Non-Planar $\mathcal{N}=4$ SYM at Four Loops and Supersum Structures

NBIA workshop
Aug 12, 2009
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0903.5348[hep-th]: Z.Bern, J.J.Carrasco, H.Ita, HJ, R.Roiban

Outline

- Motivation & introduction

- Calculating multi-loop amplitudes in $\mathcal{N} = 4$ SYM
  - Unitarity & maximal cuts
  - Special cuts $\leftrightarrow$ heuristic rules
  - Supersum structure in cuts
  - 4-loop non-planar $\mathcal{N} = 4$

- 1,2,3,4-loop UV divergences
  - Full color structure of divergences

- Conclusions
Motivation - hidden structures

Maximal SUSY theories are remarkably rich in hidden structures

- \( \mathcal{N} = 4 \) SYM – AdS/CFT, dual conformal symmetry (Yangian), integrability, BDS resummation, twistors

- \( \mathcal{N} = 8 \) SUGRA – UV finite, \( E_{7(7)} \), simplest theory?

- \( \mathcal{N} = 4 \) SYM input to \( \mathcal{N} = 8 \) SUGRA ampl. through KLT & unitarity
  \( \Rightarrow \) talks by Carrasco, Roiban

- Goal: study the less-well-understood non-planar sector of \( \mathcal{N} = 4 \) SYM
$\mathcal{N} = 4$ super-Yang-Mills

\[ \mathcal{L}_{YM} = -\frac{1}{4g^2} F_{\mu\nu}^a F^{a\mu\nu} \]

Maximal SUSY extension of YM

On-shell spectrum:

<table>
<thead>
<tr>
<th></th>
<th>$g^-$</th>
<th>$f^-$</th>
<th>$s$</th>
<th>$f^+$</th>
<th>$g^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>helicity</td>
<td>$-1$</td>
<td>$-1/2$</td>
<td>$0$</td>
<td>$1/2$</td>
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<td>$1$</td>
<td>$4$</td>
<td>$6$</td>
<td>$4$</td>
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Particles in adjoint group $G$, usually $\text{SU}(N_c)$
Unitarity Method

optical theorem

unitarity method
Bern, Dixon, Dunbar and Kosower (1994)

generalized unitarity
Bern, Dixon and Kosower

quadruple cut
(leading singularity)
Britto, Cachazo, Feng; Buchbinder, Cachazo (2004)
Cachazo and Skinner
Cachazo, Spradlin, Volovich (2008)

maximal cut
Bern, Carrasco, HJ and Kosower (2007)

2 \text{Im} = \int_{d\text{LIPS}}

⇒ talks by Dunbar, Kosower

on-shell 3-vertex

Bern, Carrasco, HJ and Kosower (2007)
Calculation Strategy

Ansatz: \[ A^{4\text{-loop}}_4 = g^{10} s t A^{\text{tree}}_4 \sum_{S_4} \sum_{i=1}^{\text{#topologies}} c_i \mathcal{I}_i \]

Separate color from kinematics:

\[ \mathcal{I}_i = C_i I_i \]

\[ C_i = f^{abc} f^{cde} \ldots f^{xyz} \]

Integrals:

\[ I_i = \int d^D l_1 d^D l_2 d^D l_3 d^D l_4 \frac{N_i(l_j, k_j)}{l_1^2 l_2^2 l_3^2 l_4^2 l_5^2 l_6^2 l_7^2 l_8^2 l_9^2 l_{10}^2 l_{11}^2 l_{12}^2 l_{13}^2} \]

Find all integral topologies & numerators!
Integral topologies

- We choose to work with only trivalent (cubic) diagram topologies
- Contact terms are absorbed into the numerator $N$

Start with 4-loop trivalent 1PI vacuum graphs

(a) 
(b) 
(c) 
(d) 
(e)

Attach 4 external legs

... 

Remove diagrams with 2, 3-point sub-graphs consistent with No-Triangle property (checked by cuts)

...gives 50 diagram topologies or integrals

⇒ talk by Carrasco
50 Integral topologies

Bern, Carrasco, Dixon, HJ, Roiban [to appear]
Fix Numerators with Maximal Cuts

- put maximum number of propagator on-shell $\rightarrow$ simplifies calculation

$$N_i = \frac{1}{stA_4^{\text{tree}}} \times$$

- systematically release cut conditions $\rightarrow$ great control of missing terms

Reconstructs the amplitude piece by piece!
$\mathcal{N}=4$ bag of tricks!

