A journey through three maximally supersymmetric theories

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Hidden Structures in Field Theory Amplitudes, NBI, August 2009
• **N=4 super Yang-Mills**
  - Dual conformal symmetry & equations for one-loop supercoefficients

• **N=8 supergravity**
  - New expressions for one-loop supercoefficients

• **N=2 super QED**
  - Four-point photon amplitude at one and two loops and approximate iterative relations
$N=4$ super Yang-Mills
Amplitudes in N=4 SYM

- Scattering amplitudes in N=4 are conveniently described in terms of on-shell superspace variables

- **Super wave-function:**  
  \[ \Phi(p, \eta) = A^+(p) + \eta^A \psi_A(p) + \frac{1}{2} \eta^A \eta^B \phi_{AB}(p) + \frac{1}{3!} \eta^A \eta^B \eta^C \epsilon_{ABCD} \bar{\psi}^D(p) \]
  
  \[ + \frac{1}{4!} \eta^A \eta^B \eta^C \eta^D \epsilon_{ABCD} A^-(p) \]
  
  \( h = +1 \quad h = +1/2 \quad h = 0 \quad h = -1/2 \quad h = -1 \)

- \( \eta^A, \ A = 1, \ldots, 4 \) fermionic variables, \( A \) is an SU(4) index

- particle content is that of N=4 super Yang-Mills

- similar to Mandelstam’s LC superfield
Package all amplitudes with a fixed external helicity assignment into a single superamplitude: (Nair; Witten)

\[ A(\{\lambda_i, \tilde{\lambda}_i, \eta_i\}) = \delta^{(4)}(\sum_i \lambda_i \tilde{\lambda}_i) \delta^{(8)}(\sum_i \eta_i \lambda_i) A(\{\lambda_i, \tilde{\lambda}_i, \eta_i\}) \]

- \( q^A_\alpha = \sum \lambda_i;\alpha \eta_i^A \) susy manifest,
- \( \bar{q}_{A,\dot{\alpha}} = \sum \tilde{\lambda}_i;\dot{\alpha} \frac{\partial}{\partial \eta_i^A} \) imposes restrictions on \( A \)
- \( \eta\)-expansion: \( p \) powers of \( \eta_i \) correspond to helicity \( h_i = 1 - p/2 \)
Examples:

\[ A = \delta^{(4)} \left( \sum_i \lambda_i \tilde{\lambda}_i \right) \delta^{(8)} \left( \sum_i \eta_i \lambda_i \right) A \]

- **All-plus & single minus amplitudes:** automatically zero
  - Too few powers of \( \eta \)'s: fermionic delta function cannot be satisfied
  - All-minus and single-plus vanish by parity

- **MHV superamplitude:**
  \[ A_{\text{MHV}} = \frac{1}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n \, 1 \rangle} \]

- Gluons \( i^- , j^- \): get factor of \( (\eta_i)^4 (\eta_j)^4 \langle i \, j \rangle^4 \)

\[ A_{\text{MHV}}^{\text{gluons}} (1^+ , \ldots , i^- , \ldots , j^- , \ldots n^+) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n \, 1 \rangle} \]

(Parke & Taylor)
Early applications:

- **Tree-level super MHV diagrams**  (Georgiou, Khoze + Glover, 2004)

- **One-loop super MHV diagrams**  (Brandhuber, Spence, GT, 2004)
  - very efficient way to perform internal supersums, e.g. one-loop MHV superamplitude:

Roiban’s seminar
Dual Conformal Symmetry
(Drummond, Henn, Korchemsky, Sokatchev)

• **Conjecture:** dual (super)conformal symmetry is a symmetry of the planar S-matrix of N=4 SYM

  ▶ planarity, maximal supersymmetry, on-shellness

  ▶ originates from Wilson loops
    - MHV amplitude/Wilson loop duality
Dual conformal symmetry: (DHKS)

- is the standard conformal symmetry acting on dual momenta \( x ' s \)
  \[
  p_i = x_i - x_{i+1}
  \]
  \[
  x_{n+1} = x_1
  \]

- symmetry is anomalous
  - ultraviolet divergences from cusps in the contour
  - UV for the Wilson loop = IR for the amplitude

- Since \( K=IPI \), enough to look at conformal inversions \( I \)
  - under a conformal inversion \( x_i^\mu \rightarrow x_i^\mu / x_i^2 \)
Extending Dual Conformal Symmetry from Wilson loops to all (non-MHV) amplitudes is very nontrivial!
Trees
Covariance of tree-level S-matrix

(Brandhuber, Heslop, GT)

- Idea: look for a computational method which respects the symmetry of the problem at the diagrammatic level. Is there one?
- YES! N=4 supersymmetric recursion relations
  - supersymmetric version of the BCFW on-shell recursion relations (Britto, Cachazo, Feng + Witten)
  - MHV diagrams more problematic
Why do we think it will work?

