A journey through three maximally supersymmetric theories

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Brandhuber, Heslop, GT	0807.4097 [hep-th]	
Brandhuber, Heslop, GT	0905.4377 [hep-th]	N=4 super Yang-Mills
Brandhuber, Heslop, GT	0906.3552 [hep-th]	
Katsaroumpas, Spence, GT	0906.0521 [hep-th]	N=8 supergravity
Brandhuber, Vincon, GT	0908.1306 [hep-th]	N=2 super QED

Hidden Structures in Field Theory Amplitudes, NBI, August 2009

N=8
N=4
N=2 QED
Livorno
Isola di Gorgona
Oasi Santa Luce
Montecatini Terme
Volterra
Rosignano
Siena
Cecina
Colline Metallifere
Sovicille
Oasi di Bolgheri
1060 le Cornate
Castagneto Carducci
M. Calvi 646
Capraia
M. Castello
Massa Marittima
Campiglia Marittima
Isola di Capraia
Oasi della Palude degli Orti-Bottagone
Piombino
Follonica
Civitella
Portoferraio
Rifugio delle Marze
Vetulonia
M. Capanne 1019
Grosseto
Castiglione della Pescaia
Isola d'Elba
Marina di Grosseto
Parco naturale della Maremma
Parco nazionale dell'Arcipelago Toscano
Isola di Pianosa
Pianosa
Formiche di Grosseto
Mandano
Riserva naturale e Oasi della Laguna di Orbetello
Mar Tirreno
Orbetello
Porto Santo Stefano
Ansedonia
635
M. Argentario
Scoglio d'Africa
Giglio Porto
Isola del Giglio
Isola di Montecristo
Riserva naturale di Montecristo
Isola di Giannutri
N
0
50
km

- N=4 super Yang-Mills
  - Dual conformal symmetry & equations for one-loop supercoefficients
- N=8 supergravity
  - New expressions for one-loop supercoefficients
- N=2 super QED
  - Four-point photon amplitude at one and two loops and approximate iterative relations

N=4 super Yang-Mills

Amplitudes in N=4 SYM

- Scattering amplitudes in N=4 are conveniently described in terms of on-shell superspace variables
- Super wave-function: (Nair)

$h = +1$ $\quad h = +1/2$ $\quad h = 0$ $\quad h = -1/2$

$$\begin{aligned}\Phi(p,\eta) \quad &= \quad A^{+}(p) + \eta^{A}\psi_{A}(p) + \frac{1}{2}\eta^{A}\eta^{B}\phi_{AB}(p) + \frac{1}{3!}\eta^{A}\eta^{B}\eta^{C}\epsilon_{ABCD}\,\bar{\psi}^{D}(p) \\ &+ \quad \frac{1}{4!}\eta^{A}\eta^{B}\eta^{C}\eta^{D}\epsilon_{ABCD}A^{-}(p)\end{aligned}$$

$h = -1$

- $\eta^{A}\,,\ \ A = 1, \ldots, 4$ fermionic variables, A is an SU(4) index
- particle content is that of N=4 super Yang-Mills
- similar to Mandelstam's LC superfield

- Package all amplitudes with a fixed external helicity assignment into a single superamplitude: (Nair;Witten)

$$\mathcal{A}(\{\lambda_i, \tilde{\lambda}_i, \eta_i\}) \;=\; \delta^{(4)}\Big(\sum_i \lambda_i \tilde{\lambda}_i\Big)\; \delta^{(8)}\Big(\sum_i \eta_i \lambda_i\Big)\; A(\{\lambda_i, \tilde{\lambda}_i, \eta_i\})$$

momentum

supermomentum

- $q^A_\alpha = \sum_i \lambda_{i;\alpha} \eta^A_i$ susy manifest,
- $\bar{q}_{A,\dot{\alpha}} = \sum_i \tilde{\lambda}_{i;\dot{\alpha}} \frac{\partial}{\partial \eta^A_i}$ imposes restrictions on A

- η-expansion: p powers of η_i correspond to helicity $h_i = 1 - p/2$

Examples:

$$\mathcal{A} \;=\; \delta^{(4)}\Big(\sum_i \lambda_i \tilde{\lambda}_i\Big)\; \delta^{(8)}\Big(\sum_i \eta_i \lambda_i\Big)\; A$$

- **All-plus & single minus amplitudes:** automatically zero
  - Too few powers of η's: fermionic delta function cannot be satisfied
  - All-minus and single-plus vanish by parity

- **MHV superamplitude:** $A_{\mathrm{MHV}} \;=\; \dfrac{1}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}$
- gluons i^-, j^- : get factor of $(\eta_i)^4 (\eta_j)^4 \, \langle i\, j\rangle^4$

$$A^{\mathrm{gluons}}_{\mathrm{MHV}}(1^+, \ldots, i^-, \ldots, j^-, \ldots n^+) \;=\; \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$$

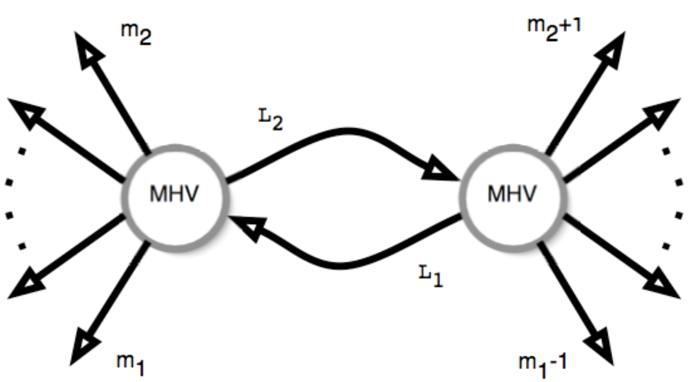
(Parke & Taylor)

Early applications:

- Tree-level super MHV diagrams (Georgiou, Khoze + Glover, 2004)

- One-loop super MHV diagrams (Brandhuber, Spence, GT, 2004)

    - very efficient way to perform internal supersums, e.g. one-loop MHV superamplitude:



Roiban's seminar

Dual Conformal Symmetry

(Drummond, Henn, Korchemsky, Sokatchev)

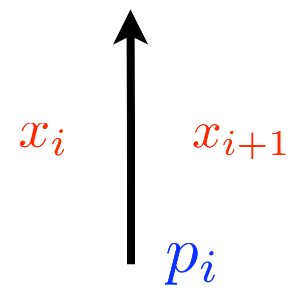
- Conjecture: dual (super)conformal symmetry is a symmetry of the planar S-matrix of N=4 SYM
  - planarity, maximal supersymmetry, on-shellness
  - originates from Wilson loops
    - MHV amplitude/Wilson loop duality

- **Dual conformal symmetry:** (DHKS)

  - is the standard conformal symmetry acting on dual momenta x's

$$p_i = x_i - x_{i+1}$$

$$x_{n+1} = x_1$$



  - symmetry is anomalous
    - ultraviolet divergences from cusps in the contour — Korchemsky's seminar
    - UV for the Wilson loop = IR for the amplitude

  - Since $K = IPI$, enough to look at conformal inversions I
    - under a conformal inversion $x_i^\mu \to x_i^\mu / x_i^2$

- Extending Dual Conformal Symmetry from Wilson loops to all (non-MHV) amplitudes is <u>very nontrivial</u> !

Trees

Covariance of tree-level S-matrix

(Brandhuber, Heslop, GT)

- Idea: look for a computational method which respects the symmetry of the problem at the diagrammatic level. Is there one ?

- YES! $N=4$ supersymmetric recursion relations

  - supersymmetric version of the BCFW on-shell recursion relations (Britto, Cachazo, Feng + Witten)

  - MHV diagrams more problematic

Why do we think it will work?

