Pure Spinor Superspace and Superstring Scattering Amplitudes

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Outline

Motivation

- Find hidden structures in superstring scattering amplitudes
- Results so far suggest that the pure spinor formalism is a good tool to use
- Clean up existing computations and express them in pure spinor superspace (only one computation for all external states related by supersymmetry)
- More recent motivation: Obtain/check the BCJ relations in a supersymmetric framework

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Computation of superstring scattering amplitudes:

- 4-pt @ 2-loop (Berkovits,C.M.)
- 4-pt @ 1-loop (Berkovits,C.M.)
- 4-pt: tree-level, 1-loop and 2-loop are proportional (C.M.)
- Anomaly, minimal $\leftrightarrow$ non-minimal (Berkovits,C.M.)
- 5-pt @ 1-loop (C.M., C. Stahn)
- 5-pt @ tree-level
  - Derivation of 5-pt BCJ relations (work in progress)

Some highlights

Pure spinor superspace representation of kinematic factors

- 4-pt at tree-level:

$$K_0 = \frac{1}{2} k_1^m k_2^n \langle (\lambda A^1)(\lambda A^2)(\lambda A^3)\mathcal{F}^4_{mn}\rangle - (k^1 \cdot k^3)\langle A^1_n (\lambda A^2)(\lambda A^3)(\lambda\gamma^n W^4)\rangle$$

- 4-pt at 1-loop:

$$K_1 = \langle (\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\rangle$$

- 4-pt at 2-loops:

$$K_2 = \langle (\lambda\gamma^{mnpqr}\lambda)\mathcal{F}^1_{mn}\mathcal{F}^2_{pq}\mathcal{F}^3_{rs}(\lambda\gamma^s W^4)\rangle$$

- Anomaly (gauge variation of 6-pt 1-loop)

$$\langle (\lambda\gamma^m W)(\lambda\gamma^n W)(\lambda\gamma^p W)(W\gamma_{mnp} W)\rangle$$

- 5-pt @ 1-loop:

$$L_{12} = -\Big\langle \Big[(\lambda A^1)(k^1 \cdot A^2) + A^1_p(\lambda\gamma^p W^2)\Big](\lambda\gamma^m W^5)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\Big\rangle$$

$$K_{25} = -\Big\langle(\lambda A^1)\Big[(\lambda\gamma^m W^2)(k^2 \cdot A^5) - \frac{1}{4}(\lambda\gamma^m\gamma^{rs}W^5)\mathcal{F}^2_{rs}\Big](\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\Big\rangle -$$

Superspace Identities

Explicit proof in pure spinor superspace that the massless 4-point amplitudes at tree-level, one-loop and two-loops are all proportional to each other (up to Mandelstam variables).

Pure Spinor Formalism, Berkovits 2000

Action (Minimal Pure Spinor Formalism)

$$S = \int d^2z \left(\frac{1}{2} \partial X^m \overline{\partial} X_m + p_\alpha \overline{\partial} \theta^\alpha - w_\alpha \overline{\partial} \lambda^\alpha \right)$$

Bosonic Pure Spinor

$$(\lambda \gamma^m \lambda) = 0$$

Pure Spinor Formalism

Some important definitions for amplitude computations:

- Lorentz current

$$N^{mn} = \frac{\alpha\prime}{4}(w\gamma^{mn}\lambda)$$

- Supersymmetric momentum

$$\Pi^m = \partial X^m + \frac{1}{2}(\theta\gamma^m\partial\theta)$$

- Supersymmetric derivative

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}(\theta\gamma^m)_\alpha\partial_m$$

Pure Spinor Formalism

- Supersymmetric Green-Schwarz constraint

$$d_\alpha = \frac{\alpha'}{2} p_\alpha - \frac{1}{2}(\gamma^m\theta)_\alpha \partial X_m - \frac{1}{8}(\gamma^m\theta)_\alpha(\theta\gamma_m\partial\theta)$$

