Grassmannian Geometry and the Analytic S-Matrix

Jacob L. Bourjaily
Niels Bohr Institute

based on work in collaboration with

N. Arkani-Hamed, F. Cachazo, A. Goncharov, A. Postnikov, and J. Trnka

[arXiv:1212.5605], [arXiv:1212.6974]

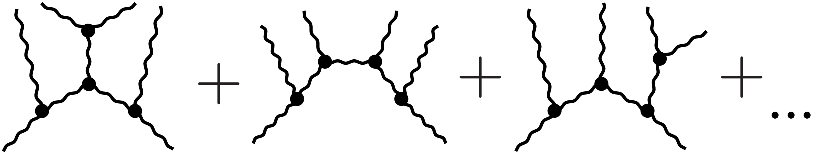
Organization and Outline

1. Spiritus Movens
    - A Simple, Practical Problem in Quantum Chromodynamics
    - The *Shocking* Simplicity of Scattering Amplitudes (a parable)
2. A New Class of Physical Observables: On-Shell Functions
    - Beyond (Mere) Scattering Amplitudes: On-Shell Functions
    - Basic Building Blocks: S-Matrices for Three Massless Particles
    - *Grassmannian* Representations of On-Shell Functions
3. The On-Shell Analytic S-Matrix: All-Loop Recursion Relations
    - Building-up Diagrams with "BCFW" Bridges
    - On-Shell (Recursive) Representations of Scattering Amplitudes
    - *Exempli Gratia*: On-Shell Manifestations of Tree Amplitudes
4. The *Combinatorial* Simplicity of Planar $\mathcal{N}=4$ SYM
    - Combinatorial Classification of On-Shell Functions in $\mathcal{N}=4$
    - Canonical Coordinates, Computation, and the Positroid Stratification

Supercomputer Computations in Quantum Chromodynamics

Consider the amplitude for two gluons to collide and produce four: $gg \to gggg$.
Before modern computers, this would have been computationally intractable

- 220 Feynman diagrams, thousands of terms



Supercollider physics

E. Eichten
Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510

I. Hinchliffe
Lawrence Berkeley Laboratory, Berkeley, California 94720

K. Lane
The Ohio State University, Columbus, Ohio 43210

C. Quigg
Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510

Eichten *et al.* summarize the motivation for exploring the 1-TeV ($=10^{12}$ eV) energy scale in elementary particle interactions and explore the capabilities of proton-(anti)proton colliders with beam energies between 1 and 50 TeV. The authors calculate the production rates and characteristics for a number of conventional processes, and discuss their intrinsic physics interest as well as their role as backgrounds to more exotic phenomena. The authors review the theoretical motivation and expected signatures for several new phenomena which may occur on the 1-TeV scale. Their results provide a reference point for the choice of machine parameters and for experiment design.

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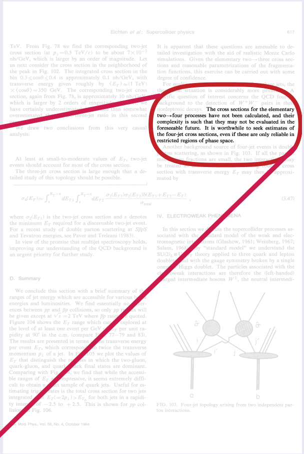
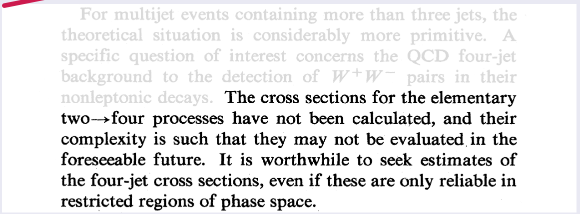

Eichten et al.: Supercollider physics 617

The cross sections for the elementary two→four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future. It is worthwhile to seek estimates of the four-jet cross sections, even if these are only reliable in restricted regions of phase space.

FIG. 103. Four-jet topology arising from two independent parton interactions.

Rev. Mod. Phys., Vol. 56, No. 4, October 1984


For multijet events containing more than three jets, the theoretical situation is considerably more primitive. A specific question of interest concerns the QCD four-jet background to the detection of W^+W^- pairs in their nonleptonic decays. The cross sections for the elementary two→four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future. It is worthwhile to seek estimates of the four-jet cross sections, even if these are only reliable in restricted regions of phase space.

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- and the world's best supercomputers
- final formula fit into 8 pages

THE CROSS SECTION FOR FOUR-GLUON PRODUCTION BY GLUON–GLUON FUSION

Stephen J. PARKE and T.R. TAYLOR

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, IL 60510 USA

Received 13 September 1985

The cross section for two-gluon to four-gluon scattering is given in a form suitable for fast numerical calculations.

of our calculation, the most powerful test does not rely on the gauge symmetry, but on the appropriate permutation symmetries. The function $A_0(p_1, p_2, p_3, p_4, p_5, p_6)$ must be symmetric under arbitrary permutations of the momenta (p_1, p_2, p_3) and separately, (p_4, p_5, p_6), whereas the function $A_2(p_1, p_2, p_3, p_4, p_5, p_6)$ must be symmetric under the permutations of (p_1, p_2, p_3, p_4) and separately, (p_5, p_6). This test is extremely powerful, because the required permutation symmetries are hidden in our supersymmetry relations, eqs. (1) and (3), and in the structure of amplitudes involving different species of particles. Another, very important test relies on the absence of the double poles of the form $(s_{ij})^{-2}$ in the cross section, as required by general arguments based on the helicity conservation. Further, in the leading $(s_{ij})^{-1}$ pole approximation, the answer should reduce to the two goes to three cross section [3, 4], convoluted with the appropriate Altarelli-Parisi probabilities [5]. Our result has succesfully passed both these numerical checks.

Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight.

We thank Keith Ellis, Chris Quigg and especially, Estia Eichten for many useful discussions and encouragement during the course of this work. We acknowledge the hospitality of Aspen Center for Physics, where this work was being completed in a pleasant, strung-out atmosphere.

References

[1] E. Eichten, I. Hinchliffe, K. Lane and C. Quigg, Rev. Mod. Phys. 56 (1984) 579
[2] Z. Kunszt, Nucl. Phys. B247 (1984) 339
[3] S.J. Parke and T.R. Taylor, Phys. Lett. 157B (1985) 81
[4] T. Gottschalk and D. Sivers, Phys. Rev. D21 (1980) 102;
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[5] G. Altarelli and G. Parisi, Nucl. Phys. B126 (1977) 298

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420 *S.J. Parke, T.R. Taylor / Four gluon production*

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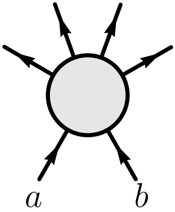
The Discovery of Incredible, Unanticipated Simplicity

They soon **guessed** a simplified form of the amplitude (checked numerically):
—which naturally suggested the amplitude for **all** multiplicity!



$$= \frac{\langle a\,b\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle\langle 5\,6\rangle\langle 6\,1\rangle}\,\delta^{2\times 2}(\lambda\cdot\widetilde{\lambda})$$

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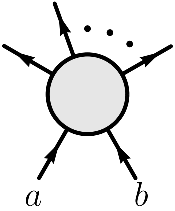

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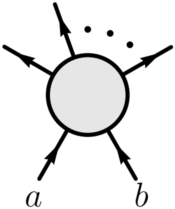

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Here, we have used **spinor variables** to describe the external momenta:

$$p_a^\mu \mapsto p_a^{\alpha\dot{\alpha}} \equiv p_a^\mu \sigma_\mu^{\alpha\dot{\alpha}} = \begin{pmatrix} p_a^0 + p_a^3 & p_a^1 - i p_a^2 \\ p_a^1 + i p_a^2 & p_a^0 - p_a^3 \end{pmatrix} \equiv \lambda_a^\alpha \widetilde{\lambda}_a^{\dot{\alpha}} \Leftrightarrow \text{“}\,a\rangle[a\,\text{”}$$

The (local) Lorentz group, $SL(2)_L \times SL(2)_R$, acts on λ_a and $\widetilde{\lambda}_a$, respectively.

Thus, Lorentz invariants must be constructed out of determinants:

$$\langle a\,b\rangle \equiv \det(\lambda_a, \lambda_b), \qquad [a\,b] \equiv \det(\widetilde{\lambda}_a, \widetilde{\lambda}_b)$$

The action of the **little group** corresponds to:

$$(\lambda_a, \widetilde{\lambda}_a) \mapsto (t_a\,\lambda_a, t_a^{-1}\widetilde{\lambda}_a) \colon \Psi_a^{h_a} \mapsto t_a^{-2h_a}\Psi_a^{h_a}$$

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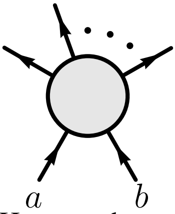

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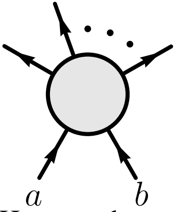

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The **Grassmannian** $G(k,n)$: the linear span of k vectors in $\mathbb{C}^n$.

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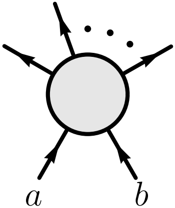

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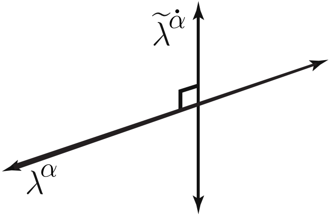
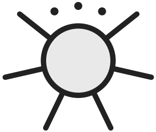
Momentum conservation becomes the geometric statement: $\lambda \subset \widetilde{\lambda}^{\perp}$ and $\widetilde{\lambda} \subset \lambda^{\perp}$.