- Rules & assumptions for $\mathcal{N}_i$ can used at intermediate steps
- Correctness of amplitude established at the end (complete set of cuts)

**Power counting & singlet maximal cuts**

- Power counting $D_c = 4 + \frac{6}{L}$ can constrain numerators
- Worst case 4 loops: 2 inverse propagators $N \sim s l_1^2 l_2^2$
- Such terms fixed by maximal cuts with **two collapsed cut lines**
- Remarkably all needed maximal cuts have corners in phase space where only gluon states propagate in loops: **susy invariant “singlet cuts”**

**Heuristic rules for numerators ↔ special cuts (that iterate)**

- **rung rule** ↔ two-particle cut
- **box substitution rule** ↔ box cut
- **diagram twist rule** ↔ (BCJ) Jacobi-like numerator & amplitude relations

*valid in $D$ dimensions*
Two-particle cut ↔ Rung Rule

- A simple property of the 2-particle cuts at one loop

\[ \sum_{N=4}^{2} \sigma_1 \sigma_2 i s_{12} s_{23} \]

Lead to easy rules for computing iterated 2-particle cuts in multiloop amps

- Inspired “rung rule” for easily finding numerator factors

Bern, Rozowsky, Yan (1997)

- rung rule is less useful for non-planar diagrams
- better use the 2-particle cut directly
Box substitution rule $\leftrightarrow$ box cut

Observation in 0705.1864 [hep-th] inspired a “box substitution rule”

- $N_i$ can be obtained easily for diagrams containing a box subdiagram
- Follows from the simplicity of the 4-point $\mathcal{N}=4$ SYM amplitude.

Known 3-loop diagrams

"Box cut":
- Allows back-of-the-envelope calculation of $N$
- Input comes from known lower loop integrals
- Result is $D$-dim, but may miss contact terms.

\[ N = s_{27}s_{25}\left(\frac{s_{12}s_{49}}{s_{27}} + \frac{s_{23}s_{48}}{s_{25}}\right) = s_{12}s_{25}s_{49} + s_{23}s_{27}s_{48} \]
Box cut

- A box cut may not involve a box subdiagram
- Isolating any 4-point $\mathcal{N}=4$ SYM loop-amplitude will do it

2-particle cuts

"box cuts"

- 44 (out of 50) of the cubic topologies have box subdiagrams or other 4-pt subdiagrams
- But, many contact terms cannot be determined by the box cut
A diagram “twist rule”

We can use the Jacobi-like numerator identity of 0805.3993 [hep-ph]
Bern, Carrasco, HJ

\[ A^\text{tree}_4(1, 2, 3, 4) = g^2 \left( \frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u} \right) \]

\[ \Rightarrow \text{talk by Bern} \]

Numerators of diagrams entering a cut are not independent

- Possible to relate non-planar topologies to planar ones
- In general, the \( N_i \) are constrained by a large linear eqn system
Some results from two-particle cuts

Simple numerator structure follow from the simplicity of the two-particle cut

$$s_{ij} = (k_i + k_j)^2$$

Bern, Carrasco, Dixon, HJ, Roiban, [to appear]
Some Results from Box Cuts

\[ s_{56}(s_{12}s_{56} - s_{23}s_{35} - l^2_6 s_{12}) \]

\[ s_{15}(s_{13}s_{56} - s_{12}s_{78} - l^2_3 s_{13}) \]

\[ s_{16}(s_{23}s_{56} - s_{12}s_{79} - l^2_9 s_{23}) + l^2_7 s_{12} \]

\[ s_{47}(s_{12}s_{67} + s_{23}s_{89}) \]

\[ s_{46}(s_{12}s_{28} - s_{23}s_{67}) \]

\[ s_{45}(s_{13}s_{37} - s_{24}s_{58}) \]

\[ s_{40}(s_{13}s_{37} - s_{24}s_{58}) \]

\[ s_{23}s_{46}s_{45} + l^2_5 s_{12}s_{13} \]

\[ s_{23}s_{23}s_{56} - s_{12}s_{29} - l^2_5 s_{23} \]

\[ + l^2_7(s_{12}s_{29} - l^2_7 s_{23}) \]

\[ + s_{12}s_{15}s_{78} \]

\[ + l^2_5(s_{12}s_{29} - l^2_7 s_{23}) \]

The numerator structure is more complicated, but still quite modest…
The most complicated numerators