- The b-ghost is a composite operator...

$$b_{\rm non-min} = \ldots + (\overline{\lambda}\gamma^{mnp} r)(d\gamma_{mnp} d) + \ldots$$

$$b = \ldots + d^4\delta'(N) + \ldots$$

Relevant OPE's

$$X^m(z,\overline{z})X^n(w,\overline{w}) \longrightarrow -\frac{1}{2}\eta^{mn}\ln|z-w|^2$$

$$N^{mn}(z)\lambda^\alpha(y) \longrightarrow \frac{\alpha\prime}{4}\frac{(\gamma^{mn}\lambda)^\alpha}{z-y}$$

$$d_\alpha(z)V(y,\theta) \longrightarrow \frac{D_\alpha V(y,\theta)}{z-y}$$

$$\Pi^m(z)V(y,\theta) \longrightarrow \frac{\partial^m V(y,\theta)}{z-y}$$

Issues of RNS and GS not present

Space-time SUSY

The pure spinor formalism has manifest space-time supersymmetry

- Scattering amplitudes will result in superspace expressions
- Only one computation for all multiplet states

Covariant BRST Quantization

$$Q_{\text{BRST}} = \oint \lambda^\alpha d_\alpha$$

Prescription for Scattering Amplitudes

- Massless On-shell Vertex Operators:
  - Unintegrated

$$V = \lambda^\alpha A_\alpha(X, \theta), \quad QV = 0$$

  - Integrated

$$U = \int dz \left(\partial\theta^\alpha A_\alpha + A_m \Pi^m + d_\alpha W^\alpha + \frac{1}{2} N^{mn} \mathcal{F}_{mn} \right), \quad QU = \partial V$$

- Where $A_\alpha(x, \theta), A_m(x, \theta), W^\alpha(x, \theta)$ and $\mathcal{F}_{mn}(x, \theta)$ are the SYM superfields

$$D_\alpha A_\beta + D_\beta A_\alpha = \gamma^m_{\alpha\beta} A_m, \quad D_\alpha A_m = (\gamma_m W)_\alpha + k_m A_\alpha$$

$$D_\alpha W^\beta = \frac{1}{4} (\gamma^{mn})_\alpha{}^\beta \mathcal{F}_{mn}, \quad D_\alpha \mathcal{F}_{mn} = 2k_{[m} (\gamma_{n]} W)_\alpha$$

θ's all over the place

SYM Superfields θ-Expansion

$$A_\alpha(x,\theta) = \frac{1}{2}a_m(\gamma^m\theta)_\alpha - \frac{1}{3}(\xi\gamma_m\theta)(\gamma^m\theta)_\alpha - \frac{1}{32}F_{mn}(\gamma_p\theta)_\alpha(\theta\gamma^{mnp}\theta) + \ldots$$

$$A_m(x,\theta) = a_m - (\xi\gamma_m\theta) - \frac{1}{8}(\theta\gamma_m\gamma^{pq}\theta)F_{pq} + \frac{1}{12}(\theta\gamma_m\gamma^{pq}\theta)(\partial_p\xi\gamma_q\theta) + \ldots$$

$$W^\alpha(x,\theta) = \xi^\alpha - \frac{1}{4}(\gamma^{mn}\theta)^\alpha F_{mn} + \frac{1}{4}(\gamma^{mn}\theta)^\alpha(\partial_m\xi\gamma_n\theta)$$
$$+\frac{1}{48}(\gamma^{mn}\theta)^\alpha(\theta\gamma_n\gamma^{pq}\theta)\partial_m F_{pq} + \ldots$$

$$\mathcal{F}_{mn}(x,\theta) = F_{mn} - 2(\partial_{[m}\xi\gamma_{n]}\theta) + \frac{1}{4}(\theta\gamma_{[m}\gamma^{pq}\theta)\partial_{n]}F_{pq} + \ldots,$$

