$
$$+3\zeta(4)e^{2\phi_B}(1+\frac{5}{3r_B^2})+\frac{\zeta(6)}{9}e^{4\phi_B}(1+r_B^{-6})\Big]$$

and

$$\frac{1}{2\cdot 4!}\Big[\frac{\zeta(3)^2}{2}e^{-2\phi_A}+\zeta(2)\zeta(3)(1+r_A^{-2})+\frac{\zeta(2)\zeta(5)r_A^4}{21}$$
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- Perturbative equality is violated at genus 1 and 2, and must be restored on including three loops, along with possible extra contributions which respect perturbative equality upto genus 3 by themselves (Berkovits'06).
- Hence, calculating the 3 loop amplitude is important, which we consider now.

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- The leading term in the low momentum expansion of the 3 loop amplitude is $D^6\mathcal{R}^4$.
- The relevant loop diagrams can be expressed in various ways, which are useful in various settings. Originally, the leading term naively seemed to be of the form $D^4\mathcal{R}^4$ but that coefficient was shown to vanish identically (Bern et. al. '07).
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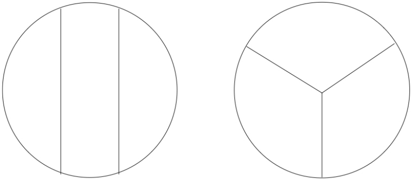
- Yet another way of expressing these the amplitude was obtained (Bern et. al. '10) where the KLT relation was manifest. We shall not use those expressions.

- The loop diagrams that arise in the amplitude all arise from attaching external vertex operators to the two skeleton diagrams



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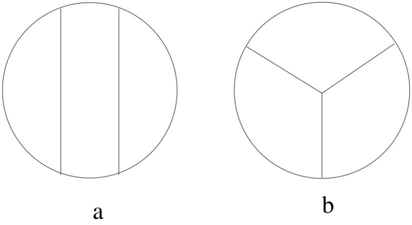


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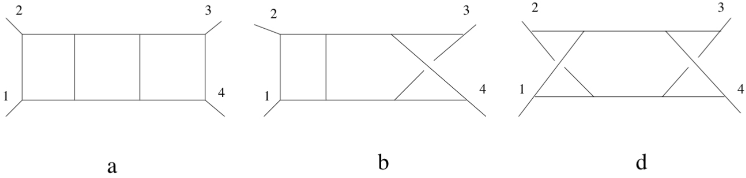
- There are 9 loop diagrams that follow, which we refer to as a to i.
- Diagrams a, b, and d follow from the ladder skeleton and are given by



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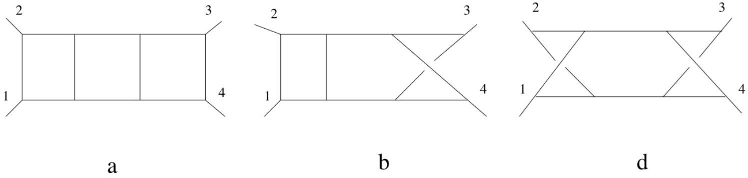
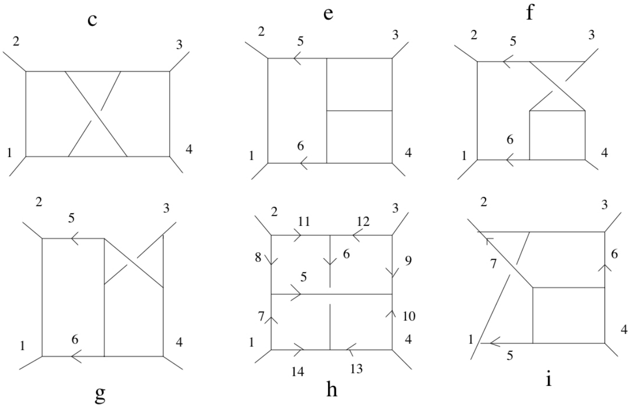


- Diagrams c, e, f, g, h and i follow from the Mercedes skeleton.

- These 6 diagrams are



- The diagrams a, b, c and d are massless $\varphi^3$ field theory diagrams, while the rest are not as they have non–trivial numerators even though the vertices are of the $\varphi^3$ type.