Related to planar contributions through the “twist rule”

Contact-term corrections determined by maximal cuts
Proof of amplitude

To do:

• Find missing contact terms not given by heuristic rules
  ✓ automate (singlet) maximal cuts
• Check correctness of cuts in $D = 4$ using a complete set of cuts
  ✓ automate general cuts, and
  ✓ include full $\mathcal{N} = 4$ supersum $\Rightarrow$ talk by Roiban
• Check correctness of amplitude using $D$-dimensional cuts
  □ automate $D$-dim cuts and superspace
  ✓ partial checks done: two-particle cuts, box cuts

Note: for 1,2,3 loops, 4-dim cuts capture the full 4pt amplitude
Supersum Structure

Convenient to use “index diagrams” to visualize the supersum structure

$\uparrow$ 4pt MHV amplitudes in a graphical notation

Bern, Carrasco, Ita, HJ, Roiban

$\Rightarrow$ talk by Roiban

For non-MHV trees $\rightarrow$ CSW expansion
Tracking the $R$ charge index

Helicity dependent part of cut

$$\langle 2|5|4 \rangle^4 + 4 \langle 2|5|4 \rangle^3 \langle 2|6|4 \rangle + 6 \langle 2|5|4 \rangle^2 \langle 2|6|4 \rangle^2 + \ldots$$

$$= \left[ \langle 2|5|4 \rangle + \langle 2|6|4 \rangle \right]^4$$

General structure of cuts

$$(A + B + C + \ldots)^\mathcal{N}, \quad \mathcal{N} = 4$$

“single $R$ index contribution” $\uparrow$

$\Rightarrow$ talk by Roiban
Supersum Example

“single R index contribution”

\[ l_2 \cdot l_3 \left[ l_4 l_5 \right] + \left[ l_4 l_1 l_2 l_5 \right] + \left[ l_4 l_2 l_1 l_5 \right] + \left[ l_4 l_1 l_3 l_5 \right] + \left[ l_4 l_3 l_1 l_5 \right] = s \left[ l_4 l_5 \right] \]

\[
\text{cut} = s^4 \left[ l_4 l_5 \right]^4 \frac{1}{\langle 3 \ 4 \rangle \langle 4 \ 3 \rangle \langle l_3 l_2 \rangle \langle l_2 l_1 \rangle \langle l_1 3 \rangle} \frac{1}{\left[ 1 \ l_5 \right] \left[ l_5 l_4 \right] \left[ l_4 l_2 \right] \left[ l_2 l_3 \right] \left[ l_3 1 \right]} \frac{1}{\left[ 2 \ l_1 \right] \left[ l_1 l_4 \right] \left[ l_4 l_5 \right] \left[ l_5 2 \right]} \]

\[ 5^4 = 625 \text{ contributions from individual states re-summed} \]
Automate Supersums

- After the supersum structure is understood, automating calculations is straightforward
- Exploit similarity of **N=4 SYM** and pure Yang Mills (QCD)

\[
\begin{align*}
\text{QCD:} & \quad A^4 + B^4 + C^4 + D^4 + E^4 + F^4 + G^4 + H^4 \\
\text{N=4 SYM:} & \quad \left[ A + B + C + D + E + F + G + H \right]^4
\end{align*}
\]

Algorithm \(\Rightarrow\) Find the (few) QCD contributions and map to N=4 SYM

\[
\begin{align*}
A &= \langle l_4 l_5 \rangle \langle l_1 l_5 \rangle \langle l_2 l_7 \rangle \langle l_1 l_3 \rangle, \\
B &= \langle l_4 l_5 \rangle \langle l_1 l_5 \rangle \langle l_7 l_1 \rangle \langle l_2 l_3 \rangle, \\
C &= \langle l_4 l_6 \rangle \langle l_1 l_7 \rangle \langle l_2 l_6 \rangle \langle l_1 l_3 \rangle, \\
\text{Algorithm} \Rightarrow \text{Find the (few) QCD contributions and map to N=4 SYM} \\
\end{align*}
\]

\[
\begin{align*}
A + B + C + D + E + F + G + H \right]^4 = \left[ s \langle l_1 l_2 \rangle \langle l_7 l_3 \rangle \right]^4
\end{align*}
\]

\(8^4 = 4096\) contributions from individual states re-summed

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UV properties
UV properties

- **N=4 SYM UV properties** are interesting due to recent studies of potential counterterms: Bossard, Howe, Stelle, 0901.4661

