High-energy QCD and Wilson lines

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JLAB & ODU

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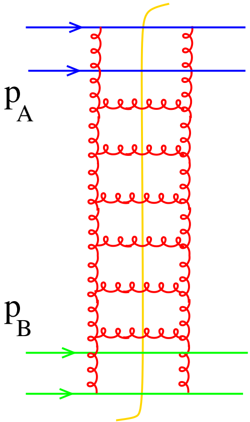
Outline

1. Introduction: BFKL pomeron in hign-energy pQCD
   - Regge limit in perturbative QCD.
   - BFKL and collider physics
2. High-energy amplitudes in N=4 SYM
   - Regge limit in a conformal theory.
   - OPE in light-ray operators *vs* high-energy OPE in Wilson lines.
   - Rapidity evolution of Wilson lines.
   - Leading order: BK equation for color dipoles.
   - NLO BK kernel.
   - An example of NLO calculation of 4-point CF
   - NLO BFKL and anomalous dimensions of light-ray operators.
   - Conclusions 1
3. High-energy scattering in QCD
   - Impact factor for virtual photon in deep inelastic scattering
   - Evolution of color dipoles in QCD
   - Argument of α_s in the BK equation (rcBK)
   - Conclusions 2

Regge limit in pQCD: the BFKL pomeron

$$s = (p_A + p_B)^2 \simeq 4E^2 \;>>\; p_A^2, p_B^2, t$$



Leading Log Approximation (LLA(x)):

$$\alpha_s \ll 1, \quad \alpha_s \ln s \sim 1$$

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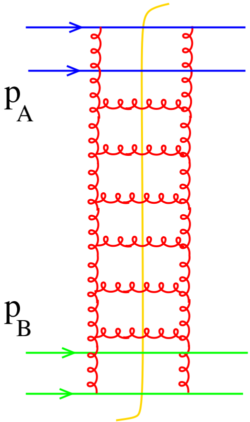

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The sum of gluon ladder diagrams gives

$$\sigma_{\text{tot}} \sim s^{12\frac{\alpha_s}{\pi}\ln 2} \qquad \text{BFKL pomeron}$$

Numerically: for DIS at HERA

$$\sigma \sim s^{0.3} = x_B^{-0.3}$$

- qualitatively OK

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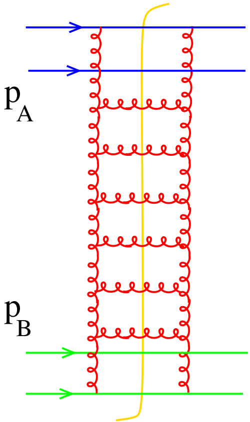

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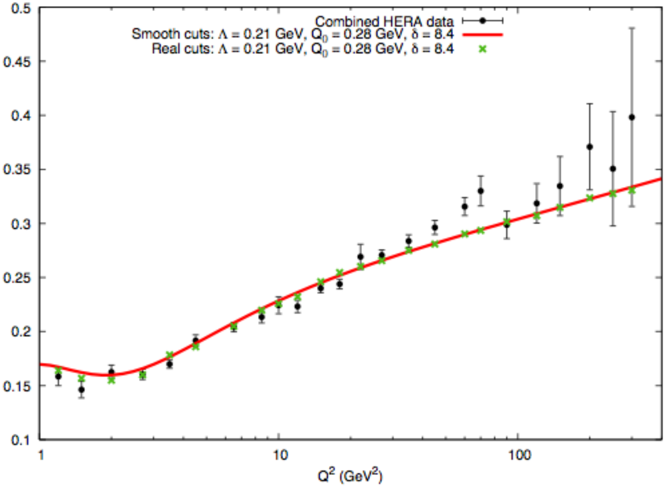
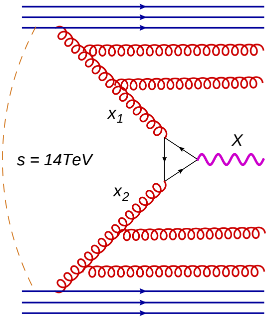
BFKL *vs* HERA data

$$F_2(x_B, Q^2) \;=\; c(Q^2)x_B^{-\lambda(Q^2)}$$



M.Hentschinski, A. Sabio Vera and C. Salas, 2010

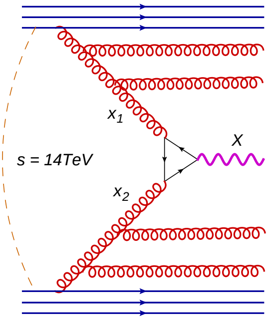
Collinear *vs* BFKL factorization in particle production



Collinear factorization (LLA(Q^2)):

$$\sigma_{pp\to X} = \int_0^1 dx_1 dx_2 D_g(x_1, m_X) D_g(x_2, m_X) \sigma_{gg\to X}$$

sum of the logs $\left(\alpha_s \ln \frac{m_X^2}{m_N^2}\right)^n, \quad \ln \frac{s}{m_X^2} \sim 1$



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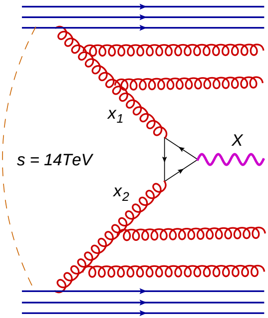
LLA(x): k_T-factorization

$$\sigma_{pp\to X} = \int dk_1^\perp dk_2^\perp g(k_1^\perp, x_A) g(k_2^\perp, x_B) \sigma_{gg\to X}$$

- sum of the logs $\left(\alpha_s \ln x_i\right)^n$, $\ln \frac{m_X^2}{m_N^2} \sim 1$

Much less understood theoretically.



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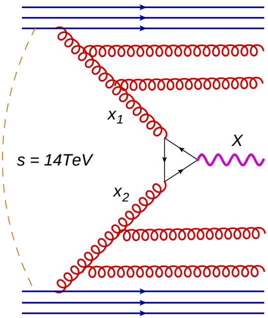
For Higgs production in the central rapidity region $x_{1.2} \sim \frac{m_H}{\sqrt{s}} \simeq 0.01$ and we know from DIS experiments that at such x_B the DGLAP formalism works pretty well $\Rightarrow$ no need for BFKL resummation



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Much less understood theoretically.

For $m_X \sim 10$GeV (like $\bar{b}b$ pair or mini-jet) collinear factorization does not seem to work well $\Rightarrow$ some kind of BFKL resummation is needed.

Structure functions of DIS are determined by matrix elements of twist-2 operators

$$\mathcal{O}_G^{(j)} = F_{\mu_1\xi} D_{\mu_2} ... D_{\mu_{j-2}} F_{\mu_j}^{\ \xi}$$

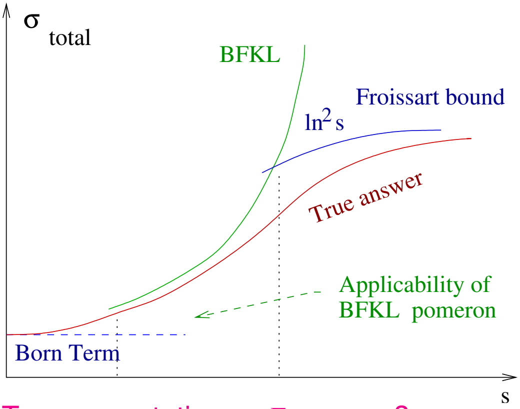
$$\mu^2 \frac{d}{d\mu^2} \mathcal{O}_G^{(j)} = \frac{\gamma_{(j)}(\alpha_s)}{4\pi} \mathcal{O}_G^{(j)}$$

BFKL gives asymptotics of $\gamma_{(j)}$ at $j \to 1$ in all orders in α_s

$$\gamma_{(j)} = \sum_n \Big(\frac{\alpha_s}{j-1}\Big)^n \Big[C^{(n)}_{\text{LO BFKL}} + \alpha_s C^{(n)}_{\text{NLO BFKL}}\Big]$$

Checked by explicit calculation of Feynman diagrams.up to 3 loops in QCD and 9 loops in $\mathcal{N} = 4$ SYM.

Integrablility of spin chains corresponding to evolution of $\mathcal{N} = 4$ SYM operators $\Rightarrow \gamma_{(j)}$ in 5 loops agrees with BFKL (Janik et al).
For all order of pert. theory: Y-system of equations (Gromov, Kazakov, Viera). Reproduces LO BFKL.



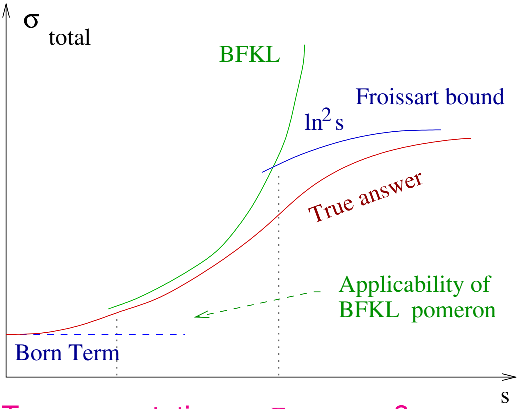
$\sigma_{\rm tot} \sim s^{12\frac{\alpha_s}{\pi}\ln 2}$ violates Froissart bound $\sigma_{\rm tot} \leq \ln^2 s$ $\Rightarrow$ pre-asymptotic behavior.

True asymptotics as $E \to \infty$ = ?