- The three loop amplitude is given by (Bern et. al. '08)

$$\begin{aligned}\mathcal{A}_4^{(3)} &= \frac{(4\pi^2)^3\kappa_{11}^8}{(2\pi)^{33}}\sum_{S_3}\Big[I^{(a)}+I^{(b)}+\frac{1}{2}I^{(c)}+\frac{1}{4}I^{(d)}+2I^{(e)} \\ &\quad +2I^{(f)}+4I^{(g)}+\frac{1}{2}I^{(h)}+2I^{(i)}\Big]\mathcal{K} \\ &\equiv \frac{(4\pi^2)^3\kappa_{11}^8}{(2\pi)^{33}} I_3\mathcal{K},\end{aligned}$$

where $S_3$ represents the 6 independent permutations of the external legs marked $\{1,2,3\}$ keeping the external leg $\{4\}$ fixed.

- The numerators $N^{(x)}$ for the various integrands in the loop diagrams are given by

$$
\begin{aligned}
N^{(a)} &= N^{(b)} = N^{(c)} = N^{(d)} = S^4, \\
N^{(e)} &= N^{(f)} = N^{(g)} = S^2 \tau_{35}\tau_{46}, \\
N^{(h)} &= \Big( S(\tau_{26}+\tau_{36}) + T(\tau_{15}+\tau_{25}) + ST \Big)^2 \\
&+ \Big( S^2(\tau_{26}+\tau_{36}) - T^2(\tau_{15}+\tau_{25}) \Big)\Big( \tau_{17}+\tau_{28}+\tau_{39}+\tau_{4,10} \Big) \\
&+ S^2(\tau_{17}\tau_{28}+\tau_{39}\tau_{4,10}) + T^2(\tau_{28}\tau_{39}+\tau_{17}\tau_{4,10}) \\
&+ U^2(\tau_{17}\tau_{39}+\tau_{28}\tau_{4,10}),
\end{aligned}
$$

- and

$$N^{(i)} = (S\tau_{45} - T\tau_{46})^2 - \tau_{27}(S^2\tau_{45} + T^2\tau_{46})$$
$$-\tau_{15}(S^2\tau_{47} + U^2\tau_{46}) - \tau_{36}(T^2\tau_{47} + U^2\tau_{45})$$
$$-l_5^2 S^2 T - l_6^2 S T^2 + \frac{l_7^2}{3} STU,$$

where

$$\tau_{ij} = -2k_i \cdot l_j \ (i \leq 4, j \geq 5).$$

- Thus we need not consider a, b, c and d.
- We shall refer to the contribution from diagram $x$ as $I^{(x)}$, and the total contribution after the $S_3$ sum as $I^{(X)}$.
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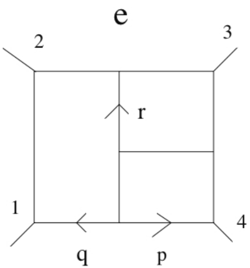
- Diagram e is given by



- In 11 uncompactified dimensions, the diagram $e$ contributes

$$I^{(e)} = -4S^2 \int \frac{d^{11}pd^{11}qd^{11}r(k_3\cdot q)(k_4\cdot q)}{q^6p^4r^2(p+q)^2(q+r)^4(p+q+r)^2}.$$

- On $\mathbb{R}^{8,1}\times T^2$, $p_M$, $q_M$ and $r_M$ decompose as $\{p_\mu, l_I/l_{11}\}$, $\{q_\mu, m_I/l_{11}\}$ and $\{r_\mu, n_I/l_{11}\}$ respectively where $p_\mu, q_\mu$ and $r_\mu$ are the 9 dimensional momenta and $l_I$, $m_I$ and $n_I$ $(I=1,2)$ are the KK momenta along $T^2$.

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- Introduce 10 Schwinger parameters for the 10 propagators. Thus their contribution to the integral becomes

$$\int_0^\infty \prod_{i=1}^{10} d\sigma^i e^{-\sum_{j=1}^{10}\sigma^j q_j^2} e^{-\left((\sigma_1+\sigma_2+\sigma_3)\mathbf{m}^2+(\sigma_4+\sigma_5)\mathbf{l}^2+\sigma_6\mathbf{n}^2\right)/l_{11}^2}$$
$$\times e^{-\left(\sigma_7(\mathbf{l}+\mathbf{m})^2+(\sigma_8+\sigma_9)(\mathbf{m}+\mathbf{n})^2+\sigma_{10}(\mathbf{l}+\mathbf{m}+\mathbf{n})^2\right)/l_{11}^2}$$

