Symmetries of scattering amplitudes in $\mathcal{N} = 4$ SYM

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Based on work in collaboration with

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Scattering amplitudes in $\mathcal{N}=4$ super Yang-Mills theory

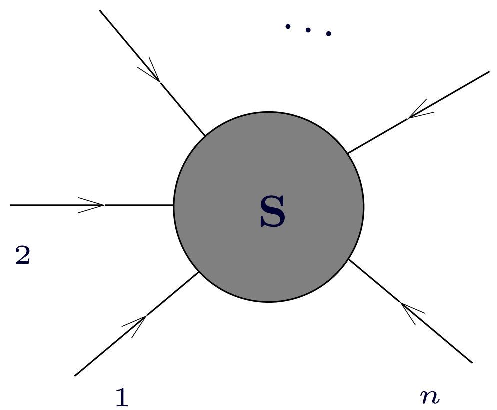
✔ Extended spectrum of asymptotic on-shell states

$$\textit{2 gluons with helicity} \quad \pm 1\,, \qquad \textit{6 scalars with helicity} \;\; 0\,, \qquad \textit{8 gaugino with helicity} \quad \pm\frac{1}{2}$$

all in the adjoint of the $SU(N_c)$ gauge group

✔ On-shell matrix elements of $S-$matrix:

$\mathcal{A}_n \;=$



- Nontrivial functions of Mandelstam variables $s_{i\ldots j}$ and 't Hooft coupling $a = g^2_{\rm YM} N_c$
- Are independent on gauge choice
- Probe (hidden) symmetries of gauge theory

Three questions in this talk:

✔ Do tree amplitudes in $\mathcal{N}=4$ SYM have hidden symmetries?

✔ How powerful are these symmetries to completely determine the scattering amplitudes?

✔ What happens to these symmetries at loop level?

Color-ordered planar MHV, NMHV,... amplitudes

✔ Color-ordered planar partial amplitudes:

$$A_n = \mathrm{tr}\left[T^{a_1}T^{a_2}\dots T^{a_n}\right] A_n^{h_1,h_2,\dots,h_n}(p_1,p_2,\dots,p_n) + \text{[Bose symmetry]}$$

✗ Quantum numbers: light-like momenta ($p_i^2 = 0$), helicity $(h_i = 0, \pm\frac{1}{2}, \pm 1)$, color ($a_i$)

✗ Amplitudes suffer from IR divergences $\mapsto$ require regularization (dim.reg. with $D = 4 - 2\epsilon$)

✔ The amplitudes are classified according to their total helicity

$$h_{\mathrm{tot}} = h_1 + \dots + h_n = \{n, n-2, n-4, \dots, -(n-2), -n\}$$

✗ $h_{\mathrm{tot}} = \pm n, \pm(n-2)$: $\mapsto$ amplitudes vanish to all loops due to supersymmetry

✗ $h_{\mathrm{tot}} = n - 4$: $\mapsto$ MHV amplitudes $A^{--+\dots+}$, $A^{-+-\dots+}$

$$A_n^{\mathrm{MHV}} = A_n^{\mathrm{MHV(tree)}}(p_i,h_i) M_n^{\mathrm{MHV}}(\{s_{ij}\};a)$$

All-loop corrections are described by a single scalar function! [Parke,Taylor]

✗ $h_{\mathrm{tot}} = n - 4 - 2p$: $\mapsto$ $\mathrm{N^pMHV}$ amplitudes $A^{---+\dots+}$, $A^{--+-\dots+}$

$$A_n^{\mathrm{N^pMHV}} = \textit{much more complicated structure compared with MHV amplitudes}$$

Use supersymmetry to combine amplitudes into superamplitudes

From MHV amplitudes to MHV superamplitude in $\mathcal{N}=4$ SYM

- ✔ On-shell helicity states in $\mathcal{N}=4$ SYM:

$$G^{\pm} \text{ (gluons } h=\pm 1), \qquad \Gamma_A,\, \bar\Gamma^A \text{ (gluinos } h=\pm\tfrac{1}{2}), \qquad S_{AB} \text{ (scalars } h=0)$$

- ✔ Can be combined into a single on-shell superstate [Mandelstam],[Brink et el]

$$\Phi(p,\eta) = G^+(p) + \eta^A\Gamma_A(p) + \frac{1}{2}\eta^A\eta^B S_{AB}(p) + \frac{1}{3!}\eta^A\eta^B\eta^C\epsilon_{ABCD}\bar\Gamma^D(p) + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D\epsilon_{ABCD}G^-(p)$$