Tree-level Amplitudes

- The prescription for tree-level amplitudes is given by

Tree-level N-point

$$\mathcal{A}_N = \langle V_1(z_1)V_2(z_2)V_3(z_3)\int dz_4 U_4(z_4)\ldots\int dz_N U_N(z_N)\rangle$$

- Computation proceeds as usual in a CFT
- Use OPE's to integrate out conformal weight 1 variables
- Then integrate out zero-modes of λ^α and θ^α

$\lambda^3\theta^5$ prescription

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle = 1$$

Pure Spinor Superspace

The computation of scattering amplitudes gives rise to pure spinor superspace (PSS) expressions:

1. 3 pure spinors λ^α and superfields such that $\langle\lambda^3\theta^5\rangle = 1$
2. Manifestly Lorentz covariant
3. Supersymmetric

- Component expansions can be computed
- Use FORM to do the boring stuff

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What makes PSS interesting?

- There is a close relation between $Q_{BRST} = \lambda^\alpha D_\alpha$ and the EOM

$$D_\alpha A_\beta + D_\beta A_\alpha = \gamma^m_{\alpha\beta} A_m$$

$$D_\alpha A_m = (\gamma_m W)_\alpha + k_m A_\alpha, \quad D_\alpha \mathcal{F}_{mn} = 2k_{[m}(\gamma_{n]} W)_\alpha$$

- Kinematic factors are on-shell, so EOM and BRST integrations by parts can be used to find non-trivial relations. Example from 4-pt at tree-level

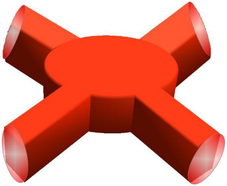
$$\langle D_\alpha(\lambda A^1)(\lambda A^2)(\lambda A^3) W_4^\alpha \rangle = -\langle (\lambda D A^1_\alpha)(\lambda A^2)(\lambda A^3) W_4^\alpha \rangle$$

$$+\langle A^1_m (\lambda A^2)(\lambda A^3)(\lambda \gamma^m W^4) \rangle$$

As BRST-exact terms decouple,

$$\langle (\lambda D A^1_\alpha)(\lambda A^2)(\lambda A^3) W_4^\alpha \rangle = \frac{1}{4} \langle (\lambda \gamma^{mn} A^1)(\lambda A^2)(\lambda A^3) \mathcal{F}^4_{mn} \rangle$$

Four gravitons at tree-level



Example

$$\mathcal{A} = \langle V^1(z_1,\overline{z}_1)V^2(z_2,\overline{z}_2)V^3(z_3,\overline{z}_3)\int_{\mathbb{C}} d^2z U^4(z,\overline{z})\rangle$$

where $V^i(z,\overline{z}) = V^i(z)\otimes\tilde{V}^i(\overline{z})e^{ik\cdot X}$ and $U(z,\overline{z}) = U(z)\otimes\tilde{U}(\overline{z})e^{ik\cdot X}$

Tree-level 4-graviton computation

Sidenote

Previous computation (Policastro, Tsimpis 2006) was done in a way that hid the simplicity of the result. Cancellations were overlooked and no simple pure spinor expression was written down for the kinematical factor.

- We have to compute

$$\langle(\lambda A^1)(z_1)(\lambda A^2)(z_2)(\lambda A^3)(z_3)\int d^2z(\Pi^m A^4_m + (dW^4) + \frac{1}{2}N^{mn}\mathcal{F}_{mn})\rangle$$

$$\otimes\text{(right-moving part)}$$

Tree-level 4-graviton computation

- Delay as long as possible explicit evaluation of pure spinor integrals!