- Look at split-helicity gluon amplitudes:

\[ 1_g^-, 2_g^- \ldots p_g^- (p + 1)_g^+ (p + 2)_g^+ \ldots n_g^+ \]

- calculated in closed form by Britto, Feng, Roiban, Spradlin, Volovich solving the BCF recursion relation

- each recursive diagram is manifestly covariant!

- E.g.:

\[
A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+) = \frac{\langle 1|2 + 3|4 \rangle^3}{t_2^{[3]} \langle 5 \ 6 \rangle \langle 6 \ 7 \rangle \langle 7 \ 1 \rangle [2 \ 3] [3 \ 4] \langle 5 |4 + 3|2 \rangle} - \frac{1}{\langle 3 \ 4 \rangle \langle 4 \ 5 \rangle \langle 6 |7 + 1|2 \rangle} \left( \frac{\langle 3|4 + 5 \rangle (6 + 7) |1 \rangle^3}{t_3^{[3]} t_5^{[3]} \langle 6 \ 7 \rangle \langle 7 \ 1 \rangle \langle 5 |4 + 3|2 \rangle} + \frac{\langle 3|2 + 1 \rangle |7 \rangle^3}{t_7^{[3]} \langle 6 \ 5 \rangle [7 \ 1] [1 \ 2]} \right)
\]

Spinor brackets above involve cyclically adjacent gluons.
• **On-shell recursion relations** (Britto, Cachazo, Feng)

  - based on **singularity structure** of tree-level amplitudes:

  \[ A = \sum_{j, h} \]

  \[ A_i \quad \text{on} \quad \text{multiparticle poles} \quad \text{(simple poles, tree level), bilinear structure} \]

  \[ h = \text{internal particles helicities} \]

  \[ P_{ij} := p_i + \cdots + p_j \]
Susy recursion relations
(Brandhuber, Heslop, GT; Arkani-Hamed, Cachazo, Kaplan)

- supersymmetric deformations of amplitudes:

\[ \hat{x}_2 := x_2 - z \lambda_1 \hat{\lambda}_2 \]

Hence, by supersymmetry:
\[ \hat{\theta}_2 := \theta_2 - z \lambda_1 \eta_2 \]

\[ \hat{p}_1 := \lambda_1 \hat{\lambda}_1 = \lambda_1 (\tilde{\lambda}_1 + z \tilde{\lambda}_2) := x_1 - \hat{x}_2 \]
\[ \hat{p}_2 := \hat{\lambda}_2 \tilde{\lambda}_2 = (\lambda_2 - z \lambda_1) \tilde{\lambda}_2 := \hat{x}_2 - x_3 \]

\[ \hat{\eta}_1 := \eta_1 + z \eta_2 \]
\[ \hat{\eta}_2 := \eta_2 \]

Amplitudes with different external states mix under the supersymmetric shifts
• Generic **superamplitude** expressed as:

\[
A = \sum_{\mathbf{P}} \int d^4 \eta \mathbf{P} \ A_L(z_{\mathbf{P}}) \frac{i}{P^2} A_R(z_{\mathbf{P}})
\]

- sum over internal species becomes a **fermionic integral**
- proof of DSC symmetry obtained by induction
  - three-point superamplitudes are the input
  - propagator + fermionic integration

• **Symmetry maintained at the diagrammatic level**

- explicit solution of recursion relations (Drummond, Henn)
• Important difference with standard BCF shifts

\[ A_{BCF}(z) = A(\lambda_1, \hat{\lambda}_1(z), \eta_1; \lambda_2(z), \tilde{\lambda}_2, \eta_2) \]

\[ A_{SUSY}(z) = A(\lambda_1, \hat{\lambda}_1(z), \hat{\eta}_1(z); \hat{\lambda}_2(z), \tilde{\lambda}_2, \eta_2) \]