Possible approaches:

- Sum all logs $\alpha_s^m \ln^n s$
- Reduce high-energy QCD to $2+1$ effective theory

Towards the high-energy QCD



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This talk: NLO corrections $\alpha_s^{n+1} \ln^n s$

Conformal four-point amplitude

$$A(x,y,x',y') = (x-y)^4(x'-y')^4 N_c^2 \langle \mathcal{O}(x)\mathcal{O}^\dagger(y)\mathcal{O}(x')\mathcal{O}^\dagger(y')\rangle$$

$\mathcal{O} = \mathrm{Tr}\{Z^2\}$ ($Z = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$) - chiral primary operator

In a conformal theory the amplitude is a function of two conformal ratios

$$A = F(R,R')$$
$$R = \frac{(x-y)^2(x'-y')^2}{(x-x')^2(y-y')^2}, \qquad R' = \frac{(x-y)^2(x'-y')^2}{(x-y')^2(x'-y)^2}$$

At large N_c

$$A(x,y,x',y') = A(\lambda; x,y,x',y') \qquad \lambda = g^2 N_c = 4\pi\alpha_s N_c \;-\; \text{‘t Hooft coupling}$$

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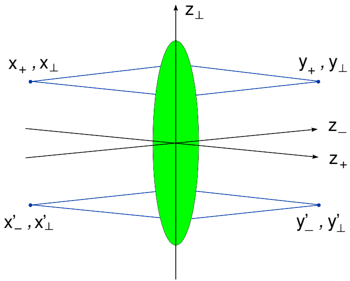
Our goal is perturbative expansion and resummation of $(\lambda \ln s)^n$ at large energies in the next-to-leading approximation

$$(\lambda \ln s)^n (c_n^{\mathrm{LO}} + c_n^{\mathrm{NLO}}\lambda)$$

Regge limit in the coordinate space

Regge limit: $x_+ \to \rho x_+,\ x'_+ \to \rho x'_+,\ y_- \to \rho' y_-,\ y'_- \to \rho' y-_-$ $\rho, \rho' \to \infty$



Full 4-dim conformal group: $A = F(R, r)$

$$R = \frac{(x-y)^2(x'-y')^2}{(x-x')^2(y-y')^2} \to \frac{\rho^2\rho'^2 x_+ x'_+ y_- y'_-}{(x-x')^2_\perp (y-y')^2_\perp} \to \infty$$

$$r = \frac{[(x-y)^2(x'-y')^2 - (x'-y)^2(x-y')^2]^2}{(x-x')^2(y-y')^2(x-y)^2(x'-y')^2}$$

$$\to \frac{[(x'-y')^2_\perp x_+ y_- + x'_+ y'_-(x-y)^2_\perp + x_+ y'_-(x'-y)^2_\perp + x'_+ y_-(x-y')^2_\perp]^2}{(x-x')^2_\perp (y-y')^2_\perp x_+ x'_+ y_- y'_-}$$

4-dim conformal group versus $SL(2,C)$

Regge limit: $x_+ \to \rho x_+,\ x'_+ \to \rho x'_+, y_- \to \rho' y_-,\ y'_- \to \rho' y-_-$
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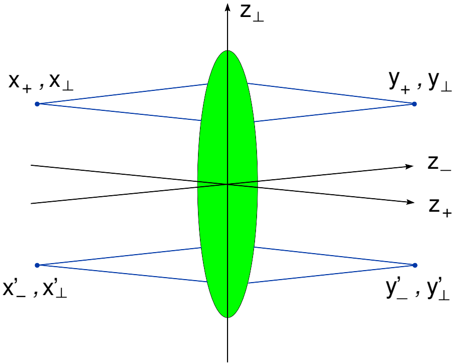

Regge limit symmetry: 2-dim conformal group $SL(2,C)$ formed from P_1, P_2, M^{12}, D, K_1 and K_2 which leave the plane $(0,0,z_\perp)$ invariant.

Pomeron in a conformal theory

$$A(x,y;x',y') \stackrel{s\to\infty}{=} \frac{i}{2}\int d\nu\, f_+(\aleph(\lambda,\nu))F(\lambda,\nu)\Omega(r,\nu)R^{\aleph(\lambda,\nu)/2}$$

L. Cornalba (2007)

$f_+(\omega) = \frac{e^{i\pi\omega}-1}{\sin\pi\omega}$ - signature factor

$$\Omega(r,\nu) = \frac{\sin\nu\rho}{\sinh\rho}, \qquad \cosh\rho = \frac{\sqrt{r}}{2}$$

- solution of the eqn $(\Box_{H_3}+\nu^2+1)\Omega(r,\nu)=0$

The dynamics is described by:
$\aleph(\lambda,\nu)$ - pomeron intercept,
and
$F(\lambda,\nu)$ - “pomeron residue”.

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Pomeron intercept $\aleph(\nu,\lambda)$ is known in two limits:

$$1. \qquad \lambda\to 0: \qquad \aleph(\nu,\lambda) = \frac{\lambda}{\pi}\chi(\nu)+\lambda^2\aleph_1(\nu)+...$$

$\chi(\nu)=2\psi(1)-\psi(\frac{1}{2}+i\nu)-\psi(\frac{1}{2}-i\nu)$ - BFKL intercept,

$\aleph_1(\nu)$ - NLO BFKL intercept Lipatov, Kotikov (2000)

$$2. \qquad \lambda\to\infty: \qquad AdS/CFT \quad \Rightarrow \quad \aleph(\nu,\lambda) = 2-\frac{\nu^2+4}{2\sqrt{\lambda}}+...$$

2 = gravition spin , next term - Brower, Polchinski, Strassler, Tan (2006)
Cornalba, Costa, Penedones (2007)

Now: two more terms ($\sim\lambda^{-1}$ and $\sim\lambda^{-3/2}$)

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The function $F(\nu,\lambda)$ in two limits:

$$1. \qquad \lambda \to 0: \qquad F(\nu,\lambda) = \lambda^2 F_0(\nu) + \lambda^3 F_1(\nu) + \ldots$$

$F_0(\nu) = \frac{\pi \sinh \pi\nu}{4\nu \cosh^3 \pi\nu}$ Cornalba, Costa, Penedones (2007)

$F_1(\nu)$ = see below G. Chirilli and I.B. (2009)

$$2. \qquad \lambda \to \infty: \qquad AdS/CFT \quad \Rightarrow \quad \aleph(\nu,\lambda) = \pi^3\nu^2\frac{1+\nu^2}{\sinh^2\pi\nu} + \ldots$$

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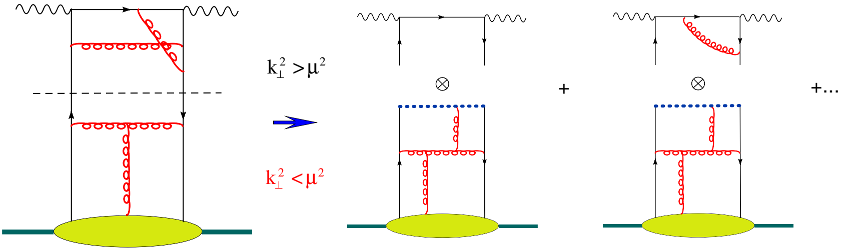
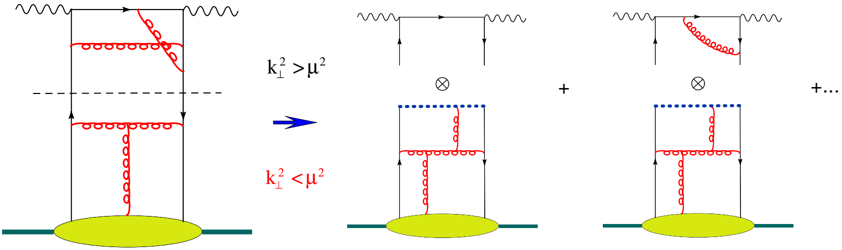
L.Cornalba (2007)

We calculate $F_1(\nu)$ (and confirm $\aleph_1(\nu)$) using the expansion of high-energy amplitudes in Wilson lines (color dipoles)



μ^2 - factorization scale (normalization point)

$k_\perp^2 \ > \ \mu^2$ - coefficient functions
$k_\perp^2 \ < \ \mu^2$ - matrix elements of light-ray operators (normalized at μ^2)



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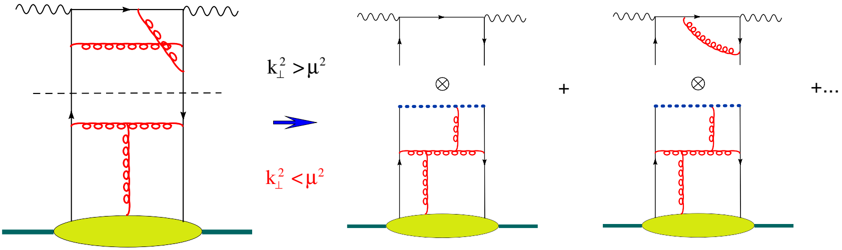
OPE in light-ray operators $(x-y)^2 \to 0$

$$T\{j_\mu(x)j_\nu(y)\} = \frac{x_\xi}{2\pi^2 x^4}\Big[1 + \frac{\alpha_s}{\pi}(\ln x^2\mu^2 + C)\Big]\bar\psi(x)\gamma_\mu\gamma^\xi\gamma_\nu[x,y]\psi(y) + O(\frac{1}{x^2})$$

$$[x,y] \equiv Pe^{ig\int_0^1 du\ (x-y)^\mu A_\mu(ux+(1-u)y)}$$ - gauge link

Light-cone expansion and DGLAP evolution in the NLO



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Renorm-group equation for light-ray operators ⇒ DGLAP evolution of parton densities $(x-y)^2 = 0$

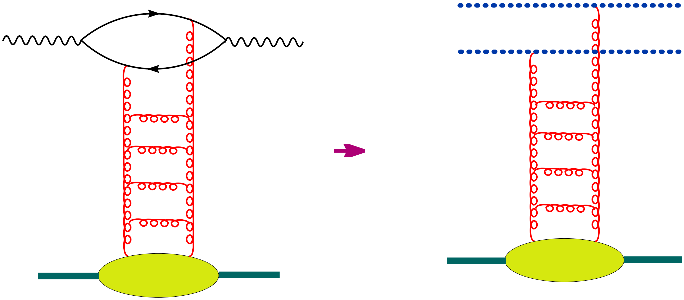
$$\mu^2 \frac{d}{d\mu^2} \bar{\psi}(x)[x,y]\psi(y) \ = \ K_{\mathrm{LO}} \bar{\psi}(x)[x,y]\psi(y) + \alpha_s K_{\mathrm{NLO}} \bar{\psi}(x)[x,y]\psi(y)$$