- ✔ Combine all MHV amplitudes into a single MHV superamplitude [Nair]

$$\mathcal{A}_n^{\rm MHV} = (\eta_1)^4(\eta_2)^4 \times A\left(G_1^- G_2^- G_3^+ \dots G_n^+\right) + (\eta_1)^4(\eta_2)^2(\eta_3)^2 \times A\left(G_1^- \bar S_2\, S_3 \dots G_n^+\right) + \dots$$

- ✔ Spinor helicity formalism: [Xu,Zhang,Chang'87]

$$p_i^2 = 0 \qquad \Leftrightarrow \qquad p_i^{\alpha\dot\alpha} \equiv p_i^\mu(\sigma_\mu)^{\alpha\dot\alpha} = \lambda_i^\alpha\,\tilde\lambda_i^{\dot\alpha} \equiv |i\rangle[i|$$

- ✔ Superamplitudes are functions of $\{\lambda_i, \tilde\lambda_i, \eta_i\}$

$$\mathcal{A}_n(\Phi_1,\Phi_2,\dots,\Phi_n) = \mathcal{A}_n(\lambda_1,\tilde\lambda_1,\eta_1;\dots\lambda_n,\tilde\lambda_n,\eta_n)$$

Tree MHV superamplitude

- All MHV amplitudes are combined into a single superamplitude (spinor notations $\langle ij\rangle = \lambda_i^\alpha \lambda_{j\alpha}$)

$$\mathcal{A}_n^{\mathrm{MHV}} = i\,\frac{\delta^{(4)}\left(\sum_{i=1}^n p_i\right)\,\delta^{(8)}\left(\sum_{i=1}^n \lambda_i^\alpha \eta_i^A\right)}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}$$

- On-shell $\mathcal{N}=4$ supersymmetry: [Nair]

$$q_\alpha^A = \sum_i \lambda_{i,\alpha}\eta_i^A\,, \quad \bar{q}_{A\,\dot\alpha} = \sum_i \tilde\lambda_{i,\dot\alpha}\frac{\partial}{\partial \eta_i^A} \qquad \Longrightarrow \qquad q_\alpha^A\,\mathcal{A}_n^{\mathrm{MHV}} = \bar{q}_{A\,\dot\alpha}\mathcal{A}_n^{\mathrm{MHV}} = 0$$

- (Super)conformal invariance [Witten]

$$k_{\alpha\dot\alpha} = \sum_i \frac{\partial^2}{\partial\lambda_i^\alpha\partial\tilde\lambda_i^{\dot\alpha}} \qquad \Longrightarrow \qquad k_{\alpha\dot\alpha}\mathcal{A}_n^{\mathrm{MHV}} = 0$$

  Much less trivial to verify for NMHV amplitudes

- In fact, (super)conformal symmetry is almost exact (due to holomorphic anomaly)

$$\bar{s}\,\mathcal{A}_n^{\mathrm{MHV}} \sim \sum_i \left(\eta_i\tilde\lambda_{i+1} - \eta_{i+1}\tilde\lambda_i\right)\delta^{(2)}(\lambda_i,\lambda_{i+1})\,\mathcal{A}_{n-1}^{\mathrm{MHV}}$$

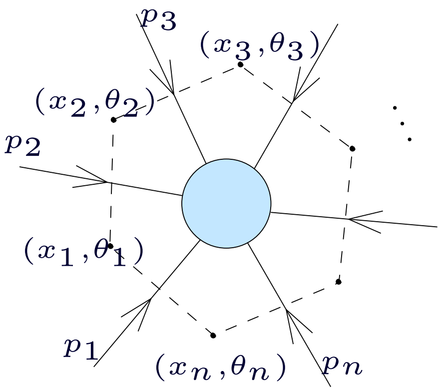
  $\bar{s}\,\mathcal{A}_n^{\mathrm{MHV}}$ is localized at collinear configurations $p_i||p_{i+1}$ [Bargheer,Beisert,Galleas,Loebbert,McLoughlin]

Dual $\mathcal{N}=4$ superconformal symmetry I

- ✔ The $\mathcal{N}=4$ superamplitudes possess a much bigger, dual superconformal symmetry

  [Drummond, Henn, GK, Sokatchev]

- ✔ Chiral dual superspace $(x_{\alpha\dot{\alpha}}\,,\,\theta^A_\alpha\,,\,\lambda_\alpha)$:



✗ $p = \sum_{i=1}^{n} p_i = 0 \quad \longrightarrow \quad p_i = x_i - x_{i+1}$

✗ $q = \sum_{i=1}^{n} \lambda_i \eta_i = 0 \quad \longrightarrow \quad \lambda_{i\,\alpha}\,\eta_i^A = (\theta_i - \theta_{i+1})^A_\alpha$

- ✔ The MHV superamplitude in the dual superspace

$$\mathcal{A}_n^{\rm MHV} = i(2\pi)^4 \frac{\delta^{(4)}\left(x_1 - x_{n+1}\right)\,\delta^{(8)}(\theta_1 - \theta_{n+1})}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}$$

