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Neutron star oscillations and instabilities: Lecture 1

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1 MOTIVATION

1.1 Why study neutron star oscillations?

Seismology is an excellent technique for studying the composition and structure of planets and stars. On Earth, seismologists study two things. Firstly, local geophysics - the properties of the rupture zone where an earthquake happened. Secondly, global geophysics - large earthquakes generate waves which spread around the planet and interfere to form global modes. The frequences and decay times of these modes provide information about the entire planet, not just the original earthquake zone.

Neutron stars have densities and magnetic fields orders of magnitude above that which we can test on Earth. Neutron star seismology therefore tests some very interesting physics!

1.2 How can we observe neutron star oscillations?

Neutron stars emit in a wide range of electromagnetic wavebands, from the radio all the way up to the gamma-ray (and even higher). Given telescopes with good enough time resolution and collecting area, we can search for NS oscillations in all of these bands.

NS are however dense, relativistic stars. This means that non-axisymmetric oscillations will generate gravitational waves - if excited to a sufficiently large amplitude they might therefore be detectable by the new generation of gravitational wave detectors as well.

1.3 Why should we care about instabilities?

Viscous processes will tend to damp out most oscillations. Clearly this is undesirable from a detection point of view! It is therefore in our interest to find oscillations that are naturally unstable (have an exponentially growing amplitude), which might be able to 'beat' viscosity and hence maintain a detectable amplitude.

Instabilities are also interesting, because if they grow fast enough they result in some spectacular phenomena - explosions and gamma-ray flares, for example. These are not only interesting phenomena in themselves, but may also trigger more long-lived types of seismic oscillation.

2 GENERAL PRINCIPLES OF MODE CALCULATION

A general recipe for modelling seismic oscillations is as follows:

(i) The first stage is to generate a dynamical model of the oscillations - writing down the various conservation equations and equations of motion that govern the star's motion. In first instance it is common to start by studying linearized perturbations.

(ii) Once the dynamics are established, the stability of the oscillation is considered. We must consider both dynamical instabilities - those that are present in the system even in the absence of a forcing mechanism - and secular instabilities driven by dissipative forces such as gravitational radiation reaction.

(iii) The next step is to consider the effect of limiting mechanisms such as viscosity. Oscillations will only be astrophysically relevant if damping is slow, or the oscillations can grow (via instabilities) faster than they are damped.

(iv) The linearized approximation is useful to identify those perturbations that can grow or reach reasonable amplitudes. To determine the amplitude at which an oscillation will saturate, however, we may have to return to the full non-linear equations.

(v) We then need to compute the emitted signal, in either gravitational or electromagnetic radiation. For EM radiation, scattering and absorption processes in the stellar atmosphere or the interstellar medium are usually important. We must also establish whether there are plausible excitation mechanisms that would set the star vibrating in the first place.

3 OSCILLATION TYPES

Different families of oscillations involve different restoring forces and different types of motion. On Earth, for example, we have body waves (P waves which depend on bulk compressibility, and S waves which also depend on the shear modulus) and interface waves at

[h!]

Physical parameter	Notation
Density	ρ
Pressure	P
Velocity (vector)	$\mathbf{v}$
Displacement (vector)	$\boldsymbol{\xi}$
Time	t
Gravitational potential	Φ
Temperature	T
Mass	M
Radius of neutron star	R
Inertial frame frequency	ω
Rotating frame frequency	ω_r

Table 1. Notation used for physical parameters

the surface of the crust (e.g. Love waves, which involve primarily horizontal motions, and Rayleigh waves, which are acousto-elastic modes).

NS are complicated systems and hence admit many different types of oscillation:

(i) f-mode and p-modes: Sound modes, depend on the compression

(ii) g-modes: Driven by thermal or composition gradients

(iii) s- and t-modes: Depend on the shear modulus in the crust, s-modes having a larger radial component than the primarily horizontal t-modes

(iv) r-modes: Modes in a rotating fluid (e.g. the neutron star core) driven by the Coriolis force

(v) Many others such as Alfven modes associated with magnetic forces, superfluid modes....

4 SOME SIMPLE OSCILLATION CALCULATIONS

Starting with a very simple Newtonian dynamical model, we can easily derive expressions for the frequencies of the first two classes (p- and g-modes). These modes are particularly interesting for gravitational wave emission because they are oscillations of the dense fluid core. In Table (1) I summarise the notation used for physical parameters.