• Large $z$ behaviour

\[ A_{SUSY}(z) \sim 1/z, \quad A_{SUGRA}(z) \sim 1/z^2 \quad \text{as} \quad z \to \infty \]

• proof uses maximal supersymmetry (Arkani-Hamed, Cachazo, Kaplan)
Loops
Simplicity of loop amplitudes

- **One loop:** amplitudes in maximally supersymmetric theories are sums of **box functions**

\[ \mathcal{A} = \sum_{i,j,k,l} C(i,j,k,l) \]

- **N=4 super Yang-Mills** (Bern, Dixon, Dunbar, Kosower)
- **N=8 supergravity (no-triangle property)** (Bern, Bjerrum-Bohr, Dixon, Dunbar, Dunbar, Ita, Perelstein, Perkins, Risager, Rozowsky; Bjerrum-Bohr + Vanhove; Arkani-Hamed, Cachazo, Kaplan)
- **n-photon amplitudes in QED, \( n \geq 8 \)** (Badger, Bjerrum-Bohr, Vanhove)

- **Infrared divergences:**
  - Work in \( D = 4 - 2\epsilon \)
Generic amplitudes are anomalous under DCS:

- only two-particle invariants appears (similarly to infrared divergent terms)

\[ K^\mu A_n^{1\text{-loop}} = -4 r_\Gamma A_n^{\text{tree}} \sum_{i=1}^{n} x_{i+1}^\mu \frac{(-x_{i+2}^2)^{-\epsilon}}{\epsilon} \]

- conjectured by DHKS from Wilson loops

- caveat: Wilson loops/amplitudes duality only applies to MHV amplitudes

- important evidence from NMHV superamplitude at one loop (DHKS)
Next goals:

I. prove the dual conformal anomaly equation

II. prove covariance of the one-loop coefficients

III. derive new relations of one-loop coefficients (loops without loops...)
I. Proof of Dual Conformal Anomaly

(Brandhuber, Heslop, GT)

• Idea: calculate discontinuities of the anomaly

  ‣ multi-particle channel discontinuities: \(( j > i+1 )\)

\[
disc_{x^2_{i,j+1}}[K^\mu A_{n-1\text{loop}}^1] = 4\epsilon \int d^D y \delta(+) (y - x_i)^2 \delta(+) (x_{j+1} - y)^2 [y^\mu \langle l_1 l_2 \rangle^4 A_{\text{tree}}^L A_{\text{tree}}^R]
\]

  ‣ integral is finite \(\times\ \epsilon\), hence multi-particle disc's vanish

  ‣ anomaly independent of multi-particle invariants
Two-particle channel discontinuities:

\[
\epsilon \\sim x_{i+1} \quad l_1 \sim p_i, \quad l_2 \sim p_{i+1}, \quad \text{hence} \quad y \sim x_{i+1}
\]

integral is \(1/\epsilon^2\) divergent. Divergences arise from a region of integration where \(l_1 \sim p_i, \ l_2 \sim p_{i+1}\), hence \(y \sim x_{i+1}\)

Pick (leading) divergence: discontinuity factorises

Emergence of tree-level amplitude \(A_{n}^{\text{tree}}\)!
Lift cut integral, to get expected dual conformal anomaly:

\[ K^\mu A^{1-\text{loop}}_n = -4 r \Gamma A^{\text{tree}}_n \sum_{i=1}^{n} x_{i+1}^\mu \frac{(-x_{i+2}^2)^{-\epsilon}}{\epsilon} \]

**Comments:**

- Analysis exposes intimate link of anomaly to IR divergences
  - singular channel cut-integral lifted to full loop integral (Kosower)
- Subleading divergences absent due to maximal susy
- Absence of holomorphic anomaly crucial: \( K^\mu A^{\text{tree}} = 0 \), with no delta-function supported terms on the RHS
  (Bargheer, Beisert, Galleas, Loebbert, McLoughlin; Korchemsky, Sokatchev)
II. Covariance of one-loop coefficients

- Same idea as for tree-level recursion:
  - look for a computational method which respects the symmetry of the problem at the diagrammatic level