where

$$q_j = \{q, q, q, p, p, r, p+q, q+r, q+r, p+q+r\}$$

and

$$\mathbf{m}^2 \equiv G^{IJ} m_I m_J.$$

- Thus

$$I^{(e)} = -\frac{4S^2}{(4\pi^2 l_{11}^2 \mathcal{V}_2)^3} \int d^9p d^9q d^9r (k_3 \cdot q)(k_4 \cdot q) \int_0^\infty \prod_{i=1}^{10} d\sigma^i \times f_P(\lambda, \sigma, \sigma_6, \sigma_7, \rho, \sigma_{10}) F_L(\lambda, \sigma, \sigma_6, \sigma_7, \rho, \sigma_{10}),$$

where we have defined

$$\sigma = \sigma_1 + \sigma_2 + \sigma_3, \quad \lambda = \sigma_4 + \sigma_5, \quad \rho = \sigma_8 + \sigma_9.$$

- Now

$$f_P(\sigma,\lambda,\mu,\rho,\nu,\theta) = e^{-\sigma p^2-\lambda q^2-\mu r^2-\rho(p+q)^2-\nu(q+r)^2-\theta(p+q+r)^2}$$

is the unintegrated momentum factor, and the lattice factor $F_L$ is given by

$$F_L(\sigma,\lambda,\mu,\rho,\nu,\theta) = \sum_{l_I,m_I,n_I} e^{-G^{IJ}\Big(\sigma l_I l_J+\lambda m_I m_J+\mu n_I n_J\Big)/l_{11}^2} \times e^{-G^{IJ}\Big(\rho(l+m)_I(l+m)_J+\nu(m+n)_I(m+n)_J+\theta(l+m+n)_I(l+m+n)_J\Big)/l_{11}^2}.$$

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- They depend on 6 "radial" Schwinger parameters.

- Using $(k_3 \cdot q)(k_4 \cdot q) \to -Sq^2/18$ in the integral, we get

$$I^{(e)} = \frac{2S^3}{9(4\pi^2 l_{11}^2 \mathcal{V}_2)^3} \int d^9p d^9q d^9r q^2 \int_0^\infty \prod_{i=1}^{10} d\sigma^i$$
$$\times f_P(\lambda, \sigma, \sigma_6, \sigma_7, \rho, \sigma_{10}) F_L(\lambda, \sigma, \sigma_6, \sigma_7, \rho, \sigma_{10}).$$

- Defining the 4 "angular" Schwinger parameters by

$$w_1 = \frac{\sigma_1}{\sigma}, \quad w_2 = \frac{\sigma_1 + \sigma_2}{\sigma}, \quad u = \frac{\sigma_4}{\lambda}, \quad v = \frac{\sigma_8}{\rho},$$

we get that

$$I^{(e)} = -\frac{S^3}{9(4\pi^2 l_{11}^2 \mathcal{V}_2)^3} \int_0^\infty d\sigma d\lambda d\mu d\rho d\nu d\theta (\lambda^2 \sigma \nu) \times F_L(\sigma, \lambda, \mu, \rho, \nu, \theta) \frac{\partial}{\partial \lambda} \mathcal{J}(\sigma, \lambda, \mu, \rho, \nu, \theta).$$

- $\mathcal{J}$ is defined by

$$\mathcal{J}(\sigma,\lambda,\mu,\rho,\nu,\theta) = \int d^9p d^9q d^9r e^{-\sigma p^2 - \lambda q^2 - \mu r^2 - \rho(p+q)^2 - \nu(q+r)^2 \ldots}$$
$$= \pi^{27/2}\Delta_3^{-9/2}(\sigma,\lambda,\mu,\rho,\nu,\theta),$$

where $\Delta_3$ is defined by

$$\Delta_3(\sigma,\lambda,\mu,\rho,\nu,\theta) = \sigma\lambda\mu + \rho\nu\theta + \sigma\mu(\rho+\nu+\theta) + \lambda\mu(\rho+\theta)$$
$$+\sigma\lambda(\nu+\theta) + \mu\nu(\rho+\theta) + \sigma\rho(\nu+\theta) + \lambda(\rho\nu+\nu\theta+\rho\theta).$$