- Look at split-helicity gluon amplitudes:

$$1_g^- \, 2_g^- \ldots p_g^- \, (p+1)_g^+ \, (p+2)_g^+ \ldots n_g^+$$

  - calculated in closed form by Britto, Feng, Roiban, Spradlin, Volovich solving the BCF recursion relation
  - each recursive diagram is manifestly covariant !
  - E.g.:

$$A(1^-,2^-,3^-,4^+,5^+,6^+,7^+) = \frac{\langle 1|2+3|4]^3}{t_2^{[3]}\langle 5\ 6\rangle\langle 6\ 7\rangle\langle 7\ 1\rangle[2\ 3][3\ 4]\langle 5|4+3|2]} - \frac{1}{\langle 3\ 4\rangle\langle 4\ 5\rangle\langle 6|7+1|2]}\left(\frac{\langle 3|(4+5)(6+7)|1\rangle^3}{t_3^{[3]}t_6^{[3]}\langle 6\ 7\rangle\langle 7\ 1\rangle\langle 5|4+3|2]} + \frac{\langle 3|2+1|7]^3}{t_7^{[3]}\langle 6\ 5\rangle[7\ 1][1\ 2]}\right)$$

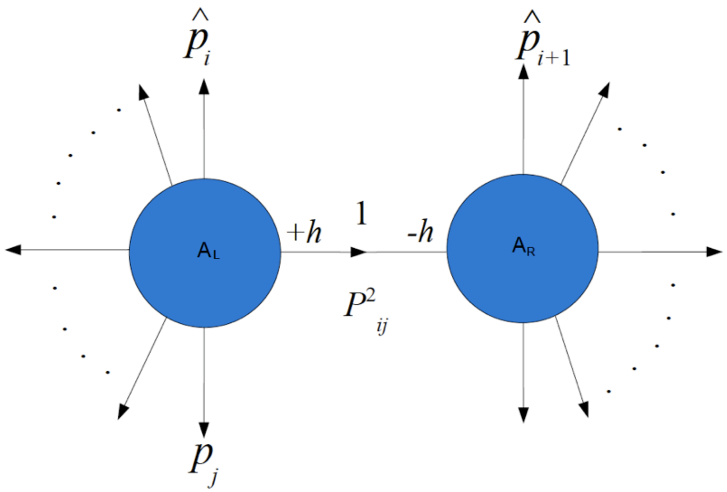
Spradlin's talk

Spinor brackets above involve cyclically adjacent gluons

- # On-shell recursion relations (Britto, Cachazo, Feng)

  - based on singularity structure of tree-level amplitudes:

$$\mathcal{A} = \sum_{j,h}$$



h = internal particles helicities

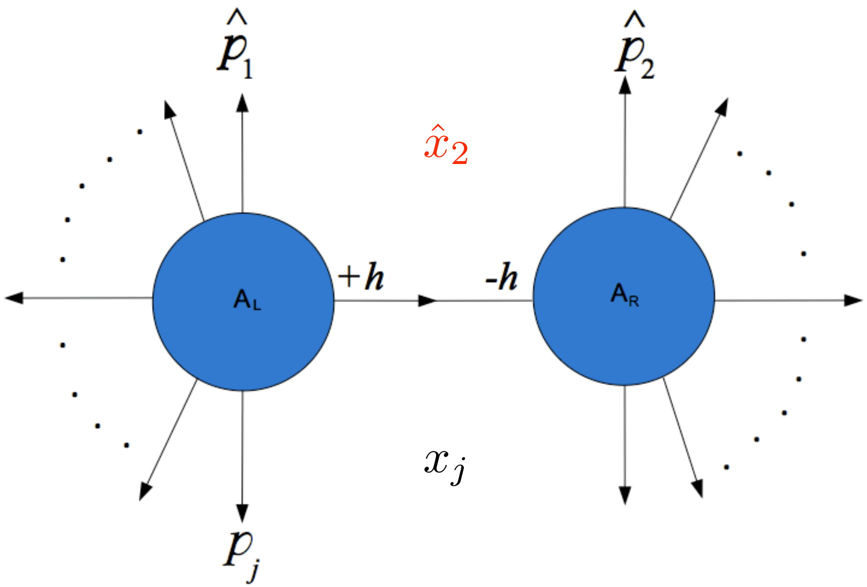
$$P_{ij} := p_i + \cdots + p_j$$

- Factorisation on multiparticle poles (simple poles, tree level), bilinear structure

Susy recursion relations

(Brandhuber, Heslop, GT; Arkani-Hamed, Cachazo, Kaplan)

- supersymmetric deformations of amplitudes:



$$\begin{aligned}\hat{p}_1 &:= \lambda_1 \hat{\tilde{\lambda}}_1 = \lambda_1(\tilde{\lambda}_1 + z\tilde{\lambda}_2) := x_1 - \hat{x}_2 \\ \hat{p}_2 &:= \hat{\lambda}_2 \tilde{\lambda}_2 = (\lambda_2 - z\lambda_1)\tilde{\lambda}_2 := \hat{x}_2 - x_3\end{aligned}$$

$$\hat{x}_2 := x_2 - z\lambda_1 \hat{\tilde{\lambda}}_2$$

Hence, by supersymmetry:

$$\hat{\theta}_2 := \theta_2 - z\lambda_1\eta_2$$

$$\Rightarrow \quad \begin{aligned}\hat{\eta}_1 &:= \eta_1 + z\eta_2 \\ \hat{\eta}_2 &:= \eta_2\end{aligned}$$

Amplitudes with different external states mix under the supersymmetric shifts

- Generic superamplitude expressed as:

$$\mathcal{A} = \sum_P \int d^4\eta_{\hat{P}} \, \mathcal{A}_L(z_P) \frac{i}{P^2} \mathcal{A}_R(z_P)$$

  - sum over internal species becomes a fermionic integral
  - proof of DSC symmetry obtained by induction
    - three-point superamplitudes are the input
    - propagator + fermionic integration

- Symmetry maintained at the diagrammatic level

  - explicit solution of recursion relations (Drummond, Henn)

Drummond's talk

- Important difference with standard BCF shifts

  - $$\mathcal{A}_{\mathrm{BCF}}(z) \;=\; \mathcal{A}(\lambda_1, \hat{\tilde{\lambda}}_1(z), \eta_1;\, \hat{\lambda}_2(z), \tilde{\lambda}_2, \eta_2)$$

  - $$\mathcal{A}_{\mathrm{SUSY}}(z) \;=\; \mathcal{A}(\lambda_1, \hat{\tilde{\lambda}}_1(z), \hat{\eta}_1(z);\, \hat{\lambda}_2(z), \tilde{\lambda}_2, \eta_2)$$

- Large z behaviour

  - $\mathcal{A}_{\mathrm{SUSY}}(z) \sim 1/z$, $\mathcal{A}_{\mathrm{SUGRA}}(z) \sim 1/z^2$ as $z \to \infty$

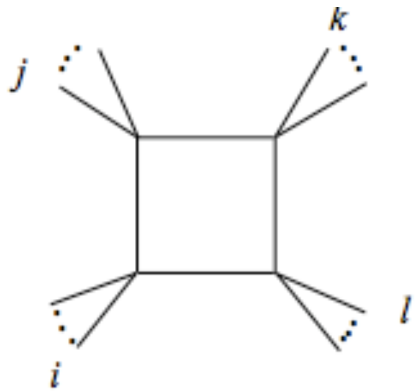
  - proof uses maximal supersymmetry (Arkani-Hamed, Cachazo, Kaplan)

Loops

Simplicity of loop amplitudes

- One loop: amplitudes in maximally supersymmetric theories are sums of box functions

$$\mathcal{A} = \sum_{i,j,k,l} \mathcal{C}(i,j,k,l)$$



  - N=4 super Yang-Mills (Bern, Dixon, Dunbar, Kosower)
  - N=8 supergravity (no-triangle property) (Bern, Bjerrum-Bohr, Dixon, Dunbar, Dunbar, Ita, Perelstein, Perkins, Risager, Rozowsky; Bjerrum-Bohr + Vanhove; Arkani-Hamed, Cachazo, Kaplan)
  - n-photon amplitudes in QED, $n \geq 8$ (Badger, Bjerrum-Bohr, Vanhove)