- ✔ $\mathcal{N}=4$ supersymmetry in the dual superspace:

$$Q_{A\,\alpha} = \sum_{i=1}^{n} \frac{\partial}{\partial\theta_i^{A\,\alpha}}\,, \qquad \bar{Q}^A_{\dot{\alpha}} = \sum_{i=1}^{n} \theta_i^{A\,\alpha}\frac{\partial}{\partial x_i^{\dot{\alpha}\alpha}}\,, \qquad P_{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i^{\dot{\alpha}\alpha}}$$

- ✔ Dual supersymmetry

$$Q_{A\,\alpha}\mathcal{A}_n^{\rm MHV} = \bar{Q}^A_{\dot{\alpha}}\mathcal{A}_n^{\rm MHV} = P_{\alpha\dot{\alpha}}\mathcal{A}_n^{\rm MHV} = 0$$

Dual $\mathcal{N}=4$ superconformal symmetry II

✔ Super-Poincaré $+$ inversion $=$ conformal supersymmetry:

✗ Inversions in the dual superspace

$$I[\lambda_i^{\alpha}] = (x_i^{-1})^{\dot\alpha\beta}\lambda_{i\beta}\,, \qquad I[\theta_i^{\alpha\,A}] = (x_i^{-1})^{\dot\alpha\beta}\theta_{i\beta}^{A}$$

✗ Neighbouring contractions are dual conformal covariant

$$I[\langle i\, i+1\rangle] = (x_i^2)^{-1}\langle i\, i+1\rangle$$

✗ Impose cyclicity, $x_{n+1}=x_1$, $\theta_{n+1}=\theta_1$, through delta functions. Then, only in $\mathcal{N}=4$,

$$I[\delta^{(4)}(x_1-x_{n+1})] = x_1^8\,\delta^{(4)}(x_1-x_{n+1})$$
$$I[\delta^{(8)}(\theta_1-\theta_{n+1})] = x_1^{-8}\,\delta^{(8)}(\theta_1-\theta_{n+1})$$

✔ The tree-level MHV superamplitude is covariant under dual conformal inversions

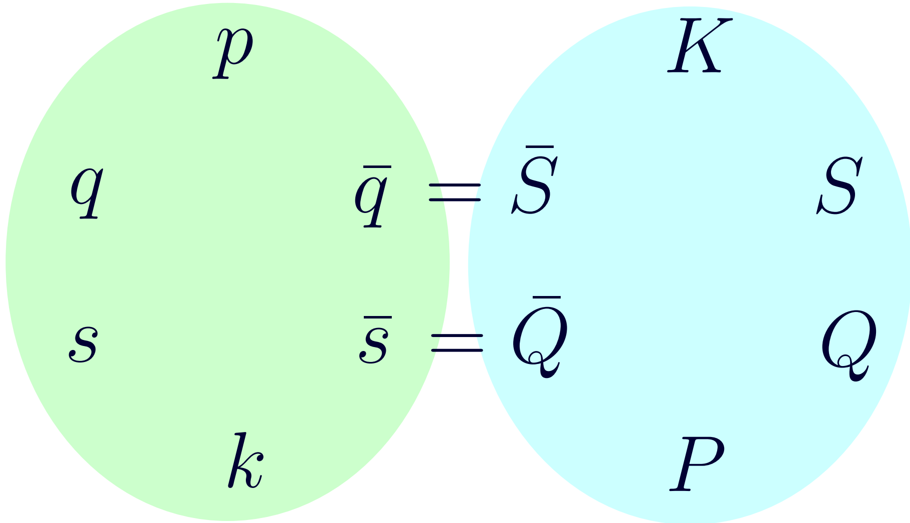
$$I\left[\mathcal{A}_n^{\rm MHV}\right] = (x_1^2 x_2^2 \dots x_n^2)\times\mathcal{A}_n^{\rm MHV}$$

✔ ***Dual superconformal covariance is a property of all tree-level superamplitudes (MHV, NMHV, N^{2}MHV,...) in $\mathcal{N}=4$ SYM theory*** [Drummond,Henn,GK,Sokatchev],[Brandhuber,Heslop,Travaglini]

Symmetries at tree amplitudes

- ✔ The relationship between conventional and dual superconformal $su(2,2|4)$ symmetries:

  [Drummond,Henn,GK,Sokatchev]



- ✔ The same symmetries appear at strong coupling from invariance of $\mathrm{AdS}_5 \times \mathrm{S}^5$ sigma model under bosonic [Kallosh,Tseytlin] + fermionic T-duality [Berkovits,Maldacena],[Beisert,Ricci,Tseytlin,Wolf]
- ✔ (Infinite-dimensional) closure of two symmetries has Yangian structure [Drummond,Henn,Plefka]
- ✔ All tree-level amplitudes are known [Drummond,Henn] from the supersymmetric generalization of the BCFW recursion relations [Brandhuber,Heslop,Travaglini],[Bianchi,Elvang,Freeman],[Arkani-Hamed,Cachazo,Kaplan]

Are tree level amplitudes completely determined by the symmetries?

