Amplitudes and the Scattering Equations

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Neils Bohr Institute, August 2014

(work with Peter Goddard, IAS Princeton)

1402.7374 [hep-th], *The Polynomial Form of the Scattering Equations*

1311.5200 [hep-th], *Proof of the Formula of Cachazo, He and Yuan for Yang-Mills Tree Amplitudes in Arbitrary Dimension*

1111.0950 [hep-th], *Complete Equivalence Between Gluon Tree Amplitudes in Twistor String Theory and in Gauge Theory*

See also

Freddy Cachazo, Song He, and Ellis Yuan (CHY)

1309.0885 [hep-th],
Scattering of Massless Particles: Scalars, Gluons and Gravitons

1307.2199 [hep-th],
Scattering of Massless Particles in Arbitrary Dimensions

1306.6575 [hep-th],
Scattering Equations and KLT Orthogonality

Edward Witten, *hep-th/0312171*,
Perturbative Gauge Theory as a String theory in Twistor Space

Nathan Berkovits, *hep-th/0402045*,
An Alternative String Theory in Twistor Space for N=4 SuperYang-Mills

Outline

- Tree amplitudes from the Scattering Equations in any dimension
- Möbius invariance and massive Scattering Equations
- Proof of the equivalence with ϕ^3 and Yang-Mills field theories
- In 4d: link variables, twistor string $\leftrightarrow$ the Scattering Equations
- Direct proof of equivalence between twistor string and field theory gluon tree amplitudes
- Polynomial form of the Scattering Equations
- Polynomial form of the Scattering Equations at One Loop

Tree Amplitudes

$$\mathcal{A}(k_1, k_2, \ldots, k_N) = \oint_{\mathcal{O}} \Psi_N(z, k, \epsilon) \prod_{a \in A}{}' \frac{1}{f_a(z,k)} \prod_{a \in A} \frac{dz_a}{(z_a - z_{a+1})^2} / d\omega,$$

$\mathcal{O}$ encircles the zeros of $f_a(z, k)$,

$$f_a(z,k) \equiv \sum_{\substack{b \in A \\ b \neq a}} \frac{k_a \cdot k_b}{z_a - z_b} = 0 \qquad \text{The Scattering Equations}$$

(Cachazo, He, Yuan 2013) . . . (Gross, Mende 1987) . . . (Fairlie, Roberts 1972)

$$k_a^2 = 0, \quad \sum_{a \in A} k_a^\mu = 0, \quad A = \{1, 2, \ldots N.\}$$

DG proved $\mathcal{A}(k_1, k_2, \ldots k_n)$ are ϕ^3 and Yang-Mills gluon field theory tree amplitudes , as conjectured by CHY.

Möbius Invariance

$$z_a \to \frac{\alpha z_a+\beta}{\gamma z_a+\delta},$$

$$\mathcal{A}(k_1, k_2, \ldots, k_N) = \oint_{\mathcal{O}} \Psi_N(z,k,\epsilon) {\prod_{a\in A}}' \frac{1}{f_a(z,k)} \prod_{a\in A} \frac{dz_a}{(z_a - z_{a+1})^2} / d\omega$$

$$\begin{aligned} {\prod_{a\in A}}' \frac{1}{f_a(z,k)} &\equiv (z_1 - z_2)(z_2 - z_N)(z_N - z_1) \prod_{\substack{a\in A \\ a\neq 1,2,N}} \frac{1}{f_a(z,k)} \\ &\to \prod_{a\in A} \frac{(\alpha\delta - \beta\delta)}{(\gamma z_a + \delta)^2} {\prod_{a\in A}}' \frac{1}{f_a(z,k)}, \end{aligned}$$

$\Psi_N(z,k,\epsilon)$ is Möbius invariant,
$\Psi_N = 1$ for ϕ^3, $\quad \Psi_N = \prod_{a\in A}(z_a - z_{a+1}) \times$ Pffafian for Yang-Mills

The integrand and the Scattering Equations are Möbius invariant (CHY).

Massive Scattering Equations $\quad \widehat{f}_a(z,k)=0, \quad k_a^2=m^2$

$U(z,k) \equiv \prod_{a<b}(z_a-z_b)^{-k_a\cdot k_b}\prod_{a\in A}(z_a-z_{a+1})^{-\frac{m^2}{2}}$ is Möbius invariant,

$$\frac{\partial U}{\partial z_a}=-\widehat{f}_a U, \quad \widehat{f}_a(z,k)=\sum_{\substack{b\in A\\ b\neq a}}\frac{k_a\cdot k_b}{z_a-z_b}+\frac{m^2}{2(z_a-z_{a+1})}+\frac{m^2}{2(z_a-z_{a-1})},$$

implying $\widehat{f}_a(z)\to\widehat{f}_a(z)\dfrac{(\gamma z_a+\delta)^2}{(\alpha\delta-\beta\gamma)}$.