OPE identity

$$\langle(\lambda A^1)(\lambda A^2)(\lambda A^3)((dW^4) + \frac{1}{2}N^{mn}\mathcal{F}_{mn})\rangle =$$

$$+\frac{\alpha'}{2(z_1 - z_4)}\langle A^1_m(\lambda A^2)(\lambda A^3)(\lambda\gamma^m W^4)\rangle - (1 \leftrightarrow 2) + (1 \leftrightarrow 3)$$

Proof: Instead of using OPEs of p_α and ∂X^m individually in

$$d_\alpha = \frac{\alpha'}{2}p_\alpha - \frac{1}{2}(\gamma^m\theta)_\alpha\partial X_m - \frac{1}{8}(\gamma^m\theta)_\alpha(\theta\gamma_m\partial\theta)$$

use OPE of d_α, SYM EOM and BRST cohomology,

$$d_\alpha(\lambda A) \to D_\alpha(\lambda A) = -(\lambda D)A_\alpha + (\lambda\gamma^m)_\alpha A_m$$

Tree-level 4-graviton computation

- The amplitude becomes

$$\mathcal{A} = \text{const} \int d^2 z_4 \left(\frac{F_1}{z_1 - z_4} + \frac{F_2}{z_2 - z_4} \right) \otimes \left(\frac{\tilde{F}_1}{\overline{z}_1 - \overline{z}_4} + \frac{\tilde{F}_2}{\overline{z}_2 - \overline{z}_4} \right)$$

$$\cdot |z_4|^{-\alpha' t/2} |1 - z_4|^{-\alpha' u/2}$$

where

$$F_1 = i k^1_m \langle (\lambda A^1)(\lambda A^2)(\lambda A^3) A^4_m \rangle + \langle A^1_m (\lambda A^2)(\lambda A^3)(\lambda \gamma^m W^4) \rangle$$

$$F_2 = i k^2_m \langle (\lambda A^1)(\lambda A^2)(\lambda A^3) A^4_m \rangle - \langle (\lambda A^1) A^2_m (\lambda A^3)(\lambda \gamma^m W^4) \rangle$$

Tree-level 4-graviton computation

- Using the general formula

$$\int d^2 z z^A (1-z)^B \bar{z}^{\tilde{A}} (1-\bar{z})^{\tilde{B}} = 2\pi \frac{\Gamma(1+A)\Gamma(1+B)}{\Gamma(2+A+B)} \cdot \frac{\Gamma(-1-\tilde{A}-\tilde{B})}{\Gamma(-\tilde{A})\Gamma(-\tilde{B})}$$

one can pull some Mandelstam invariants from the Gamma functions to obtain

$$\mathcal{A} = K_0 \tilde{K}_0 \frac{\Gamma(-\alpha' t/4)\Gamma(-\alpha' u/4)\Gamma(-\alpha' s/4)}{\Gamma(1+\alpha' s/4)\Gamma(1+\alpha' t/4)\Gamma(1+\alpha' u/4)}$$

where

$$K_0 = uF_1 - tF_2$$

is given by

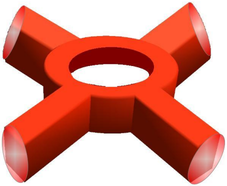
Tree-level 4-graviton result

$$K_0 = \langle \partial^n(\lambda A^1)\partial^m(\lambda A^2)(\lambda A^3)\mathcal{F}^4_{mn}\rangle$$
$$+\langle(\partial_p A^1_m)(\lambda A^2)\partial^p(\lambda A^3)(\lambda\gamma^m W^4)\rangle$$
$$+\langle(\lambda A^1)(\partial_p A^2_m)\partial^p(\lambda A^3)(\lambda\gamma^m W^4)\rangle$$

- Simplify it even more..(CM 2008)

$$K_0 = -\langle(\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\rangle$$

Massless 4-point one-loop amplitude



Prescription

$$\mathcal{A}_N = \langle \mathcal{N} \left(\int \mu \cdot b \right) V_1(z_1) \int U_2 \int U_3 \int U_4 \rangle$$