Broadening the Class of Physically Meaningful Functions

We are interested in the class of functions involving **only** observable quantities



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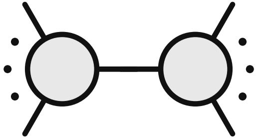

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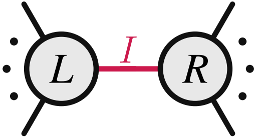

Internal Particles: locality dictates that we multiply each amplitude, and unitarity dictates that we marginalize over unobserved states—integrating over the Lorentz-invariant phase space ("LIPS") for each particle I, and summing over the possible states (helicities, masses, colours, etc.).

$$\sum_{\text{states } I} \int d^3\text{LIPS}_I \; \mathcal{A}_L(\ldots, I) \times \mathcal{A}_R(I, \ldots)$$

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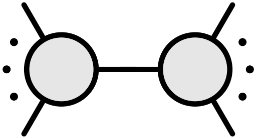

On-Shell Functions: networks of amplitudes, $\mathcal{A}_v$, connected by any number of internal particles, $i \in I$, forming a graph Γ called an "on-shell diagram".

$$f_\Gamma \equiv \prod_{i \in I} \Bigg(\sum_{\substack{h_i, c_i,\\ m_i, \cdots}} \int d^3\text{LIPS}_i \Bigg) \prod_v \mathcal{A}_v$$

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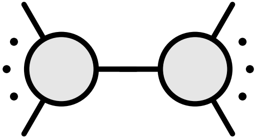

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Counting Constraints:

$$\widehat{n}_\delta \equiv 4\times n_V - 3\times n_I - 4 \;=\; \text{number of excess } \delta\text{-functions}$$
$$(= \text{minus number of remaining integrations})$$

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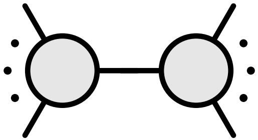

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$$\widehat{n}_\delta \equiv 4\times n_V - 3\times n_I - 4 \begin{array}{lcl} > 0 & \Rightarrow & (\widehat{n}_\delta)\text{ kinematical constraints} \\ = 0 & \Rightarrow & \text{ordinary (rational) function} \\ < 0 & \Rightarrow & (-\widehat{n}_\delta)\text{ non-trivial integrations} \end{array}$$

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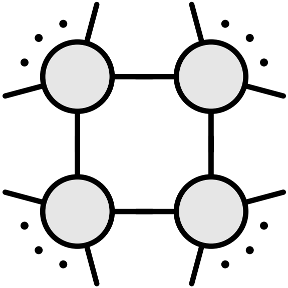

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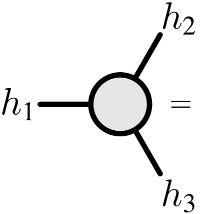
Momentum conservation and Poincarè-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).



$$= f(\lambda_1\widetilde{\lambda}_1, \lambda_2\widetilde{\lambda}_2, \lambda_3\widetilde{\lambda}_3)\delta^{2\times 2}(\lambda\cdot\widetilde{\lambda})$$

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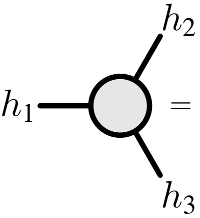

$$= f(\lambda_1\widetilde{\lambda}_1, \lambda_2\widetilde{\lambda}_2, \lambda_3\widetilde{\lambda}_3)\delta^{2\times 2}(\lambda\cdot\widetilde{\lambda}) \Rightarrow \begin{cases} \lambda^{\perp} \equiv \left(\langle 23\rangle \; \langle 31\rangle \; \langle 12\rangle\right) \supset \widetilde{\lambda} \\ \lambda \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix} \\ \\ \text{or} \\ \\ \widetilde{\lambda} \equiv \begin{pmatrix} \widetilde{\lambda}_1^{\dot{1}} & \widetilde{\lambda}_2^{\dot{1}} & \widetilde{\lambda}_3^{\dot{1}} \\ \widetilde{\lambda}_1^{\dot{2}} & \widetilde{\lambda}_2^{\dot{2}} & \widetilde{\lambda}_3^{\dot{2}} \end{pmatrix} \\ \widetilde{\lambda}^{\perp} \equiv \left([23] \; [31] \; [12]\right) \supset \lambda \end{cases}$$

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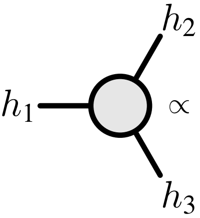

$$\propto \begin{cases} f(\lambda_1, \lambda_2, \lambda_3) \\ \\ f(\widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{\lambda}_3) \end{cases}$$

$$\lambda^{\perp} \equiv \big(\langle 23\rangle \, \langle 31\rangle \, \langle 12\rangle\big) \supset \widetilde{\lambda}$$

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or

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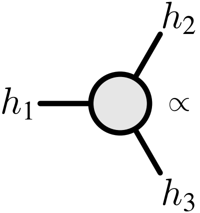

$$\propto \left\{ \begin{array}{l} \langle 12\rangle^{h_3-h_1-h_2}\langle 23\rangle^{h_1-h_2-h_3}\langle 31\rangle^{h_2-h_3-h_1} \\ \\ \\ [12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[31]^{h_3+h_1-h_2} \end{array} \right.$$

$$\lambda^{\perp} \equiv \left(\langle 23\rangle\;\langle 31\rangle\;\langle 12\rangle\right) \supset \tilde{\lambda}$$

$$\lambda \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix}$$

or

$$\tilde{\lambda} \equiv \begin{pmatrix} \tilde{\lambda}_1^{\dot{1}} & \tilde{\lambda}_2^{\dot{1}} & \tilde{\lambda}_3^{\dot{1}} \\ \tilde{\lambda}_1^{\dot{2}} & \tilde{\lambda}_2^{\dot{2}} & \tilde{\lambda}_3^{\dot{2}} \end{pmatrix}$$

$$\tilde{\lambda}^{\perp} \equiv \left([23]\;[31]\;[12]\right) \supset \lambda$$

Building Blocks: the S-Matrix for Three Massless Particles

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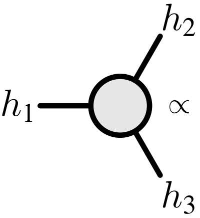

$$\propto \begin{cases} \langle 12\rangle^{h_3-h_1-h_2}\langle 23\rangle^{h_1-h_2-h_3}\langle 31\rangle^{h_2-h_3-h_1} \xrightarrow[\langle a\,b\rangle\to\mathcal{O}(\epsilon)]{} \mathcal{O}\left(\epsilon^{-(h_1+h_2+h_3)}\right) \\[2em] [12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[31]^{h_3+h_1-h_2} \xrightarrow[[a\,b]\to\mathcal{O}(\epsilon)]{} \mathcal{O}\left(\epsilon^{(h_1+h_2+h_3)}\right) \end{cases}$$

$$\lambda^{\perp} \equiv \left(\langle 23\rangle\;\langle 31\rangle\;\langle 12\rangle\right) \supset \widetilde{\lambda}$$

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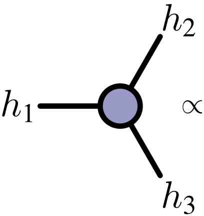

$$\propto \quad \langle 12\rangle^{h_3-h_1-h_2}\langle 23\rangle^{h_1-h_2-h_3}\langle 31\rangle^{h_2-h_3-h_1}$$
$$h_1+h_2+h_3\leq 0$$

$$\lambda^{\perp}\equiv\left(\langle 23\rangle\;\langle 31\rangle\;\langle 12\rangle\right)\supset\widetilde{\lambda}$$
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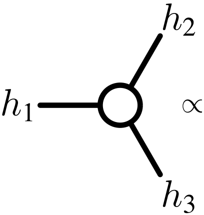
or



$$\propto \quad [12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[31]^{h_3+h_1-h_2}$$
$$h_1+h_2+h_3\geq 0$$

$$\widetilde{\lambda}\equiv\begin{pmatrix}\widetilde{\lambda}_1^{\dot{1}} & \widetilde{\lambda}_2^{\dot{1}} & \widetilde{\lambda}_3^{\dot{1}}\\ \widetilde{\lambda}_1^{\dot{2}} & \widetilde{\lambda}_2^{\dot{2}} & \widetilde{\lambda}_3^{\dot{2}}\end{pmatrix}$$
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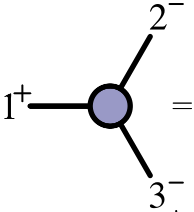
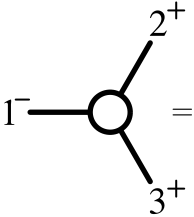


$$= \frac{\langle 2\,3\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle}\,\delta^{2\times 2}(\lambda\cdot\widetilde{\lambda})$$



$$= \frac{[2\,3]^4}{[1\,2]\,[2\,3]\,[3\,1]}\,\delta^{2\times 2}(\lambda\cdot\widetilde{\lambda})$$

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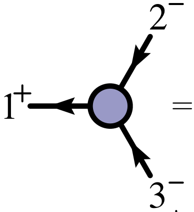

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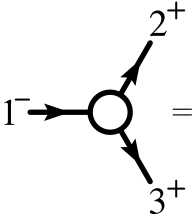

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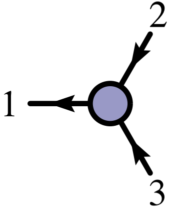
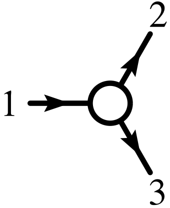


$$= \frac{\langle 2\,3\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle}\,\delta^{2\times 2}(\lambda\cdot\widetilde{\lambda}) \equiv \mathcal{A}_3(+,-,-)$$



$$= \frac{[2\,3]^4}{[1\,2]\,[2\,3]\,[3\,1]}\,\delta^{2\times 2}(\lambda\cdot\widetilde{\lambda}) \equiv \mathcal{A}_3(-,+,+)$$

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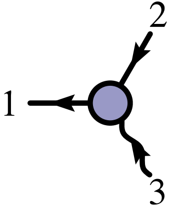
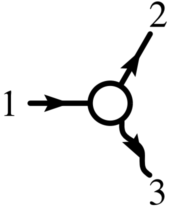


$$= \quad \frac{\langle 3\,1\rangle\langle 2\,3\rangle^3}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle}\,\delta^{2\times 2}(\lambda\cdot\widetilde{\lambda}) \equiv \mathcal{A}_3\left(+\tfrac{1}{2},-\tfrac{1}{2},-\right)$$



$$= \quad \frac{[3\,1][2\,3]^3}{[1\,2]\,[2\,3]\,[3\,1]}\,\delta^{2\times 2}(\lambda\cdot\widetilde{\lambda}) \equiv \mathcal{A}_3\left(-\tfrac{1}{2},+\tfrac{1}{2},+\right)$$