- Thus, finally we get

$$I^{(E)} = -\frac{2\sigma_3}{9(4\pi^2 l_{11}^2 \mathcal{V}_2)^3} \int_0^\infty d\Upsilon (\lambda^2 \sigma \nu) F_L(\sigma, \lambda, \mu, \rho, \nu, \theta) \times \frac{\partial}{\partial \lambda} \mathcal{J}(\sigma, \lambda, \mu, \rho, \nu, \theta),$$

where

$$d\Upsilon \equiv d\sigma d\lambda d\mu d\rho d\nu d\theta.$$

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- The total contribution is

$$I_3 = \frac{2\pi^{27/2}}{(4\pi^2 l_{11}^2 \mathcal{V}_2)^3}\sigma_3 \int_0^\infty \frac{d\Upsilon}{\Delta_3^{11/2}} F_L \Big[\lambda^2\sigma\nu\frac{\partial}{\partial\lambda} + \lambda^2\sigma\mu\frac{\partial}{\partial\lambda}$$
$$+2\sigma^2\lambda\mu\frac{\partial}{\partial\sigma} + \sigma\lambda\mu\theta\frac{\partial}{\partial\theta} + \sigma\lambda\rho\nu\Big(2\frac{\partial}{\partial\sigma} + \frac{\partial}{\partial\rho}\Big)\Big]\Delta_3$$
$$+\frac{\pi^{27/2}}{3(4\pi^2 l_{11}^2 \mathcal{V}_2)^3}\sigma_3 \int_0^\infty d\Upsilon \Delta_3^{-7/2} F_L.$$

- To simplify the expression, we need to express the integrals in a manner thats respect the symmetries of the underlying Mercedes skeleton.
- To see this, consider the Mercedes skeleton with a specific choice of the 6 Schwinger parameters. The symmetry can also be seen from the dual regular tetrahedron which follows from replacing each face of the Mercedes skeleton (which is the wheel graph of order 4) with a vertex of the regular tetrahedron, such that every edge of the regular tetrahedron is parametrized by the link of the Mercedes skeleton it cuts.

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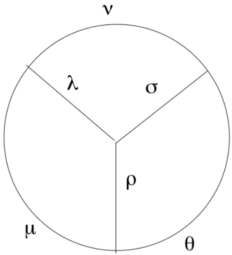
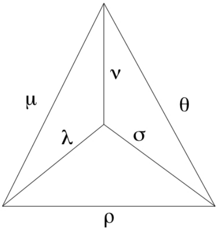
- To see this, consider the Mercedes skeleton with a specific choice of the 6 Schwinger parameters. The symmetry can also be seen from the dual regular tetrahedron which follows from replacing each face of the Mercedes skeleton (which is the wheel graph of order 4) with a vertex of the regular tetrahedron, such that every edge of the regular tetrahedron is parametrized by the link of the Mercedes skeleton it cuts.

- The dual diagrams are



- Thus the symmetry group is the set of discrete transformations which interchange the 4 vertices of either diagram, keeping the links between the vertices intact. This is the symmetric group $S_4$, the automorphism group of the wheel graph of order 4, as well as the regular tetrahedron. It is now easy to see how the Schwinger parameters transform under the action of each of the 24 elements of $S_4$.

- We can now express $I_3$ is a manifestly $S_4$ invariant way. $\Delta_3$ and $F_L$ are invariant, and the various other factors have to be replaced by invariant combinations inside the integral.

- For example,

$$\sigma\lambda\mu\theta\frac{\partial}{\partial\theta} \rightarrow \frac{D_2}{12},$$

where

$$\begin{aligned} D_2 &= \sigma\lambda\mu\theta\Big(\frac{\partial}{\partial\sigma}+\frac{\partial}{\partial\lambda}+\frac{\partial}{\partial\mu}+\frac{\partial}{\partial\theta}\Big) \\ &+\theta\rho\lambda\nu\Big(\frac{\partial}{\partial\theta}+\frac{\partial}{\partial\rho}+\frac{\partial}{\partial\lambda}+\frac{\partial}{\partial\nu}\Big) \\ &+\mu\nu\rho\sigma\Big(\frac{\partial}{\partial\mu}+\frac{\partial}{\partial\nu}+\frac{\partial}{\partial\rho}+\frac{\partial}{\partial\sigma}\Big). \end{aligned}$$

- The final answer simplifies enormously and

$$I_3 = \frac{5\pi^{27/2}}{6(4\pi^2 l_{11}^2 \mathcal{V}_2)^3} \sigma_3 \int_0^\infty d\Upsilon \Delta_3^{-7/2} F_L.$$