We will treat the stellar material as a dense nuclear fluid. The basic equations governing the system are the continuity equation, the Euler or Navier-Stokes equations, the Poisson equation for the gravitational potential, and an equation of state. The conservation of mass is expressed in the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{1}$$

Newton's laws of motion applied to a viscous fluid in a gravitational potential Φ give rise to the Navier-Stokes equations:

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\nabla P - \nabla\Phi + \frac{1}{\rho}\nabla\cdot\tau, \tag{2}$$

where τ is the viscous stress tensor and $d/dt = \partial/\partial t + \mathbf{v}\cdot\nabla$ is the total time derivative. To simplify our modelling, we will begin by considering dynamics in the absence of viscosity. In this case the Navier-Stokes equations simplify to the Euler equations:

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\nabla P - \nabla\Phi. \tag{3}$$

The gravitational potential is determined by the Poisson equation,

$$\nabla^2\Phi = 4\pi G\rho. \tag{4}$$

The final governing equation is perhaps the most complicated. We need to specify an equation of state that relates pressure to density. The true equation of state depends on the nuclear physics within the neutron star, something that is very uncertain. Here we will use a simple polytropic equation of state

$$P = K\rho^{\Gamma}, \tag{5}$$

where $\Gamma = 1 + 1/n$, n being the polytropic index. An index of $n = 1$ to 1.5 is a reasonable approximation for a neutron star.

We are going to use linearized perturbation theory, in which oscillations are treated as small perturbations of a background configuration that is in hydrostatic equilibrium. The background equilibrium configuration of the star is given by the four equations (1), (3), (4), (5), with the partial time derivatives set to zero.

Let us now consider perturbations to this equilibrium. There are two different kinds of fluid perturbation. Eulerian perturbations, denoted by a δ, are the changes in a variable at a particular point in space. If Q is a property of the perturbed flow and Q_0 the same property in the unperturbed equilibrium flow, then

$$\delta Q \equiv Q(\mathbf{x}, t) - Q_0(\mathbf{x}, t). \tag{6}$$

Lagrangian perturbations, denoted by a Δ, are the changes in a variable for a particular mass element moving in the flow. If the displacement of the fluid element is $\boldsymbol{\xi}$, then

$$\Delta Q \equiv Q(\mathbf{x}+\boldsymbol{\xi}(t),t) - Q_0(\mathbf{x},t). \tag{7}$$

The two types of perturbation are related by

$$\Delta = \delta + \boldsymbol{\xi}\cdot\nabla. \tag{8}$$

Linearizing in the perturbations (which we assume to be small), the perturbed continuity equation is

$$\frac{\partial \delta\rho}{\partial t} + (\delta\mathbf{v}\cdot\nabla)\rho + (\mathbf{v}\cdot\nabla)\delta\rho + \delta\rho(\nabla\cdot\mathbf{v}) + \rho(\nabla\cdot\delta\mathbf{v}) = 0. \tag{9}$$

The perturbed Euler equations are

$$\frac{\partial \delta\mathbf{v}}{\partial t} + (\delta\mathbf{v}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\delta\mathbf{v} = \frac{\delta\rho}{\rho^2}\nabla P - \frac{1}{\rho}\nabla\delta P - \nabla\delta\Phi, \tag{10}$$

and the perturbed Poisson equation for the gravitational potential is

$$\nabla^2\delta\Phi = 4\pi G\delta\rho. \tag{11}$$

The next step is to specify an equation of state for the perturbations. This need not necessarily be the same as the equation of state obeyed by the equilibrium background model, as different physical processes may dominate on the short timescales associated with the perturbations. It is usual to assume that the perturbations are adiabatic, so that the equation of state relating the Lagrangian perturbations of pressure and density is

$$\frac{\Delta P}{P} = \Gamma_1\frac{\Delta\rho}{\rho}, \tag{12}$$

where Γ_1, the adiabatic index, need not be the same as the background index Γ. In terms of the Eulerian perturbations this can be written as

$$\frac{\delta P}{P} = \Gamma_1\frac{\delta\rho}{\rho} + \Gamma_1\boldsymbol{\xi}\cdot\left[\nabla\log\rho - \frac{1}{\Gamma_1}\nabla\log P\right]. \tag{13}$$

The magnitude of the vector quantity in the square brackets is the Schwarzschild discriminant A_s. If we were to restrict our attention to barotropic perturbations, we would set A_s to zero. This corresponds to setting $\Gamma_1 = \Gamma$ so that the perturbations obey the same equation of state as the background star. Physically, this amounts to neglecting internal stratification caused by temperature or composition gradients - this would exclude the g-modes.