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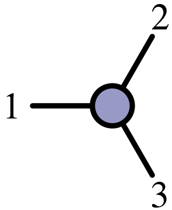
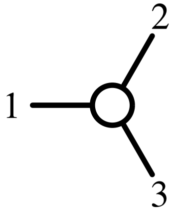


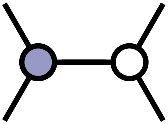
$$= \frac{\delta^{2\times4}(\lambda\cdot\widetilde{\eta})}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle}\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda}) \equiv \mathcal{A}_3^{(2)}$$



$$= \frac{\delta^{1\times4}(\widetilde{\lambda}^{\perp}\cdot\widetilde{\eta})}{[1\,2]\,[2\,3]\,[3\,1]}\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda}) \equiv \mathcal{A}_3^{(1)}$$

Amalgamating Diagrams from Three-Particle Amplitudes

On-shell diagrams constructed out of three-particle vertices are well-defined to **all orders of perturbation theory**, generating a large class of functions:



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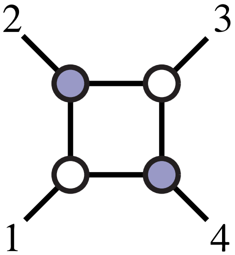

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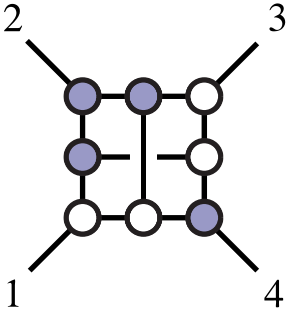

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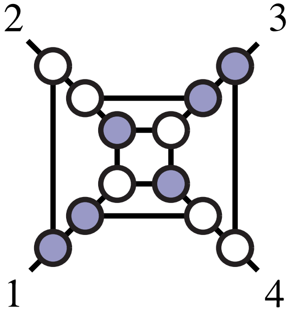

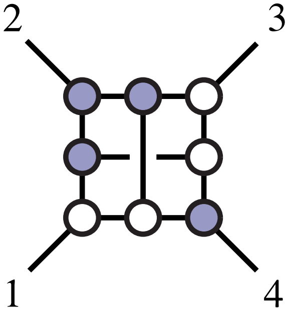

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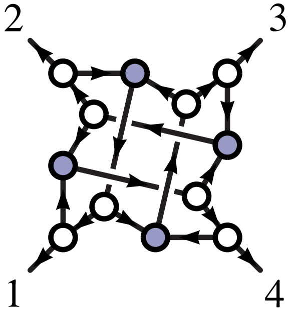

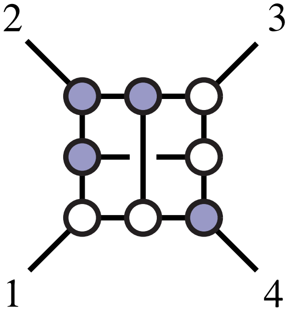

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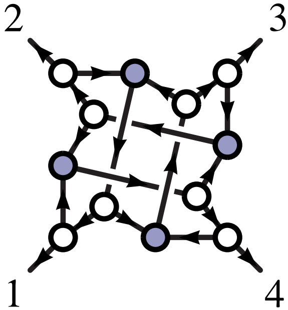

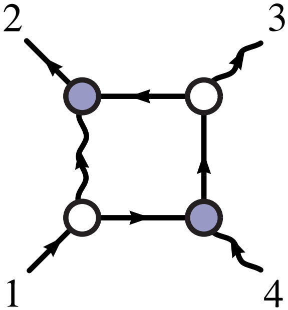

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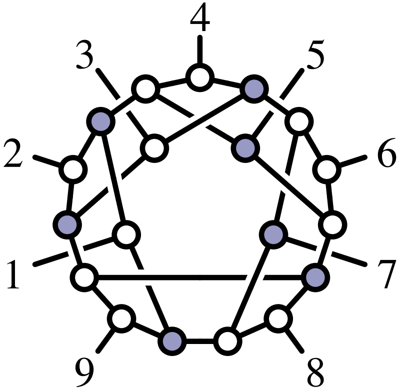
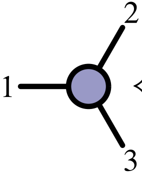


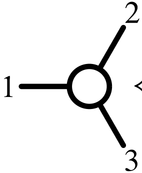
$$= \frac{(\langle 91\rangle\langle 23\rangle\langle 46\rangle - \langle 16\rangle\langle 34\rangle\langle 29\rangle)^2 \quad \delta^{2\times 2}\big(\lambda\cdot\widetilde{\lambda}\big)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 67\rangle\langle 78\rangle\langle 81\rangle\langle 14\rangle\langle 42\rangle\langle 29\rangle\langle 96\rangle\langle 63\rangle\langle 39\rangle\langle 91\rangle}$$

Grassmannian Representations of Three-Point Amplitudes

In order to **linearize** momentum conservation at each three-particle vertex, (and to specify *which* of the solutions to three-particle kinematics to use) we introduce **auxiliary** $B\in G(2,3)$ and $W\in G(1,3)$ for each vertex:



$$\Leftrightarrow B\equiv\begin{pmatrix} b_1^1 & b_2^1 & b_3^1 \\ b_1^2 & b_2^2 & b_3^2 \end{pmatrix}$$



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$$\mathcal{A}_3^{(2)}=\frac{\delta^{2\times4}(\lambda\cdot\widetilde{\eta})}{\langle12\rangle\langle23\rangle\langle31\rangle}\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})\equiv\int\frac{d^{2\times3}B}{\mathrm{vol}(GL_2)}\,\frac{\delta^{2\times4}(B\cdot\widetilde{\eta})}{(12)(23)(31)}\,\delta^{2\times2}(B\cdot\widetilde{\lambda})\,\delta^{1\times2}(\lambda\cdot B^{\perp})$$

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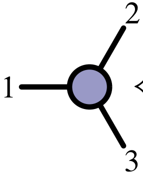

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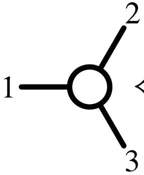

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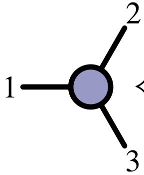
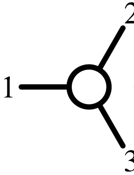


$$\Leftrightarrow B \equiv \begin{pmatrix} 1 & 0 & b_3^1 \\ 0 & 1 & b_3^2 \end{pmatrix}$$



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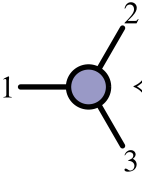

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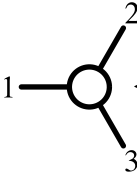

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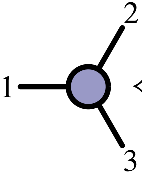

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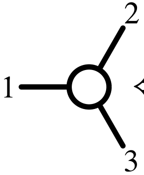

$$\Leftrightarrow W \equiv \begin{pmatrix} w_1^1 & w_2^1 & 1 \end{pmatrix}$$

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$$\mathcal{A}_3^{(1)} = \frac{\delta^{1\times4}(\widetilde{\lambda}^{\perp}\cdot\widetilde{\eta})}{[12][23][31]}\delta^{2\times2}(\lambda\cdot\widetilde{\lambda}) \equiv \int \frac{d\,w_1^1}{w_1^1}\wedge\frac{d\,w_2^1}{w_2^1}\;\delta^{1\times4}(W\cdot\widetilde{\eta})\;\;\delta^{1\times2}(W\cdot\widetilde{\lambda})\,\delta^{2\times2}(\lambda\cdot W^{\perp})$$

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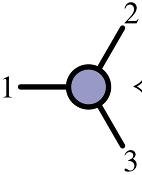

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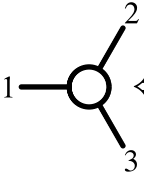

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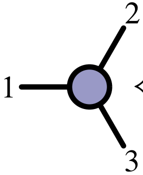

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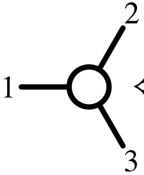

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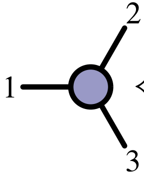

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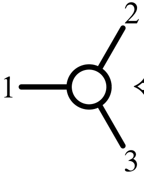

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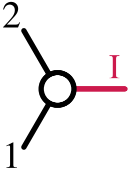
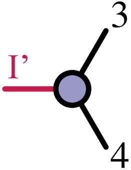


$$\begin{array}{ccc} 1 & 2 & \mathrm{I} \end{array}$$
$$\begin{pmatrix} 1 & w_2 & w_{\mathrm{I}} \end{pmatrix}$$



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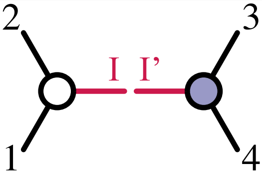

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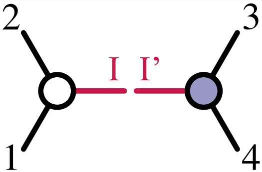

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & \mathrm{I} & \mathrm{I'} & 3 & 4 \end{array} \\ \left(\begin{array}{cccccc} 1 & w_2 & w_{\mathrm{I}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & b_4^1 \\ 0 & 0 & 0 & 0 & 1 & b_4^2 \end{array}\right) \end{array}$$

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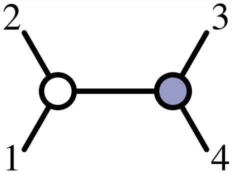

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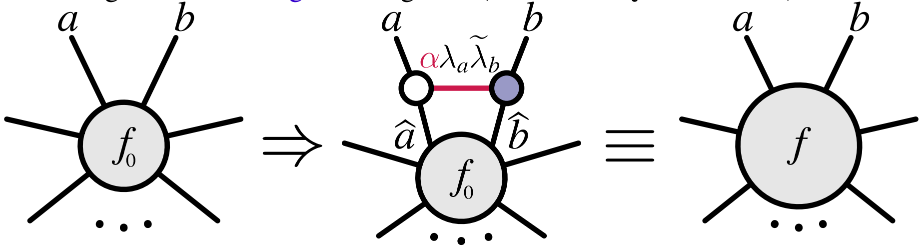
Building-Up On-Shell Diagrams with "BCFW" Bridges