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- Perform a Poisson resummation in $F_L$ to go from KK momentum to winding modes. To start with, we write $F_L$ as

$$F_L = \sum_{k_{\alpha I}} e^{-G^{IJ} G^{\alpha\beta} k_{\alpha I} k_{\beta J}/l_{11}^2},$$

where the KK integers $k_{\alpha I}$ are defined by $k_{\alpha I} = \{l_I, m_I, n_I\}$ for $\alpha = 1, 2, 3$. Also the symmetric matrix $G^{\alpha\beta}$ $(\alpha, \beta = 1, 2, 3)$ has entries (of dimension $l_{11}^2$)

$$G^{\alpha\beta} = \begin{pmatrix} \sigma+\rho+\theta & \rho+\theta & \theta \\ \rho+\theta & \lambda+\rho+\nu+\theta & \nu+\theta \\ \theta & \nu+\theta & \mu+\nu+\theta \end{pmatrix}.$$

$(\det G^{\alpha\beta} = \Delta_3.)$

- After Poisson resumming,

$$F_L = \frac{(\pi l_{11}^2 \mathcal{V}_2)^3}{\Delta_3} \sum_{\hat{k}^{\alpha I}} e^{-\pi^2 l_{11}^2 G_{IJ} G_{\alpha\beta} \hat{k}^{\alpha I} \hat{k}^{\beta J}},$$

where the winding mode integers $\hat{k}^{\alpha I}$ are defined by $\hat{k}^{\alpha I} = \{\hat{l}^I, \hat{m}^I, \hat{n}^I\}$ for $\alpha = 1, 2, 3$.

- Define

$$\hat{\sigma} = \frac{\sigma}{\Delta_3^{2/3}}, \quad \hat{\lambda} = \frac{\lambda}{\Delta_3^{2/3}}, \quad \hat{\mu} = \frac{\mu}{\Delta_3^{2/3}},$$
$$\hat{\rho} = \frac{\rho}{\Delta_3^{2/3}}, \quad \hat{\nu} = \frac{\nu}{\Delta_3^{2/3}}, \quad \hat{\theta} = \frac{\theta}{\Delta_3^{2/3}},$$

such that the hatted parameters have dimension $l_{11}^{-2}$.

- Thus

$$\hat{\Delta}_3(\hat{\sigma}, \hat{\lambda}, \hat{\mu}, \hat{\rho}, \hat{\nu}, \hat{\theta}) = \hat{\sigma}\hat{\lambda}\hat{\mu} + \ldots = \Delta_3^{-1}(\sigma, \lambda, \mu, \rho, \nu, \theta).$$

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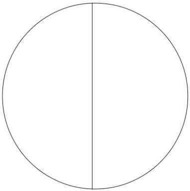
- The measure transforms in a simple way as

$$d\Upsilon = \hat{\Delta}_3^{-4} d\hat{\sigma} d\hat{\lambda} d\hat{\mu} d\hat{\rho} d\hat{\nu} d\hat{\theta} \equiv \hat{\Delta}_3^{-4} d\hat{\Upsilon},$$

finally leading to

$$I_3 = \frac{5\pi^{21/2}}{6 \cdot 64} \sigma_3 \int_0^\infty d\hat{\Upsilon} \hat{\Delta}_3^{1/2} \sum_{\hat{k}^{\alpha I}} e^{-\pi^2 l_{11}^2 G_{IJ} G_{\alpha\beta} \hat{k}^{\alpha I} \hat{k}^{\beta J}}.$$

- Is there an underlying auxiliary geometry like the two loop case? We proceed by analogy with the two loop case to make an educated guess.
- The unique two loop skeleton diagram is



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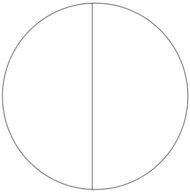

- Hence there are 3 Schwinger parameters, and $F_L$ involves a sum over two sets of integers $m_I$ and $n_I$. The 3 Schwinger parameters were traded off for the parameters of an auxiliary $T^2$, having moduli its volume, and complex structure. The Poisson resummed integers $\hat{m}_I$ and $\hat{n}_I$ correspond to non–trivial winding numbers along the two non–contractible cycles of $T^2$.
- The integral over the complex structure modulus of $T^2$ (which parametrises the coset space $U(1)\backslash SL(2,\mathbb{R})$) is $SL(2,\mathbb{Z})$ invariant for the $D^4\mathcal{R}^4$ term.