Massless 4-point one-loop amplitude

- This amplitude was computed with the minimal pure spinor formalism (Berkovits 2004)

$$K_1 = \int d^{16}\theta (\epsilon T^{-1})^{((\alpha\beta\gamma))}_{[\rho_1 \dots \rho_{11}]} \theta^{\rho_1} \dots \theta^{\rho_{11}} (\gamma_{mnpqr})_{\beta\gamma} \times$$

$$\Big[A_{1\alpha}(\theta)(W_2(\theta)\gamma^{mnp} W^3(\theta))\mathcal{F}_4^{qr}(\theta) \Big]$$

and shown to agree with the RNS and GS results (C.M. 2005)

$$K_1 = \langle (\lambda A)(\lambda \gamma^m W)(\lambda \gamma^n W)\mathcal{F}_{mn} \rangle = t_8 F^4 + \dots$$

- Computed also in the non-minimal pure spinor formalism (Berkovits 2005, Berkovits & C.M. 2006)

How to get it (Non-minimal)

- One can compute it quickly by using symmetry alone
- Recall the regulator (schematically)

$$N = \exp\left(-(\lambda\bar{\lambda}) - (r\theta) - (w\bar{w}) + (sd)\right)$$

- Zero modes:
  - s^α has 11 zero-modes
  - d_α has 16 zero-modes
- There is only one way to get a non-vanishing result
- N contributes 11 s and 11 d, the b-ghost 2 d and the external vertices 3 d's.
- Therefore one gets (λA) and $(dW)^3$ from the external vertices and $(\bar{\lambda}\gamma^{mnp}r)(d\gamma_{mnp}d)$ from the b-ghost

Unique contraction

There is only one Lorentz invariant contraction for these fields

$$\langle(\bar{\lambda}\gamma_{mnp}D)(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)(\lambda\gamma^p W)\rangle$$

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Equivalence with MPS

- The (old) PS result was

$$K_1 = \langle (\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)\mathcal{F}_{mn}\rangle$$

- The NMPS answer was argued to be proportional to MPS (Berkovits, 2005), and shown using U(5)-covariant notations (Berkovits, C.M., 2006)
- Covariant proof later (C.M., 2008)

$$\langle (\bar\lambda\gamma_{mnp}D)(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)(\lambda\gamma^p W)\rangle$$

$$= 40\langle (\lambda\bar\lambda)(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)\mathcal{F}_{mn}\rangle$$

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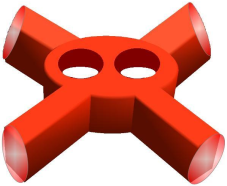
Massless 4-point two-loop amplitude



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Massless 4-point two-loop amplitude

- Can be computed quickly using zero-mode saturation (Berkovits, 2005)
- $(\overline{\lambda}\gamma_{mnp}D)^3$ from b-ghosts and W^4 from external vertices
- Using $DW = F$ one gets $K =< FFFW >$, which has a unique Lorentz invariant contraction

$$K_2 = \langle(\lambda\gamma^{mnpqr}\lambda)\mathcal{F}^1_{mn}\mathcal{F}^2_{pq}\mathcal{F}^3_{rs}(\lambda\gamma^s W^4)\rangle$$

Relating tree-level, one-loop and two-loops

- Beautiful pure spinor superspace expressions for the kinematical factors at tree-level, one- and two-loops
- Direct comparison with RNS through component expansions
- PS survived the (non-trivial) tests
- Can one exploit the PS superspace a bit more?
- Yes. There are superspace identities linking the 4-pt amplitudes (C.M. 2008)

Massless four-point identities

$$K_0 = -\langle (\lambda A^1)(\lambda\gamma^n W^2)(\lambda\gamma^m W^3)\mathcal{F}^4_{mn}\rangle = -K_1$$

$$K_2 = \langle (\lambda\gamma^{mnpqr}\lambda)\mathcal{F}^1_{mn}\mathcal{F}^2_{pq}\mathcal{F}^3_{rs}(\lambda\gamma^s W^4)\rangle =$$

$$-16(k^1\cdot k^2)\langle (\lambda A^2)(\lambda\gamma^r W^1)(\lambda\gamma^s W^4)\mathcal{F}^3_{rs}\rangle = -16(k^1\cdot k^2)K_1$$