Very complex on-shell diagrams can be constructed by successively adding "BCFW" bridges to diagrams (an **extremely** useful tool!):



Adding the bridge has the effect of shifting the momenta p_a and p_b flowing into the diagram f_0 according to:

$$\lambda_a\widetilde{\lambda}_a \mapsto \lambda_{\widehat{a}}\widetilde{\lambda}_{\widehat{a}} = \lambda_a\big(\widetilde{\lambda}_a - \alpha\widetilde{\lambda}_b\big) \quad \text{and} \quad \lambda_b\widetilde{\lambda}_b \mapsto \lambda_{\widehat{b}}\widetilde{\lambda}_{\widehat{b}} = \big(\lambda_b + \alpha\lambda_a\big)\widetilde{\lambda}_b,$$

introducing a new parameter α, in terms of which we may write:

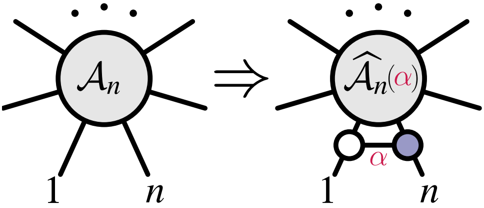
$$f(\ldots,a,b,\ldots) = \frac{d\alpha}{\alpha} f_0(\ldots,\widehat{a},\widehat{b},\ldots)$$

The Analytic Boot-Strap: All-Loop Recursion Relations

Consider adding a BCFW bridge to the full n-particle scattering amplitude the undeformed amplitude $\mathcal{A}_n$ is recovered as the **residue** about $\alpha=0$:

$$\mathcal{A}_n = \widehat{\mathcal{A}}_n(\alpha\to 0) \propto \oint\limits_{\alpha=0} \frac{d\alpha}{\alpha}\, \widehat{\mathcal{A}}_n(\alpha)$$

We can use **Cauchy's theorem** to trade the residue about $\alpha=0$ for (minus) the sum of residues away from the origin:



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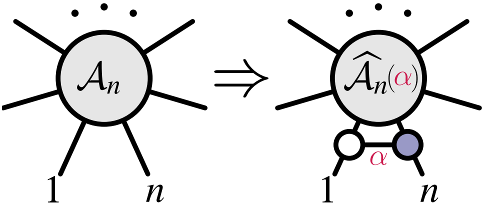

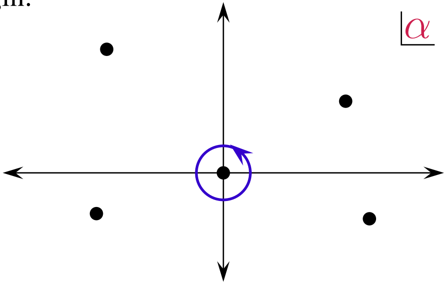

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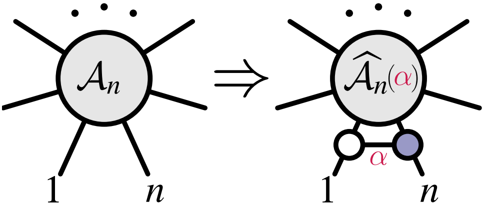

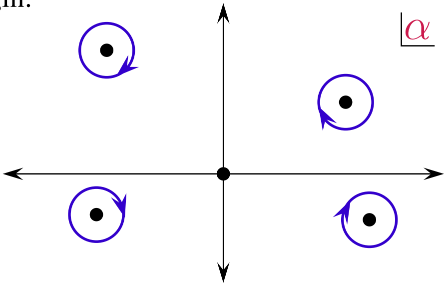

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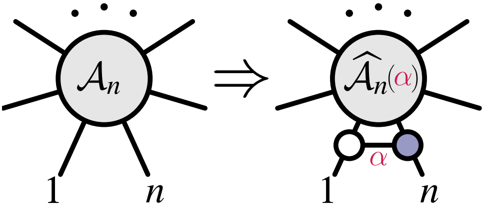

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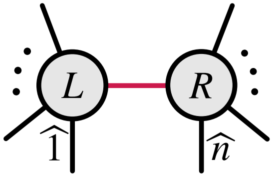

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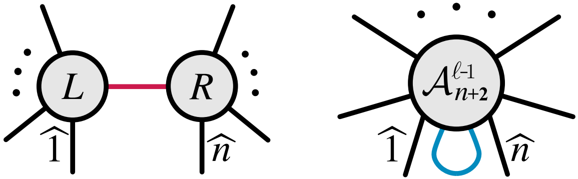

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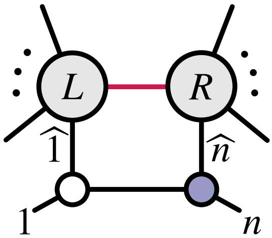

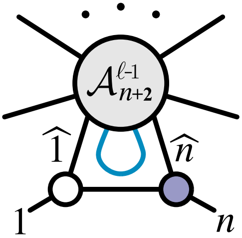

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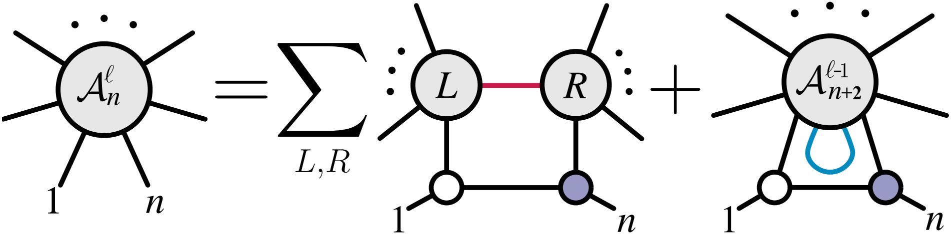
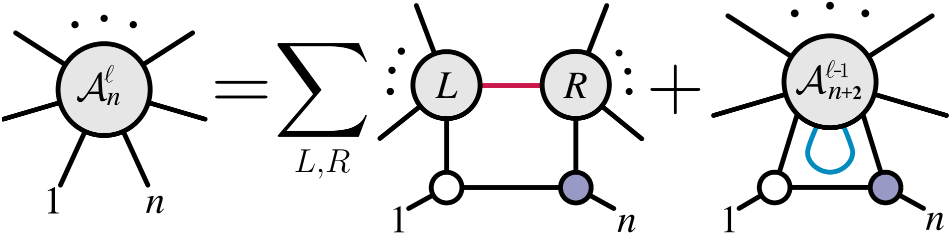


The Analytic Boot-Strap: All-Loop Recursion Relations

Forward-limits and loop-momenta:

the familiar "off-shell" loop-momentum is represented by on-shell data as:

$$\ell \equiv \lambda_I \widetilde{\lambda}_I + \alpha \lambda_1 \widetilde{\lambda}_n \quad \text{with} \quad d^4\ell = d^3 LIPS_I \; d\alpha \; \langle 1\, I \rangle \, [n\, I]$$



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The BCFW recursion relations realize an incredible fantasy: it **directly** gives the **Parke-Taylor** formula for all amplitudes with $k=2$, $\mathcal{A}_n^{(2)}$!

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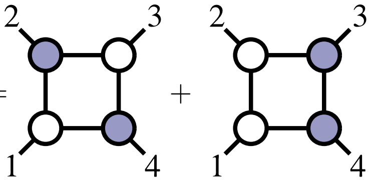
$\mathcal{A}_4^{(2)} =$ 

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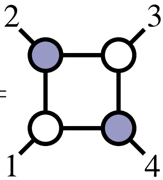
$\mathcal{A}_4^{(2)} =$ 

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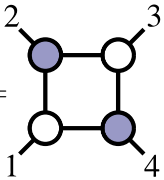

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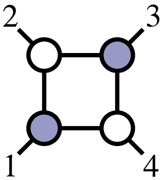

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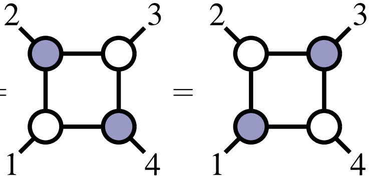
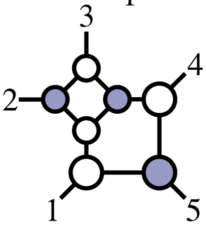
$\mathcal{A}_4^{(2)} =$ 

$\mathcal{A}_5^{(2)} =$ 

Exempli Gratia: On-Shell Representations of Amplitudes

The BCFW recursion relations realize an incredible fantasy: it **directly** gives the **Parke-Taylor** formula for all amplitudes with $k=2$, $\mathcal{A}_n^{(2)}$!