The equilibrium background model is symmetric with respect to the angular coordinate φ. The resulting perturbations are either axisymmetric (independent of φ) or non-axisymmetric with φ dependence $\exp(im\varphi)$, m being a non-zero integer. We will focus on non-axisymmetric perturbations, because non-axisymmetry is essential for gravitational wave emission. The non-axisymmetric perturbations are often classified into various families of modes that are (loosely) associated with different restoring forces. We will now introduce the various types of oscillation and some of their defining characteristics. We will assume an oscillatory time dependence given by $\exp(-i\omega t)$, where ω is the frequency.

We will start by considering the perturbations of a non-rotating ($\mathbf{v} = 0$, spherically symmetric) star. In Newtonian perturbation theory, perturbations of spherically symmetric stars are classified as being either spheroidal or toroidal[1]. Spheroidal modes have perturbations of the form

$$\delta\mathbf{v} = r\sum_{lm}\left(S_l^m Y_l^m, H_l^m \frac{\partial Y_l^m}{\partial\theta}, \frac{H_l^m}{\sin\theta}\frac{\partial Y_l^m}{\partial\varphi}\right), \tag{14}$$

where $Y_l^m(\theta,\varphi)$ are the spherical harmonics and we are working in spherical polar coordinates (r,θ,φ). The coefficients S_l^m and H_l^m are functions of r. Toroidal modes have perturbations of the form

$$\delta\mathbf{v} = r\sum_{lm}\left(0, \frac{T_l^m}{\sin\theta}\frac{\partial Y_l^m}{\partial\varphi}, -T_l^m\frac{\partial Y_l^m}{\partial\theta}\right), \tag{15}$$

where the coefficient T_l^m is a function of r.

In a non-rotating star, if we assume a normal mode time dependence, the perturbed continuity and Euler equations (equations (9) and (10)) become

$$-i\omega\delta\rho + (\delta\mathbf{v}\cdot\nabla)\rho + \rho(\nabla\cdot\delta\mathbf{v}) = 0 \tag{16}$$

[1] In relativistic perturbation studies, these classes are referred to as polar and axial, the two classes transforming differently under parity.

and

$$-i\omega\delta\mathbf{v} = \frac{\delta\rho}{\rho^2}\nabla P - \frac{1}{\rho}\nabla\delta P - \nabla\delta\Phi. \tag{17}$$

For $\omega \neq 0$ there are three families of solutions to these equations: the p-modes, the g-modes and the f-modes. All three families have spheroidal eigenfunctions. Solving the perturbation equations under the assumption that the perturbation has a radial dependence $\exp(ikr)$ where k is the radial wavenumber, one obtains a dispersion relation that relates k and ω:

$$k^2 = \frac{(L_l^2 - \omega^2)(N^2 - \omega^2)}{c_s^2\omega^2}, \tag{18}$$

where the local Lamb frequency L_l is given by

$$L_l^2(r) = \frac{l(l+1)c_s^2}{r^2}, \tag{19}$$

c_s being the sound speed. The local Brunt-Väisälä frequency N is given by

$$N^2(r) = -gA_s, \tag{20}$$

g being the local gravitational acceleration.

In the high frequency limit, $\omega^2 \gg (N^2, L_l^2)$, equation (18) becomes

$$\omega^2 = c_s^2k^2. \tag{21}$$

These are acoustic waves, the high frequency limit of the family of oscillations known as the p-modes. The p-modes are high frequency oscillations whose eigenfunctions have radial nodes, for which pressure is the dominant restoring force. In a neutron star the fundamental p-mode has a frequency of several kHz.

In the low frequency limit, $\omega^2 \ll (N^2, L_l^2)$, equation (18) becomes

$$\omega^2 = \frac{l(l+1)N^2}{r^2k^2}. \tag{22}$$

The family of oscillations with this limiting behaviour are known as the g-modes. Buoyancy caused by a temperature or composition gradient provides the dominant restoring force. For a typical neutron star, the thermal g-modes should have frequencies in the range 0.1 to 10 Hz. The Brunt-Väisälä frequency associated with composition stratification is higher, and composition g-modes are predicted to have frequencies of around 100 Hz. If we restrict attention to barotropic perturbations, for which $A_s = 0$, the g-modes form a degenerate zero-frequency set.

The f-modes have character intermediate between the p- and the g-modes, but they do not have radial nodes in their eigenfunctions. The f-mode frequency for an incompressible star is given by

$$\omega^2 = \frac{2l(l-1)}{2l+1}\frac{GM}{R^3}. \tag{23}$$

This expression is a reasonable approximation of frequency for more realistic equations of state. The f-mode frequencies for a neutron star lie below the p-modes but above the g-modes, around 2kHz.