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Quick look at the 2-loop proof

- One begins by noticing that

$$(\lambda\gamma^{mnpqr}\lambda)\mathcal{F}^1_{mn}\mathcal{F}^2_{pq}\mathcal{F}^3_{rs}(\lambda\gamma^s W^4) =$$

$$-4Q\left[(\lambda\gamma^r\gamma^{mn}W^2)(\lambda\gamma^s W^4)\mathcal{F}^1_{mn}\mathcal{F}^3_{rs}\right]$$

$$-8k^1_m(\lambda\gamma_n W^1)(\lambda\gamma^r\gamma^{mn}W^2)(\lambda\gamma^s W^4)\mathcal{F}^3_{rs},$$

- Pure spinor constraint and EOM $k^1_m(\gamma^m W^1)_\alpha = 0$ imply

$$k^1_m(\lambda\gamma_n W^1)(\lambda\gamma^r\gamma^{mn}W^2) = -2k^1_m(\lambda\gamma^r W^1)(\lambda\gamma^m W^2)$$

- Using $(\lambda\gamma_m W^2) = QA^2_m - ik^2_m(\lambda A^2)$ and $\langle(\lambda\gamma^r W^1)Q(A^m_2)(\lambda\gamma^s W^4)\mathcal{F}^3_{rs}\rangle = 0$ one gets the desired result

$$\langle(\lambda\gamma^{mnpqr}\lambda)\mathcal{F}^1_{mn}\mathcal{F}^2_{pq}\mathcal{F}^3_{rs}(\lambda\gamma^s W^4)\rangle$$

$$= -16(k^1\cdot k^2)\langle(\lambda A^2)(\lambda\gamma^r W^1)(\lambda\gamma^s W^4)\mathcal{F}^3_{rs}\rangle$$

Tree-level...

- The proof that

$$K_0 = \langle \partial^n(\lambda A^1)\partial^m(\lambda A^2)(\lambda A^3)\mathcal{F}^4_{mn}\rangle$$
$$+\langle(\partial_p A^1_m)(\lambda A^2)\partial^p(\lambda A^3)(\lambda\gamma^m W^4)\rangle$$
$$+\langle(\lambda A^1)(\partial_p A^2_m)\partial^p(\lambda A^3)(\lambda\gamma^m W^4)\rangle$$

is equal to

$$K_0 = -\langle(\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)F^4_{mn}\rangle = -\frac{1}{3}K_1,$$

is not so straightforward and requires many BRST integrations by parts of carefully chosen terms etc. But it can be done (C.M. 2008)

Massless amplitudes for 5-pts

- Are things "simple" for 5-pts?
- The 5-pt at 1-loop was computed with the NMPS and a nice answer was obtained (C.M., C. Stahn, 2009)
- However, not so straigthforward

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Massless amplitudes for 5-pts

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Massless amplitudes for 5-pts

- There are 4 ways to saturate d_α zero modes:

$$\frac{1}{2}\Big\langle \frac{\Pi^m(z_0)}{(\lambda\bar\lambda)}(\bar\lambda\gamma_m d)(\lambda A^1)(dW^2)(dW^3)(dW^4)(dW^5)\Big\rangle$$

$$-\frac{1}{16}\Big\langle \frac{(r\gamma_{mnp}r)}{(\lambda\bar\lambda)^3}(\bar\lambda\gamma^m d)N^{np}(z_0)(\lambda A^1)(dW^2)(dW^3)(dW^4)(dW^5)\Big\rangle$$