Four steps of an OPE

- To factorize an amplitude into a product of coefficient functions and matrix elements of relevant operators.
- To find the evolution equations of the operators with respect to factorization scale.
- To solve these evolution equations.
- To convolute the solution with the initial conditions for the evolution and get the amplitude

DIS at high energy: relevant operators

- At high energies, particles move along straight lines ⇒ the amplitude of $\gamma^* A \to \gamma^* A$ scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):



$$A(s) = \int \frac{d^2 k_\perp}{4\pi^2} I^A(k_\perp) \langle B | \mathrm{Tr}\{U(k_\perp) U^\dagger(-k_\perp)\} | B \rangle$$

$$U(x_\perp) = \mathrm{Pexp}\Big[ig \int_{-\infty}^{\infty} du\ n^\mu A_\mu(un + x_\perp)\Big] \qquad \text{Wilson line}$$

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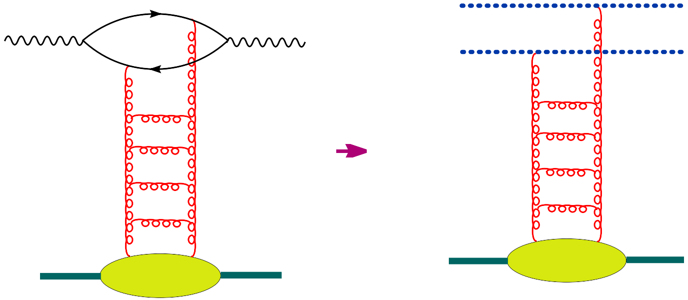

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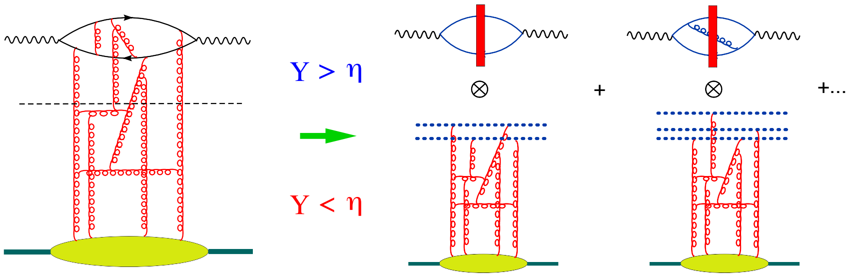
Formally, ➔ means the operator expansion in Wilson lines

Rapidity factorization



η - rapidity factorization scale

Rapidity Y > η - coefficient function ("impact factor")

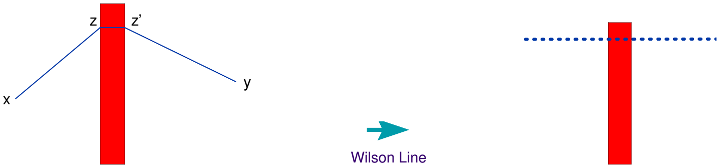
Rapidity Y < η - matrix elements of (light-like) Wilson lines with rapidity divergence cut by η

$$U^\eta_x = \mathrm{Pexp}\Big[ig\int_{-\infty}^{\infty} dx^+ A^\eta_+(x_+, x_\perp)\Big]$$

$$A^\eta_\mu(x) = \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$

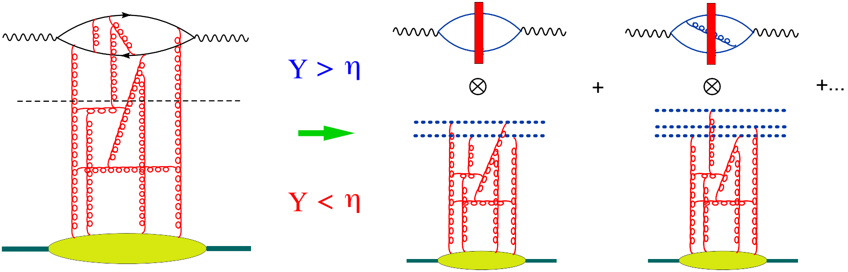
Spectator frame: propagation in the shock-wave background.

Each path is weighted with the gauge factor $Pe^{ig\int dx_\mu A^\mu}$. Quarks and gluons do not have time to deviate in the transverse space ⇒ we can replace the gauge factor along the actual path with the one along the straight-line path.



[$x \to z$: free propagation]×
[$U^{ab}(z_\perp)$ - instantaneous interaction with the $\eta < \eta_2$ shock wave]×
[$z \to y$: free propagation]

High-energy expansion in color dipoles

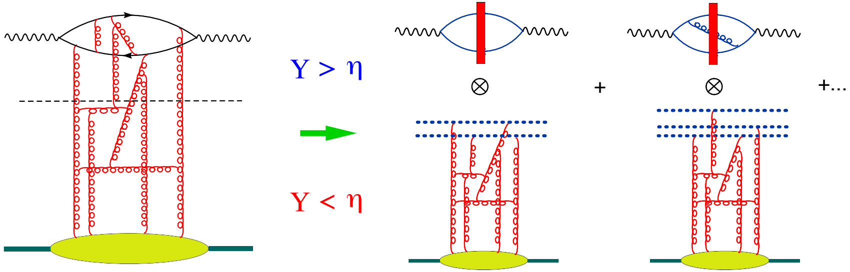


The high-energy operator expansion is

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2 \; I^{\rm LO}_{\mu\nu}(z_1,z_2,x,y)\,{\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

$$+ \text{ NLO contribution}$$

High-energy expansion in color dipoles



η - rapidity factorization scale

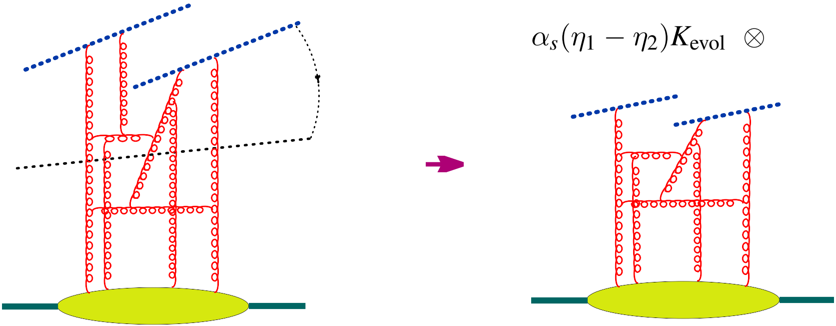
Evolution equation for color dipoles

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\} &= \frac{\alpha_s}{2\pi^2}\int d^2z \frac{(x-y)^2}{(x-z)^2(y-z)^2}[\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\}\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\} \\
&- N_c\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\}] + \alpha_s K_{\mathrm{NLO}}\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\} + O(\alpha_s^2)
\end{aligned}
$$

(Linear part of $K_{\mathrm{NLO}} = K_{\mathrm{NLO\ BFKL}}$)

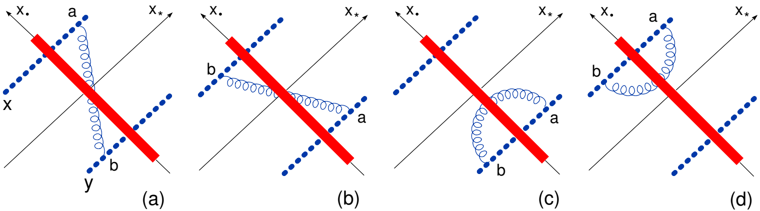
Evolution equation for color dipoles

To get the evolution equation, consider the dipole with the rapidies up to η_1 and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidities up to η_2).



$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + \ldots \quad \Rightarrow$$

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}}$$



$$U^{ab}_z = \mathrm{Tr}\{t^a U_z t^b U^\dagger_z\} \quad \Rightarrow (U_x U^\dagger_y)^{\eta_1} \rightarrow (U_x U^\dagger_y)^{\eta_1} + \alpha_s(\eta_1 - \eta_2)(U_x U^\dagger_z U_z U^\dagger_y)^{\eta_2}$$

$\Rightarrow$ Evolution equation is non-linear

Non linear evolution equation

$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

BK equation

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) \;=\; \frac{\alpha_s N_c}{2\pi^2}\int\frac{d^2z\;(x-y)^2}{(x-z)^2(y-z)^2}\Big\{\hat{\mathcal{U}}(x,z)+\hat{\mathcal{U}}(z,y)-\hat{\mathcal{U}}(x,y)-\hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\Big\}$$

I. B. (1996), Yu. Kovchegov (1999)
Alternative approach: JIMWLK (1997-2000)

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LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

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Alternative approach: JIMWLK (1997-2000)

LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

LLA for DIS in sQCD $\Rightarrow$ BK eqn (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$, $\alpha_s A^{1/3} \sim 1$)

(s for semiclassical)

Conformal invariance of the BK equation

Formally, a light-like Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \;=\; \mathrm{Pexp}\left\{ ig\int_{-\infty}^{\infty} dx^+ \, A_+(x^+, x_\perp) \right\}$$

is invariant under inversion (with respect to the point with $x^- = 0$).