- Infrared divergences:
  - Work in $D = 4 - 2\epsilon$

Dual conformal anomaly at one loop

- Generic amplitudes are anomalous under DCS:

$$K^{\mu}\mathcal{A}_n^{1-\mathrm{loop}} \;=\; -4\,r_\Gamma\,\mathcal{A}_n^{\mathrm{tree}}\sum_{i=1}^{n} x_{i+1}^{\mu}\,\frac{\left(-x_{ii+2}^2\right)^{-\epsilon}}{\epsilon}$$

$$x_{ii+2} \;:=\; p_i + p_{i+1} \qquad\qquad r_\Gamma \;:=\; \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}$$

- only two-particle invariants appears (similarly to infrared divergent terms)

▸ conjectured by DHKS from Wilson loops

- caveat: Wilson loops/amplitudes duality only applies to MHV amplitudes
- important evidence from NMHV superamplitude at one loop (DHKS)

Next goals:

I. prove the dual conformal anomaly equation

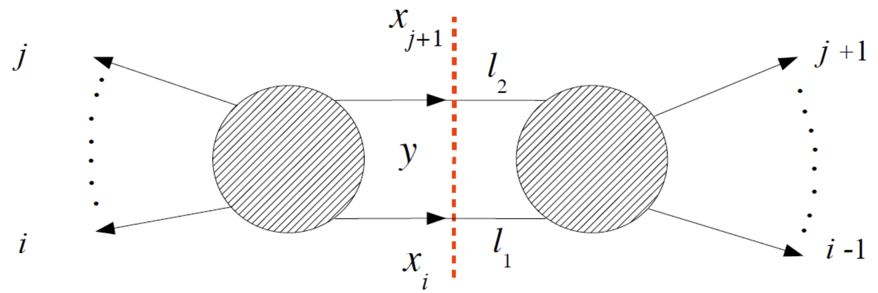
II. prove covariance of the one-loop coefficients

III. derive new relations of one-loop coefficients
(loops without loops...)

I. Proof of Dual Conformal Anomaly

(Brandhuber, Heslop, GT)

- Idea: calculate discontinuities of the anomaly
  - multi-particle channel discontinuities: $(j > i+1)$

ϵ 

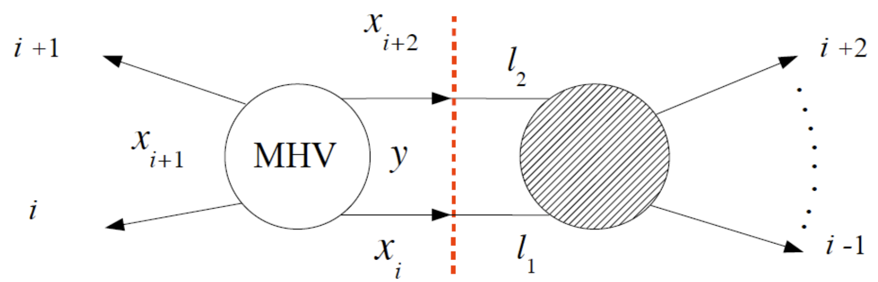
$$\mathrm{disc}_{x^2_{i\,j+1}}\left[K^\mu \mathcal{A}_n^{1-\mathrm{loop}}\right] = 4\,\epsilon \int d^D y\, \delta^{(+)}\big((y-x_i)^2\big)\, \delta^{(+)}\big((x_{j+1}-y)^2\big)\, \Big[y^\mu \langle l_1 l_2\rangle^4 \mathcal{A}_L^{\mathrm{tree}} \mathcal{A}_R^{\mathrm{tree}}\Big]$$

$$\qquad\qquad\qquad\qquad l_1 \qquad\qquad\qquad\qquad l_2$$

  - integral is finite X ϵ, hence multi-particle disc's vanish
  - anomaly independent of multi-particle invariants

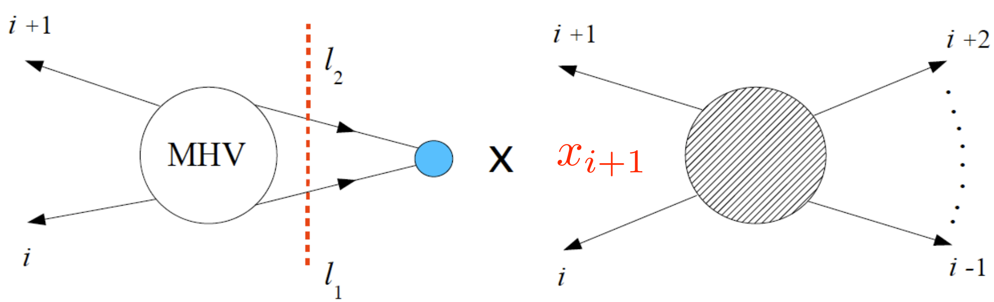
- Two-particle channel discontinuities:

ϵ



- integral is $1/\epsilon^2$ divergent. Divergences arise from a region of integration where $l_1 \sim p_i\,, \quad l_2 \sim p_{i+1}\,,$ hence $y \sim x_{i+1}$

- Pick (leading) divergence: discontinuity factorises



Emergence of tree-level amplitude $\mathcal{A}_n^{\rm tree}$!

- Lift cut integral, to get expected dual conformal anomaly:

$$K^\mu \mathcal{A}_n^{1-\text{loop}} = -4\, r_\Gamma\, \mathcal{A}_n^{\text{tree}} \sum_{i=1}^{n} x_{i+1}^\mu \frac{(-x_{ii+2}^2)^{-\epsilon}}{\epsilon}$$

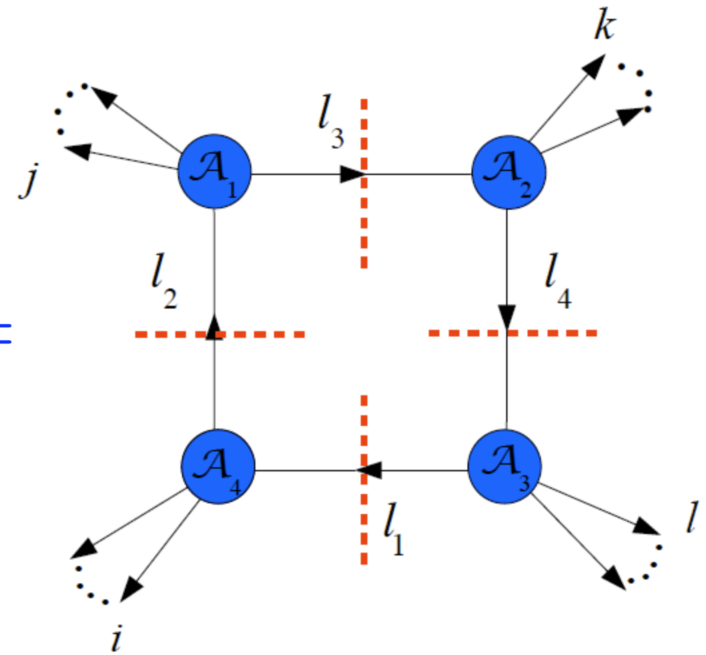
- Comments:
  - Analysis exposes intimate link of anomaly to IR divergences
    - singular channel cut-integral lifted to full loop integral (Kosower)
  - Subleading divergences absent due to maximal susy
  - Absence of holomorphic anomaly crucial: $K^\mu \mathcal{A}^{\text{tree}} = 0$, with no delta-function supported terms on the RHS (Bargheer, Beisert, Galleas, Loebbert, McLoughlin; Korchemsky, Sokatchev)

II. Covariance of one-loop coefficients

- Same idea as for tree-level recursion:
  - look for a computational method which respects the symmetry of the problem at the diagrammatic level
- Quadruple cuts (Britto, Cachazo, Feng)
  - calculate one by one coefficients of one-loop boxes
  - appropriate N=4 supersymmetric generalisation