$$\frac{1}{96}\Big\langle \frac{(\bar\lambda\gamma_{mnp}r)}{(\lambda\bar\lambda)^2}(d\gamma^{mnp}\hat d(z_0))(\lambda A^1)(dW^2)(dW^3)(dW^4)(dW^5)\Big\rangle$$

$$\frac{1}{192}\Big\langle \frac{(\bar\lambda\gamma_{mnp}r)}{(\lambda\bar\lambda)^2}(d\gamma^{mnp}d)(\lambda A^1)(dW^2)(dW^3)(dW^4)$$

$$\times\big(A^5_q\Pi^q + (\hat d W^5) + \frac{1}{2}N^{mn}\mathcal{F}^5_{mn})\Big\rangle + \text{cycl}(2345)$$

5-pt at 1-loop

- One can show that the OPE's involving the b-ghost are total derivatives and vanish
- The non-vanishing contribution is similar to the 4-pt

$$(\overline{\lambda}\gamma_{mnp}D)(\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)(\lambda\gamma^p W^4)\times$$

$$\times\left(A^5_q\Pi^q + (\hat{d}W^5) + \frac{1}{2}N^{mn}\mathcal{F}^5_{mn}\right)$$

- After computing the OPE's and acting with the derivative D_α and using lots of manipulations (BRST-exact, PS constraint, gamma matrix ids, EOM of SYM etc) and dropping total derivative terms one realizes that. . .
- The computation is the same as

$$\langle(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)\mathcal{F}_{mn}\int U\rangle$$

- Simple answer (notice how $\overline{\lambda}$ is gone!)

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5-pts at tree-level

- The RNS computation of the tree-level 5-pt amplitude is messy (Medina et al., 2002, 2005)
- Can one streamline it using pure spinors?
- Yes, the supersymmetric answer is simple

$$A_5 = T\, A_{YM}(\theta) + K_3 A_{F^4}$$

where T and K_3 have known momenta expansions and

$$A_{YM} = \frac{\tilde{L}_{2131}}{\alpha_{12}\alpha_{45}} - \frac{\tilde{L}_{3424}}{\alpha_{34}\alpha_{51}} - \frac{\tilde{L}_{2334}}{\alpha_{23}\alpha_{51}} - \frac{\tilde{L}_{2331}}{\alpha_{23}\alpha_{45}} - \frac{L_{2134}}{\alpha_{12}\alpha_{34}} - \frac{L_{23}}{\alpha_{23}}$$

$$A_{F^4}(\theta) =$$

$$+L_{2331}\left(\frac{\alpha_{12}}{\alpha_{45}} + \frac{\alpha_{51}}{\alpha_{23}}\right) + L_{2334}\left(\frac{\alpha_{34}}{\alpha_{51}} + \frac{\alpha_{45}}{\alpha_{23}}\right) + L_{2134}\left(\frac{\alpha_{45}}{\alpha_{12}} + \frac{\alpha_{51}}{\alpha_{34}}\right)$$

$$+L_{3424}\left(\frac{\alpha_{12}}{\alpha_{34}} + \frac{\alpha_{23}}{\alpha_{51}}\right) - L_{2131}\left(\frac{\alpha_{34}}{\alpha_{12}} + \frac{\alpha_{23}}{\alpha_{45}}\right),$$

$$+L_{2431} - L_{2331} - L_{2334} - L_{2134}$$

$$+\frac{L_{23}}{\alpha_{23}}(\alpha_{13}\alpha_{24} - \alpha_{12}\alpha_{34} - \alpha_{23}\alpha_{34} - \alpha_{12}\alpha_{23}) + L_{23}\left(\frac{\alpha_{12}\alpha_{13}}{\alpha_{45}} + \frac{\alpha_{34}\alpha_{24}}{\alpha_{51}}\right)$$

can be simplified to

$$A_{F^4}(\theta) = \frac{L_{12}}{\alpha_{12}} + \frac{K_{23}}{\alpha_{23}} + \frac{K_{34}}{\alpha_{34}} + \frac{K_{45}}{\alpha_{45}} + \frac{L_{51}}{\alpha_{51}}$$