In addition to the finite frequency modes, the non-rotating star also possesses a subset of zero-frequency modes that obey the conditions $\delta \mathbf{v} \neq 0$ and $\delta P = \delta\rho = \delta\Phi = 0$. There are both toroidal and spheroidal modes in this set. All toroidal perturbations satisfy the conditions without restriction, but the zero-frequency spheroidal perturbations must satisfy the additional constraint

$$\frac{d}{dr}(\rho r^3 S_l^m) - l(l+1)\rho r^2 H_l^m = 0. \tag{24}$$

For barotropic perturbations, the g-modes fall into this zero-frequency spheroidal class.

Let us now consider what happens when we add rotation. For the p-, g- and f-modes, rotation breaks the degeneracy in m so that modes with different m become distinct. Eigenfunctions are no longer purely spheroidal but contain some toroidal component, coupled in such a way as to preserve behaviour under parity.

Rotation also breaks the degeneracy of the zero-frequency modes, giving rise to the set of inertial modes. The Coriolis force is the dominant restoring force for these modes, and they have inertial frame frequencies that lie in the range $-(2+m)\Omega < \omega < (2-m)\Omega$. The subset of inertial modes that reduce to purely toroidal perturbations in the non-rotating limit are termed the r-modes. These modes are of particular significance for gravitational wave emission when we consider secular stability. As additional pieces of physics are added to the model, the spectrum of oscillations gets progressively richer.

5 STELLAR INSTABILITIES

Gravitational wave emission from neutron star oscillations will be strongest if the oscillations can grow to a large amplitude. We are therefore interested in identifying oscillations that are unstable. Within the framework of linearized perturbation theory this means identifying normal modes with an exponentially growing time dependence (so that ω has an imaginary part). Although the linearized approximation will eventually break down in a system with a growing mode, it is a useful tool in identifying those perturbations that have the potential to grow to reasonable amplitudes. Due consideration must however be given to damping and saturation mechanisms.

Instabilities are classified as being either dynamical or secular. Dynamical instabilities, which are present even in the absence of dissipation, tend to operate on short timescales. Secular instabilities, which tend to operate on longer timescales, are driven by some dissipative mechanism such as gravitational radiation reaction.

5.1 Dynamical instabilities

One dynamical instability of particular importance is the bar mode instability[2]. This sets in high T/W, where T/W is the ratio of kinetic to potential energy - in other words, at high rotation rates. The low T/W instabilities, by contrast, are dynamical instabilities that develop at lower rotation rates in the presence of strong differential rotation. They are unstable shear modes, similar in some ways to the well-known Kelvin-Helmholtz instabilities[3]. Both types of dynamical instability may be important for newly-born neutron stars immediately after core collapse.

5.2 Secular instability to the emission of gravitational waves

The perturbation equations set out before do not contain dissipative terms. Unstable perturbations found using these equations are therefore dynamical instabilities. So do we need to include the dissipative terms in our governing equations in order to investigate secular instabilities? Fortunately, thanks to the pioneering work of Chandrasekhar, Friedman and Schutz (hereafter CFS), the answer to this question is no.

[2] See http://numrel.aei.mpg.de/Visualisations/Archive/Oscillations/barmode.html for a simulation showing the development of a bar mode.

[3] See http://www.astro.princeton.edu/ jstone/tests/kh/kh.html for some nice simulations of K-H instabilities at the interface of two fluids with different velocities.

Chandrasekhar, in 1970 was the first to demonstrate that gravitational radiation would give rise to a secular instability in the Maclaurin spheroids (rotating self-gravitating uniform density ellipsoids). Friedman and Schutz, in two papers in 1978, then proved that a secular instability to any radiation (gravitational or otherwise) was generic for inviscid rotating stars. In doing so they derived an instability criterion that could be used to determine whether a solution to the non-dissipative perturbation equations would be secularly unstable to the emission of gravitational radiation.