    (Drummond, Henn, Korchemsky, Sokatchev)

$$\mathcal{C}(i,j,k,l) \;=$$



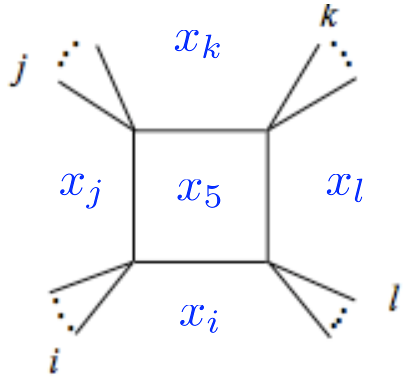
- Coefficient of each box function obtained by gluing four tree-level superamplitudes
  - quadruple cut diagram inherits symmetry properties from tree-level superamplitudes entering the cut
- One-loop coefficients are covariant
  (Brandhuber, Heslop, GT)

III. New relations for one-loop coefficients

(Brandhuber, Heslop, GT)

- One-loop amplitudes:

$$\mathcal{A}^{1-\text{loop}} = \sum_{i,j,k,l} \mathcal{C}(i,j,k,l)$$

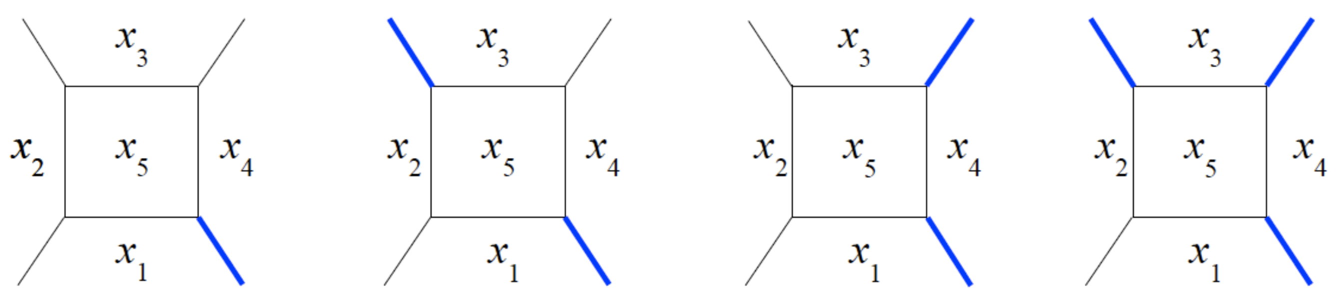


$$F = -i(4\pi)^{D/2} \int \frac{d^D x_5}{(2\pi)^D} \frac{\sqrt{R}}{x_{5i}^2 x_{5j}^2 x_{5k}^2 x_{5l}^2}$$

$$\sqrt{R} = x_{ik}^2 x_{jl}^2 - x_{jk}^2 x_{li}^2$$

$$D = 4 - 2\epsilon$$

Boxes:



1m 2me 2mh 3m

III. New relations for one-loop coefficients

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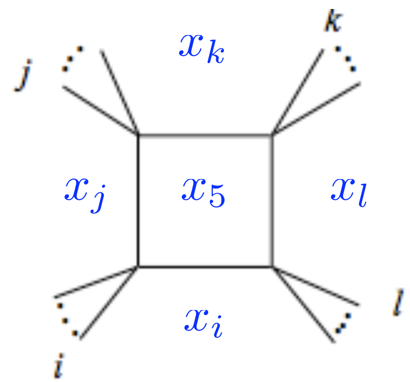

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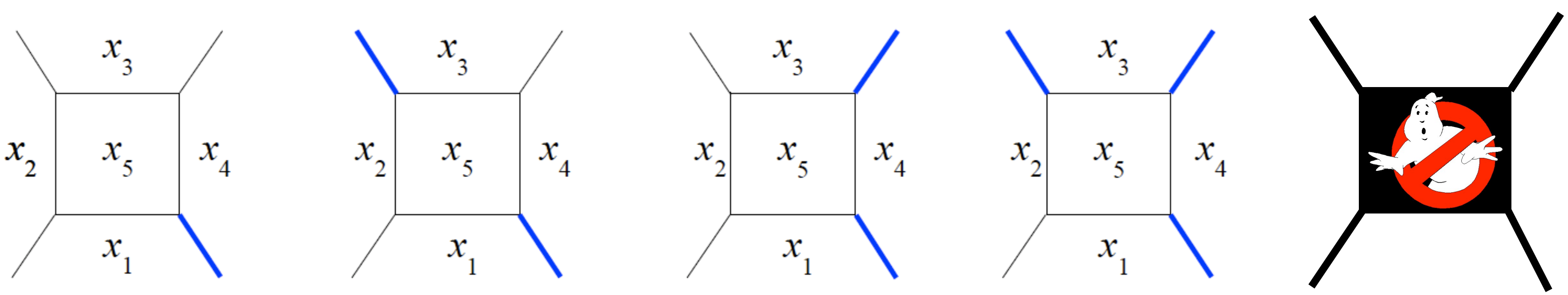


1m 2me 2mh 3m 4m

- Supercoefficients $C\,(i,j,k,l)$ are covariant $\Rightarrow$

$$K^\mu \mathcal{A}^{1-\text{loop}} = \sum_{i,j,k,l} \mathcal{C}(i,j,k,l)\; K^\mu F_{ijkl}$$

- Under a special conformal transformation

$$K^\mu F \;\sim\; \epsilon \int \frac{d^{4-2\epsilon} x_5}{(2\pi)^{4-2\epsilon}} \; \frac{\sqrt{R}}{x_{5i}^2 x_{5j}^2 x_{5k}^2 x_{5l}^2} \, x_5^\mu$$

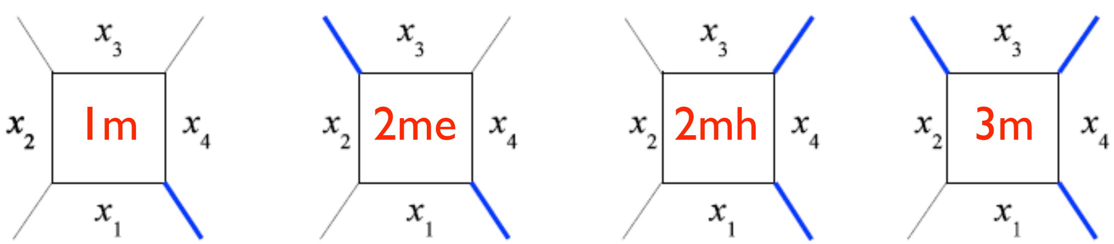
- Anomaly appears because of IR divergences

- Standard Passarino-Veltman reduction to calculate the integral above

  - result expressed in terms of scalar boxes and triangles

Box functions and their anomalies

$$
\begin{aligned}
K^\mu F^{0\mathrm{m}} &= 2\epsilon \left[(x_1+x_3)^\mu x_{24}^2 J(x_{24}^2) \;+\; (x_2+x_4)^\mu x_{13}^2 J(x_{13}^2) \right] \;, \\
K^\mu F_1^{1\mathrm{m}} &= -2\epsilon \Big\{ -x_1^\mu x_{24}^2 J(x_{24}^2, x_{41}^2) + x_2^\mu \Big[x_{41}^2 J(x_{24}^2, x_{41}^2) - x_{13}^2 J(x_{13}^2) \Big] \\
&\quad + \; x_3^\mu \Big[x_{41}^2 J(x_{13}^2, x_{41}^2) - x_{24}^2 J(x_{24}^2) \Big] - x_4^\mu x_{13}^2 J(x_{13}^2, x_{41}^2) \Big\} \;, \\
K^\mu F_{13}^{2\mathrm{me}} &= -2\epsilon \Big\{ x_1^\mu \Big[-\, x_{24}^2 J(x_{24}^2, x_{41}^2) + x_{23}^2 J(x_{13}^2, x_{23}^2) \Big] + x_2^\mu \Big[-\, x_{13}^2 J(x_{13}^2, x_{23}^2) + x_{41}^2 J(x_{24}^2, x_{41}^2) \Big] \\
&\quad + \; x_3^\mu \Big[-\, x_{24}^2 J(x_{24}^2, x_{23}^2) + x_{41}^2 J(x_{13}^2, x_{41}^2) \Big] + x_4^\mu \Big[-\, x_{13}^2 J(x_{13}^2, x_{41}^2) + x_{23}^2 J(x_{24}^2, x_{23}^2) \Big] \Big\} \;, \\
K^\mu F_{14}^{2\mathrm{mh}} &= -2\epsilon \Big\{ -x_1^\mu x_{24}^2 J(x_{24}^2, x_{41}^2) \\
&\quad + \; x_2^\mu \Big[x_{41}^2 J(x_{24}^2, x_{41}^2) - x_{13}^2 J(x_{13}^2) + x_{34}^2 J(x_{24}^2, x_{34}^2) \Big] \\
&\quad - \; x_3^\mu x_{24}^2 J(x_{24}^2, x_{34}^2) \Big] \Big\} \;, \\
K^\mu F_{134}^{3\mathrm{m}} &= -2\epsilon \Big\{ x_1^\mu \Big[x_{23}^2 J(x_{23}^2, x_{13}^2) - x_{24}^2 J(x_{24}^2, x_{41}^2) \Big] + x_2^\mu \Big[x_{41}^2 J(x_{41}^2, x_{24}^2) - x_{13}^2 J(x_{13}^2, x_{23}^2) \Big] \Big\} \;.
\end{aligned}
$$