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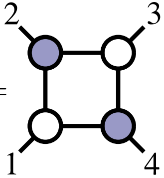
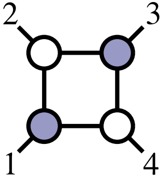
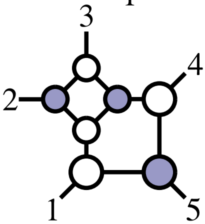
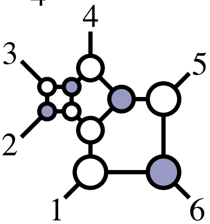
$\mathcal{A}_4^{(2)} =$ 

$=$ 

$\mathcal{A}_5^{(2)} =$ 

$\mathcal{A}_6^{(2)} =$ 

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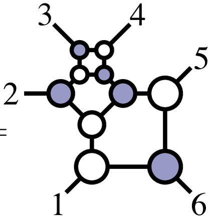
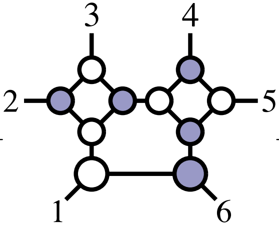
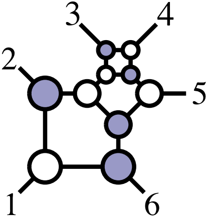
$\mathcal{A}_6^{(3)} =$  $+$  $+$ 

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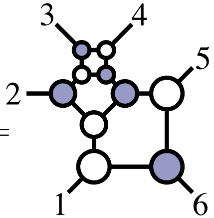
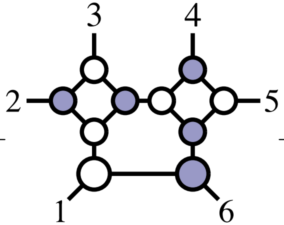
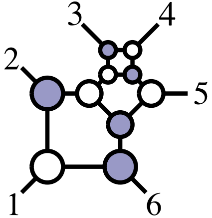
$\mathcal{A}_6^{(3)} =$  $+$  $+$ 

Observations regarding recursed representations of scattering amplitudes:

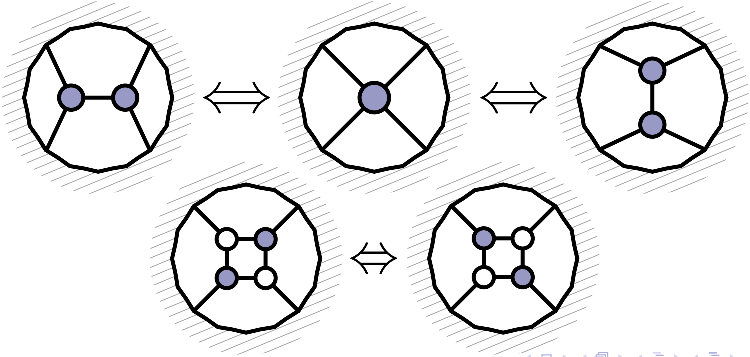
- varying recursion 'schema' can generate *many* 'BCFW formulae'
- on-shell diagrams can often be related in surprising ways

How can we characterize and systematically compute on-shell diagrams?

Combinatorial Characterization of On-Shell Diagrams

On-shell diagrams can be altered without changing their associated functions

- chains of equivalent three-particle vertices can be arbitrarily connected
- any four-particle 'square' can be drawn in its two equivalent ways



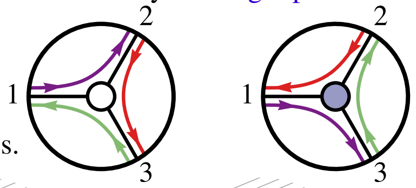
Combinatorial Characterization of On-Shell Diagrams

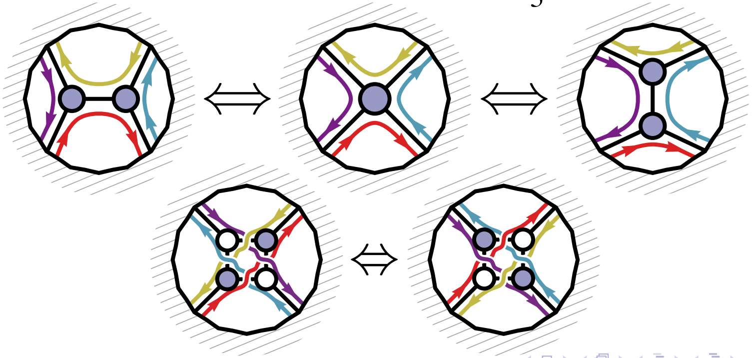
These moves leave invariant a permutation defined by ‘left-right paths’

Starting from any leg a, turn:

- *left* at each white vertex;
- *right* at each blue vertex.

Let $\sigma(a)$ denote where path terminates.

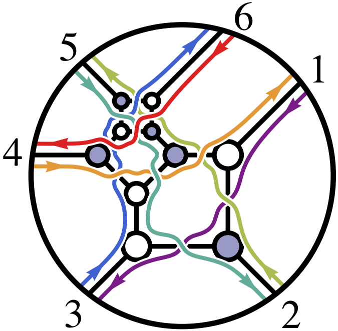


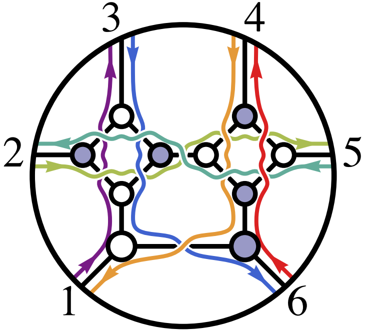


Combinatorial Characterization of On-Shell Diagrams

These moves leave invariant a permutation defined by 'left-right paths'.

Recall that different contributions to $\mathcal{A}_6^{(3)}$ were related by rotation:





left-right permutation σ

$$\sigma:\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 5 & 6 & 7 & 8 & 10 \end{pmatrix}$$

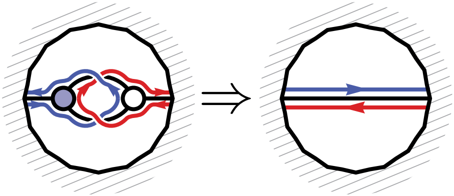
Combinatorial Characterization of On-Shell Diagrams

Notice that the merge and square moves leave the number of 'faces' of an on-shell diagram invariant. Diagrams with different numbers of faces can be related by 'reduction'—also known as 'bubble deletion':

Bubble-deletion does not, however, relate 'identical' on-shell diagrams:

- it leaves behind an overall factor of $d\alpha/\alpha$ in the on-shell function
- and it alters the corresponding left-right path permutation

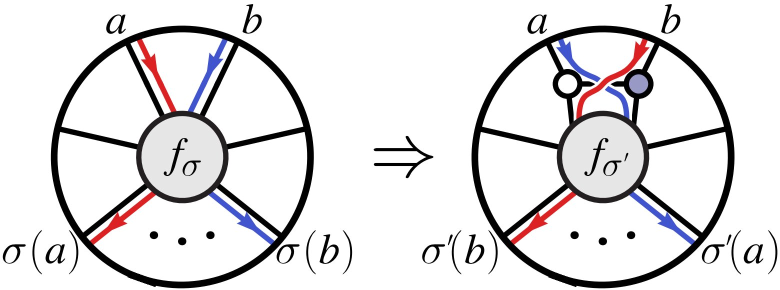
Such factors of $d\alpha/\alpha$ arising from bubble deletion encode loop integrands!



Canonical Coordinates for Computing On-Shell Functions

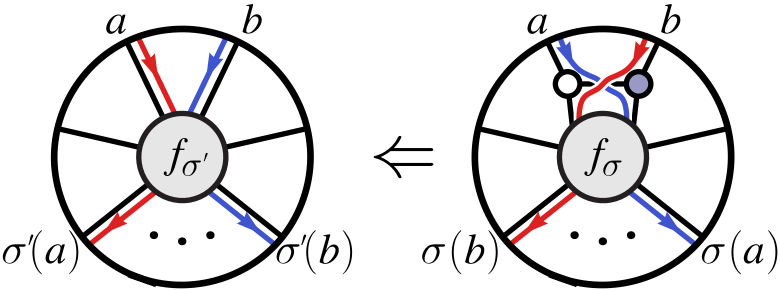
Recall that attaching 'BCFW bridges' can lead to very rich on-shell diagrams.

Conveniently, adding a BCFW bridge acts very nicely on permutations: it merely transposes the images of σ!



Canonical Coordinates for Computing On-Shell Functions

Recall that attaching 'BCFW bridges' can lead to very rich on-shell diagrams.
Read the other way, we can 'peel-off' bridges and thereby decompose a permutation into transpositions according to $\sigma = (a\, b) \circ \sigma'$



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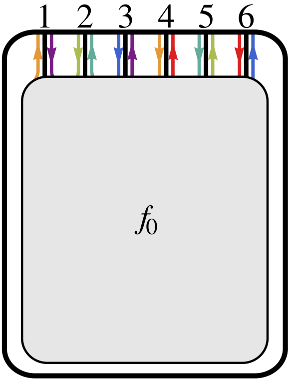

'Bridge' Decomposition

$$\begin{array}{ccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & & \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \tau \\ f_0 & \{3 & 5 & 6 & 7 & 8 & 10\} & & \end{array}$$

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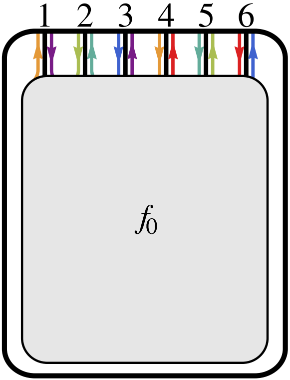

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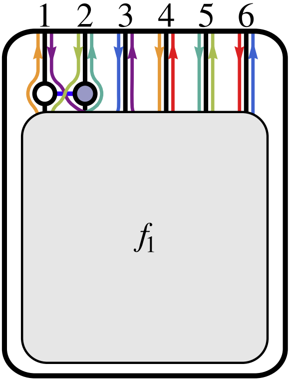
$$f_0 = \frac{d\alpha_1}{\alpha_1} f_1$$



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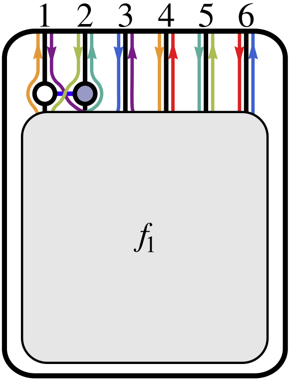

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 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \\
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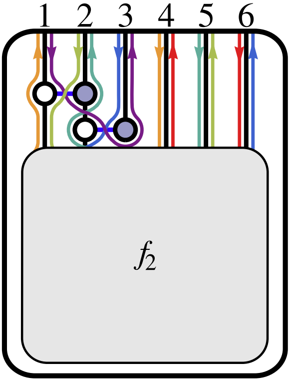

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 & & & & & & & (2\,3) \\
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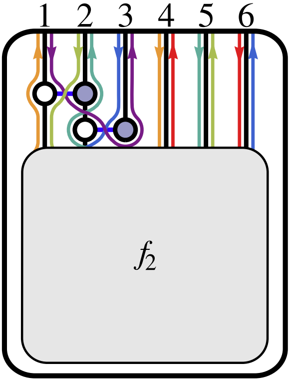