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$(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow$ after the inversion $x_\perp \to x_\perp / x_\perp^2$ and $x^+ \to x^+ / x_\perp^2$

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$\Rightarrow$The dipole kernel is invariant under the inversion $V(x_\perp) = U(x_\perp / x_\perp^2)$

$$\frac{d}{d\eta}\text{Tr}\{V_x V_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int \frac{d^2 z}{\not{z}^4} \frac{(x-y)^2 \;\not{z}^4}{(x-z)^2(z-y)^2}[\text{Tr}\{V_x V_z^\dagger\}\text{Tr}\{V_z V_y^\dagger\} - N_c \text{Tr}\{V_x V_y^\dagger\}]$$

SL(2,C) for Wilson lines

$$\hat{S}_- \equiv \frac{i}{2}(K^1 + iK^2), \quad \hat{S}_0 \equiv \frac{i}{2}(D + iM^{12}), \quad \hat{S}_+ \equiv \frac{i}{2}(P^1 - iP^2)$$

$$[\hat{S}_0, \hat{S}_\pm] = \pm\hat{S}_\pm, \quad \frac{1}{2}[\hat{S}_+, \hat{S}_-] = \hat{S}_0,$$

$$[\hat{S}_-, \hat{U}(z,\bar{z})] = z^2\partial_z\hat{U}(z,\bar{z}), \quad [\hat{S}_0, \hat{U}(z,\bar{z})] = z\partial_z\hat{U}(z,\bar{z}), \quad [\hat{S}_+, \hat{U}(z,\bar{z})] = -\partial_z\hat{U}(z,\bar{z})$$

$z \equiv z^1 + iz^2, \bar{z} \equiv z^1 + iz^2, \quad U(z_\perp) = U(z,\bar{z})$

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Conformal invariance of the evolution kernel

$$\frac{d}{d\eta}[\hat{S}_-, \mathrm{Tr}\{U_x U_y^\dagger\}] = \frac{\alpha_s N_c}{2\pi^2}\int dz\, K(x,y,z)[\hat{S}_-, \mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_y^\dagger\}]$$

$$\Rightarrow \left[x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z}\right]K(x,y,z) = 0$$

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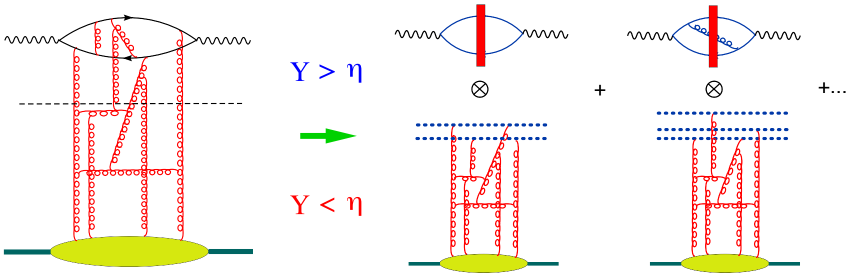
In the leading order - OK. In the NLO - ?

Expansion of the amplitude in color dipoles in the NLO



The high-energy operator expansion is

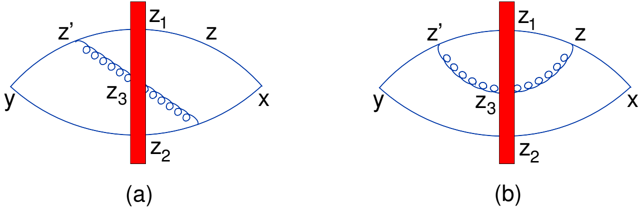
$$T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} = \int d^2z_1 d^2z_2\ I^{\rm LO}(z_1,z_2)\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

$$+\int d^2z_1 d^2z_2 d^2z_3\ I^{\rm NLO}(z_1,z_2,z_3)[\frac{1}{N_c}\mathrm{Tr}\{T^n\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^n\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - \mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

In the leading order - conf. invariant impact factor

$$I_{\rm LO} = \frac{x_+^{-2}y_+^{-2}}{\pi^2\mathcal{Z}_1^2\mathcal{Z}_2^2}, \qquad \mathcal{Z}_i \equiv \frac{(x-z_i)^2_\perp}{x_+} - \frac{(y-z_i)^2_\perp}{y_+} \qquad CCP, 2007$$

NLO impact factor



$$I^{\rm NLO}(x,y;z_1,z_2,z_3;\eta) \;=\; -\,I^{\rm LO}\times\frac{\lambda}{\pi^2}\frac{z_{13}^2}{z_{12}^2 z_{23}^2}\Big[\ln\frac{\sigma s}{4}\mathcal{Z}_3-\frac{i\pi}{2}+C\Big]$$

The NLO impact factor is not Möbius invariant $\Rightarrow$ the color dipole with the cutoff η is not invariant

However, if we define a composite operator (a - analog of μ^{-2} for usual OPE)

$$[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\rm conf} \;=\; \mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}$$
$$+\,\frac{\lambda}{2\pi^2}\int d^2z_3\;\frac{z_{12}^2}{z_{13}^2z_{23}^2}[\mathrm{Tr}\{T^n\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^n\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\ln\frac{az_{12}^2}{z_{13}^2z_{23}^2}\;+\;O(\lambda^2)$$

the impact factor becomes conformal in the NLO.

Operator expansion in conformal dipoles

$$T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} = \int d^2z_1 d^2z_2\ I^{\rm LO}(z_1,z_2){\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}^{\rm conf}$$
$$+ \int d^2z_1 d^2z_2 d^2z_3\ I^{\rm NLO}(z_1,z_2,z_3)[\frac{1}{N_c}{\rm Tr}\{T^n\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^n\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - {\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

$$I^{\rm NLO} = -I^{\rm LO}\frac{\lambda}{2\pi^2}\int dz_3\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[\ln\frac{z_{12}^2e^{2\eta}as^2}{z_{13}^2z_{23}^2}\mathcal{Z}_3^2 - i\pi + 2C\Big]$$

The new NLO impact factor is conformally invariant
$\Rightarrow$ ${\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}^{\rm conf}$ is Möbius invariant

We think that one can construct the composite conformal dipole operator order by order in perturbation theory.

Analogy: when the UV cutoff does not respect the symmetry of a local operator, the composite local renormalized operator in must be corrected by finite counterterms order by order in perturbaton theory.

Definition of the NLO kernel

In general

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + \alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + O(\alpha_s^3)$$

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$$\alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = \frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} - \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + O(\alpha_s^3)$$

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We calculate the “matrix element” of the r.h.s. in the shock-wave background

$$\langle\alpha_s^2 K_{\rm NLO}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle = \frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle - \langle\alpha_s K_{\rm LO}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle + O(\alpha_s^3)$$

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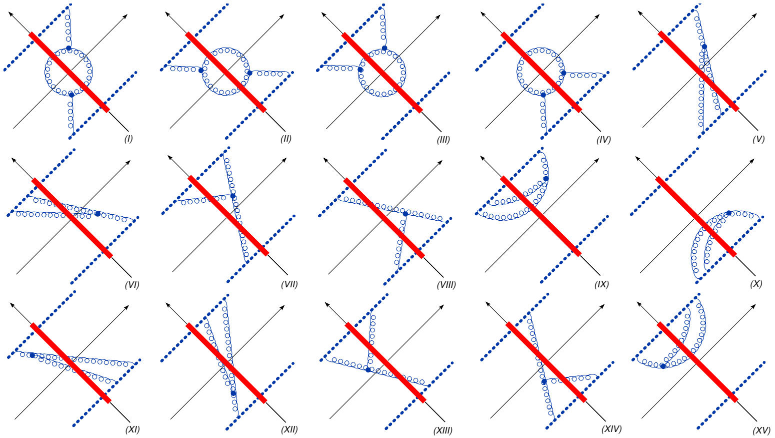
Subtraction of the (LO) contribution (with the rigid rapidity cutoff)
$\Rightarrow$ $\left[\frac{1}{v}\right]_+$ prescription in the integrals over Feynman parameter v

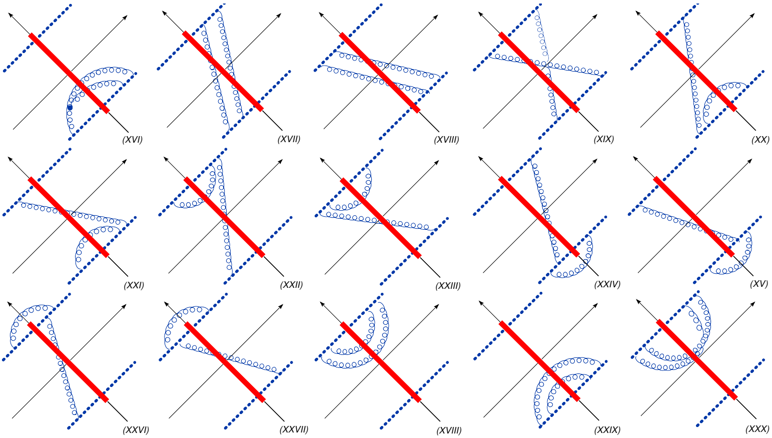
Typical integral

$$\int_0^1 dv\, \frac{1}{(k-p)_\perp^2 v + p_\perp^2(1-v)}\left[\frac{1}{v}\right]_+ = \frac{1}{p_\perp^2}\ln\frac{(k-p)_\perp^2}{p_\perp^2}$$

Gluon part of the NLO BK kernel: diagrams



Diagrams for 1→3 dipoles transition



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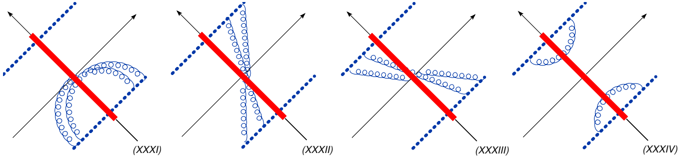
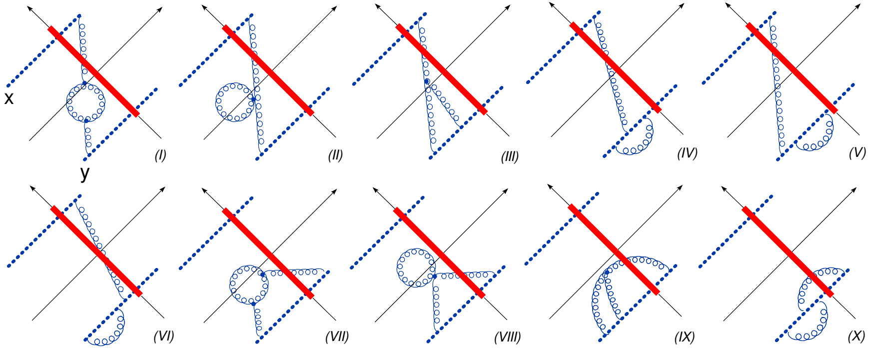
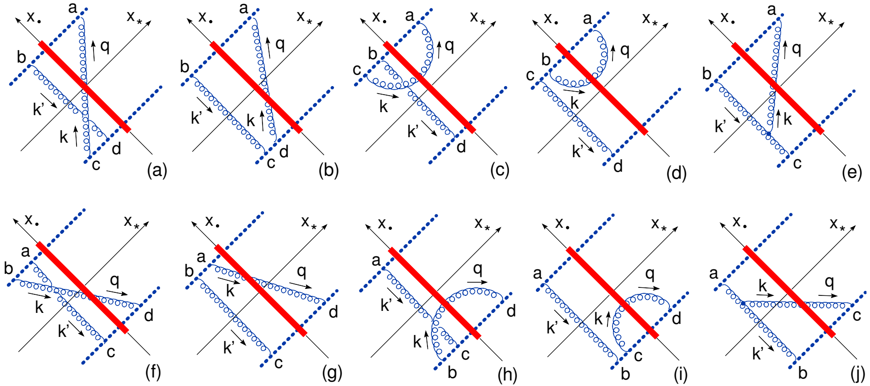