$$J(a) := \frac{r_\Gamma}{\epsilon^2} \frac{(-a)^{-\epsilon}}{(-a)}$$ 1m triangle $(1/\epsilon^2)$

$$J(a,b) := \frac{r_\Gamma}{\epsilon^2} \frac{(-a)^{-\epsilon} - (-b)^{-\epsilon}}{(-a) - (-b)}$$ 2m triangle $(1/\epsilon)$

- Summarising:

$$K^\mu \mathcal{A}_n^{1-\text{loop}} = 4\,\epsilon\,\mathcal{A}_n^{\text{tree}} \sum_{i=1}^{n} x_{i+1}^\mu \left[x_{ii+2}^2 J(x_{ii+2}^2)\right]$$ anomaly eq, proved earlier...

$$= \sum_{i,j,k,l} \mathcal{C}(i,j,k,l)\; K^\mu F_{ijkl}$$ from covariance of coeff's...

- Last line can be written as:

$$+2\,\epsilon \sum_{i=1}^{n} \sum_{k=i+2}^{i+n-2} x_{i-1}^\mu \left[x_{ii-2}^2\, J(x_{ii-2}^2)\right] \sum_{j=i+1}^{i+n-3} \mathcal{C}(i,j,i-2,i-1)$$

1m triangle
$(1/\epsilon)$

$$-2\,\epsilon \sum_{i=1}^{n} \sum_{k=i+2}^{i+n-3} \mathcal{E}(i,k)\,\left[x_{i-1}^\mu\, x_{ik}^2 \;-\; x_i^\mu\, x_{i-1\,k}^2\right] J(x_{ik}^2, x_{i-1\,k}^2)$$

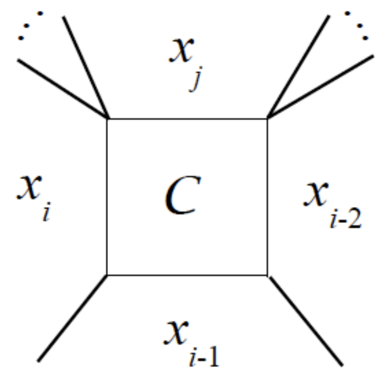
2m triangle
(finite)

where $$\mathcal{E}(i,k) := \sum_{j=k+1}^{i+n-2} \mathcal{C}(i,k,j,i-1) - \sum_{j=i+1}^{k-1} \mathcal{C}(i,j,k,i-1)$$

- Next, we require these two results to be identical

- 1m triangle coeff't: special combination of 2mh and 1m box coefficients, $\sim \mathcal{A}^{\text{tree}}$ (Roiban, Spradlin, Volovich)

$$\sum_{j=i+1}^{i+n-3} \mathcal{C}(i,j,i-2,i-1) \;=\; \sum_{j=i+1}^{i+n-3}$$



$$= \; 2\,\mathcal{A}_n^{\text{tree}}$$

for $i = 1, \ldots, n$

  - main inspiration for the BCF recursion relation

- This gives precisely the anomaly

- 2m triangle: $n(n\text{-}4)$ new relations among coefficients

$$\mathcal{E}(i,k) \, = \, 0 \, , \qquad i = 1 \ldots n \, , \quad k = i+2, \ldots, i+n-3$$

$$\mathcal{E}(i,k) := \sum_{j=k+1}^{i+n-2} \mathcal{C}(i,k,j,i-1) - \sum_{j=i+1}^{k-1} \mathcal{C}(i,j,k,i-1)$$

- These are new predictions obtained from dual conformal invariance

- Determine 1m, 2me, and half of the 2mh coeff's

n $\quad$ $n(n\text{-}5)/2$ $\quad$ $(1/2)\, n(n\text{-}5)$

Loops without loops !

Comment

- Compare to standard IR consistency conditions

$$\mathcal{A}_n^{1-\text{loop}}|_{\text{IR}} \sim -\mathcal{A}_n^{\text{tree}} \sum_{i=1}^{n} \frac{(-x_{ii+2}^2)^{-\epsilon}}{\epsilon^2}$$

  - n [2-particle] + $n(n\text{-}5)/2$ [multi-particle] $=$ $n(n\text{-}3)/2$ relations

- Can be used to predict all (all but one) 1m and 2me box coefficients for n odd (n even)

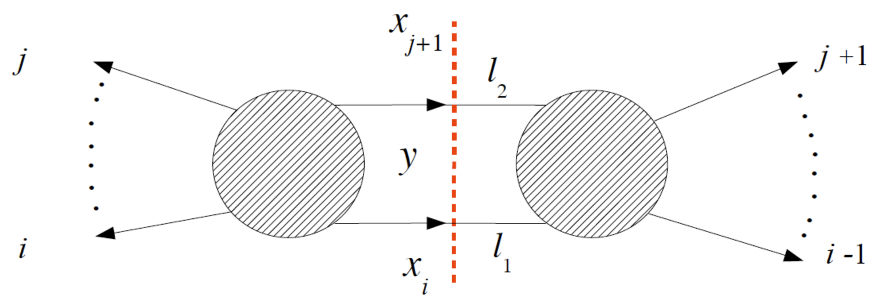
(Bern, Dixon, Kosower)

- Conformal equations predict $n(n\text{-}5)/2$ further coefficients

  - 1/2 of 2mh box coefficients

Dual Q-bar anomaly

- Two kind of contributions: IR and holomorphic
  - multi-particle channel discontinuity of the IR piece:

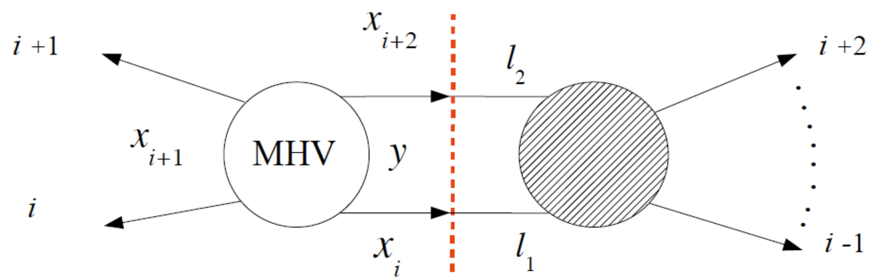
ϵ 

  - IR contribution is finite X $\epsilon \;\; \rightarrow \;\; 0$
  - holomorphic anomaly contribution is non-universal
    (Korchemsky, Sokatchev)

Korchemsky's seminar

- Two-particle channels discontinuities:

ϵ



- analysis similar to dual conformal case. Get universal

$$\bar{Q}^A_{\dot\alpha}\, \mathcal{A}_n^{1-\text{loop}} \;\sim\; \mathcal{A}_n^{\text{tree}} \sum_{i=1}^{n} \frac{(-x^2_{ii+2})^{-\epsilon}}{\epsilon}\, \frac{\eta_i^A \tilde\lambda_{i+1\,\dot\alpha} - \eta^A_{i+1}\tilde\lambda_{i\dot\alpha}}{[ii+1]} + \cdots$$