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	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
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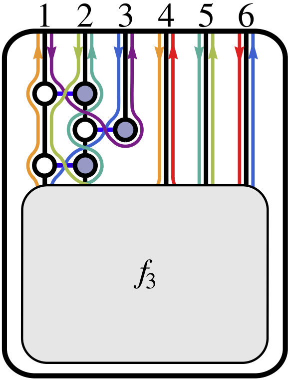

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	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
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f_1	{5	3	6	7	8	10}	$(2\,3)$
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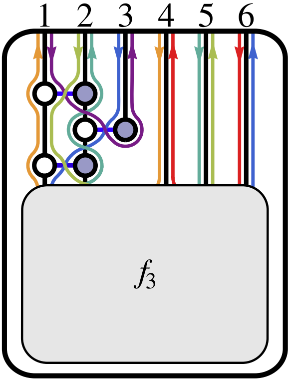

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f_0	{3	5	6	7	8	10}	
							$(1\,2)$
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							$(2\,3)$
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							$(2\,4)$

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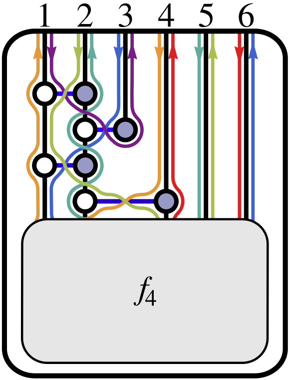

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							(1 2)
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							(2 3)
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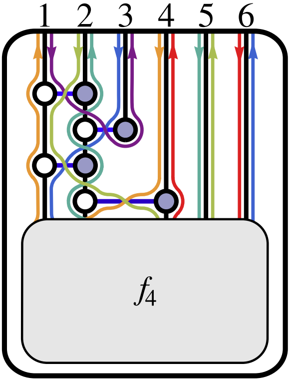

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	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	$(1\,2)$
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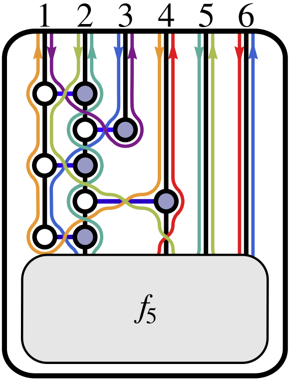

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	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
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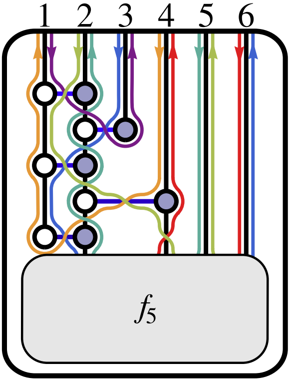

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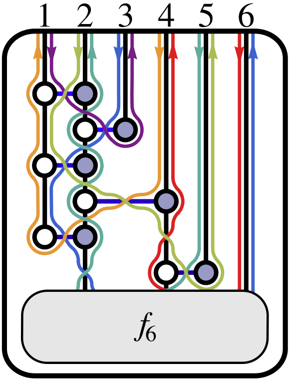

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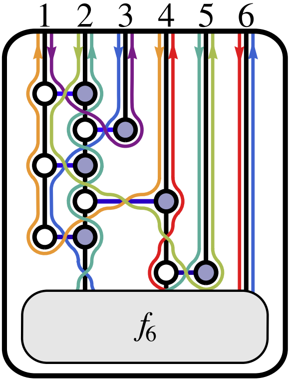

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f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau\equiv(a\,b)$ such that $\sigma(a)<\sigma(b)$:

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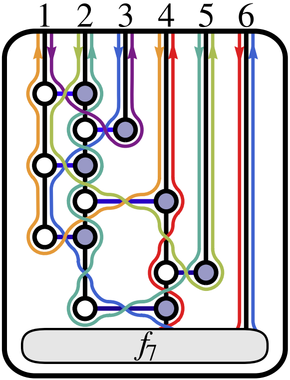

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	

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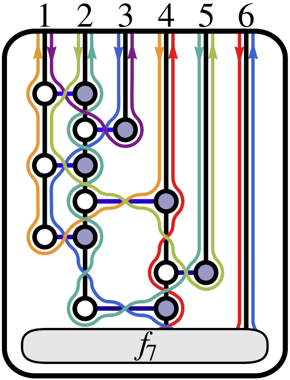

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)

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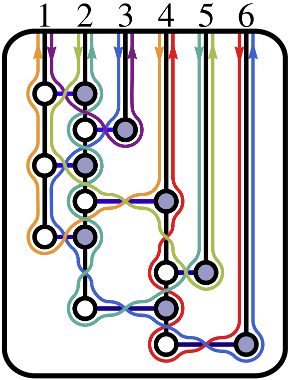

‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
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f_8	{7	8	3	10	5	6 }	

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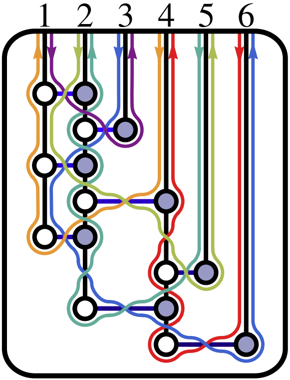

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
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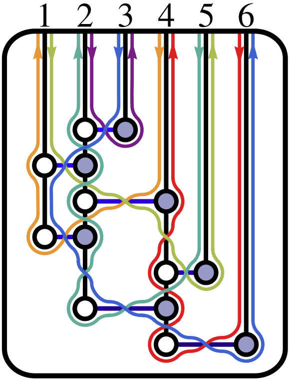

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
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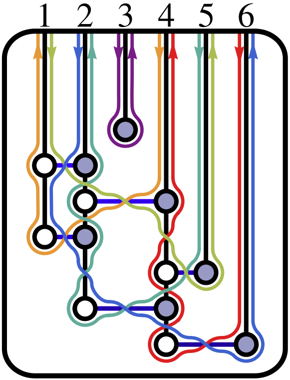

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
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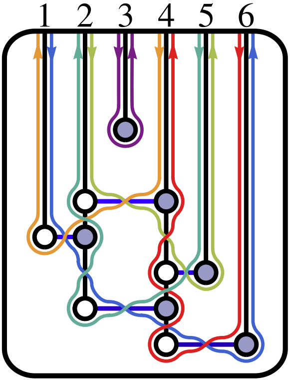

‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
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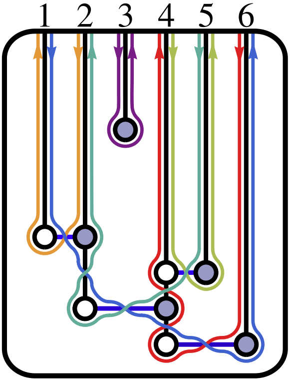

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
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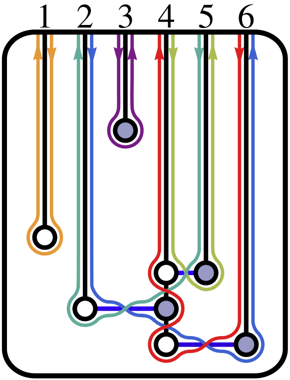

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
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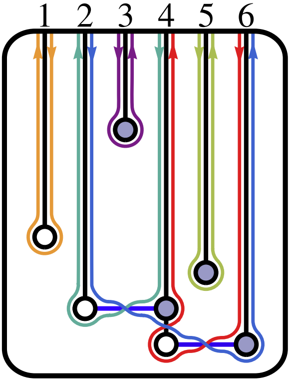

'Bridge' Decomposition

$$\begin{array}{lcccccccc}
 & & 1 & 2 & 3 & 4 & 5 & 6 & \\
 & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \\
f_6 & \{ & 7 & 6 & 3 & 8 & 5 & 10 & \} \\
 & & & & & & & & (2\,4) \\
f_7 & \{ & 7 & 8 & 3 & 6 & 5 & 10 & \} \\
 & & & & & & & & (4\,6) \\
f_8 & \{ & 7 & 8 & 3 & 10 & 5 & 6 & \}
\end{array}$$

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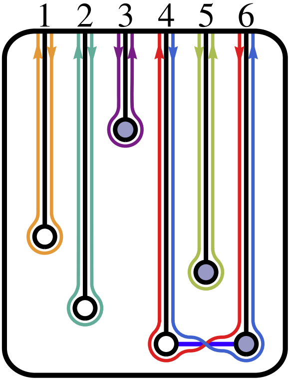

'Bridge' Decomposition

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \end{array}$$

$$\begin{array}{l} f_7\,\{7\;\;8\;\;3\;\;6\;\;5\;\;10\} \\ f_8\,\{7\;\;8\;\;3\;\;10\;\;5\;\;6\,\} \end{array}(4\,6)$$

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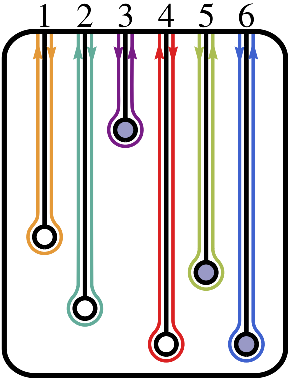

'Bridge' Decomposition

1	2	3	4	5	6	τ
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	

$f_8\;\{7\quad 8\quad 3\quad 10\quad 5\quad 6\,\}$

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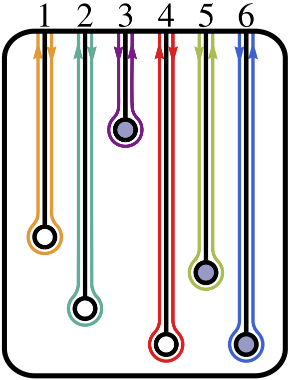

‘Bridge’ Decomposition

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \tau \end{array}$$

$$f_8\ \{7\ \ 8\ \ 3\ \ 10\ \ 5\ \ 6\ \}$$

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$$f_8 = \prod_{a=\sigma(a)+n}\Big(\delta^4(\widetilde{\eta}_a)\delta^2(\widetilde{\lambda}_a)\Big)\prod_{b=\sigma(b)}\Big(\delta^2(\lambda_b)\Big)$$

'Bridge' Decomposition

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \end{array}$$

$f_8\ \{7\ \ 8\ \ 3\ \ 10\ \ 5\ \ 6\ \}$

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$$C \equiv \begin{array}{c} \\ \end{array}\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$
$$C \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