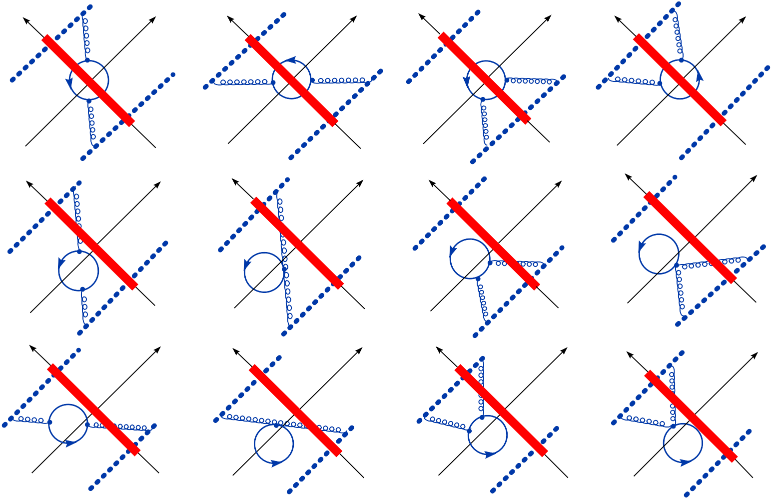
"Running coupling" diagrams



1 → 2 dipole transition diagrams



Gluino and scalar loops



Evolution equation for color dipole in $\mathcal{N}=4$ (I.B. and G. Chirilli)

$$
\begin{aligned}
&\frac{d}{d\eta}\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\\
&= \frac{\alpha_s}{\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1-\frac{\alpha_sN_c}{4\pi}\Big[\frac{\pi^2}{3}+2\ln\frac{z_{13}^2}{z_{12}^2}\ln\frac{z_{23}^2}{z_{12}^2}\Big]\Big\}\\
&\times[\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\\
&-\frac{\alpha_s^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\ \frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\\
&\times\mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}(\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_4}-\hat U^\eta_{z_3})^{bb'}
\end{aligned}
$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha<\sigma=e^{2\eta}$ in the rapidity of Wilson lines.

$$
\begin{aligned}
&\frac{d}{d\eta}\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\\
&= \frac{\alpha_s}{\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1-\frac{\alpha_sN_c}{4\pi}\Big[\frac{\pi^2}{3}+2\ln\frac{z_{13}^2}{z_{12}^2}\ln\frac{z_{23}^2}{z_{12}^2}\Big]\Big\}\\
&\times\,[\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\\
&-\frac{\alpha_s^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\,\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\\
&\times\,\mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}(\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_4}-\hat U^\eta_{z_3})^{bb'}
\end{aligned}
$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha<\sigma=e^{2\eta}$ in the rapidity of Wilson lines.

For the conformal composite dipole the result is Möbius invariant

Evolution equation for composite conformal dipoles in $\mathcal{N}=4$

$$\frac{d}{d\eta}\big[\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\big]^{\rm conf}$$

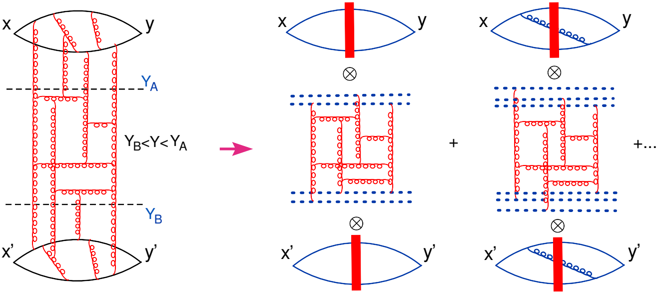
$$= \frac{\alpha_s}{\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big[1-\frac{\alpha_s N_c}{4\pi}\frac{\pi^2}{3}\Big]\big[\mathrm{Tr}\{T^a\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^a\hat{U}_{z_3}\hat{U}^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\big]^{\rm conf}$$

$$-\frac{\alpha_s^2}{4\pi^4}\int d^2z_3 d^2z_4\frac{z_{12}^2}{z_{13}^2 z_{24}^2 z_{34}^2}\Big\{2\ln\frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2}+\Big[1+\frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2-z_{14}^2 z_{23}^2}\Big]\ln\frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2}\Big\}$$

$$\times\ \mathrm{Tr}\{[T^a,T^b]\hat{U}^\eta_{z_1}T^{a'}T^{b'}\hat{U}^{\dagger\eta}_{z_2}+T^bT^a\hat{U}^\eta_{z_1}[T^{b'},T^{a'}]\hat{U}^{\dagger\eta}_{z_2}\}[(\hat{U}^\eta_{z_3})^{aa'}(\hat{U}^\eta_{z_4})^{bb'}-(z_4\to z_3)]$$

Now Möbius invariant!

NLO Amplitude in $\mathcal{N}$=4 SYM theory: factorization in rapidity

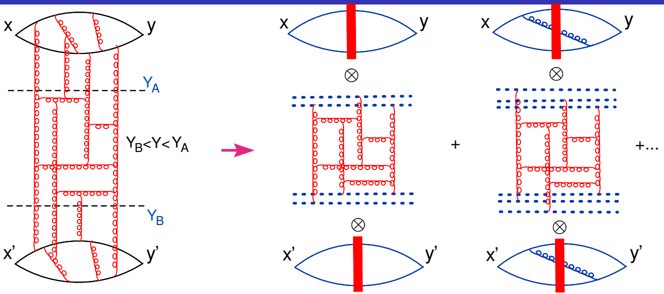


$$(x-y)^4(x'-y')^4\langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}^\dagger(y)\hat{\mathcal{O}}(x')\hat{\mathcal{O}}^\dagger(y')\}\rangle$$
$$= \int d^2z_{1\perp}d^2z_{2\perp}d^2z'_{1\perp}d^2z'_{2\perp}\,\mathrm{IF}^{a_0}(x,y;z_1,z_2)[\mathrm{DD}]^{a_0,b_0}(z_1,z_2;z'_1,z'_2)\mathrm{IF}^{b_0}(x',y';z'_1,z'_2)$$

$a_0 = \frac{x_+y_+}{(x-y)^2}$, $b_0 = \frac{x'_-y'_-}{(x'-y')^2}$ $\Leftrightarrow$ impact factors do not scale with energy

$\Rightarrow$ all energy dependence is contained in $[\mathrm{DD}]^{a_0,b_0}$ ($a_0b_0 = R$)

NLO Amplitude in $\mathcal{N}$=4 SYM theory: factorization in rapidity



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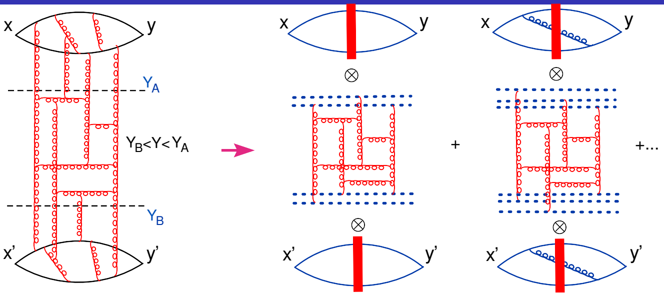
Impact factor ($\mathcal{R} = \frac{(x-y)^2z_{12}^2}{x_+y_+\mathcal{Z}_1\mathcal{Z}_2}$ - conf. ratio)

$$\mathrm{IF}^{a_0} = \int d\nu\int dz_0\,\mathcal{R}^{\frac{1}{2}+i\nu}\Big(1+\frac{\alpha_sN_c}{\pi}\Big[\frac{2\pi^2}{3}+\frac{4\chi(\nu)-8}{1+4\nu^2}-\frac{2\pi^2}{\cosh^2\pi\nu}\Big]\Big)\mathcal{U}^{a_0}(z_0,\nu)$$

$$\mathcal{U}^a(z_0,\nu) = \frac{1}{\pi^2}\int\frac{d^2z_1d^2z_2}{z_{12}^4}\Big(\frac{z_{12}^2}{z_{10}^2z_{20}^2}\Big)^{\frac{1}{2}-i\nu}\mathcal{U}(z_1,z_2), \qquad \mathcal{U}\equiv 1-\frac{1}{N_c}\mathrm{Tr}\{U_{z_1}U^\dagger_{z_2}\}$$

NLO Amplitude in $\mathcal{N}$=4 SYM theory: factorization in rapidity



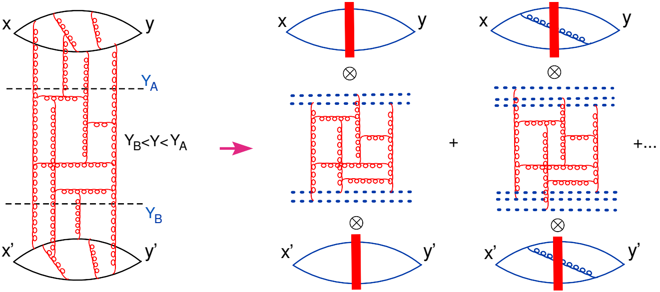
$$(x-y)^4(x'-y')^4\langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}^\dagger(y)\hat{\mathcal{O}}(x')\hat{\mathcal{O}}^\dagger(y')\}\rangle$$

$$= \int d^2z_{1\perp}d^2z_{2\perp}d^2z'_{1\perp}d^2z'_{2\perp}\,\mathrm{IF}^{a_0}(x,y;z_1,z_2)[\mathrm{DD}]^{a_0,b_0}(z_1,z_2;z'_1,z'_2)\mathrm{IF}^{b_0}(x',y';z'_1,z'_2)$$