- "..." stand for a single holomorphic anomaly contribution

- $\bar{Q}^A_{\dot\alpha}\, \mathcal{A}_4^{\text{tree}} = 0$ identically (no contact terms)
(Bargheer, Beisert, Galleas, Loebbert, McLoughlin)

N=8 Supergravity

- On-shell recursion relations for tree-level gravity amplitudes (Bedford, Brandhuber, Spence, GT; Cachazo, Svrcek)
  - large-z behaviour (Arkani-Hamed, Kaplan; Benincasa, Boucher-Veronneau, Cachazo)
- Amplitudes' calculation vastly simplified
  - $\mathcal{A}_{\rm GR}(1^+2^+3^-) = [\mathcal{A}_{\rm YM}(1^+2^+3^-)]^2$, $\mathcal{A}_{\rm GR}(1^-2^-3^+) = [\mathcal{A}_{\rm YM}(1^-2^-3^+)]^2$
  - EH Lagrangian (and its Feynman rules) not needed

Bryce S. DeWitt , Phys. Rev. 162:1239-1256,1967

corrections. For the Yang-Mills field it takes the form

$$V_{(\alpha\gamma''{}^{\sigma''})\beta'} \to -ic_{\alpha\beta\gamma}p'^{\sigma} = -ic_{\alpha\gamma\beta}(p^{\sigma}+p''^{\sigma}). \quad (2.3)$$

The propagators for the normal and fictitious quanta are, respectively,

$$G \to \gamma^{\alpha\beta}\eta_{\mu\nu}/p^2, \quad (2.4)$$

$$\hat{G} \to \gamma^{\alpha\beta}/p^2, \quad (2.5)$$

with p^2 being understood to have the usual small negative imaginary part.

The corresponding quantities for the gravitational field are much more complicated. In this case we shall employ the momentum-index combinations $p\mu\nu$, $p'\sigma'\tau'$, $p''\rho''\lambda''$, $p'''\iota'''\kappa'''$. The vertices must not only be symmetric in each index pair but must also remain unchanged under arbitrary permutations of the momentum-index triplets. At least 171 separate terms are required in the complete expression for S_3 in order to exhibit this full symmetry, and for S_4 the number is 2850. However, these numbers can be greatly reduced by counting only the combinatorially distinct terms[2] and leaving it understood that the appropriate symmetrizations are to be carried out. In this way S_3 is reduced to 11 terms and S_4 to 28 terms, as follows:

$$\frac{\delta^3 S}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma'\tau'}\delta\varphi_{\rho''\lambda''}} \to$$

$$\begin{aligned}&\mathrm{Sym}\big[-\tfrac{1}{4}P_3(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda})-\tfrac{1}{4}P_6(p^{\sigma}p^{\tau}\eta^{\mu\nu}\eta^{\rho\lambda})+\tfrac{1}{4}P_3(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda})+\tfrac{1}{2}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda})+P_3(p^{\sigma}p^{\lambda}\eta^{\mu\nu}\eta^{\tau\rho})\\&-\tfrac{1}{2}P_3(p^{\tau}p'^{\mu}\eta^{\nu\sigma}\eta^{\rho\lambda})+\tfrac{1}{2}P_3(p^{\rho}p'^{\lambda}\eta^{\mu\sigma}\eta^{\nu\tau})+\tfrac{1}{2}P_6(p^{\rho}p^{\lambda}\eta^{\mu\sigma}\eta^{\nu\tau})+P_6(p^{\sigma}p'^{\lambda}\eta^{\tau\mu}\eta^{\nu\rho})+P_3(p^{\sigma}p'^{\mu}\eta^{\tau\rho}\eta^{\lambda\nu})\\&\qquad-P_3(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\mu})\big],\end{aligned} \quad (2.6)$$

← 3-point vertex: 171 terms

$$\frac{\delta^4 S}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma'\tau'}\delta\varphi_{\rho''\lambda''}\delta\varphi_{\iota'''\kappa'''}} \to$$

$$\begin{aligned}&\mathrm{Sym}\big[-\tfrac{1}{8}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})-\tfrac{1}{8}P_{12}(p^{\sigma}p^{\tau}\eta^{\mu\nu}\eta^{\rho\lambda}\eta^{\iota\kappa})-\tfrac{1}{4}P_6(p^{\sigma}p'^{\mu}\eta^{\nu\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})+\tfrac{1}{8}P_6(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})\\&+\tfrac{1}{4}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})+\tfrac{1}{4}P_{12}(p^{\sigma}p^{\tau}\eta^{\mu\nu}\eta^{\rho\iota}\eta^{\lambda\kappa})+\tfrac{1}{2}P_6(p^{\sigma}p'^{\mu}\eta^{\nu\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})-\tfrac{1}{4}P_6(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})\\&+\tfrac{1}{4}P_{24}(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda}\eta^{\iota\kappa})+\tfrac{1}{4}P_{24}(p^{\sigma}p^{\tau}\eta^{\mu\rho}\eta^{\nu\lambda}\eta^{\iota\kappa})+\tfrac{1}{4}P_{12}(p^{\rho}p'^{\lambda}\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\iota\kappa})+\tfrac{1}{2}P_{24}(p^{\sigma}p'^{\rho}\eta^{\tau\mu}\eta^{\nu\lambda}\eta^{\iota\kappa})\\&-\tfrac{1}{2}P_{12}(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\mu}\eta^{\iota\kappa})-\tfrac{1}{2}P_{12}(p^{\sigma}p'^{\mu}\eta^{\tau\rho}\eta^{\lambda\nu}\eta^{\iota\kappa})+\tfrac{1}{2}P_{12}(p^{\sigma}p^{\rho}\eta^{\tau\lambda}\eta^{\mu\nu}\eta^{\iota\kappa})-\tfrac{1}{2}P_{24}(p\cdot p'\eta^{\mu\nu}\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\sigma})\\&-P_{12}(p^{\sigma}p^{\tau}\eta^{\nu\rho}\eta^{\lambda\iota}\eta^{\kappa\mu})-P_{12}(p^{\rho}p'^{\lambda}\eta^{\nu\iota}\eta^{\kappa\sigma}\eta^{\tau\mu})-P_{24}(p_{\sigma}p'^{\rho}\eta^{\tau\iota}\eta^{\kappa\mu}\eta^{\nu\lambda})-P_{12}(p^{\rho}p'^{\iota}\eta^{\lambda\sigma}\eta^{\tau\mu}\eta^{\nu\kappa})\\&+P_6(p\cdot p'\eta^{\nu\rho}\eta^{\lambda\sigma}\eta^{\tau\iota}\eta^{\kappa\mu})-P_{12}(p^{\sigma}p^{\rho}\eta^{\mu\nu}\eta^{\tau\iota}\eta^{\kappa\lambda})-\tfrac{1}{2}P_{12}(p\cdot p'\eta^{\mu\rho}\eta^{\nu\lambda}\eta^{\sigma\iota}\eta^{\tau\kappa})-P_{12}(p^{\sigma}p^{\rho}\eta^{\tau\lambda}\eta^{\mu\iota}\eta^{\nu\kappa})\\&\qquad-P_6(p^{\rho}p'^{\iota}\eta^{\lambda\kappa}\eta^{\mu\sigma}\eta^{\nu\tau})-P_{24}(p^{\sigma}p'^{\rho}\eta^{\tau\mu}\eta^{\nu\iota}\eta^{\kappa\lambda})-P_{12}(p^{\sigma}p'^{\mu}\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\nu})+2P_6(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\mu})\big].\end{aligned} \quad (2.7)$$

← 4-point vertex: 2850 terms

The "Sym" standing in front of these expressions indicates that a symmetrization is to be performed on each index pair $\mu\nu$, $\sigma\tau$, etc. The symbol P indicates that a summation is to be carried out over all distinct permutations of the momentum-index triplets, and the subscript gives the number of permutations required in each case.