'Bridge' Decomposition

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \end{array}$$

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$$f_8=\delta^{3\times4}\big(C\cdot\widetilde{\eta}\big)\delta^{3\times2}\big(C\cdot\widetilde{\lambda}\big)\delta^{2\times3}\big(\lambda\cdot C^{\perp}\big)$$

$$C\equiv\begin{array}{c}\begin{array}{cccccc}1&2&3&4&5&6\end{array}\\\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&1&0&0\end{pmatrix}\end{array}$$

'Bridge' Decomposition

$$\begin{array}{cccccccc}1&2&3&4&5&6&&\\\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&&\tau\end{array}$$

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'Bridge' Decomposition

$$\begin{array}{cccccc|c}1&2&3&4&5&6&\\\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\tau\end{array}$$

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$$f_7=\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}\big(C\cdot\widetilde{\eta}\big)\delta^{3\times2}\big(C\cdot\widetilde{\lambda}\big)\delta^{2\times3}\big(\lambda\cdot C^{\perp}\big)$$

$$C\equiv\begin{array}{c}\begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ \left(\begin{array}{cccccc}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&1&0&\alpha_8\end{array}\right)\end{array}$$

$$(4\,6):\quad c_6\mapsto c_6+\alpha_8\,c_4$$

'Bridge' Decomposition

$$\begin{array}{ccccccc}1&2&3&4&5&6&\\\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\tau\end{array}$$

$$\begin{array}{l}f_7\ \{7\ \ 8\ \ 3\ \ 6\ \ 5\ \ 10\}\\ f_8\ \{7\ \ 8\ \ 3\ \ 10\ \ 5\ \ 6\}\end{array}(4\,6)$$

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$$f_6=\frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}(C\cdot\widetilde{\eta})\,\delta^{3\times2}(C\cdot\widetilde{\lambda})\,\delta^{2\times3}(\lambda\cdot C^{\perp})$$

$$C\equiv\begin{array}{c}\\ \\ \end{array}\begin{array}{cccccc}1&2&3&4&5&6\end{array}$$
$$C\equiv\left(\begin{array}{cccccc}1&0&0&0&0&0\\0&1&0&\alpha_7&0&0\\0&0&0&1&0&\alpha_8\end{array}\right)$$

$$(2\,4):\quad c_4\mapsto c_4+\alpha_7\,c_2$$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	$(4\,6)$
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:

$$f_0 = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4}\frac{d\alpha_5}{\alpha_5}\frac{d\alpha_6}{\alpha_6}\frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8} f_8$$

$$f_5 = \frac{d\alpha_6}{\alpha_6}\frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}(C\cdot\widetilde{\eta})\,\delta^{3\times2}(C\cdot\widetilde{\lambda})\,\delta^{2\times3}(\lambda\cdot C^{\perp})$$

$$C \equiv \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{pmatrix} \end{array}$$

$$(4\,5):\quad c_5 \mapsto c_5 + \alpha_6\, c_4$$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	$(4\,6)$
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau\equiv(a\,b)$ such that $\sigma(a)<\sigma(b)$:

$$f_0=\frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4}\frac{d\alpha_5}{\alpha_5}\frac{d\alpha_6}{\alpha_6}\frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8}f_8$$

$$f_4=\frac{d\alpha_5}{\alpha_5}\cdots\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}\big(C\cdot\widetilde{\eta}\big)\delta^{3\times2}\big(C\cdot\widetilde{\lambda}\big)\delta^{2\times3}\big(\lambda\cdot C^{\perp}\big)$$

$$C\equiv\begin{array}{c}\begin{array}{cccccc}1 & 2 & 3 & 4 & 5 & 6\end{array}\\ \left(\begin{array}{cccccc}1 & \alpha_5 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & \alpha_7 & \alpha_6\alpha_7 & 0\\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8\end{array}\right)\end{array}$$

$$(1\,2):\quad c_2\mapsto c_2+\alpha_5\,c_1$$

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau\equiv(a\,b)$ such that $\sigma(a)<\sigma(b)$:

$$f_0=\frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4}\frac{d\alpha_5}{\alpha_5}\frac{d\alpha_6}{\alpha_6}\frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8}f_8$$

$$f_3=\frac{d\alpha_4}{\alpha_4}\cdots\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}(C\cdot\widetilde{\eta})\,\delta^{3\times2}(C\cdot\widetilde{\lambda})\,\delta^{2\times3}(\lambda\cdot C^{\perp})$$

$$C\equiv\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}\\ \left(\begin{array}{cccccc} 1 & \alpha_5 & 0 & \alpha_4\alpha_5 & 0 & 0\\ 0 & 1 & 0 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0\\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)\end{array}$$

$$(2\,4):\quad c_4\mapsto c_4+\alpha_4\,c_2$$

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:

$$f_0 = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4}\frac{d\alpha_5}{\alpha_5}\frac{d\alpha_6}{\alpha_6}\frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8} f_8$$

$$f_2 = \frac{d\alpha_3}{\alpha_3}\cdots\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}\big(C\cdot\widetilde{\eta}\big)\delta^{3\times2}\big(C\cdot\widetilde{\lambda}\big)\delta^{2\times3}\big(\lambda\cdot C^{\perp}\big)$$

$$C \equiv \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{pmatrix} 1 & (\alpha_3+\alpha_5) & 0 & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & 0 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{pmatrix} \end{array}$$

$$(1\,2):\quad c_2 \mapsto c_2 + \alpha_3\, c_1$$

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	$(4\,6)$
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau\equiv(a\,b)$ such that $\sigma(a)<\sigma(b)$:

$$f_0=\frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4}\frac{d\alpha_5}{\alpha_5}\frac{d\alpha_6}{\alpha_6}\frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8}f_8$$

$$f_1=\frac{d\alpha_2}{\alpha_2}\cdots\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}\big(C\cdot\widetilde{\eta}\big)\delta^{3\times2}\big(C\cdot\widetilde{\lambda}\big)\delta^{2\times3}\big(\lambda\cdot C^{\perp}\big)$$

$$C\equiv\begin{array}{c}\begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ \left(\begin{array}{cccccc}1&(\alpha_3+\alpha_5)&\alpha_2(\alpha_3+\alpha_5)&\alpha_4\alpha_5&0&0\\0&1&\alpha_2&(\alpha_4+\alpha_7)&\alpha_6\alpha_7&0\\0&0&0&1&\alpha_6&\alpha_8\end{array}\right)\end{array}$$

$$(2\,3):\quad c_3\mapsto c_3+\alpha_2\,c_2$$

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6 }	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau\equiv(a\,b)$ such that $\sigma(a)<\sigma(b)$:

$$f_0=\frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4}\frac{d\alpha_5}{\alpha_5}\frac{d\alpha_6}{\alpha_6}\frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8}f_8$$

$$f_0=\frac{d\alpha_1}{\alpha_1}\cdots\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}\big(C\cdot\widetilde{\eta}\big)\delta^{3\times2}\big(C\cdot\widetilde{\lambda}\big)\delta^{2\times3}\big(\lambda\cdot C^{\perp}\big)$$

$$C\equiv\begin{array}{c}\begin{array}{cccccc}1 & 2 & 3 & 4 & 5 & 6\end{array}\\ \left(\begin{array}{cccccc}1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0\\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0\\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8\end{array}\right)\end{array}$$

$$(1\,2):\quad c_2\mapsto c_2+\alpha_1\,c_1$$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	$(4\,6)$
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau\equiv(a\,b)$ such that $\sigma(a)<\sigma(b)$:

$$f_0=\frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4}\frac{d\alpha_5}{\alpha_5}\frac{d\alpha_6}{\alpha_6}\frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8}f_8$$

$$f_0=\frac{d\alpha_1}{\alpha_1}\cdots\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}\big(C\cdot\widetilde{\eta}\big)\delta^{3\times2}\big(C\cdot\widetilde{\lambda}\big)\delta^{2\times3}\big(\lambda\cdot C^{\perp}\big)$$

$$C\equiv\begin{array}{c}\begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ \begin{pmatrix}1&(\alpha_1+\alpha_3+\alpha_5)&\alpha_2(\alpha_3+\alpha_5)&\alpha_4\alpha_5&0&0\\0&1&\alpha_2&(\alpha_4+\alpha_7)&\alpha_6\alpha_7&0\\0&0&0&1&\alpha_6&\alpha_8\end{pmatrix}\end{array}$$

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
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Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau\equiv(a\,b)$ such that $\sigma(a)<\sigma(b)$:

$$f_0=\frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4}\frac{d\alpha_5}{\alpha_5}\frac{d\alpha_6}{\alpha_6}\frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8}f_8$$

$$f_0=\frac{d\alpha_1}{\alpha_1}\cdots\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}\big(C\cdot\widetilde{\eta}\big)\delta^{3\times2}\big(C\cdot\widetilde{\lambda}\big)\delta^{2\times3}\big(\lambda\cdot C^{\perp}\big)$$

$$C\equiv\begin{array}{c}\begin{array}{cccccc}1 & 2 & 3 & 4 & 5 & 6\end{array}\\ \left(\begin{array}{cccccc}1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0\\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0\\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8\end{array}\right)\end{array}$$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6}	

Canonical Coordinates and the Manifestation of the *Yangian*

All on-shell diagrams, in terms of canonical coordinates, take the form:

$$f = \int \frac{d\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{d\alpha_d}{\alpha_d} \;\delta^{k\times 4}\big(C(\vec{\alpha})\cdot\widetilde{\eta}\big)\,\delta^{k\times 2}\big(C(\vec{\alpha})\cdot\widetilde{\lambda}\big)\,\delta^{2\times(n-k)}\big(\lambda\cdot C(\vec{\alpha})^{\perp}\big)$$

Measure-preserving diffeomorphisms leave the function invariant, but—via the δ-functions—can be recast variations of the kinematical data.

The *Yangian* corresponds to those diffeomorphisms that simultaneously preserve the measures of *all* on-shell diagrams.