The basis of the instability criterion is a quantity called the canonical energy, E_c, that is conserved in a non-dissipative system.

$$
\begin{aligned}
E_c(\boldsymbol{\xi}) \quad = \quad & \frac{1}{2}\int\left[\rho\frac{\partial\xi^i}{\partial t}\frac{\partial\xi_i}{\partial t} - \rho v^j\nabla_j\xi^i v^k\nabla_k\xi_i + \Gamma P\left(\nabla_i\xi^i\right)^2 + 2\xi^i\nabla_i P\nabla_j\xi^j\right. \\
& \left. +\xi^i\xi^j\left(\nabla_i\nabla_j P + \rho\nabla_i\nabla_j\Phi\right) - \frac{1}{4\pi G}\nabla_i\delta\Phi\nabla^i\delta\Phi\right] dV, \qquad (25)
\end{aligned}
$$

where the perturbations are assumed to be real. One can also define a conserved canonical angular momentum J_c,

$$
J_c = -\int\rho\frac{\partial\xi^i}{\partial\varphi}\left(\frac{\partial\xi_i}{\partial t} + v^j\nabla_j\xi_i\right) dV. \qquad (26)
$$

Note that equations (25) and (26) are expressed in a coordinate basis, rather than in an orthonormal basis (unlike most of my other equations).

Friedman & Schutz then introduced radiation reaction terms to the perturbation equations, and showed that in a radiating system $dE_c/dt < 0$. Thus if $E_c > 0$ for given initial data, the system radiates energy until $E_c = 0$ and the perturbation has died away. The perturbation is therefore stable. If on the other hand $E_c < 0$ for given initial data, it will become increasingly negative, and the perturbation will be secularly unstable. One subtle point - when calculating E_c we use equation (25), which is strictly only valid for the non-dissipative case. In turns out that we can use this expression for E_c because radiation reaction is treated as a small perturbation on the non-dissipative system. Friedman & Schutz went on to demonstrate that for both isentropic and non-isentropic stars one can always define non-axisymmetric initial data (for high enough m) that will have a negative E_c. The question is then whether its growth rate will be sufficiently high to beat the various damping mechanisms such as viscosity.

In order to calculate E_c we need to know the Lagrangian displacement $\boldsymbol{\xi}$ associated with the perturbation. Some care is needed in doing this, so refer to the original papers if you want to do it right! But essentially we solve the non-dissipative equations, compute E_c for given initial data, and from there can tell whether the perturbation would be secularly unstable if we had solved the more complex dissipative problem.

In fact we can makes things a little bit simpler. For the normal modes of a uniformly rotating system, the secular instability condition $E_c < 0$ can be simplified to the following: *a mode is secularly unstable to the emission of radiation if it is retrograde in the rotating frame but prograde in the inertial frame*. Why this should be so can be understood from the following argument.

Firstly we note that for normal modes, there is a simple relation between E_c, J_c and the mode pattern speed in the inertial frame, σ_p.

$$\frac{E_c}{J_c} = \sigma_p. \tag{27}$$

Consider the modes of a rotating star. In the rotating frame, some of the modes propagate in the sense of rotation of the star, whilst others propagate in the other direction. The canonical angular momentum of the first class is positive, whilst that of the second class is negative (this follows from equation (26)). If the star is rotating slowly, the first class will appear to an inertial observer to be moving forward on the star, with pattern speed $\sigma_p > 0$. The second class will appear to be moving backwards on the star, with $\sigma_p < 0$. From equation (27) it follows that $E_c > 0$ for both classes and the perturbations are stable.

The rotation rate of the star is now increased to the point where a mode that moves backwards in the rotating frame appears to an inertial observer to be moving forwards. At this point the pattern speed σ_p changes sign, becoming positive. But the canonical angular momentum is defined with respect to the rotating frame, so J_c is still negative for this mode. E_c is therefore negative, and the perturbation is unstable. Most modes are unstable only above some threshold rotation rate. The r-modes are unusual in that they are unstable at *any* rotation rate.

In discussing instability to gravitational radiation, Friedman & Schutz concentrated on modes that had a non-zero frequency in the limit of zero rotation such as the f-modes. This was a reasonable step because these modes are associated with strong density perturbations and hence strong gravitational wave emission via the dominant mass quadrupole moment. Unfortunately, modelling of the f-mode suggested that only at very rapid rotation would the low m modes with the highest growth rates be CFS unstable, due to the limiting effects of viscosity.

Modes such as the r-modes, which emit gravitational waves primarily via the weaker current quadrupole moment, were not thought to be strong gravitational wave sources and were not addressed by Friedman & Schutz. They escaped attention until 1998, when Andersson showed that that the r-modes of a uniformly rotating perfect fluid star would be unstable for all rates of stellar rotation. The fact that the r-modes, in contrast to the f-modes, could be unstable at any rotation rate has given them new importance as a gravitational wave source.