Expressions (2.6) and (2.7) can be obtained in a straightforward manner by repeated functional differentiation of the Einstein action. This procedure, however, is exceedingly laborious. A more efficient (but still lengthy) method is to make use of the hierarchy of identities (II, 17.31). It is a remarkable fact that once $S_2{}^0$ is known all the higher vertex functions, and hence the complete action functional itself, are determined by the general coordinate invariance of the theory. It is convenient, in the actual computation of the vertices via (II, 17.31), to invent diagrammatic schemes for displaying the combinatorics of indices. Since each reader will devise the scheme which suits him best we shall not shackle him by describing one here. We also make no attempt to display S_5 or any higher vertices.

The vertex $V_{(\alpha i)\beta}$ has the following form for the gravitational field:

$$V_{(\mu^{\sigma''\tau''})\nu'} \to$$
$$\tfrac{1}{2}\mathrm{Sym}\big[2p''_{\mu}p'^{\sigma}\delta_{\nu}{}^{\tau}-p''_{\mu}p'_{\nu}\eta^{\sigma\tau} +(p_{\nu}p'^{\sigma}-p'_{\nu}p^{\sigma})\delta_{\mu}{}^{\tau}+p\cdot p'\delta_{\mu}{}^{\sigma}\delta_{\nu}{}^{\tau}\big], \quad (2.8)$$

where the momentum-index combinations are $p\mu$, $p'\nu'$, $p''\sigma''\tau''$, and the symmetrization is to be performed on the index pair $\sigma\tau$. The propagators for the normal and fictitious quanta are given by

$$G \to (\eta_{\mu\sigma}\eta_{\nu\tau}+\eta_{\mu\tau}\eta_{\nu\sigma}-\eta_{\mu\nu}\eta_{\sigma\tau})/p^2, \quad (2.9)$$

$$\hat{G} \to \eta^{\mu\nu}/p^2. \quad (2.10)$$

[2] The choice of terms is not completely unique since momentum conservation may be used to replace a given term by other terms. We give here what we believe (but have not proved) to be the expressions containing the smallest number of terms.

Gravity (and YM) amplitudes are much simpler than what one would expect from Feynman rules !

Recent developments

- New expression for MHV amplitudes (Elvang, Freedman)

$$\mathcal{M}_n^{\mathrm{MHV}} = \sum_{\mathcal{P}(2,\ldots,n-1)} [A_n^{\mathrm{MHV}}(1,\ldots,n)]^2\, G^{\mathrm{MHV}}(1,\ldots,n)$$

  - intriguing sum of squares of YM amplitudes

Spradlin's talk

- N=8 recursion relation solved (Drummond, Spradlin, Volovich, Wen)

$$\mathcal{M}_n = \sum_{\mathcal{P}(2,\ldots,n-1)} [A_n^{\mathrm{MHV}}(1,\ldots,n)]^2 \sum_{\{\alpha\}} [R_\alpha(\lambda_i,\tilde{\lambda}_i;\eta_i)]^2\, G_\alpha(\lambda_i,\tilde{\lambda}_i)$$

No η's here

- Solution parallels that for N=4 SYM ! (Drummond, Henn)

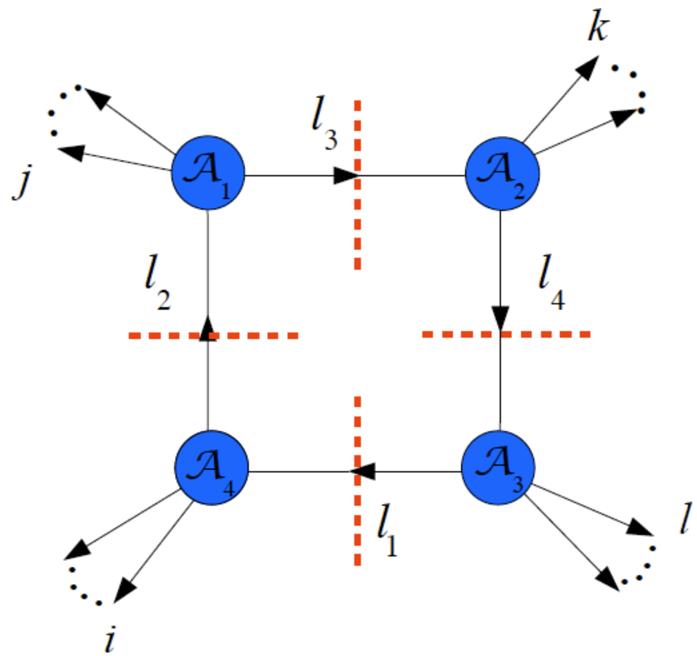
$$\mathcal{A}_n^{\mathcal{N}=4}(1,\ldots,n) = A_n^{\mathrm{MHV}}(1,\ldots,n) \sum_{\{\alpha\}} R_\alpha(\lambda_i,\tilde{\lambda}_i;\eta_i)$$

G = bosonic "dressing function", R = dual superconformal invariant

- Derive one-loop box coefficients of N=8 superamplitudes

  - use quadruple cuts

$$\mathcal{C}(i,j,k,l) \; =$$



  - input: tree-level superamplitudes obtained by solving the N=8 supersymmetric recursion relation

  - remarkable simplicity of the tree-level R-functions feeds loops

One-loop box coefficients

(Hall; Katsaroumpas, Spence, GT)

- Main result: pattern echoes tree level

$$\mathcal{C}_n^{\mathcal{N}=8} = \sum_{\mathcal{P}(2,\dots,n-1)} [\mathcal{C}^{\mathcal{N}=4}_{\mathrm{MHV};n}(1,\dots,n)]^2 \sum_{\{\alpha\}} [\tilde{R}_\alpha(\lambda_i,\tilde{\lambda}_i;\eta_i)]^2\, \tilde{G}_\alpha(\lambda_i,\tilde{\lambda}_i)$$

  - $\tilde{R}$ superconformal invariant; $\tilde{G}$ bosonic

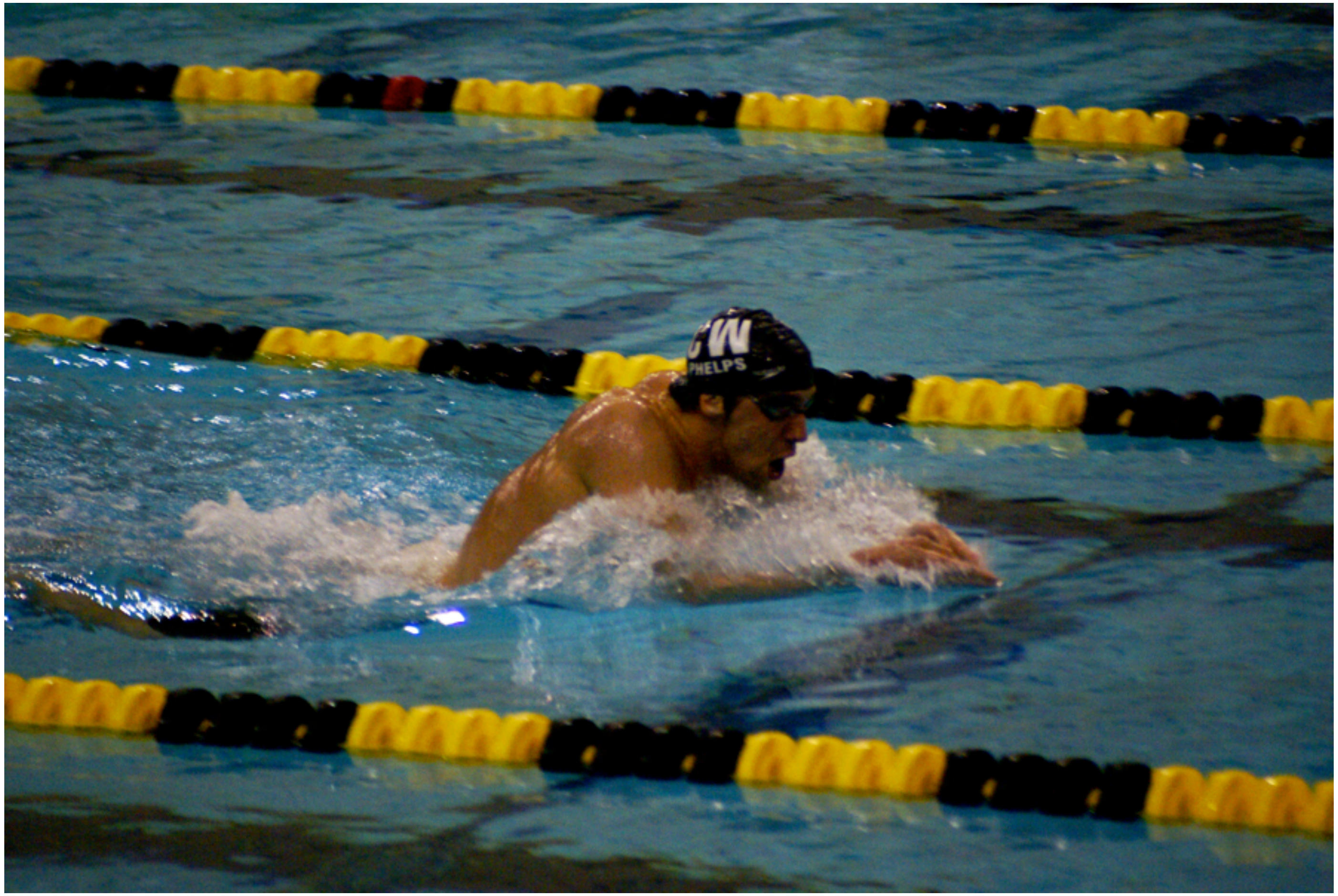
- Checks:
  - MHV amplitudes (Bern, Dixon, Perelstein, Rozowsky), up to 22 legs
  - six-point NMHV amplitudes (Bern,Bjerrum-Bohr, Dunbar; Bjerrum-Bohr, Dunbar, Ita)