The infinitesimal transformations $\delta z_a=\epsilon_1+\epsilon_2 z_a+\epsilon_3 z_a^2$,

$U(z+\delta z)\sim U(z)+\dfrac{\partial U}{\partial z_a}\delta z_a$, so the $\widehat{f}_a$ satisfy the three relations

$$\sum_{a\in A}\widehat{f}_a=0, \quad \sum_{a\in A}z_a\widehat{f}_a=0, \quad \sum_{a\in A}z_a^2\,\widehat{f}_a=0.$$

There are $N-3$ independent Scattering Equations $\widehat{f}_a=0$.

Fixing $z_1=\infty, z_2=1, z_N=0$, there are $N-3$ variables,

and generally $(N-3)!$ solutions $z_a(k)$. $\quad \hat{f}=f$ when $m^2=0$.

Total Amplitudes

For example, $N=4$,

$$A^{abcd}(k_1,k_2,k_3,k_4) = g^2\Big(\ f_{abe}f_{ecd}\,\frac{n_s}{s} + f_{bce}f_{ead}\,\frac{n_t}{t} + f_{cae}f_{ebd}\,\frac{n_u}{u}\Big)$$

$$= g^2\Big(\big(tr(T_aT_bT_cT_d) + tr(T_dT_cT_bT_a)\big)\,A(1234)$$

$$+ \big(tr(T_aT_cT_dT_b) + tr(T_bT_dT_cT_a)\big)\,A(1342)$$

$$+ \big(tr(T_aT_dT_bT_c) + tr(T_cT_bT_dT_a)\big)\,A(1423)\Big),$$

$$n_s = \big(\epsilon_1\cdot\epsilon_2(k_1-k_2)_\alpha + 2\epsilon_1\cdot k_2\epsilon_{2\alpha} - 2\epsilon_2\cdot k_1\epsilon_{1\alpha}\big)$$

$$\times\big(\epsilon_3\cdot\epsilon_4(k_3-k_4)^\alpha + 2\epsilon_3\cdot k_4\epsilon_4^\alpha - 2\epsilon_4\cdot k_3\epsilon_3^\alpha\big)$$

$$+\ \big(\epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4 - \epsilon_1\cdot\epsilon_4\epsilon_2\cdot\epsilon_3\big)\,s,$$

$$A(1234) = \frac{n_s}{s} + \frac{n_t}{t}. \qquad s=(k_1+k_2)^2, t=(k_2+k_3)^2, u=(k_1+k_3)^2$$

$$\mathcal{A}(k_1,k_2,k_3,k_4) = A(1234).$$

A Single Scalar Field, Massless ϕ^3

A single massless scalar field, $\Psi_N = 1$.

$$\mathcal{A}^{\phi}(k_1, k_2, \ldots, k_N) = \oint_{\mathcal{O}} \prod_{a \in A}{}' \frac{1}{f_a(z,k)} \prod_{a \in A} \frac{dz_a}{(z_a - z_{a+1})^2} / d\omega$$

$$\mathcal{A}^{\phi}(k_1, k_2, k_3, k_4) = \frac{1}{s} + \frac{1}{t},$$

$$A^{\text{total}} = \mathcal{A}^{\phi}(k_1, k_2, k_3, k_4) + \mathcal{A}^{\phi}(k_1, k_3, k_2, k_4) + \mathcal{A}^{\phi}(k_1, k_4, k_2, k_3)$$

$$= 2\left(\frac{1}{s} + \frac{1}{t} + \frac{1}{u}\right)$$

Proof of the Formula of CHY for Massless ϕ^3

$$A_N^\phi(\zeta) = A_N^\phi(k_1, k_2 + \zeta\ell, k_3, \ldots, k_{N-1}, k_N - \zeta\ell),$$

For $\ell^2 = \ell \cdot k_2 = \ell \cdot k_N = 0$, these shifted, ordered field theory tree amplitudes have simple poles in ζ, and $A_N^\phi(\zeta) \to 0$ as $\zeta \to \infty$.

$$A_N^\phi(\zeta) = -\sum_i \frac{\mathrm{Res}_{\zeta_i} A_N^\phi}{\zeta_i - \zeta}$$