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- Recall that

$$F_L = \sum_{k_{\alpha I}} e^{-G^{IJ} G^{\alpha\beta} k_{\alpha I} k_{\beta J} / l_{11}^2}$$

indeed has the correct structure if $G^{\alpha\beta}$ is identified with the inverse metric of the $T^3$.

- To do so, we identify

$$\frac{G^{\alpha\beta}}{l_{11}^2} = V_3^{-2/3} \begin{pmatrix} 1/L^2 & A_1/L^2 & A_2/L^2 \\ A_1/L^2 & A_1^2/L^2 + L/T_2 & A_1 A_2/L^2 + LT_1/T_2 \\ A_2/L^2 & A_1 A_2/L^2 + LT_1/T_2 & A_2^2/L^2 + L|T|^2/T_2, \end{pmatrix}$$

where $V_3$ is the volume of $T^3$, and $L, T_1, T_2, A_1, A_2$ are the 5 shape moduli.

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$$= \frac{4V_3^4}{l_{11}^{15}L^4T_2^2} dV_3 dL dT_1 dT_2 dA_1 dA_2 \sum_{\hat{k}^{\alpha I}} e^{-\pi^2\mathcal{V}_2 V_3^{2/3}\hat{G}_{IJ}\hat{G}_{\alpha\beta}\hat{k}^{\alpha I}\hat{k}^{\beta J}}.$$

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- Also $G_{\alpha\beta} = V_2^{2/3}\hat{G}_{\alpha\beta}$, where

$$\hat{G}_{\alpha\beta} = \begin{pmatrix} L^2 + \frac{|A_1 T - A_2|^2}{LT_2} & \frac{(A_2 T_1 - A_1 |T|^2)}{LT_2} & \frac{(A_1 T_1 - A_2)}{LT_2} \\ \frac{(A_2 T_1 - A_1 |T|^2)}{LT_2} & \frac{|T|^2}{LT_2} & -\frac{T_1}{LT_2} \\ \frac{(A_1 T_1 - A_2)}{LT_2} & -\frac{T_1}{LT_2} & (LT_2)^{-1}, \end{pmatrix}$$

is the metric of $T^3$ of unit volume.

- Now
$$d\mu \equiv \frac{1}{L^4 T_2^2} dL dT_1 dT_2 dA_1 dA_2$$
is $SL(3,\mathbb{Z})$ invariant.
- Thus $I_3$ is proportional to
$$l_{11}^{-15} \int V_3^4 dV_3 d\mu \sum_{\hat{k}^{\alpha I}} e^{-\pi^2 \mathcal{V}_2 V_3^{2/3} \hat{G}_{IJ} \hat{G}_{\alpha\beta} \hat{k}^{\alpha I} \hat{k}^{\beta J}}.$$

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- Finally

$$I_3 = \frac{5\cdot 27\pi^{21/2}}{16 l_{11}^{15}}\sigma_3 \int_0^\infty dV_3 V_3^4 \int_{\mathcal{F}_3} d\mu \sum_{\hat{k}^{\alpha I}} e^{-\pi^2 \mathcal{V}_2 V_3^{2/3}\hat{G}_{IJ}\hat{G}_{\alpha\beta}\hat{k}^{\alpha I}\hat{k}^{\beta J}}$$

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- Assuming that the UV divergences arise from the boundaries of moduli space, we get that

$$\mathcal{A}_4^{(3)} = \frac{\kappa_{11}^8}{(2\pi)^{33}} \cdot \frac{\mathcal{K}\sigma_3}{l_{11}^{15}} \Big[ h_1 (\Lambda l_{11})^{15} + h_2 (\Lambda l_{11})^8 \mathcal{V}_2^{-7/2} E_{7/2} + h_3 (\Lambda l_{11})^6 \mathcal{V}_2^{-9/2} E_{9/2} + h_4 \pi^{11/2} (\Lambda l_{11})^3 \zeta(5)\zeta(6) \mathcal{V}_2^{-6} \Big],$$

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- The terms involving $E_{7/2}$ and $E_{9/2}$ are completely cancelled by two loop $\Lambda^8$ and $\Lambda^6$ counterterms respectively, without leaving any finite remainder as they are inconsistent with string perturbation theory. (Any finite remainder would lead to terms of the form $e^{-8\phi_B/3}$ and $e^{-4\phi_B}$ respectively at weak coupling in the IIB theory.)