- Quadruple cuts (Britto, Cachazo, Feng)
  - calculate one by one coefficients of one-loop boxes
  - appropriate N=4 supersymmetric generalisation
    (Drummond, Henn, Korchemsky, Sokatchev)
Coefficient of each box function obtained by gluing **four tree-level superamplitudes**

- quadruple cut diagram inherits symmetry properties from tree-level superamplitudes entering the cut

- **One-loop coefficients are covariant**
  (Brandhuber, Heslop, GT)
III. New relations for one-loop coefficients

(Brandhuber, Heslop, GT)

● One-loop amplitudes:

\[ \mathcal{A}^{1-\text{loop}} = \sum_{i,j,k,l} C(i, j, k, l) \]

\[ F = -i(4\pi)^{D/2} \int \frac{d^D x_5}{(2\pi)^D} \frac{\sqrt{R}}{x_5^2 x_i^2 x_j^2 x_l^2} \]

\[ \sqrt{R} = x_{ik}x_{jl} - x_{jk}x_{li} \]

\[ D = 4 - 2\epsilon \]

Boxes:

1m  2me  2mh  3m
III. New relations for one-loop coefficients

(Brandhuber, Heslop, GT)

- One-loop amplitudes:

\[ \mathcal{A}^{1\text{-loop}} = \sum_{i,j,k,l} C(i,j,k,l) \]

\[ F = -i(4\pi)^{D/2} \int \frac{d^D x_5}{(2\pi)^D} \frac{\sqrt{R}}{x_{5i}^2 x_{5j}^2 x_{5k}^2 x_{5l}^2} \]

\[ \sqrt{R} = x_{ik}^2 x_{jl}^2 - x_{jk}^2 x_{li}^2 \]

\[ D = 4 - 2\epsilon \]

Boxes:

1m  2me  2mh  3m  4m
• **Supercoefficients** $C(i, j, k, l)$ are covariant $\Rightarrow$

$$K^\mu A_{1\text{-loop}} = \sum_{i,j,k,l} C(i, j, k, l) K^\mu F_{ijkl}$$

• **Under a special conformal transformation**

$$K^\mu F \sim \epsilon \int \frac{d^{4-2\epsilon} x_5}{(2\pi)^{4-2\epsilon}} \frac{\sqrt{R}}{x_5^2 x_5^2 x_5^2 x_5^2} x_5^\mu$$

• **Anomaly appears because of IR divergences**

• **Standard Passarino-Veltman reduction to calculate the integral above**

  $$\text{result expressed in terms of scalar boxes and triangles}$$
Box functions and their anomalies

$$K^\mu F^{0m} = 2\epsilon \left[ (x_1 + x_3)^\mu x_{24}^2 J(x_{24}^2) + (x_2 + x_4)^\mu x_{13}^2 J(x_{13}^2) \right],$$

$$K^\mu F^1_1 = -2\epsilon \left\{ -x_1^\mu x_{24}^2 J(x_{24}^2, x_{41}^2) + x_2^\mu x_{41}^2 J(x_{24}^2, x_{41}^2) - x_{13}^2 J(x_{13}^2) \right\},$$

$$K^\mu F^{2me}_{13} = -2\epsilon \left\{ x_1^\mu \left[ -x_{24}^2 J(x_{24}^2, x_{41}^2) + x_{23}^2 J(x_{13}^2, x_{23}^2) \right] + x_2^\mu \left[ -x_{13}^2 J(x_{13}^2, x_{23}^2) + x_{41}^2 J(x_{24}^2, x_{41}^2) \right] \right\},$$

$$K^\mu F^{2mh}_{14} = -2\epsilon \left\{ -x_1^\mu x_{24}^2 J(x_{24}^2, x_{41}^2) + x_2^\mu \left[ x_{41}^2 J(x_{24}^2, x_{41}^2) - x_{13}^2 J(x_{13}^2) + x_{34}^2 J(x_{24}^2, x_{34}^2) \right] \right\},$$

$$K^\mu F^{3m}_{134} = -2\epsilon \left\{ x_1^\mu \left[ x_{23}^2 J(x_{23}^2, x_{13}^2) - x_{24}^2 J(x_{24}^2, x_{41}^2) \right] + x_2^\mu \left[ x_{41}^2 J(x_{41}^2, x_{24}^2) - x_{13}^2 J(x_{13}^2, x_{23}^2) \right] \right\}.$$