The poles ζ_i occur where $(\pi_m^\zeta)^2 = 0$ or $(\bar{\pi}_m^\zeta)^2 = 0$, *i.e.* at

$$\zeta = s_m/2\pi_m \cdot \ell \equiv \zeta_m^L, \text{ and } \zeta = -\bar{s}_m/2\bar{\pi}_m \cdot \ell \equiv \zeta_m^R, \quad 3 \le m \le N-1,$$

with residues given by

$$\mathrm{Res}_{\zeta_m^R} A_N^\phi = A_m^\phi(k_1, k_2^{\zeta_m^R}, k_3, \ldots, k_{m-1}, -\bar{\pi}_m^{\zeta_m^R}) \frac{1}{2\bar{\pi}_m \cdot \ell}$$
$$\times A_{N-m+2}^\phi(\bar{\pi}_m^{\zeta_m^R}, k_m, \ldots, k_{N-1}, k_N^{\zeta_m^R}),$$

$\pi_m \equiv -k_2 - k_3 - \ldots - k_m, \quad \bar{\pi}_m \equiv -k_m - k_3 - \ldots - k_N;\ s_m = \pi_m^2,\ \bar{s}_m = \bar{\pi}_m^2.$

$$A^\phi(k_1, k_2, \dots, k_N) = A_N^\phi(\zeta = 0)$$

$$= -2 \sum_{m=3}^{N-1} \left[\frac{2\pi_m \cdot \ell}{s_m} \mathrm{Res}_{\zeta_m^L} A_N^\phi - \frac{2\bar{\pi}_m \cdot \ell}{\bar{s}_m} \mathrm{Res}_{\zeta_m^R} A_N^\phi \right] \quad *$$

which determines $A^\phi(k_1, \dots k_N)$ for $N > 3$ from $A^\phi(k_1, k_2, k_3) = 1$.

Our proof is to show $A^\phi = \mathcal{A}^\phi$ satisfies $*$.

$$\mathcal{A}_N^\phi(\zeta) \sim \oint \frac{\prod_{a=3}^{N-2} z_a \prod_{a=4}^{N-1} (1 - z_a)}{(1 - z_3) z_{N-1}} \prod_{b=5}^{N-1} \prod_{a=3}^{b-2} (z_a - z_b)^2 \prod_{a=3}^{N-1} \frac{dz_a}{f_a(z, \zeta)}$$

A pole at ζ_m^R comes from the integration region $z_a \to 0$, $m \le a \le N-1$. Let $z_a = x_a z_m$, $z_m \to 0$,

$$\prod_{a=3}^{N-1} dz_a = \prod_{a=3}^{m-1} dz_a \; dz_m \; \prod_{a=m+1}^{N-1} dx_a,$$

$$\begin{aligned}\mathrm{Res}_{\zeta_m^R}\mathcal{A}_N^{\phi} &= \mathcal{A}_m^{\phi}(k_1, k_2^{\zeta_m^R}, k_3, \ldots, k_{m-1}, -\bar{\pi}_m^{\zeta_m^R})\frac{1}{2\bar{\pi}_m \cdot \ell} \\ &\quad \times \mathcal{A}_{N-m+2}^{\phi}(\bar{\pi}_m^{\zeta_m^R}, k_m, \ldots, k_{N-1}, k_N^{\zeta_m^R}),\end{aligned}$$

Similarly for $\mathrm{Res}_{\zeta_m^L}\mathcal{A}_N^{\phi}$.

So proving the formula for $\mathcal{A}^{\phi}(k_1, \ldots, k_N)$ by induction.

Proof for Pure Gauge Theory

$$\mathcal{A}_N^{\rm YM}(\zeta) \sim \oint \Psi_N^o \frac{\prod_{a=3}^{N-2} z_a \prod_{a=4}^{N-1}(1-z_a)}{(1-z_3)z_{N-1}} \prod_{b=5}^{N-1}\prod_{a=3}^{b-2}(z_a-z_b)^2 \prod_{a=3}^{N-1}\frac{dz_a}{f_a(z,\zeta)}$$

where the only difference from the scalar case is Ψ_N^o, which is related to the Pfaffian of the antisymmetric matrix M_N with the 2nd and Nth rows and columns removed,

$$\Psi_N^o = (-1)^N \mathrm{Pf}\,\mathrm{M}_N(z;k^\zeta;\epsilon^\zeta)_{(2,N)} \prod_{a=1}^{N}(z_a - z_{a+1}),$$

$$\det \mathrm{M} \equiv (\mathrm{Pf}\;\; \mathrm{M})^2,$$

$$\epsilon_2^{\zeta+} = \bar\ell - 2(\zeta/k_2\cdot k_N)k_N, \;\; \epsilon_2^{\zeta-} = \ell; \;\; \epsilon_4^{\zeta\pm},$$

$$\bar\ell^2 = \bar\ell\cdot k_2 = \bar\ell\cdot k_N = 0, \ell\cdot\bar\ell = 2.$$