  ⇒ talk by Howe

- **Planar amplitudes** have established divergences in critical dimensions:

  \[ D_c = 8 \quad L = 1 \]
  \[ D_c = 4 + 6/L \quad L = 2, 3, 4 \]

- We wish to determine the **full color dependence** of the UV divergences

- In gauge group **SU(N_c)**, using color structures:

  \[ \text{Tr}_{ijkl} \equiv \text{Tr}(T^{a_i}T^{a_j}T^{a_k}T^{a_l}) \]
  \[ \text{Tr}_{ij} \equiv \text{Tr}(T^{a_i}T^{a_j}) = \delta^{a_i a_j} \]
1,2-Loop Amplitudes

1-loop: \[ \mathcal{K} \left( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
2
\end{array}
\begin{array}{c}
3
\end{array}
\end{array}
\begin{array}{c}
1
\begin{array}{c}
4
\end{array}
\end{array}
\end{array}
+ \begin{array}{c}
\begin{array}{c}
3
\end{array}
\begin{array}{c}
4
\end{array}
\end{array}
\begin{array}{c}
1
\begin{array}{c}
2
\end{array}
\end{array}
\end{array}
+ \begin{array}{c}
\begin{array}{c}
4
\end{array}
\begin{array}{c}
2
\end{array}
\end{array}
\begin{array}{c}
1
\begin{array}{c}
3
\end{array}
\end{array}
\end{array} \right) \]

Green, Schwarz, Brink (1982)

2-loop: \[ \mathcal{K} \left( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
2
\end{array}
\begin{array}{c}
3
\end{array}
\end{array}
\begin{array}{c}
1
\begin{array}{c}
4
\end{array}
\end{array}
\end{array}
+ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
s
\end{array}
\begin{array}{c}
3
\end{array}
\end{array}
\begin{array}{c}
1
\begin{array}{c}
4
\end{array}
\end{array}
\end{array}
+ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
s
\end{array}
\begin{array}{c}
2
\end{array}
\end{array}
\begin{array}{c}
1
\begin{array}{c}
3
\end{array}
\end{array}
\end{array}
+ \text{perms} \right) \]


\[ s = (k_1 + k_2)^2 \]

Helicity containing prefactor: \[ \mathcal{K} = stA^\text{tree}_4 \]

Color factors: dress each diagram with \[ f^{abc} = \text{Tr}(T^a[T^b,T^c]) \]
1 loop: \( D_c = 8 - 2\varepsilon \)

\[
A_4^{(1)}(1, 2, 3, 4)|_{\text{pole}} = -\frac{g^4 K}{6(4\pi)^4\varepsilon}\left[N_c\left(\text{Tr}_{1324} + \text{Tr}_{1423} + \text{Tr}_{1243} + \text{Tr}_{1342} + \text{Tr}_{1234} + \text{Tr}_{1432}ight) + 6\left(\text{Tr}_{12}\text{Tr}_{34} + \text{Tr}_{14}\text{Tr}_{23} + \text{Tr}_{13}\text{Tr}_{24}\right)\right]
\]

Counterterms \( \sim \) \( \text{Tr} \ F^4 \quad \text{Tr} \ F^2 \quad \text{Tr} \ F^2 \)

2 loops: \( D_c = 7 - 2\varepsilon \)

\[
A_4^{(2)}(1, 2, 3, 4)|_{\text{pole}} = \frac{g^6 \pi K}{20(4\pi)^7\varepsilon}\left[N_c^2 + 20\right]\left(\text{Tr}_{1324} + \text{Tr}_{1423}\right)
\quad + s_{23}\left(\text{Tr}_{1243} + \text{Tr}_{1342}\right) + s_{13}\left(\text{Tr}_{1234} + \text{Tr}_{1432}\right)
\quad - 20N_c\left(s_{12}\text{Tr}_{12}\text{Tr}_{34} + s_{23}\text{Tr}_{14}\text{Tr}_{23} + s_{13}\text{Tr}_{13}\text{Tr}_{24}\right)\]

Counterterms \( \sim \) \( \partial^2 \text{Tr} \ F^4 \quad \partial^2\left[\text{Tr} \ F^2\right]^2 \)