$J(a) := \frac{r_\Gamma}{\epsilon^2} \frac{(-a)^{-\epsilon}}{(-a)}$  \quad Im triangle $(1/\epsilon^2)$

$J(a, b) := \frac{r_\Gamma}{\epsilon^2} \frac{(-a)^{-\epsilon} - (-b)^{-\epsilon}}{(-a) - (-b)}$  \quad 2m triangle $(1/\epsilon)$
• **Summarising:**

\[
K^\mu A^{1-\text{loop}}_n = 4 \epsilon A^{\text{tree}}_n \sum_{i=1}^{n} x_{i+1}^{\mu} \left[ x_{ii+2}^2 J(x_{ii+2}^2) \right] = \sum_{i,j,k,l} C(i, j, k, l) K^\mu F_{ijkl}
\]

anomaly eq, proved earlier...

from covariance of coeff’s...

• **Last line can be written as:**

\[
+2 \epsilon \sum_{i=1}^{n} \sum_{k=i+2}^{i+n-2} x_{i-1}^{\mu} \left[ x_{ii-2}^2 J(x_{ii-2}^2) \right] \sum_{j=i+1}^{i+n-3} C(i, j, i-2, i-1)
\]

1m triangle \((1/\epsilon)\)

\[
-2 \epsilon \sum_{i=1}^{n} \sum_{k=i+2}^{i+n-3} \mathcal{E}(i, k) \left[ x_{i-1}^{\mu} x_{ik}^2 - x_{i}^{\mu} x_{i-1k}^2 \right] J(x_{ik}^2, x_{i-1k}^2)
\]

2m triangle \((\text{finite})\)

where \( \mathcal{E}(i, k) := \sum_{j=k+1}^{i+n-2} C(i, k, j, i-1) - \sum_{j=i+1}^{k-1} C(i, j, k, i-1) \)
• Next, we require these two results to be identical

• \( \text{Im triangle coeff'\text{'t}: special combination of } 2m_h \text{ and } \text{Im box coefficients, } \sim A^{\text{tree}} \) (Roiban, Spradlin, Volovich)

\[
\sum_{j=i+1}^{i+n-3} C(i, j, i-2, i-1) = \sum_{j=i+1}^{i+n-3} C(i, j, i-2, i-1) = 2A_n^{\text{tree}}
\]

for \( i = 1, \ldots, n \)

• main inspiration for the BCF recursion relation

• This gives precisely the anomaly
• 2m triangle: \( n(n-4) \) new relations among coefficients

\[
\mathcal{E}(i, k) = 0, \quad i = 1 \ldots n, \quad k = i + 2, \ldots, i + n - 3
\]

\[
\mathcal{E}(i, k) := \sum_{j=k+1}^{i+n-2} C(i, k, j, i - 1) - \sum_{j=i+1}^{k-1} C(i, j, k, i - 1)
\]

• These are new predictions obtained from dual conformal invariance

• Determine 1m, 2me, and half of the 2mh coeff’s

\[
n \quad n(n-5)/2 \quad \quad (1/2) n(n-5)
\]

Loops without loops!
Comment

• Compare to standard IR consistency conditions

\[ A_{n \text{-loop}}^{\text{IR}} \sim - A_{n \text{tree}} \sum_{i=1}^{n} \frac{(-x_{ii+2}^2)^{-\epsilon}}{\epsilon^2} \]

\[ n \text{ [2-particle]} + n(n-5)/2 \text{ [multi-particle]} = n(n-3)/2 \text{ relations} \]

• Can be used to predict all (all but one) \text{Im} and \text{2me} box coefficients for \( n \text{ odd} \) (\( n \text{ even} \))

(Bern, Dixon, Kosower)

• Conformal equations predict \( n(n-5)/2 \) further coefficients

\[ 1/2 \text{ of } 2\text{mh box coefficients} \]
Dual Q-bar anomaly

- Two kind of contributions: IR and holomorphic
  - multi-particle channel discontinuity of the IR piece:
    - IR contribution is finite $X \epsilon \rightarrow 0$
  - holomorphic anomaly contribution is non-universal
    (Korchemsky, Sokatchev)