All singularities in Ψ_N^o are canceled by the numerator. Ψ_N^o factorizes at the poles in the integrand $\zeta_m^{L,R}$, since the Pfaffian does. As $z_m \to 0$,

$$\mathrm{Pf}\ \mathrm{M}_N(k_1, \ldots, k_N; \epsilon_1, \ldots, \epsilon_N; z_3, \ldots, z_{N-1})_{(2,N)}$$
$$\sim \sum_s \mathrm{Pf},\ \mathrm{M}_m(k_1, \ldots, k_{m-1}, -\bar{\pi}_m; \epsilon_1, \ldots, \epsilon_{m-1}, \epsilon^s; z_3, \ldots, z_{m-1})_{(2,m)}$$
$$\times\, \mathrm{Pf}\ \mathrm{M}_{N-m+2}(\bar{\pi}_m, k_m, \ldots, k_N; \epsilon^s, \epsilon_m, \ldots, \epsilon_N; x_{m+1}, \ldots, x_{N-1})_{(1,N-m+2)},$$

and

$$\prod_{a=2}^{N-1}(z_a - z_{a+1}) \to z_{m-1} z_m^{N-m} \prod_{a=2}^{m-2}(z_a - z_{a+1}) \prod_{a=m}^{N-1}(x_a - x_{a+1})$$

This demonstrates that $\mathcal{A}_N^{\mathrm{YM}}(\zeta = 0)$ satisfies the BCFW recurrence relation, so that $\mathcal{A}^{\mathrm{YM}}(k_1, \ldots k_N)$, computed from the scattering equations, are equal to the Yang Mills field theory tree amplitudes.

Twistor String Theory (4d)

$k^{\mu}\sigma_{\mu\,\alpha\dot{\alpha}} \equiv k_{\alpha\dot{\alpha}} = \pi_{\alpha}\bar{\pi}_{\dot{\alpha}},$ Conjugate twistor variables

$$Z = \begin{pmatrix} \pi^{\alpha} \\ \omega^{\dot{\alpha}} \end{pmatrix}, \qquad W = \begin{pmatrix} \bar{\omega}_{\alpha} \\ \bar{\pi}_{\dot{\alpha}} \end{pmatrix}, \qquad W \cdot Z = \bar{\omega}_{\alpha}\pi^{\alpha} + \bar{\pi}_{\dot{\alpha}}\omega^{\dot{\alpha}},$$

and twistor string worldsheet fields, $Z(\rho) = \begin{pmatrix} \lambda^{\alpha}(\rho) \\ \mu^{\dot{\alpha}}(\rho) \end{pmatrix}.$

Fourier transform gluon vertex operators according to helicity:

$$V_{+}^{A}(W,\rho) = \int \frac{d\kappa}{\kappa} e^{i\kappa W\cdot Z(\rho)} J^{A},$$

$$V_{-}^{A}(Z,\rho) = \int \kappa^{3} d\kappa\, \delta^{4}(\kappa Z(\rho) - Z)\, J^{A}\, \psi^{1} \ldots \psi^{4}.$$

$$\text{Tree } M^{\epsilon_1\ldots\epsilon_N} = \int \langle 0| e^{(n-1)q_0} \prod_{s\in\mathcal{N}} \delta^{4}(\kappa_s Z(\rho_s) - Z_s)$$
$$\times \exp\left\{ i \sum_{j\in\mathcal{P}} \kappa_j W_j \cdot Z(\rho_j) \right\} |0\rangle \prod_{a=1}^{N} \frac{d\rho_a d\kappa_a}{\kappa_a} \prod_{s\in\mathcal{N}} \kappa_s^{4}$$
$$\times \prod_{r<s;r,s\in\mathcal{N}} (\rho_r - \rho_s)^4 \langle 0| J^{A_1}(\rho_1) J^{A_2}(\rho_2) \ldots J^{A_N}(\rho_N) |0\rangle \Big/ dg$$

$$\delta^4(\kappa_s Z(\rho_s) - Z_s)$$

$Z(\rho) = \mathcal{Z}_0 + \mathcal{Z}_{-1}\rho + \cdots + \mathcal{Z}_{-n+1}\rho^{n-1}$, polynomial of order $n-1$, so

$$Z(\rho) = \sum_{s\in\mathcal{N}} \frac{1}{\kappa_s} Z_s \prod_{r\neq s; r\in\mathcal{N}} \frac{\rho-\rho_r}{\rho_s-\rho_r}, \qquad \text{where } \kappa_s Z(\rho_s) = Z_s.$$

The positive helicity vertices become

$$e^{i\sum_{j\in\mathcal{P}}\kappa_j W_j\cdot Z(\rho_j)} = e^{i\sum_{j\in\mathcal{P}}\sum_{s\in\mathcal{N}} c_{js} W_j\cdot Z_s}$$

where $c_{js} = \frac{\kappa_j}{\kappa_s}\prod_{r\neq s; r\in\mathcal{N}} \frac{\rho_j-\rho_r}{\rho_s-\rho_r} = \frac{\lambda_j}{\lambda_s(\rho_j-\rho_s)}$ are the link variables.