On-Shell Structures of *General* Quantum Field Theories

$$f_\Gamma \equiv \int d\Omega_C \; \delta^{k\times\mathcal{N}}(C\cdot\widetilde{\eta})\,\delta^{k\times 2}(C\cdot\widetilde{\lambda})\,\delta^{2\times(n-k)}(\lambda\cdot C^{\perp})$$

On-Shell Physics

- on-shell diagrams
- physical symmetries
  - trivial symmetries (identities)

$\Longleftrightarrow$

Grassmannian Geometry

- $\{$strata $C\in G(k,n)$, volume-form $\Omega_C\}$
- volume-preserving diffeomorphisms
  - cluster coordinate mutations

On-Shell Structures of *General* Quantum Field Theories

$$f_\Gamma \equiv \int d\Omega_C \;\delta^{k\times\mathcal{N}}(C\cdot\widetilde{\eta})\,\delta^{k\times 2}(C\cdot\widetilde{\lambda})\,\delta^{2\times(n-k)}(\lambda\cdot C^{\perp})$$

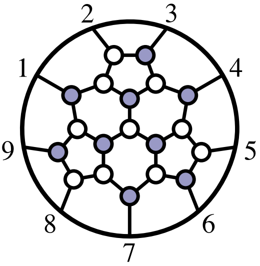
On-Shell Physics: planar $\mathcal{N}=4$

- on-shell diagrams
  - bi-colored, **un**directed, **planar**
- physical symmetries
  - trivial symmetries (identities)

$\Longleftrightarrow$

Grassmannian Geometry

- $\{$strata $C\in G(k,n)$, volume-form $\Omega_C\}$
  - positroid variety , $\left(\prod_i \frac{d\alpha_i}{\alpha_i}\right)$
- volume-preserving diffeomorphisms
  - cluster coordinate mutations



$$C \equiv \begin{pmatrix} 1 & \alpha_8 & \alpha_5{+}\alpha_8\alpha_{14} & \alpha_5\alpha_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \alpha_{10} & \alpha_4{+}\alpha_{10}\alpha_{13} & \alpha_4\alpha_7 & 0 & 0 \\ \alpha_3\alpha_9 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_6 & \alpha_3{+}\alpha_6\alpha_{12} \\ \alpha_9 & 0 & \alpha_1 & \alpha_1\alpha_{11} & 0 & \alpha_1\alpha_2 & \alpha_1\alpha_2\alpha_7 & 0 & 1 \end{pmatrix}$$

$$\Omega_C \equiv \left(\frac{d\alpha_1}{\alpha_1}\wedge\cdots\wedge\frac{d\alpha_{14}}{\alpha_{14}}\right)$$

On-Shell Structures of *General* Quantum Field Theories

$$f_\Gamma \equiv \int d\Omega_C \; \delta^{k\times\mathcal{N}}(C\cdot\widetilde{\eta})\,\delta^{k\times 2}(C\cdot\widetilde{\lambda})\,\delta^{2\times(n-k)}(\lambda\cdot C^\perp)$$

On-Shell Physics: planar $\mathcal{N}=4$

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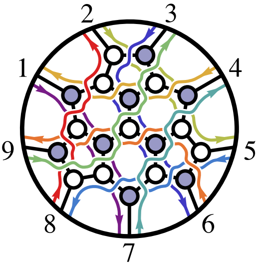


$$C \equiv \begin{pmatrix} 1 & \alpha_8 & \alpha_5+\alpha_8\alpha_{14} & \alpha_5\alpha_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \alpha_{10} & \alpha_4+\alpha_{10}\alpha_{13} & \alpha_4\alpha_7 & 0 & 0 \\ \alpha_3\alpha_9 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_6 & \alpha_3+\alpha_6\alpha_{12} \\ \alpha_9 & 0 & \alpha_1 & \alpha_1\alpha_{11} & 0 & \alpha_1\alpha_2 & \alpha_1\alpha_2\alpha_7 & 0 & 1 \end{pmatrix}$$

$$\Omega_C \equiv \left(\frac{d\alpha_1}{\alpha_1}\wedge\cdots\wedge\frac{d\alpha_{14}}{\alpha_{14}}\right)$$

On-Shell Structures of *General* Quantum Field Theories

$$f_\Gamma \equiv \int d\Omega_C \;\delta^{k\times\mathcal{N}}(C\cdot\widetilde{\eta})\,\delta^{k\times 2}(C\cdot\widetilde{\lambda})\,\delta^{2\times(n-k)}(\lambda\cdot C^{\perp})$$

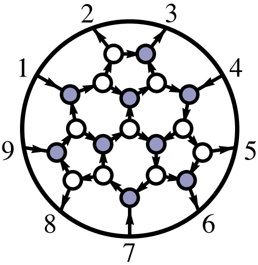
On-Shell Physics: planar $\mathcal{N}<4$

- on-shell diagrams
  - bi-colored, **directed** , **planar**
- physical symmetries: **?**
  - trivial symmetries (identities)

$\Longleftrightarrow$

Grassmannian Geometry

- $\{$strata $C\in G(k,n)$, volume-form $\Omega_C\}$
  - positroid variety , $\left(\prod_i \frac{d\alpha_i}{\alpha_i}\right)\times\mathcal{J}^{\mathcal{N}-4}$
- volume-preserving diffeomorphisms
  - cluster coordinate mutations



$$C \equiv \begin{pmatrix} 1 & \alpha_8 & \alpha_5+\alpha_8\alpha_{14} & \alpha_5\alpha_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \alpha_{10} & \alpha_4+\alpha_{10}\alpha_{13} & \alpha_4\alpha_7 & 0 & 0 \\ \alpha_3\alpha_9 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_6 & \alpha_3+\alpha_6\alpha_{12} \\ \alpha_9 & 0 & \alpha_1 & \alpha_1\alpha_{11} & 0 & \alpha_1\alpha_2 & \alpha_1\alpha_2\alpha_7 & 0 & 1 \end{pmatrix}$$

$$\Omega_C \equiv \left(\frac{d\alpha_1}{\alpha_1}\wedge\cdots\wedge\frac{d\alpha_{14}}{\alpha_{14}}\right)\times(1)^{\mathcal{N}-4}$$

On-Shell Structures of *General* Quantum Field Theories

$$f_\Gamma \equiv \int d\Omega_C\ \delta^{k\times\mathcal{N}}(C\cdot\widetilde{\eta})\,\delta^{k\times 2}(C\cdot\widetilde{\lambda})\,\delta^{2\times(n-k)}(\lambda\cdot C^\perp)$$

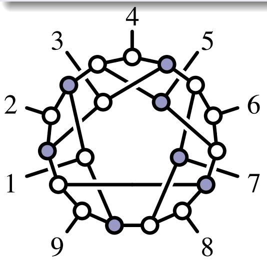
On-Shell Physics: non-planar $\mathcal{N}=4$

- on-shell diagrams
  - bi-colored, **un**directed, **non**-planar
- physical symmetries: **?**
  - trivial symmetries (identities)

$\Longleftrightarrow$

Grassmannian Geometry

- $\{$strata $C\in G(k,n)$, volume-form $\Omega_C\}$
  - cluster variety(**?**), $\left(\prod_i \frac{d\alpha_i}{\alpha_i}\right)$
- volume-preserving diffeomorphisms
  - cluster coordinate mutations



$$C^\perp \equiv \begin{pmatrix} \alpha_1 & 1 & 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_3 & 1 & 0 & \alpha_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_5 & 1 & \alpha_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \alpha_7 & \alpha_8 \\ 0 & 0 & 0 & 0 & 0 & \alpha_9 & 1 & \alpha_{10} & 0 \\ \alpha_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_{12} \\ 0 & \alpha_{13} & \alpha_{14} & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Omega_C \equiv \left(\frac{d\alpha_1}{\alpha_1}\wedge\cdots\wedge\frac{d\alpha_{14}}{\alpha_{14}}\right)$$

On-Shell Structures of *General* Quantum Field Theories

$$f_\Gamma \equiv \int d\Omega_C\; \delta^{k\times\mathcal{N}}(C\cdot\widetilde{\eta})\,\delta^{k\times 2}(C\cdot\widetilde{\lambda})\,\delta^{2\times(n-k)}(\lambda\cdot C^\perp)$$

On-Shell Physics: non-planar $\mathcal{N}<4$

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  - bi-colored, **directed** , **non**-planar
- physical symmetries: **?**
  - trivial symmetries (identities)

$\Longleftrightarrow$

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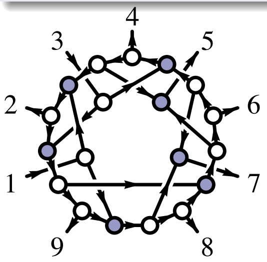
  - cluster variety(**?**), $\left(\prod_i \frac{d\alpha_i}{\alpha_i}\right)\times\mathcal{J}^{\mathcal{N}-4}$
- volume-preserving diffeomorphisms
  - cluster coordinate mutations



$$C^\perp \equiv \begin{pmatrix} \alpha_1 & 1 & 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_3 & 1 & 0 & \alpha_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_5 & 1 & \alpha_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \alpha_7 & \alpha_8 \\ 0 & 0 & 0 & 0 & 0 & \alpha_9 & 1 & \alpha_{10} & 0 \\ \alpha_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_{12} \\ 0 & \alpha_{13} & \alpha_{14} & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Omega_C \equiv \left(\frac{d\alpha_1}{\alpha_1}\wedge\cdots\wedge\frac{d\alpha_{14}}{\alpha_{14}}\right)\times\Big(1+\alpha_2\alpha_4\alpha_{13}(\alpha_8+\alpha_7\alpha_{12})\Big)^{\mathcal{N}-4}$$

Open Directions for Further Research

- **Classifying On-Shell Functions for General Quantum Field Theories**
  - non-planar $\mathcal{N}=4$ SYM, $\mathcal{N}=8$ SUGRA, planar $\mathcal{N}<4$, QCD, …
- **Verifying the All-Loop Recursion Relations Beyond Planar $\mathcal{N}=4$**
- **Evaluating Amplitudes** (beyond the leading order)
  - regularization, renormalization, …
  - *directly* evaluating the terms generated by recursion
  - better understanding the (motivic?) structure of loop-amplitudes
- **A Purely Geometric Definition of Scattering Amplitudes**?
  - extending the *amplituhedron* to more general quantum field theories
- …

Closing Words: Lessons Learned at TASI 2014

A Contribution to the 40-Particle Scattering Amplitude