Dipole-dipole scattering $\quad \chi(\gamma)\equiv 2C-\psi(\gamma)-\psi(1-\gamma),\ \gamma\equiv\frac{1}{2}+i\nu$

$$[DD]=\int d\nu\int dz_0\ \Big(\frac{z_{12}^2}{z_{10}^2z_{20}^2}\Big)^{\frac{1}{2}+i\nu}\Big(\frac{z_{12}^2}{z_{10}^2z_{20}^2}\Big)^{\frac{1}{2}-i\nu}D(\frac{1}{2}+i\nu;\lambda)R^{\omega(\nu)/2}$$

$$D(\gamma;\lambda)\ =\ \frac{\Gamma(-\gamma)\Gamma(\gamma-1)}{\Gamma(1+\gamma)\Gamma(2-\gamma)}\Big\{1+\frac{\alpha_sN_c}{2\pi}\Big[\frac{4\chi(\nu)}{1+4\nu^2}-\frac{\pi^2}{3}+i\pi\frac{N_c^2-4}{2N_c^2}\Big]+O(g^4)\Big\}$$



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$$= \int d^2z_{1\perp}d^2z_{2\perp}d^2z'_{1\perp}d^2z'_{2\perp}\,\mathrm{IF}^{a_0}(x,y;z_1,z_2)[\mathrm{DD}]^{a_0,b_0}(z_1,z_2;z'_1,z'_2)\mathrm{IF}^{b_0}(x',y';z'_1,z'_2)$$

Result : (G.A. Chirilli and I.B.)

$$F(\nu) \;=\; \frac{N_c^2}{N_c^2-1}\frac{4\pi^4\alpha_s^2}{\cosh^2\pi\nu}\left\{1+\frac{\alpha_sN_c}{\pi}\left[\frac{\pi^2}{2}-\frac{2\pi^2}{\cosh^2\pi\nu}-\frac{8}{1+4\nu^2}+i\pi\frac{N_c^2-4}{N_c^2}\right]\right\}$$

Gluon operators of leading twist

$$\mathcal{O}_n^g \equiv F^a_{-i}\nabla_-^{n-2}F^{ai}_-$$

Anomalous dimension (in gluodynamics)

$$\gamma_n = \frac{2}{\pi}\alpha_s N_c\Big[-\frac{1}{n(n-1)} - \frac{1}{(n+1)(n+2)} + \psi(n+1) + \gamma_E - \frac{11}{12}\Big] + O(\alpha_s^2)$$

NLO BFKL and anomalous dimensions of light-ray operators

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BFKL gives $\gamma(n, \alpha_s)$ at the non-physical point $n \to 1$

$$\gamma_n = \Big[A_n^{\text{LO BFKL}} + \omega B_n^{\text{NLO BFKL}} + \ldots\Big]\Big(\frac{\alpha_s N_c}{\pi\omega}\Big)^n \qquad \omega \equiv n-1$$

LO: Jaroszewicz (1982), NLO: Lipatov, Fadin, Camici, Ciafaloni (1998)

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$\omega = n-1 \to 0$ corresponds to $F^a_{-i}\nabla_-^{\omega-1}F_-^{ai}$ - some non-local operator.

Q: which one? A: gluon light-ray (LR) operator

Light-ray operators and parton densities

Gluon light-ray (LR) operator of twist 2

$$F^a_{-i}(x'_+ + x_\perp)[x'_+, x_+]^{ab} F^{b\ i}_-(x_+ + x_\perp)$$

Forward matrix element - gluon parton density

$$z^\mu z^\nu \langle p|F^a_{\mu\xi}(z)[z,0]^{ab} F^{b\xi}_\nu(0)|p\rangle^\mu \stackrel{z^2=0}{=} 2(pz)^2 \int_0^1 dx_B\ x_B D_g(x_B,\mu)\cos(pz)x_B$$

Evolution equation (in gluodynamics)

$$\mu^2 \frac{d}{d\mu^2} F^a_{-i}(x'_+ + x_\perp)[x'_+, x_+]^{ab} F^{b\ i}_-(x_+ + x_\perp)$$
$$= \int_{x_+}^{x'_+} dz'_+ \int_{x_+}^{z'_+} dz_+\ K(x'_+, x_+; z'_+, z_+; \alpha_s) F^a_{-i}(z'_+ + x_\perp)[z'_+, z_+]^{ab} F^{b\ i}_-(z_+ + x_\perp)$$

"Forward" LR operator

$$F(L_+, x_\perp)\ =\ \int dx_+\ F^a_{-i}(L_+ + x_+ + x_\perp)[L_+ + x_+, x_+]^{ab} F^{bi}_-(x_+ + x_\perp)$$

Light-ray operators and parton densities

Expansion in ("forward") local operators

$$F(L_+,x_\perp) = \sum_{n=2}^{\infty} \frac{L_+^{n-2}}{(n-2)!}\mathcal{O}_n^g(x_\perp), \qquad \mathcal{O}_n^g \equiv \int dx_+ F^a_{-i}\nabla_-^{n-2}F_-^{ai}(x_+,x_\perp)$$

Evolution equation for $F(L_+,x_\perp)$

$$\mu\frac{d}{d\mu}F(L_+,x_\perp) = \int_0^1 du\, K_{gg}(u,\alpha_s)F(uL_+,x_\perp)$$

$$\Rightarrow \quad \gamma_n(\alpha_s) = -\int_0^1 du\, u^{n-2}K_{gg}(u,\alpha_s) \qquad \mu\frac{d}{d\mu}\mathcal{O}_n^g = -\gamma_n(\alpha_s)\mathcal{O}_n^g$$

$u^{-1}K_{gg}$ - DGLAP kernel

$$u^{-1}K_{gg}(u) = \frac{2\alpha_s N_c}{\pi}\Big(\bar{u}u + \Big[\frac{1}{\bar{u}u}\Big]_+ - 2 + \frac{11}{12}\delta(\bar{u})\Big) + \text{higher orders in } \alpha_s$$

Conformal LR operator ($j = \frac{1}{2} + i\nu$)

$$F^{\mu}(L_+, x_\perp) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} (L_+)^{-\frac{3}{2}+i\nu} \mathcal{F}^{\mu}_{\frac{1}{2}+i\nu}(x_\perp)$$

$$\mathcal{F}^{\mu}_{j}(x_\perp) = \int_0^{\infty} dL_+ \, L_+^{1-j} F^{\mu}(L_+, x_\perp)$$

Evolution equation for “forward” conformal light-ray operators

$$\Rightarrow \mu^2 \frac{d}{d\mu^2} \mathcal{F}_j(z_\perp) = \int_0^1 du \, K_{gg}(u, \alpha_s) u^{j-2} \mathcal{F}_j(z_\perp)$$

$\Rightarrow \gamma_j(\alpha_s)$ is an analytical continuation of $\gamma_n(\alpha_s)$

Supermultiplet of LR operators

Gluino and scalar LR operators

$$\Lambda(L_+,x_\perp) \;=\; \frac{i}{2}\int dx'_+\;[\bar\lambda^a(L_++x_++x_\perp)[x'_++x_+,x_+]^{ab}\sigma_-\nabla_-\lambda^b(x_++x_\perp)+\text{c.c}]$$

$$\Phi(L_+,x_\perp) \;=\; \int dx'_+\;\phi^{a,I}(L_++x_++x_\perp)[x'_++x_+,x_+]^{ab}\nabla_-^2\phi^{b,I}(x_++x_\perp)$$

$$\Lambda_j(x_\perp) \;=\; \int_0^\infty dL_+\;L_+^{-j+1}\Lambda(L_+,x_\perp),\qquad \Phi_j(x_\perp) \;=\; \int_0^\infty dL_+\;L_+^{-j+1}\Phi(L_+,x_\perp)$$

SU_4 singlet LR operators.

$$S_{1j}(x_\perp) \;=\; F_j(x_\perp)+\frac{j-1}{8}\Lambda_j(x_\perp)-\frac{j(j-1)}{8}\Phi_j(x_\perp)$$

$$S_{2j}(x_\perp) \;=\; F_j(x_\perp)-\frac{1}{8}\Lambda_j(x_\perp)+\frac{j(j+1)}{24}\Phi_j(x_\perp)$$

$$S_{3j}(x_\perp) \;=\; F_j(x_\perp)-\frac{j+2}{4}\Lambda_j(x_\perp)-\frac{(j+1)(j+2)}{8}\Phi_j(x_\perp)$$

All operators have the same anomalous dimension

$$\gamma_j^{S_1}(\alpha_s)\equiv\gamma_j(\alpha_s)=\frac{2\alpha_s}{\pi}N_c[\psi(j-1)+C]+O(\alpha_s^2),\qquad \gamma_j^{S_2}=\gamma_{j+1}^{S_1},\qquad \gamma_j^{S_3}=\gamma_{j+2}^{S_1}$$