Fourier transforming to momentum space,

$$\mathcal{M}^{\epsilon_1\ldots\epsilon_N} = \langle r_1, r_n\rangle^2(\rho_{r_1}-\rho_{r_n})^2 \int \prod_{j\in\mathcal{P}} \delta^2\left(\pi_j - \sum_{r\in\mathcal{N}} c_{jr}\pi_r\right) \times \prod_{s\in\mathcal{N}'} \delta^2\left(\bar{\pi}_s + \sum_{i\in\mathcal{P}} \bar{\pi}_i c_{is}\right) \times \prod_{a=1}^{N} \frac{1}{(\rho_a-\rho_{a+1})} \prod_{\substack{a=1\\ a\neq r_1, r_n}}^{N} \frac{d\kappa_a d\rho_a}{\kappa_a}.$$

4-dimensional momenta $k_{a\alpha\dot{\alpha}} = \pi_{a\alpha}\bar{\pi}_{a\dot{\alpha}}, \quad 1 \le a \le N; \alpha, \dot{\alpha} = 1, 2.$
$\{a \in A : a = i \in \mathcal{P},\ r \in \mathcal{N}, \quad m + n = N\}, \quad \rho_a \equiv z_a.$

Link variables $c_{ir} \equiv \frac{\lambda_i}{\lambda_r(z_i - z_r)}$ satisfy:

$$\pi_i^\alpha = \sum_{r\in\mathcal{N}} c_{ir}\pi_r^\alpha, \quad -\bar{\pi}_{r\dot{\alpha}} = \sum_{i\in\mathcal{P}} \bar{\pi}_{i\dot{\alpha}} c_{ir}.$$

BCFW in Twistor String Theory link variables

$$\mathcal{M}_{mn}(\zeta) = K_{mn} \oint_{\mathcal{O}} F_{mn}(c(\zeta)) \prod_{a=2}^{m-1} \prod_{b=2}^{n-1} \frac{dc_{i_a r_b}}{\mathcal{C}_{ab}(c(\zeta))},$$

$$\mathcal{M}_{mn}(\zeta) = \sum_{z_i} \frac{\mathcal{M}_{mn}^{\zeta_i}}{\zeta - \zeta_i}, \qquad \mathcal{M}_{mn} = \mathcal{M}_{mn}(0) = -\sum_{z_i} \frac{1}{\zeta_i}\mathcal{M}_{mn}^{\zeta_i}.$$

Analysis of the poles and residues proves BCFW, demonstrating equivalence between the twistor string amplitudes and Yang Mills.

Twistor String Equations imply the Scattering Equations

$$2\sum_b \frac{k_i \cdot k_b}{z_i - z_b} = \sum_r \frac{\langle \pi_i, \pi_r\rangle[\bar\pi_i, \bar\pi_r]}{z_i - z_r} + \sum_j \frac{\langle \pi_i, \pi_j\rangle[\bar\pi_i, \bar\pi_j]}{z_i - z_j}$$

$$\sum_r \frac{\langle \pi_i, \pi_r\rangle[\bar\pi_i, \bar\pi_r]}{z_i - z_r} = -\sum_{rsj} \frac{c_{is}\langle \pi_s, \pi_r\rangle[\bar\pi_i, \bar\pi_j]c_{jr}}{z_i - z_r}$$

$$= \tfrac{1}{2}\sum_{rsj} \frac{\lambda_i\lambda_j\langle \pi_r, \pi_s\rangle[\bar\pi_i, \bar\pi_j](z_r - z_s)}{\lambda_r\lambda_s(z_i - z_r)(z_i - z_s)(z_j - z_r)(z_j - z_s)}$$

$$\sum_j \frac{\langle \pi_i, \pi_j\rangle[\bar\pi_i, \bar\pi_j]}{z_i - z_j} = \sum_{rsj} c_{ir}c_{js}\frac{\langle \pi_r, \pi_s\rangle[\bar\pi_i, \bar\pi_j]}{z_i - z_j}$$

$$= -\tfrac{1}{2}\sum_{rsj} \frac{\lambda_i\lambda_j\langle \pi_r, \pi_s\rangle[\bar\pi_i, \bar\pi_j](z_r - z_s)}{\lambda_r\lambda_s(z_i - z_r)(z_j - z_s)(z_i - z_s)(z_j - z_r)}.$$

So $\sum_b \frac{k_i \cdot k_b}{z_i - z_b} = 0$, and similarly $\sum_b \frac{k_r \cdot k_b}{z_r - z_b} = 0$.