Korchemsky’s seminar
Two-particle channels discontinuities:

- Analysis similar to dual conformal case. Get "..." stand for a single holomorphic anomaly contribution

\[
Q^A_{\dot{\alpha}} \mathcal{A}^{1\text{-loop}}_n \sim \mathcal{A}^{\text{tree}}_n \sum_{i=1}^n \frac{(-x_{ii+2}^2)^{-\epsilon}}{\epsilon} \frac{\eta^A_i \tilde{\lambda}_{i+1} \dot{\alpha} - \eta^A_{i+1} \tilde{\lambda}_i \dot{\alpha}}{[ii + 1]} + \ldots
\]

- "..." stand for a single holomorphic anomaly contribution

\[Q^A_{\dot{\alpha}} \mathcal{A}^{\text{tree}}_4 = 0\] identically (no contact terms)

(Bargheer, Beisert, Galleas, Loebbert, McLoughlin)
N=8  Supergravity
• On-shell recursion relations for tree-level gravity amplitudes (Bedford, Brandhuber, Spence, GT; Cachazo, Svrcek)

  • large-$\mathcal{Z}$ behaviour (Arkani-Hamed, Kaplan; Benincasa, Boucher-Veronneau, Cachazo)

• Amplitudes’ calculation vastly simplified

  \[ \mathcal{A}_{\text{GR}}(1^{+}2^{+}3^{-}) = [\mathcal{A}_{\text{YM}}(1^{+}2^{+}3^{-})]^2, \quad \mathcal{A}_{\text{GR}}(1^{-}2^{-}3^{+}) = [\mathcal{A}_{\text{YM}}(1^{-}2^{-}3^{+})]^2 \]

  • EH Lagrangian (and its Feynman rules) not needed
corrections. For the Yang-Mills field it takes the form
\[ V_{\text{YM}} = \gamma \quad \text{with} \quad \gamma = \gamma_{\mu
u} p^\mu p^\nu. \]

The propagators for the normal and fictitious quanta are, respectively,
\[ G \rightarrow \gamma_{\mu
u} p^\mu, \quad (2.4) \]
\[ G \rightarrow \gamma_{\mu
nu} p^\mu, \quad (2.5) \]
with \( p^\mu \) being understood to have the usual small negative imaginary part.

The corresponding quantities for the gravitational
\[ \delta^3 S \]

\[ \delta^4 S \]

3-point vertex: \( 171 \) terms

4-point vertex: \( 2850 \) terms

\[ \text{gravity (and YM) amplitudes are much simpler than what one would expect from Feynman rules!} \]
Recent developments

- New expression for MHV amplitudes (Elvang, Freedman)

\[ \mathcal{M}_n^{\text{MHV}} = \sum_{\mathcal{P}(2,\ldots,n-1)} [A_n^{\text{MHV}}(1,\ldots,n)]^2 G^{\text{MHV}}(1,\ldots,n) \]

\[ \alpha \] intriguing sum of squares of YM amplitudes

- \( N=8 \) recursion relation solved (Drummond, Spradlin, Volovich, Wen)

\[ \mathcal{M}_n = \sum_{\mathcal{P}(2,\ldots,n-1)} [A_n^{\text{MHV}}(1,\ldots,n)]^2 \sum_\{\alpha\} [R_\alpha(\lambda_i, \tilde{\lambda}_i; \eta_i)]^2 G_\alpha(\lambda_i, \tilde{\lambda}_i) \]

\[ \alpha \] No \( \eta \)'s here

- Solution parallels that for \( N=4 \) SYM ! (Drummond, Henn)

\[ A_n^{N=4}(1,\ldots,n) = A_n^{\text{MHV}}(1,\ldots,n) \sum_\{\alpha\} R_\alpha(\lambda_i, \tilde{\lambda}_i; \eta_i) \]

\[ G = \text{bosonic “dressing function”}, \quad R = \text{dual superconformal invariant} \]
Derive one-loop box coefficients of N=8 superamplitudes

- use quadruple cuts

\[ C(i, j, k, l) = \]

- input: tree-level superamplitudes obtained by solving the N=8 supersymmetric recursion relation