Light-ray operators

Light-ray operators in (+) and (-) directions:

$$\Phi^+(L_+, x_\perp) \;=\; \int dx'_+ \;\nabla_-\phi^{a,I}(L_+ + x_+ + x_\perp)[x'_+ + x_+, x_+]^{ab}\nabla_-\phi^{b,I}(x_+ + x_\perp)$$
$$\Phi_j^+(x_\perp) \;=\; \int_0^\infty dL_+ \;L_+^{-j+1}\Phi(L_+, x_\perp)$$
$$\Phi^-(L_-, x_\perp) \;=\; \int dx'_- \;\nabla_+\phi^{a,I}(L_- + x_- + x_\perp)[x'_- + x_-, x_-]^{ab}\nabla_+\phi^{b,I}(x_- + x_\perp)$$
$$\Phi_j^-(x_\perp) \;=\; \int_0^\infty dL_- \;L_+^{-j+1}\Phi(L_-, x_\perp)$$

and similarly for Λ_j's and G_j's

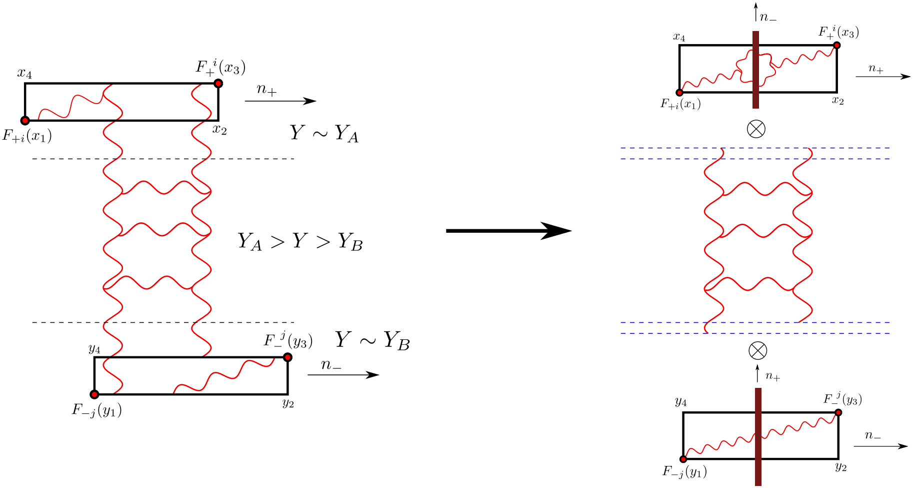
A general formula for the correlation function of two LR operators reads

$$\langle S^+_{j=\frac{3}{2}+i\nu}(x_\perp) S^-_{j'=\frac{3}{2}+i\nu'}(y_\perp)\rangle = \frac{\delta(\nu-\nu')a(j,\alpha_s)(\mu^2)^{-\gamma(j,\alpha_s)}}{((x-y)^2_\perp)^{j+1+\gamma(j,\alpha_s)}}$$

We need to reproduce it and find $\gamma(j,\alpha_s)$ at $j\to 1$.

Correlator of two "Wilson frames"

"Wilson frame": light-ray operator with point-splitting in the transverse direction



$$\langle S_+^{2+\omega_1}(x_{1\perp},x_{3\perp})S_-^{2+\omega_2}(y_{1\perp},y_{3\perp})\rangle \xrightarrow[x_{13\perp},\, y_{13\perp}\to 0]{} \delta(\omega_1-\omega_2)\Upsilon(\tilde\gamma)\frac{(x^2_{13\perp})^{\frac{\tilde\gamma}{2}-\frac{\omega}{2}}(y^2_{13\perp})^{\frac{\tilde\gamma}{2}-\frac{\omega}{2}}}{((x-y)^2_\perp)^{2+\tilde\gamma}},$$

$$\Upsilon(\tilde\gamma) = -N^2g^4\frac{2^{-1-2\tilde\gamma}\pi}{\tilde\gamma^2\Gamma^2(1-\frac{\tilde\gamma}{2})\Gamma^2(\frac{1}{2}+\frac{\tilde\gamma}{2})\sin(\pi\tilde\gamma)\hat\aleph'(\tilde\gamma)}$$

$\tilde\gamma = -1+2i\nu$ is the solution of $\omega = \hat\aleph(\tilde\gamma)$
We use the point-splitting regularization in the orthogonal direction for our light-ray operators $\Rightarrow$ cutoffs are defined as $\Lambda_x = \frac{1}{|x_{13\perp}|}$ and $\Lambda_y = \frac{1}{|y_{13\perp}|}$
Rewrite

$$\langle S_+^{2+\omega_1}(x_{1\perp},x_{3\perp})S_-^{2+\omega_2}(y_{1\perp},y_{3\perp})\rangle \xrightarrow[x_{13\perp},\, y_{13\perp}\to 0]{} \delta(\omega_1-\omega_2)\Upsilon(\gamma+\omega)\frac{(x^2_{13\perp})^{\frac{\gamma}{2}}(y^2_{13\perp})^{\frac{\gamma}{2}}}{((x-y)^2_\perp)^{2+\omega+\gamma}}.$$

The anomalous dimension $\gamma = \tilde\gamma - \omega$ satisfies

$$\omega = \hat\aleph(\gamma+\omega) = \hat\aleph(\gamma) + \hat\aleph'(\gamma)\hat\aleph(\gamma) + o(g^4).$$

- Lipatov-Fadin formula

Result

$$\gamma_j = -2\frac{\alpha_s N_c}{\pi(j-1)} \ + \ [0+\zeta(3)(j-1)]\big(\frac{\alpha_s N_c}{\pi(j-1)})\big)^3 + ...$$

is an anomalous dimension of light-ray operator $F\nabla^{j-2}F(x_\perp)$

Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.

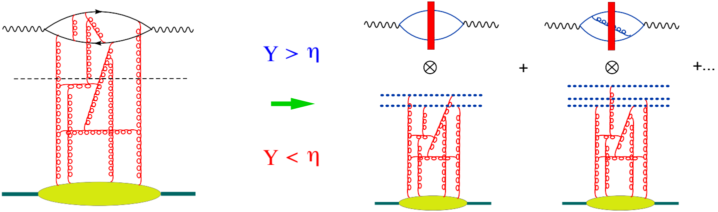
Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.
- The NLO BK kernel in for the evolution of conformal composite dipoles in $\mathcal{N} = 4$ SYM is Möbius invariant in the transverse plane.
- The NLO BK kernel agrees with NLO BFKL equation.
- The correlation function of four Z^2 operators is calculated at the NLO BFKL order.
- NLO BFKL gives anomalous dimensions of light-ray operators $F_{+i}\nabla^{\omega-1}F_{+}{}^{i}$ as $\omega \to 0$ (in all orders in α_s)

Conclusion and Outlook

1. Outlook
   - In QCD
   - 3-point CF of $\langle F\nabla^{\omega_1-1}F(x_{1_\perp})\ F\nabla^{\omega_2-1}F(x_{2_\perp})\ F\nabla^{\omega_3-1}F(x_{3_\perp})\rangle$ as $\omega_1,\omega_2,\omega_3\to 0$ (joint project with V. Kazakov and E. Sobko):

In QCD



DIS structure function $F_2(x)$: photon impact factor + evolution of color dipoles+ initial conditions for the small-x evolution

Photon impact factor in the LO

$$(x-y)^4 T\{\hat{\bar{\psi}}(x)\gamma^\mu\hat{\psi}(x)\hat{\bar{\psi}}(y)\gamma^\nu\hat{\psi}(y)\} = \int \frac{d^2z_1 d^2z_2}{z_{12}^4} I^{\rm LO}_{\mu\nu}(z_1,z_2) {\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

$$I^{\rm LO}_{\mu\nu}(z_1,z_2) = \frac{\mathcal{R}^2}{\pi^6(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}\frac{\partial^2}{\partial x^\mu\partial y^\nu}\Big[(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)-\frac{1}{2}\kappa^2(\zeta_1\cdot\zeta_2)\Big].$$

$$\kappa \equiv \frac{1}{\sqrt{s}x^+}\Big(\frac{p_1}{s}-x^2p_2+x_\perp\Big)-\frac{1}{\sqrt{s}y^+}\Big(\frac{p_1}{s}-y^2p_2+y_\perp\Big)$$

$$\zeta_i \equiv \Big(\frac{p_1}{s}+z_{i\perp}^2p_2+z_{i\perp}\Big), \qquad \mathcal{R} \equiv \frac{\kappa^2(\zeta_1\cdot\zeta_2)}{2(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}$$

Composite "conformal" dipole $[\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]_{a_0}$ - same as in $\mathcal{N}=4$ case.

$$
\begin{aligned}
&(x-y)^4 T\{\bar{\hat{\psi}}(x)\gamma^\mu\hat{\psi}(x)\bar{\hat{\psi}}(y)\gamma^\nu\hat{\psi}(y)\} \\
&= \int \frac{d^2z_1 d^2z_2}{z_{12}^4}\Big\{I^{\mu\nu}_{\mathrm{LO}}(z_1,z_2)\big[1+\frac{\alpha_s}{\pi}\big][\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]_{a_0} \\
&+ \int d^2z_3\Big[\frac{\alpha_s}{4\pi^2}\frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big(\ln\frac{\kappa^2(\zeta_1\cdot\zeta_3)(\zeta_1\cdot\zeta_3)}{2(\kappa\cdot\zeta_3)^2(\zeta_1\cdot\zeta_2)} - 2C\Big)I^{\mu\nu}_{\mathrm{LO}} + I^{\mu\nu}_2\Big] \\
&\qquad\times [\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\}\mathrm{tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_2}\} - N_c\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]_{a_0}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
(I_2)_{\mu\nu}(z_1,z_2,z_3) &= \frac{\alpha_s}{16\pi^8}\frac{\mathcal{R}^2}{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}\Big\{\frac{(\kappa\cdot\zeta_2)}{(\kappa\cdot\zeta_3)}\frac{\partial^2}{\partial x^\mu\partial y^\nu}\Big[-\frac{(\kappa\cdot\zeta_1)^2}{(\zeta_1\cdot\zeta_3)} \\
&+\frac{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}{(\zeta_2\cdot\zeta_3)} + \frac{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_3)(\zeta_1\cdot\zeta_2)}{(\zeta_1\cdot\zeta_3)(\zeta_2\cdot\zeta_3)} - \frac{\kappa^2(\zeta_1\cdot\zeta_2)}{(\zeta_2\cdot\zeta_3)}\Big] \\
&+\frac{(\kappa\cdot\zeta_2)^2}{(\kappa\cdot\zeta_3)^2}\frac{\partial^2}{\partial x^\mu\partial y^\nu}\Big[\frac{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_3)}{(\zeta_2\cdot\zeta_3)} - \frac{\kappa^2(\zeta_1\cdot\zeta_3)}{2(\zeta_2\cdot\zeta_3)}\Big] + (\zeta_1\leftrightarrow\zeta_2)\Big\}
\end{aligned}
$$