$2k_a \cdot k_b = \langle \pi_a, \pi_b\rangle[\bar\pi_a, \bar\pi_b]$; $\quad \langle \pi_a, \pi_b\rangle \equiv \pi_{a\alpha}\pi_b^{\alpha}$, $[\bar\pi_a, \bar\pi_b] \equiv \bar\pi_{a\dot\alpha}\pi_b^{\dot\alpha}$

Polynomial Form for the Scattering Equations

For a subset $U \subset A$,

$$k_U \equiv \sum_{a \in U} k_a, \qquad z_U \equiv \prod_{b \in U} z_b,$$

then the Scattering Equations

$$\sum_{\substack{b \in A \\ b \neq a}} \frac{k_a \cdot k_b}{z_a - z_b} = 0$$

are equivalent to the homogeneous polynomial equations

$$\sum_{\substack{U \subset A \\ |U| = m}} k_U^2 z_U = 0, \quad 2 \leq m \leq N - 2,$$

where the sum is over all $\frac{N!}{m!(N-m)!}$ subsets $U \subset A$ with m elements.

Proof of the Polynomial Form of the Scattering Equations

$$p^\mu(z) \equiv \sum_{a\in A} \frac{k_a^\mu}{z - z_a}, \quad \sum_a k_a^\mu = 0, \quad k_a^2 = 0,$$

$$p^2(z) = \sum_{a,b} \frac{k_a \cdot k_b}{(z - z_a)(z - z_b)} = \frac{1}{2} \sum_a \frac{1}{z - z_a} \sum_{b\neq a} \frac{k_a \cdot k_b}{(z_a - z_b)} = 0$$

$$2p^2(z) \prod_{c\in A} (z - z_c) = \sum_{a,b\in A} 2k_a \cdot k_b \prod_{\substack{c\in A \\ c\neq a,b}} (z - z_c)$$

$$= \sum_{m=0}^{N-2} z^{N-m-2} \sum_{\substack{U\subset A \\ |U|=m}} z_U \sum_{\substack{S\subset \overline{U} \\ |S|=2}} k_S^2 = 0$$

where $\overline{U} = \{b \in A : b \notin U\}$. Using $\sum_{\substack{S\subset \overline{U} \\ |S|=2}} k_S^2 = k_{\overline{U}}^2 = k_U^2$, then

$\widetilde{h}_m \equiv \sum_{\substack{U\subset A \\ |U|=m}} k_U^2 z_U = 0.$

The Scattering Equations are the Unique Polynomial Equations that are Möbius Invariant

L_{-1} denotes the generator of translations,

$$L_{-1} = -\sum_{a\in A} \frac{\partial}{\partial z_a}, \qquad L_{-1}\, \tilde{h}_m = -(N-m-1)\tilde{h}_{m-1}\,,$$

L_1, special conformal transformations

$$L_1 = -\sum_{a\in A} z_a^2 \frac{\partial}{\partial z_a} + \Sigma_1^A, \qquad L_1\, \tilde{h}_m = (m-1)\tilde{h}_{m+1},$$

L_0, scale transformations

$$L_0 = -\sum_{a\in A} z_a \frac{\partial}{\partial z_a} + \frac{N}{2}, \qquad L_0\, \tilde{h}_m = (\tfrac{1}{2}N - m)\tilde{h}_m\,,$$

so that $\quad [L_1, L_{-1}] = 2L_0, \qquad [L_0, L_{\pm 1}] = \mp L_{\pm 1}.$

The $\tilde{h}_m$, $2 \le m \le N-2$, form an $(N-3)$-dimensional multiplet of the Möbius algebra, *i.e.* a representation of 'Möbius spin' $\frac{1}{2}N - 2$. The equations $\tilde{h}_m(z_1, \ldots, z_n) = \sum_{\substack{U\subset A \\ |U|=m}} k_U^2 z_U = 0$ determine a discrete set of points (up to Möbius invariance).