As expected, both single and double trace terms appears
The 3-loop amplitude

\[ S_4 \left[ C^{(a)} I^{(a)} + C^{(b)} I^{(b)} + \frac{1}{2} C^{(c)} I^{(c)} + \frac{1}{4} C^{(d)} I^{(d)} + 2 C^{(e)} I^{(e)} + 2 C^{(f)} I^{(f)} + 4 C^{(g)} I^{(g)} + \frac{1}{2} C^{(h)} I^{(h)} + 2 C^{(i)} I^{(i)} \right] \]

Finite in \( D = 6 \)

Divergent in \( D = 6 \)

\[ s_{ij} = (k_i + k_j)^2 \]
\[ \tau_{ij} = 2k_i \cdot l_j \]
3-Loop UV Divergence

3 loops: $D_c = 6 - 2\epsilon$

$$A_4^{(3)}(1, 2, 3, 4)\big|_{\text{pole}} = -\frac{g^8 K}{3(4\pi)^9 \epsilon} (N_c^3 + 36 \zeta(3) N_c) \left[ s_{12} (\text{Tr}_{1324} + \text{Tr}_{1423}) 
+ s_{23} (\text{Tr}_{1243} + \text{Tr}_{1342}) + s_{13} (\text{Tr}_{1234} + \text{Tr}_{1432}) \right]$$

Counterterms $\sim \partial^2 \text{Tr} F^4$ $\partial^2 [\text{Tr} F^2]^2$

Remakably the double-trace contributions are finite in $D=6$
4-loop UV vacuum integrals

(before cancellations between different topologies)

In the assembled amplitude only 3 integrals contribute
UV integral evaluation

• Vacuum integrals factorize into product of 1-loop integral with UV pole and a finite 3-loop propagator (2pt) integral

• Finite 3-loop integrals reduce to master integrals using integration by parts (IBP), a la MINCER. Chetyrkin, Tkachov (1981)

• Most nontrivial integral is nonplanar master integral, for which we only have numerical results (obtained using Gegenbauer polynomial x-space technique) Chetyrkin, Tkachov (1981); Bekavac, hep-ph/0505174

\[
V_1 = \frac{1}{(4\pi)^{11}} \varepsilon \left[ \frac{512}{5} \Gamma^4\left(\frac{3}{4}\right) - \frac{2048}{105} \Gamma^3\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{4}\right) \right] + O(1)
\]

\[
V_2 = \frac{1}{(4\pi)^{11}} \varepsilon \left[ -\frac{4352}{105} \Gamma^4\left(\frac{3}{4}\right) + \frac{832}{105} \Gamma^3\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{4}\right) \right] + O(1)
\]

\[
V_8 = \frac{1}{(4\pi)^{11}} \frac{4}{21} \frac{1}{\Gamma\left(\frac{3}{4}\right)} \frac{V_8^{\text{fin}}}{\varepsilon} \quad V_8^{\text{fin}} = 1.428452926283(3)
\]

Evaluated in critical dimension \(D_c = 4 + 6/4 = 11/2\)
4-Loop UV Divergence

Combining UV poles of integrals with color factors

\[
A_4^{(4)}(1, 2, 3, 4)|_{\text{pole}} = -6g^{10}\kappa N_c^2 \left[ N_c^2 V_1 + 12(V_1 + 2V_2 + V_3) \right] \\
\times \left[ s_{12}(\text{Tr}_{1324} + \text{Tr}_{1423}) + s_{23}(\text{Tr}_{1243} + \text{Tr}_{1342}) \\
+ s_{13}(\text{Tr}_{1234} + \text{Tr}_{1432}) \right]
\]

- Again the double-trace contributions are finite in \(D_c = 11/2\)
- Also \((N_c)^0\) term is finite
- Absence of double-trace and \((N_c)^0\) terms at 3 and 4 loops calls out for explanation.
- Related to better UV behavior of colorless theories?

\[\Rightarrow\text{talk by Vanhove}\]
Conclusions

• Full color 4-point 4-loop amplitude has been computed in $\mathcal{N}=4$ super-Yang-Mills theory

• Tools: rung rule, box cut, twist rules, maximal cuts, and generalized cuts with full susy multiplet

• “Index diagrams” introduced to clarify supersum structure in cuts, paving the way for automated calculations

• $L = 4$ UV divergence have been extracted, and compared with results for $L = 1,2,3$

• Double-trace and $(N_c)^0$ terms drop out after $L = 2$

• Future studies of the IR information is possible ... once technology is developed for doing non-planar 4-point integrals (even numerically) in $D = 4 - 2\varepsilon$ at $L = 3,4$