Once a secularly unstable mode has been identified, we need to compute the growth timescale. To do this we can use post-Newtonian multipole expansions to estimate the effect of gravitational radiation back reaction. The gravitational wave luminosity associated with a mode of oscillation, measured in the rotating frame, is

$$\left.\frac{dE}{dt}\right|_{\mathrm{gw}} = -\omega_r \sum_{l=2}^{\infty} N_l \omega^{2l+1}(|\delta D_{lm}|^2 + |\delta J_{lm}|^2), \tag{28}$$

where

$$N_l = \frac{4\pi G}{c^{2l+1}} \frac{(l+1)(l+2)}{l(l-1)[(2l+1)!!]^2}. \tag{29}$$

The mass multipoles δD_{lm} are given by

$$\delta D_{lm} = \int \delta\rho r^l Y_l^{m*} dV, \tag{30}$$

while the current multipoles δJ_{lm} are given by

$$\delta J_{lm} = \frac{2}{c}\left[\frac{l}{l+1}\right]^{\frac{1}{2}} \int r^l(\rho\delta\mathbf{v} + \delta\rho\mathbf{v}) \cdot \mathbf{Y}_{lm}^{B*} dV. \tag{31}$$

with

$$\mathbf{Y}_{lm}^{B} = \sqrt{l(l+1)}\left(0, \frac{1}{\sin\theta}\frac{\partial Y_l^m}{\partial\varphi}, -\frac{\partial Y_l^m}{\partial\theta}\right) \tag{32}$$

Conservation of mass precludes the existence of gravitational monopoles. The mass dipole moment cannot radiate without violating conservation of momentum, while the current dipole moment cannot radiate without violating conservation of angular momentum. The lowest radiative multipoles are therefore the mass and current quadrupole moments.

For modes associated with large density perturbations, such as the f-, p- and g-modes, mass quadrupole emission will dominate, because for a given l the current multipoles are $\sim 1/c$ smaller than the mass multipoles. The contribution from higher multipoles will also be weaker, because emission weakens as l rises. For the inertial modes, however, the situation can change. The density perturbation associated with these modes is far smaller than the velocity perturbations, and the current multipoles may be comparable in strength to the mass multipoles. For the $l = m = 2$ r-modes of a barotropic star, emission is dominated by the current quadrupole moment (see Section 6).

Having identified the dominant emission multipole for a given mode of oscillation (using the eigenfunctions computed for the non-dissipative system) we return to equation (28) and compute the gravitational wave luminosity $dE/dt|_{\rm gw}$. We then estimate the timescale associated with the rate of energy loss via gravitational waves, τ_g, by assuming that the unstable eigenfunctions grow as $\exp(t/|\tau_g|)$. The mode energy depends on the square of the perturbation, hence we can write

$$\left.\frac{dE}{dt}\right|_{\rm gw} = -\frac{2E}{|\tau_g|}. \tag{33}$$

Knowing E and $dE/dt|_{\rm gw}$ for a given perturbation we can identify τ_g.

5.3 Damping mechanisms

An unstable mode (dynamical or secular) will only be astrophysically relevant if its growth time is shorter than the timescale on which the mode is damped. The major damping mechanisms in a neutron star are due to viscosity, the two main types being shear viscosity and beta-decay induced bulk viscosity. Other mechanisms such as hyperon bulk viscosity, and damping associated with the formation of a neutron star crust, have also been discussed in the literature.

Shear viscosity is dissipation caused by scattering as particles move within the star, neutron-neutron scattering being the dominant process. Shear viscosity is the main viscous process for temperatures from a few times 10^9 K down to 10^9 K (below this temperature we expect the neutrons to form a superfluid).

At higher temperatures, the main viscous mechanism is bulk viscosity. Bulk viscosity arises because the pressure and density fluctuations associated with the mode drive the star out of beta equilibrium (the state in which beta decays are exactly balanced by inverse beta decays). Energy is dissipated in the form of neutrinos as weak nuclear reactions attempt to re-establish this equilibrium.

The damping timescales associated with the viscous mechanisms are estimated in the same way as the growth time for radiation back reaction. We start with calculations of the viscous processes which give the rate of energy dissipation for a given mechanism, $dE/dt|_d$. Assuming the perturbation varies as $\exp(-t/\tau_d)$, τ_d being the dissipation timescale, we obtain

$$\left.\frac{dE}{dt}\right|_d = -\frac{2E}{\tau_d}. \tag{34}$$

This enables us to calculate the dissipation timescale for a given perturbation.

6 A BIT MORE ON THE R-MODES

In this section I outline the key characteristics of the r-modes, using the same type of simple dynamical model used in previous sections, to show why they are particularly prone to gravitational radiation driven secular instability (Section 5.2).