Amplitudes in terms of Polynomial Constraints

$$\mathcal{A}_N = \oint_{\mathcal{O}} \Psi_N(z,k) \frac{z_2}{z_{N-1}} \prod_{m=1}^{N-3} \frac{1}{h_m(z,k)} \prod_{2 \le a < b \le N-1} (z_a - z_b) \prod_{a=2}^{N-2} \frac{z_a dz_{a+1}}{(z_a - z_{a+1})^2}.$$

$$h_m = \lim_{z_1 \to \infty} \frac{\widetilde{h}_{m+1}}{z_1} = \frac{1}{m!} \sum_{\substack{a_1, a_2, \ldots, a_m \neq 1, N \\ a_i \text{ uneq.}}} k^2_{1 a_1 \ldots a_m} z_{a_1} z_{a_2} \ldots z_{a_m}, \qquad 1 \le m \le N-3,$$

The $N-3$ polynomial equations $h_m = 0$, of degree m,
linear in each z_a individually,
are equivalent to the Scattering Equations.

By Bézout's theorem, they determine (N-3)! solutions for the ratios of the $z_2, \ldots, z_{N-1}$.

Solutions to the Scattering Equations

$N = 4$

$$h_1 = k_{12}^2 z_2 + k_{13}^2 z_3 = 0, \qquad z_3/z_2 = -k_{12}^2/k_{13}^2 = -k_1 \cdot k_2/k_1 \cdot k_3.$$

$N = 5$

$$h_1 = k_{12}^2 z_2 + k_{13}^2 z_3 + k_{14}^2 z_4 = 0,$$
$$h_2 = k_{123}^2 z_2 z_3 + k_{124}^2 z_2 z_4 + k_{134}^2 z_3 z_4 = 0,$$

eliminating z_4 yields a quadratic equation for z_3/z_2.
This can be written as

$$\begin{vmatrix} h_1 & h_2 \\ \frac{\partial h_1}{\partial z_2} & \frac{\partial h_2}{\partial z_2} \end{vmatrix} = 0.$$

$N = 6$ write $(x, y, z, u) = (z_2, z_3, z_4, z_5)$

$$h_1 = k_{12}^2 x + k_{13}^2 y + k_{14}^2 z + k_{15}^2 = 0,$$
$$h_2 = k_{123}^2 xy + k_{124}^2 xz + k_{134}^2 yz + k_{125}^2 xu + k_{135}^2 yu + k_{145}^2 zu = 0,$$
$$h_3 = k_{1234}^2 xyz + k_{1235}^2 xyu + k_{1245}^2 xzu + k_{1345}^2 yzu = 0,$$

eliminating x, y yields a sextic equation for z/u.
This can be written

$$\begin{vmatrix} h_1 & h_2 & h_3 & 0 & 0 & 0 \\ h_1^x & h_2^x & h_3^x & 0 & 0 & 0 \\ h_1^y & h_2^y & h_3^y & h_1 & h_2 & h_3 \\ h_1^{xy} & h_2^{xy} & h_3^{xy} & h_1^x & h_2^x & h_3^x \\ 0 & 0 & 0 & h_1^y & h_2^y & h_3^y \\ 0 & 0 & 0 & h_1^{xy} & h_2^{xy} & h_3^{xy} \end{vmatrix} = 0,$$

$$h_m^x = \frac{\partial h_m}{\partial x}, \qquad h_m^{xy} = \frac{\partial^2 h_m}{\partial x \partial y}, \qquad \text{etc.}$$

The one-loop Scattering Equations

$$P^\mu(\nu,\tau) \equiv p^\mu + \sum_{a\in A} k_a^\mu \zeta(\nu-\nu_a) = k^\mu + \frac{1}{2}\sum_{a\in A} k_a^\mu \frac{\wp'(\nu)+\wp'(\nu_a)}{\wp(\nu)-\wp(\nu_a)},$$

where $p^\mu = k^\mu + \sum_{a\in A} k_a^\mu \zeta(\nu_a)$.

$P^\mu(\nu,\tau)$ is defined on the torus,
$P^\mu(\nu+1,\tau) = P^\mu(\nu+\tau,\tau) = P^\mu(\nu,\tau)$, when $\sum_{a\in A} k_a^\mu = 0$.
$P(\nu,\tau)^2$ has no poles when $k_a^2 = 0$ and

$$f_a = p\cdot k_a + \sum_{b\in A, b\neq a} k_a\cdot k_b\,\zeta(\nu_a-\nu_b) = 0, \qquad a\in A.$$

Then $P(\nu,\tau)^2 = k^2$ is a constant. For modular invariance, $k^2=0$.