- Summarising:
  - Tree-level amplitudes in N=8 supergravity expressed as sums of squares of amplitudes in N=4 SYM...
  - ...Similar properties for the coefficients of one-loop amplitudes
- Various incarnations of the idea that

$$\text{Gravity} = (\text{Yang-Mills})^2$$

- Intriguing appearance of dual superconformal expressions

A quick swim to



N=2 super QED

Why QED ?

- Because (we think) it's simple...
  - ...but it took 50 years to calculate two-loop amplitudes...
  - multi-photon amplitudes are free of divergences
- Because it's similar to (other) theories we like
  - No-triangle property for $n > 6$ one-loop n-photon amplitudes (Badger, Bjerrum-Bohr, Vanhove)
  - N=2 SQED four-photon amplitude at one and two loops are maximally transcendental... (Binoth, Glover, Marquard, Van der Bij)
  - ...and can be obtained from N<2 amplitudes by suppressing non maximally transcendental terms

Four-photon amplitudes

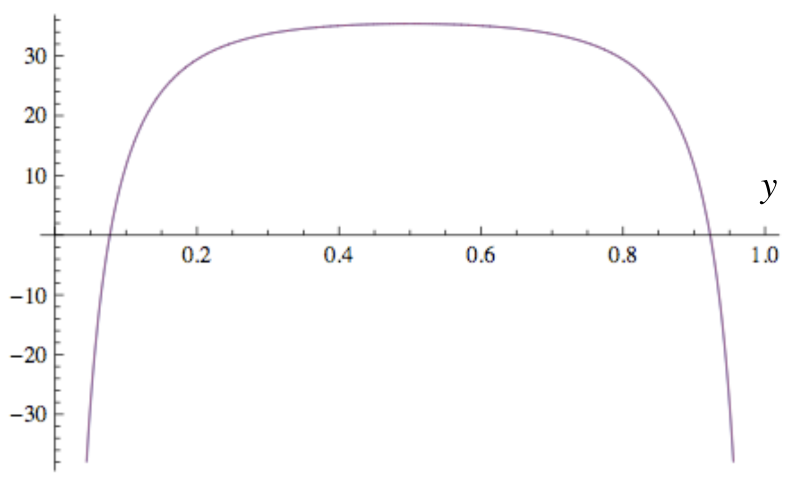
- Tree level: zero
- One loop: $\mathcal{M}_4^{(1)} = -4\left[(\log(x) - \log(y))^2 + \pi^2\right]$ (Karplus & Neuman, 1950)
    - x=-s/t , y = -u/t = 1-x
- Two loops: (Binoth, Glover, Marquard, van der Bij, 2002; Bern, De Freitas, Dixon, Ghinculov, Wong, 2001)

$$
\begin{aligned}
\mathrm{Re}\left[\mathcal{M}_4^{(2)}\right] \quad = \quad & -16\,\mathrm{Li}_4(y) - 16\,\mathrm{Li}_4(x) + 8(\log(x) + \log(y))\left(\mathrm{Li}_3(x) + \mathrm{Li}_3(y)\right) \\
& +4\log^2(x)^2\log^2(y) - \frac{4}{3}\log(x)\log(y)\pi^2 \\
& -\frac{1}{24}\left[\mathcal{M}_4^{(1)}\right]^2 - \frac{\pi^2}{3}\mathcal{M}_4^{(1)} + \frac{2}{45}\pi^4
\end{aligned}
$$

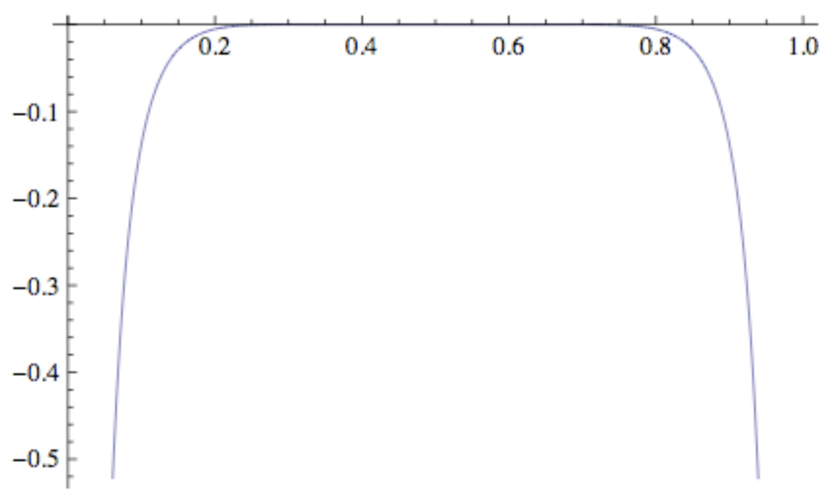
Approximate iterative structures

(Brandhuber, Vincon, GT)

- **Try** $[\mathcal{M}_4^{(2)}]_{\rm ansatz} = b\,[\mathcal{M}_4^{(1)}]^2 + c\,\mathcal{M}_4^{(1)} + d$
  - determine b, c, d in some appropriate way (e.g. least squares)



Ansatz & two-loop amplitude



"Remainder" function

$$\mathcal{R}_4(y) := {\rm Re}\Big[\mathcal{M}_4^{(2)}\Big] - \Big(b\,[\mathcal{M}_4^{(1)}]^2 + c\,\mathcal{M}_4^{(1)} + d\Big)$$

- Comments:
  - An attempt at approximating the two-loop four-photon amplitude in N=1 super QED with an ansatz similar to that used earlier in N=2 is much more crude
  - maximally supersymmetric QED amplitudes are special !

Conclusions

"...something deeper than what we know underlies quantum field theory..."

Open questions

- How many one-loop coefficients can be determined from symmetry considerations alone? Yangians?
- Derive the anomaly & conformal equations at higher loops
- Derive anomaly equations for other generators of dual superconformal group, specifically $\bar{Q}$
- Origin of dual superconformal symmetry in field theory?
- Understand the implications of the structure found for N=8 supergravity amplitudes and one-loop coefficients

and many more...