NLO impact factor for DIS

$$I^{\mu\nu}(q,k_\perp) = \frac{N_c}{32}\int\frac{d\nu}{\pi\nu}\frac{\sinh\pi\nu}{(1+\nu^2)\cosh^2\pi\nu}\Big(\frac{k_\perp^2}{Q^2}\Big)^{\frac{1}{2}-i\nu}$$
$$\times\Big\{\Big[\Big(\frac{9}{4}+\nu^2\Big)\Big(1+\frac{\alpha_s}{\pi}+\frac{\alpha_s N_c}{2\pi}\mathcal{F}_1(\nu)\Big)P_1^{\mu\nu} + \Big(\frac{11}{4}+3\nu^2\Big)\Big(1+\frac{\alpha_s}{\pi}+\frac{\alpha_s N_c}{2\pi}\mathcal{F}_2(\nu)\Big)P_2^{\mu\nu}\Big]$$

$$P_1^{\mu\nu} = g^{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \qquad P_2^{\mu\nu} = \frac{1}{q^2}\Big(q^\mu - \frac{p_2^\mu q^2}{q\cdot p_2}\Big)\Big(q^\nu - \frac{p_2^\nu q^2}{q\cdot p_2}\Big)$$

$$\mathcal{F}_{1(2)}(\nu) = \Phi_{1(2)}(\nu) + \chi_\gamma\Psi(\nu),$$
$$\Psi(\nu) \equiv \psi(\bar\gamma) + 2\psi(2-\gamma) - 2\psi(4-2\gamma) - \psi(2+\gamma), \qquad \gamma\equiv\frac{1}{2}+i\nu$$

$$\Phi_1(\nu) = F(\gamma) + \frac{3\chi_\gamma}{2+\bar\gamma\gamma} + 1 + \frac{25}{18(2-\gamma)} + \frac{1}{2\bar\gamma} - \frac{1}{2\gamma} - \frac{7}{18(1+\gamma)} + \frac{10}{3(1+\gamma)^2}$$
$$\Phi_2(\nu) = F(\gamma) + \frac{3\chi_\gamma}{2+\bar\gamma\gamma} + 1 + \frac{1}{2\bar\gamma\gamma} - \frac{7}{2(2+3\bar\gamma\gamma)} + \frac{\chi_\gamma}{1+\gamma} + \frac{\chi_\gamma(1+3\gamma)}{2+3\bar\gamma\gamma}$$

$$F(\gamma) = \frac{2\pi^2}{3} - \frac{2\pi^2}{\sin^2\pi\gamma} - 2C\chi_\gamma + \frac{\chi_\gamma-2}{\bar\gamma\gamma}$$

NLO evolution of composite "conformal" dipoles in QCD

I. B. and G. Chirilli

$$
\begin{aligned}
a\frac{d}{da}[\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}]^{\rm comp}_a \;=\;& \frac{\alpha_s}{2\pi^2}\int d^2z_3\,\Big([\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}]^{\rm comp}_a\\
&\times\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\Big(b\ln z_{12}^2\mu^2+b\frac{z_{13}^2-z_{23}^2}{z_{13}^2z_{23}^2}\ln\frac{z_{13}^2}{z_{23}^2}+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big]\\
&+\frac{\alpha_s}{4\pi^2}\int\frac{d^2z_4}{z_{34}^4}\Big\{\Big[-2+\frac{z_{23}{}^2z_{23}^2+z_{24}^2z_{13}^2-4z_{12}^2z_{34}^2}{2(z_{23}{}^2z_{23}^2-z_{24}^2z_{13}^2)}\ln\frac{z_{23}{}^2z_{23}^2}{z_{24}^2z_{13}^2}\Big]\\
&\times[\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}\{U_{z_4}U^\dagger_{z_2}\}-\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}U_{z_4}U^\dagger_{z_2}U_{z_3}U^\dagger_{z_4}\}-(z_4\to z_3)]\\
&+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[2\ln\frac{z_{12}^2z_{34}^2}{z_{23}{}^2z_{23}^2}+\Big(1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}{}^2z_{23}^2}\Big)\ln\frac{z_{13}^2z_{24}^2}{z_{23}{}^2z_{23}^2}\Big]\\
&\times[\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}\mathrm{tr}\{U_{z_4}U^\dagger_{z_2}\}-\mathrm{tr}\{U_{z_1}U^\dagger_{z_4}U_{z_3}U^\dagger_{z_2}U_{z_4}U^\dagger_{z_3}\}-(z_4\to z_3)]\Big\}
\end{aligned}
$$

$$b=\tfrac{11}{3}N_c-\tfrac{2}{3}n_f$$

$K_{\rm NLO\ BK}$ = Running coupling part + Conformal "non-analytic" (in j) part + Conformal analytic ($\mathcal{N}=4$) part

Linearized $K_{\rm NLO\ BK}$ reproduces the known result for the forward NLO BFKL kernel.

Argument of coupling constant

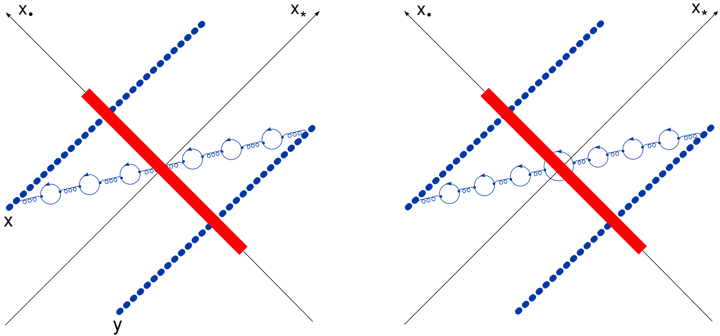
$$\frac{d}{d\eta}\hat{\mathcal{U}}(z_1, z_2) =$$

$$\frac{\alpha_s(?_\perp)N_c}{2\pi^2}\int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big\{\hat{\mathcal{U}}(z_1, z_3) + \hat{\mathcal{U}}(z_3, z_2) - \hat{\mathcal{U}}(z_1, z_2) - \hat{\mathcal{U}}(z_1, z_3)\hat{\mathcal{U}}(z_3, z_2)\Big\}$$

Argument of coupling constant

$$\frac{d}{d\eta}\hat{\mathcal{U}}(z_1,z_2) =$$
$$\frac{\alpha_s(?_\perp)N_c}{2\pi^2}\int dz_3\,\frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big\{\hat{\mathcal{U}}(z_1,z_3)+\hat{\mathcal{U}}(z_3,z_2)-\hat{\mathcal{U}}(z_1,z_2)-\hat{\mathcal{U}}(z_1,z_3)\hat{\mathcal{U}}(z_3,z_2)\Big\}$$

Renormalon-based approach: summation of quark bubbles



$$-\tfrac{2}{3}n_f \to b = \tfrac{11}{3}N_c - \tfrac{2}{3}n_f$$

Argument of coupling constant (rcBK)

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat U_{z_1}\hat U^\dagger_{z_2}\} = \frac{\alpha_s(z_{12}^2)}{2\pi^2}\int d^2z\,[\mathrm{Tr}\{\hat U_{z_1}\hat U^\dagger_{z_3}\}\mathrm{Tr}\{\hat U_{z_3}\hat U^\dagger_{z_2}\} - N_c\mathrm{Tr}\{\hat U_{z_1}\hat U^\dagger_{z_2}\}]$$

$$\times\Big[\frac{z_{12}^2}{z_{13}^2 z_{23}^2} + \frac{1}{z_{13}^2}\Big(\frac{\alpha_s(z_{13}^2)}{\alpha_s(z_{23}^2)} - 1\Big) + \frac{1}{z_{23}^2}\Big(\frac{\alpha_s(z_{23}^2)}{\alpha_s(z_{13}^2)} - 1\Big)\Big] + \ldots$$

I.B.; Yu. Kovchegov and H. Weigert (2006)

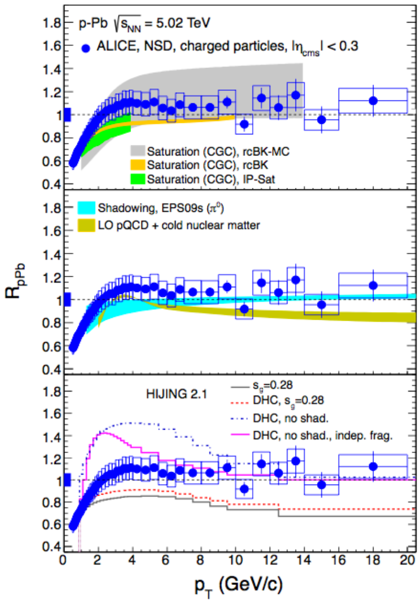
When the sizes of the dipoles are very different the kernel reduces to:

$$\frac{\alpha_s(z_{12}^2)}{2\pi^2}\frac{z_{12}^2}{z_{13}^2 z_{23}^2} \qquad |z_{12}| \ll |z_{13}|, |z_{23}|$$

$$\frac{\alpha_s(z_{13}^2)}{2\pi^2 z_{13}^2} \qquad |z_{13}| \ll |z_{12}|, |z_{23}|$$

$$\frac{\alpha_s(z_{23}^2)}{2\pi^2 z_{23}^2} \qquad |z_{23}| \ll |z_{12}|, |z_{13}|$$

$\Rightarrow$ the argument of the coupling constant is given by the size of the smallest dipole.



ALICE arXiv:1210.4520

Nuclear modification factor

$$R^{pPb}(p_T) = \frac{d^2N^{pPb}_{\rm ch}/d\eta dp_T}{\langle T_{pPb}\rangle d^2\sigma^{\rm pp}_{\rm ch}/d\eta dp_T}$$

$N^{pPb} \equiv$ charged particle yield in p-Pb collisions.

Conclusions 2

- BK equation is a correct form of non-linear GLR equation for parton saturation
- The NLO BK kernel in QCD is a sum of running-coupling part, $\mathcal{N} = 4$ part and additional conformal part.
- The NLO BK kernel agrees with NLO BFKL equation.
- NLO photon impact factor is calculated.
- Argument of α_s in the BK equation is the size of the smallest dipole.