- The term diverging as $\Lambda^3$ has to be cancelled by a 1 loop counterterm. This counter term can be calculated, and involves the $\mathcal{R}^4$ counter term with coefficient $c_1$, as well as a 5 point counter term in the same supermultiplet with coefficient proportional to $\hat{c}_1$.

- Including this, we get

$$\mathcal{A}_4^{(3)} + \delta\mathcal{A}_4^{(3),1-loop}$$
$$= (2\pi^8 l_{11}^{15})\mathcal{K}\sigma_3\Big[\frac{\pi^{5/2}h_4}{6}(\Lambda l_{11})^3 + c_1 + \hat{c}_1\Big]\frac{l_{11}^6\zeta(2)\zeta(5)\mathcal{V}_2^{-6}}{16\cdot 105\pi^2}.$$

- There was mismatch in the genus 1 contribution including contributions unto 2 loops. Adding these terms to the one above, in the IIA and IIB theory we get

$$
\begin{aligned}
&(2\pi^8 l_{11}^{15} r_B)\mathcal{K} r_B l_s^6 \sigma_3 \frac{\zeta(2)\zeta(5)}{16\cdot 21}\Big[\frac{1}{3r_B^6}+\frac{r_B^4}{5\pi^2}\Big(\frac{\pi^{5/2}h_4}{6}(\Lambda l_{11})^3 \\
&+c_1+\hat{c}_1\Big)\Big] \\
&= (2\pi^8 l_{11}^{15} r_A^{-1})\mathcal{K} r_A l_s^6 \sigma_3 \frac{\zeta(2)\zeta(5)}{16\cdot 21}\Big[\frac{r_A^4}{3}+\frac{1}{5\pi^2 r_A^6}\Big(\frac{\pi^{5/2}h_4}{6}(\Lambda l_{11})^3 \\
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- Equality of the genus one amplitude in the IIA and IIB theories gives

$$\frac{\pi^{5/2} h_4}{6} (\Lambda l_{11})^3 + c_1 + \hat{c}_1 = \frac{5\pi^2}{3}.$$

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- TO SUMMARIZE:
- Including all contributions unto 3 loops, we get the interaction

$$l_{11}^5 \int d^9x \sqrt{-G^{(9)}} \mathcal{V}_2 D^6 \mathcal{R}^4 F(\Omega, \bar{\Omega})$$

in the effective action, where

$$F(\Omega, \bar{\Omega}) = \frac{4}{21}\zeta(2) E_{5/2}(\Omega, \bar{\Omega}) \mathcal{V}_2^{3/2} + \mathcal{E}(\Omega, \bar{\Omega}) \mathcal{V}_2^{-3} + 4\zeta(2) E_{3/2}(\Omega, \bar{\Omega}) \mathcal{V}_2^{-3/2} + 24\zeta(4) + \frac{8}{21}\zeta(2)\zeta(5)\mathcal{V}_2^{-6}.$$

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- Ignoring exponentially suppressed contributions, in the IIA and IIB effective action, we get

$$
\begin{aligned}
& l_s^5 \int d^9x \sqrt{-g^B} r_B D^6 \mathcal{R}^4 \Big[ 4\zeta(3)^2 e^{-2\phi^B} + 8\zeta(2)\zeta(3)(1 + r_B^{-2}) \\
& + \frac{8}{21}\zeta(2)\zeta(5)(r_B^4 + r_B^{-6}) + 24\zeta(4) e^{2\phi^B}(1 + \frac{5}{3} r_B^{-2} + r_B^{-4}) \\
& + \frac{8}{9}\zeta(6) e^{4\phi^B}(1 + r_B^{-6}) \Big] \\
& = l_s^5 \int d^9x \sqrt{-g^A} r_A D^6 \mathcal{R}^4 \Big[ 4\zeta(3)^2 e^{-2\phi^A} + 8\zeta(2)\zeta(3)(1 + r_A^{-2}) \\
& + \frac{8}{21}\zeta(2)\zeta(5)(r_A^4 + r_A^{-6}) + 24\zeta(4) e^{2\phi^A}(1 + \frac{5}{3} r_A^{-2} + r_A^{-4}) \\
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- Perturbative equality unto genus 3 is true. Agrees with an earlier result of mine (Basu '07).
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