6.1 Mode characteristics

Previously, we saw that in the non-rotating limit the r-modes reduce to a zero-frequency degenerate set of toroidal perturbations. In this section we will derive their properties for a slowly-rotating star (working in spherical polar coordinates $[r, \theta, \varphi]$) by assuming that they are, to leading order in Ω, still purely toroidal. This means that to leading order only δv_θ and δv_φ will be non-zero. Spheroidal contributions, and the associated physical variables $\delta v_r, \delta\rho$ and δP appear at higher order. In the limit as $\Omega \to 0$ all of the physical variables and the frequency must tend to zero.

Making these assumptions we expand in terms of the small parameter Ω^2:

$$
\begin{aligned}
\omega &\approx \Omega\omega_0(1+\Omega^2\omega_1) && (35)\\
\delta v_\theta &\approx \Omega(\delta v_\theta^0+\Omega^2\delta v_\theta^1) && (36)\\
\delta v_\varphi &\approx \Omega(\delta v_\varphi^0+\Omega^2\delta v_\varphi^1) && (37)\\
\delta v_r &\approx \Omega(\Omega^2\delta v_r^1) && (38)\\
\delta\rho &\approx \Omega^2\delta\rho_1 && (39)\\
\delta P &\approx \Omega^2\delta P_1. && (40)
\end{aligned}
$$

The rotation of the star will lead to centrifugal flattening, but when the rotation is slow the departure from sphericity is small, and to first order this effect can be neglected. The perturbation of the gravitational potential is also neglected, an approximation that is known as the Cowling approximation. We will consider barotropic perturbations. Setting $\Gamma_1 = \Gamma$ in equation (12) we obtain the leading order perturbed equation of state

$$\frac{\delta P_1}{P} = \Gamma\frac{\delta\rho_1}{\rho}. \tag{41}$$

To leading order, the perturbed continuity equation, equation (9), yields

$$\frac{\partial}{\partial\theta}\left(\delta v_\theta^0\sin\theta\right) + im\delta v_\varphi^0 = 0. \tag{42}$$

The solutions to equation (42) are, as expected, a set of toroidal eigenfunctions,

$$
\begin{aligned}
\delta v_\theta^0 &= imT_l^m(r)\frac{Y_l^m(\cos\theta)}{\sin\theta}\\
\delta v_\varphi^0 &= -T_l^m(r)\frac{\partial}{\partial\theta}Y_l^m(\cos\theta). && (43)
\end{aligned}
$$

To find the leading order frequency coefficient ω_0 we use the perturbed Euler equation, equation (10). To leading order the θ and φ components of this equation are

$$i(m-\omega_0)\delta v_\theta^0 - 2\cos\theta\delta v_\varphi^0 = -\frac{1}{\rho r}\frac{\partial\delta P_1}{\partial\theta} \tag{44}$$

$$2\cos\theta\delta v_\theta^0 + i(m-\omega_0)\delta v_\varphi^0 = -\frac{im\delta P_1}{\rho r\sin\theta}. \tag{45}$$

Eliminating δP_1 between equations (44) and (45) and using equation (43), we obtain the vorticity equation for the perturbations.

$$\frac{T_l^m}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial Y_l^m}{\partial\theta}\right]+\left[\frac{2m}{m-\omega_0}-\frac{m^2}{\sin^2\theta}\right]T_l^m Y_l^m=0. \tag{46}$$

Comparing to Legendre's equation, we can solve for ω_0. We find that to leading order the r-mode frequency is

$$\omega=\omega_0\Omega=m\Omega\left[\frac{(l+2)(l-1)}{l(l+1)}\right]. \tag{47}$$

To determine the amplitude, T_l^m, consider the r component of the perturbed Euler equation at leading order,

$$-2\sin\theta\delta v_\varphi^0=\frac{\delta\rho_1}{\rho^2}\frac{dP}{dr}-\frac{1}{\rho}\frac{\partial\delta P_1}{\partial r}. \tag{48}$$

Using equation (45) to eliminate δP_1, and equations (41) and (43), equation (48) becomes

$$\left[T_l^m+\frac{T_l^m}{l(l+1)}+\frac{r}{l(l+1)}\frac{dT_l^m}{dr}\right]\sin\theta\frac{\partial Y_l^m}{\partial\theta}=\left[T_l^m+r\frac{dT_l^m}{dr}\right]\cos\theta Y_l^m. \tag{49}$$