where L_{1j} and K_{ij} are the 5-pt one-loop kinematic structures

- The tree-level 5-pt amplitude prescription requires the computation of

$$\langle V^1(z_1)V^4(z_4)V^5(z_5)U^2(z_2)U^3(z_3)\rangle,$$

where V^I and U^I are the unintegrated and integrated vertices

- The Bern-Carrasco-Johansson 5-pt kinematic relations follow from the fact that it doesn't matter the order in which the OPE's are computed (work in progress with Vanhove)

Supersymmetric BCJ relations

- Eliminating first z_2 followed by z_3 one gets

$$\frac{L_{2131}}{z_{21}z_{31}} + \frac{L_{2134}}{z_{21}z_{34}} - \frac{L_{2434}}{z_{24}z_{34}} - \frac{L_{2431}}{z_{24}z_{31}} + \frac{L_{2331}}{z_{23}z_{31}} - \frac{L_{2334}}{z_{23}z_{34}} + \frac{L_{2314}}{z_{23}^2}$$

while in reverse order,

$$\frac{L_{3121}}{z_{31}z_{21}} + \frac{L_{3124}}{z_{31}z_{24}} - \frac{L_{3424}}{z_{34}z_{24}} - \frac{L_{3421}}{z_{34}z_{21}} + \frac{L_{3221}}{z_{32}z_{21}} - \frac{L_{3224}}{z_{32}z_{24}} + \frac{L_{3214}}{z_{32}^2}$$

- As U^I is bosonic, they must be equal. Using

$$\frac{1}{z_{23}z_{31}} + \frac{1}{z_{32}z_{21}} = \frac{1}{z_{21}z_{31}}$$

and $L_{3221} = -L_{2331}$ one gets

Supersymmetric BCJ relations

$$L_{2131} - L_{3121} + L_{2331} = 0, \quad L_{2434} - L_{3424} + L_{2334} = 0,$$
$$(L_{2134} = -L_{3421}, \quad L_{2431} = -L_{3124})$$

- Comparing the BCJ definition of n_j and the PS result for A_{YM} one realizes that they correspond to

$$n_4 - n_1 + n_{15} = 0, \quad n_5 - n_2 + n_{11} = 0$$

and

$$n_8 - n_6 + n_9 = 0$$

follows from the first one by $2 \leftrightarrow 4$

Supersymmetric BCJ relations

- Using PSS manipulations one can check them explicitly

$$L_{2331} = A^1_m \mathcal{F}^2_{mn} (\lambda\gamma^n W^3)(\lambda A^4)(\lambda A^5) - \frac{1}{2}(\lambda\gamma_m W^1)(W^2\gamma^m W^3)(\lambda A^4)(\lambda A$$

$$+ \big[A^1_m(\lambda\gamma^m W^3) + (\lambda A^1)(k^1 \cdot A^3)\big](k^3 \cdot A^2)(\lambda A^4)(\lambda A^5)\rangle - (2 \leftrightarrow 3)$$

$$L_{2131} = + \big[A^1_m(\lambda\gamma^m W^2) + (\lambda A^1)(k^1 \cdot A^2)\big](\lambda A^4)(\lambda A^5)((k^1 + k^2)\cdot A^3)$$

$$+ \big[A^1_m(\lambda\gamma^m W^3)(k^1 \cdot A^2) - A^{1\,m}(\lambda\gamma^n W^3)\mathcal{F}^2_{mn} - (\lambda\gamma^m W^3)(W^1\gamma^m W^2)\big](\lambda A$$

and L_{3121} is obtained from the above by $2 \leftrightarrow 3$ (total derivative terms were omitted)

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- Everything in the PS formalism is supersymmetric, so we have obtained supersymmetric BCJ relations (work in progress)

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