- remarkable simplicity of the tree-level $R$-functions feeds loops
Main result: pattern echoes tree level

\[ C_n^{\mathcal{N}=8} = \sum_{\mathcal{P}(2,\ldots,n-1)} [C_{\mathcal{N}=4}^{\text{MHV};n(1,\ldots,n)}]^2 \sum_{\{\alpha\}} [\tilde{R}_\alpha(\lambda_i, \tilde{\lambda}_i; \eta_i)]^2 \tilde{G}_\alpha(\lambda_i, \tilde{\lambda}_i) \]

- \( \tilde{R} \) superconformal invariant; \( \tilde{G} \) bosonic

Checks:

- **MHV amplitudes** (Bern, Dixon, Perelstein, Rozowsky), up to 22 legs
- six-point NMHV amplitudes (Bern, Bjerrum-Bohr, Dunbar; Bjerrum-Bohr, Dunbar, Ita)
• **Summarising:**

  ‣ Tree-level amplitudes in N=8 supergravity expressed as sums of squares of amplitudes in N=4 SYM...

  ‣ ...Similar properties for the coefficients of one-loop amplitudes

• **Various incarnations of the idea that**

  \[ \text{Gravity} = (\text{Yang-Mills})^2 \]

• **Intriguing appearance of dual superconformal expressions**
A quick swim to

N=2 super QED
Why QED?

- Because (we think) it’s simple...
  - ...but it took 50 years to calculate two-loop amplitudes...
  - multi-photon amplitudes are free of divergences

- Because it’s similar to (other) theories we like
  - No-triangle property for $n > 6$ one-loop $n$-photon amplitudes
    (Badger, Bjerrum-Bohr, Vanhove)
  - N=2 SQED four-photon amplitude at one and two loops are maximally transcendental...
    (Binoth, Glover, Marquard, Van der Bij)
  - ...and can be obtained from N<2 amplitudes by suppressing non maximally transcendental terms
Four-photon amplitudes

- **Tree level:** zero

- **One loop:** \( \mathcal{M}_4^{(1)} = -4 \left[ (\log(x) - \log(y))^2 + \pi^2 \right] \) (Karplus & Neuman, 1950)
  
  \[ x = -\frac{s}{t}, \ y = -\frac{u}{t} = 1 - x \]

- **Two loops:** (Binoth, Glover, Marquard, van der Bij, 2002; Bern, De Freitas, Dixon, Ghinculov, Wong, 2001)
  
  \[
  \text{Re} \left[ \mathcal{M}_4^{(2)} \right] = -16 \text{Li}_4(y) - 16 \text{Li}_4(x) + 8(\log(x) + \log(y)) (\text{Li}_3(x) + \text{Li}_3(y)) \\
  + 4 \log^2(x)^2 \log^2(y) - \frac{4}{3} \log(x) \log(y) \pi^2 \\
  - \frac{1}{24} \left[ \mathcal{M}_4^{(1)} \right]^2 - \frac{\pi^2}{3} \mathcal{M}_4^{(1)} + \frac{2}{45} \pi^4
  \]
Approximate iterative structures
(Brandhuber, Vincon, GT)

- **Try** \([\mathcal{M}_4^{(2)}]_{\text{ansatz}} = b [\mathcal{M}_4^{(1)}]^2 + c \mathcal{M}_4^{(1)} + d\)

- determine \(b, c, d\) in some appropriate way (e.g. least squares)

\[
R_4(y) := \text{Re}[\mathcal{M}_4^{(2)}] - \left( b [\mathcal{M}_4^{(1)}]^2 + c \mathcal{M}_4^{(1)} + d \right)
\]
• Comments:

- An attempt at approximating the two-loop four-photon amplitude in $N=1$ super QED with an ansatz similar to that used earlier in $N=2$ is much more crude.

- Maximally supersymmetric QED amplitudes are special!
Conclusions

“...something deeper than what we know underlies quantum field theory...”
Open questions

- How many one-loop coefficients can be determined from symmetry considerations alone? Yangians?
- Derive the anomaly & conformal equations at higher loops
- Derive anomaly equations for other generators of dual superconformal group, specifically $Q$
- Origin of dual superconformal symmetry in field theory?
- Understand the implications of the structure found for N=8 supergravity amplitudes and one-loop coefficients

and many more...