$$\sum_{a\in A} f_a = 0.$$

The one-loop Scattering Equations are the N equations:

$$f_a = 0, \qquad k^2 = 0.$$

see also Adamo, Casali and Skinner 1312.3828 [hep-th],

Gross and Mende, Nucl. Phys. B ***303****, 407 (1988).*

The Weierstrass $\wp(\nu, \tau)$ function,

$$\wp(\nu+1) = \wp(\nu), \quad \wp(\nu+\tau) = \wp(\nu),$$

$$\wp(\nu) = -\zeta'(\nu), \quad \zeta(\nu) = \frac{\theta_1'(\nu,\tau)}{\theta_1(\nu,\tau)} + 2\eta(\tau)\nu, \quad \eta(\tau) = -\frac{\theta_1'''(0,\tau)}{6\,\theta_1'(0,\tau)},$$

$$\zeta(\nu+1) = \zeta(\nu) + 2\eta(\tau); \qquad \zeta(\nu+\tau) = \zeta(\nu) - 2\pi i + 2\eta(\tau)\tau.$$

Modular Properties:

$$\zeta(\nu, \tau+1) = \zeta(\nu, \tau), \qquad \zeta(\frac{\nu}{\tau}, -\frac{1}{\tau}) = \tau\,\zeta(\nu,\tau),$$

$$\wp(\nu, \tau+1) = \wp(\nu, \tau), \qquad \wp(\frac{\nu}{\tau}, -\frac{1}{\tau}) = \tau^2\,\wp(\nu,\tau),$$

$$\wp'(\nu, \tau+1) = \wp'(\nu, \tau), \qquad \wp'(\frac{\nu}{\tau}, -\frac{1}{\tau}) = \tau^3\,\wp'(\nu,\tau).$$

The Polynomial Form of the one-loop Scattering Equations

For a subset $U \subset A$,

$$k_U \equiv \sum_{a \in U} k_a, \qquad \wp_U \equiv \prod_{b \in U} \wp_b,$$

where $\wp_b = \wp(\nu_b, \tau)$, then the one-loop Scattering Equations

$$f_a = k \cdot k_a + \frac{1}{2} \sum_{b \neq a} k_a \cdot k_b \frac{\wp'(\nu_a) + \wp'(\nu_b)}{\wp(\nu_a) - \wp(\nu_b)} = 0$$

are equivalent to the one-loop polynomial equations

$$\mathcal{A}_m = \sum_{|U|=m} k \cdot k_{\bar{U}} \wp_U + \tfrac{1}{2} \sum_{|U|=m-1} \sum_{a \in \overline{U}} k_a \cdot k_U \wp'_a \wp_U = 0, \quad 1 \leq m \leq N-1.$$

$$\mathcal{A}_1 = -\sum_a k \cdot k_a \wp_a = 0,$$

$$\mathcal{A}_2 = -\sum_{a \neq b} k \cdot k_a \wp_a \wp_b + \frac{1}{2} \sum_{a \neq b} k_a \cdot k_b \wp'_a \wp_b = 0,$$

$$\mathcal{A}_3 = -\frac{1}{2} \sum_{a,b,c \text{ unequal}} k \cdot k_a \wp_a \wp_b \wp_c + \frac{1}{2} \sum_{a,b,c \text{ unequal}} k_a \cdot k_b \wp'_a \wp_b \wp_c = 0$$

etc.

Proof of the Polynomial Form of the one-loop Scattering Equations

For $M_{ma} = \sum_{\substack{|U|=m \\ U\not\ni a}} \wp_U, \qquad 1 \le m \le N-1, \qquad M_{0a} = 1,$

$\sum_{a=1}^{N} M_{ma} f_a = \mathcal{A}_m, \qquad 0 \le m \le N-1, \qquad \mathcal{A}_0 = \sum_a k \cdot k_a,$

so that $f_a = 0$ implies $\mathcal{A}_m = 0$.

The inverse matrix is

$M^{-1}_{am} = (-1)^m \wp_a^{N-m-1} \prod_{b\neq a} (\wp_a - \wp_b)^{-1}$ implying

$$f_a = \sum_{m=0}^{N-1} M^{-1}_{am} \mathcal{A}_m,$$

so that $\mathcal{A}_m = 0$ implies $f_a = 0$.

Comments

The polynomial form of the tree level Scattering Equations facilitates computation of their solutions $z_a(k)$, due to the linearity of the equations in the individual variables z_a. Bézout's theorem provides an explanation for the $(N-3)!$ solutions.

The Scattering Equations can be generalized to massive particles, enabling the description of tree amplitudes for massive ϕ^3 theory.

In four dimensions, the Scattering Equations and the twistor string equations are closely related.

The proofs make it certain that both the twistor string and the Scattering Equations approach are equivalent to gauge field theory at tree level.

The polynomial form of the one-loop Scattering Equations is given.