We use the following relations of spherical harmonics:

$$\sin\theta\frac{\partial Y_l^m}{\partial\theta}=lQ_{l+1}Y_{l+1}^m-(l+1)Q_lY_{l-1}^m \tag{50}$$

and

$$\cos\theta Y_l^m=Q_{l+1}Y_{l+1}^m+Q_lY_{l-1}^m, \tag{51}$$

where

$$Q_l=\left[\frac{(l+m)(l-m)}{(2l+1)(2l-1)}\right]^{1/2}. \tag{52}$$

Substituting into equation (49) we obtain an equation in terms of Y_{l+1}^m and Y_{l-1}^m. Because we have assumed that the mode is represented by a single spherical harmonic (equation (43)), we can decouple this resulting equation into two separate equations corresponding to the coefficients of the two different spherical harmonics:

$$Q_l\left[T_l^m+\frac{1}{l}\frac{d}{dr}(rT_l^m)\right]=0, \tag{53}$$

and

$$Q_{l+1}\left[T_l^m-\frac{1}{l+1}\frac{d}{dr}(rT_l^m)\right]=0. \tag{54}$$

Equation (53) can only be satisfied if $Q_l = 0$, which requires $l = m$. This means that a barotropic star has only one r-mode solution for each value of l, that for which $l = m$. In this case equation (54) becomes

$$(l+1)T_l^l - \frac{d}{dr}(rT_l^l) = 0. \tag{55}$$

Thus $T_l^l \sim r^l$, which means that the r-mode motion will be strongest near the surface of the star. To leading order, we have

$$\begin{aligned} \delta v_\theta &= im\Omega r^l \frac{Y_l^m}{\sin\theta} \\ \delta v_\varphi &= -\Omega r^l \frac{\partial Y_l^m}{\partial\theta}. \end{aligned} \tag{56}$$

Comparing to equation (15), and introducing a dimensionless mode amplitude α, we can rewrite equation (56) for $l = m$ as

$$\delta\mathbf{v} = \alpha\Omega R \left(\frac{r}{R}\right)^l \mathbf{Y}_{ll}^B e^{-i\omega t}. \tag{57}$$

Equation (57) can be used to show that at leading order the fluid elements move in ellipses on the surfaces of constant pressure, with the amplitude of the ellipses being determined by the mode amplitude α.

6.2 The r-mode instability

Equation (47) gives the inertial frame frequency of the r-mode to leading order. The pattern speed in the inertial frame, σ_p, is

$$\sigma_p = \Omega\left[\frac{(l+2)(l-1)}{l(l+1)}\right]. \tag{58}$$

All r-modes with $l > 1$ are prograde in the inertial frame. The pattern speed in the rotating frame, σ_{pr}, is

$$\sigma_{pr} = \sigma_p - \Omega = -\frac{2\Omega}{l(l+1)}. \tag{59}$$

The r-modes are therefore retrograde in the rotating frame. The r-modes of a perfect fluid star therefore meet the condition for instability to the emission of gravitational radiation outlined in Section 5.2. The fact that this is true for any rotation rate (in contrast to the f-modes, which require a certain minimum rotation rate for instability), implies that unstable r-modes could be a potentially strong source of gravitational waves.

In Section 5.2 we saw that the strongest gravitational wave emission should come from the lowest l unstable multipoles. The $l = m = 2$ r-mode should therefore be the strongest emitter. But is it the mass or current quadrupole that dominates? To leading order the r-modes are purely toroidal, with $l = m$. The density perturbations, which enter at higher order, are spheroidal. These spheroidal components have $l = m + 1$, because toroidal and spheroidal components couple in such a way as to preserve behaviour under parity. In addition, the density perturbation is of order Ω^2 while the velocity perturbations are of order Ω. It is clear from equations (30) and (31) that the leading order contribution to the gravitational wave emission comes from the current multipole not the mass multipole[4]. The dominant multipole is the $l = m = 2$ current multipole.

7 FURTHER READING

The r-mode instability in rotating neutron stars, N. Andersson, K.D. Kokkotas & V. Ferrari, International Journal of Modern Physics D, 10, 381-441 (2001), arXiv:gr-qc/001010

TOPICAL REVIEW: Gravitational waves from instabilities in relativistic stars, N. Andersson, Classical and Quantum Gravity, 20, 105-+ (2003), arXiv:astro-ph/0211057

Secular instability of rotating neutron stars, J.L.Friedman & B.F. Schutz, Astrophysical Journal, 222, 281-296 (1978)

[4] Note that this may not be true for the r-modes of non-barotropic stars.