Invariants of both symmetries

- ✔ The 'ratio' of two tree superamplitudes

$$\mathcal{A}_n = \mathcal{A}_n^{\rm MHV} R_n = \mathcal{A}_n^{\rm MHV}\left[R_n^{\rm MHV} + R_n^{\rm NMHV} + \ldots\right]$$

- ✔ The ratio R_n–functions are invariants of ***both*** conventional (g) and dual (G) symmetries:

$$g \cdot R_n^{\rm N^pMHV} = G \cdot R_n^{\rm N^pMHV} = 0\,,$$

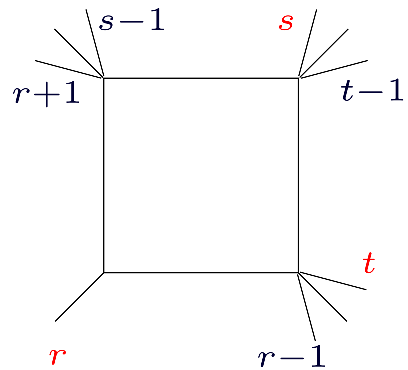
$$R_n^{\rm N^pMHV} = \text{ Polynomial in } \theta\text{'s of degree } 4p$$

General solution is unknown, classification of $\rm N^pMHV$ superinvariants remains to be done

- ✗ MHV superinvariants $(p=0)$ are trivial: $R_n^{\rm MHV} = \text{const}$
- ✗ NMHV superinvariants $(p=1)$ are nontrivial:

[Drummond,Henn,GK,Sokatchev]

$$R_{rst}(x,\lambda,\theta) = \frac{\langle s-1 s\rangle\langle t-1 t\rangle\delta^{(4)}(\langle r|x_{rs}x_{st}|\theta_{tr}\rangle + \langle r|x_{rt}x_{ts}|\theta_{sr}\rangle)}{x_{st}^2\langle r|x_{rs}x_{st}|t-1\rangle\langle r|x_{rs}x_{st}|t\rangle\langle r|x_{rt}x_{ts}|s-1\rangle\langle r|x_{rt}x_{ts}|s\rangle} \quad \leftrightarrow$$



Supersymmetric extenstion of three-mass box coefficients

[Bern,Dixon,Kosower]

How powerful are the symmetries?

General expression for the NMHV ratio function dictated by the symmetries

$$g \cdot R_n^{\rm NMHV} = G \cdot R_n^{\rm NMHV} = 0 \qquad \mapsto \qquad R_n^{\rm NMHV} = \sum_{r,s,t} c_{rst} R_{rst} \quad \text{(with } c_{rst} \text{ arbitrary!)}$$

- ✔ The combined action of conventional and dual superconformal symmetries *is not sufficient* to fix all the freedom in the tree-level amplitudes
- ✔ Additional information needed comes from the analytic properties of tree amplitudes: [GK, Sokatchev]
  - ✗ 'Physical' poles in multi-particle invariant masses $(p_s + \ldots + p_{t-1})^2 = x_{st}^2 = 0$
  - ✗ Free from spurious singularities
  - ✗ Correct collinear factorization $p_i || p_{i+1}$

Analytical properties of $R_{rst}-$invariants:

$$R_{rst} \sim \Big(\underbrace{x_{st}^2}_{\text{physical pole}} \times \underbrace{\langle r|x_{rs}x_{st}|t-1\rangle\langle r|x_{rs}x_{st}|t\rangle\langle r|x_{rt}x_{ts}|s-1\rangle\langle r|x_{rt}x_{ts}|s\rangle}_{\text{spurious poles}} \Big)^{-1}$$

Kinematical configuration corresponding to spurious pole at $\langle r|x_{rs}x_{st}|t\rangle = 0$:

$$-x_{rt}^2 x_{r+1,s}^2 x_{t+1,s}^2 + x_{r+1,t}^2 x_{rs}^2 x_{t+1,s}^2 - x_{r+1,t+1}^2 x_{rs}^2 x_{ts}^2 + x_{r,t+1}^2 x_{r+1,s}^2 x_{ts}^2 = 0$$

Spurious poles should cancel inside $R_n^{\rm NMHV}$*!*

Cancellation of spurious poles

- ✔ Each invariant R_{rst} has four sets of spurious poles

$$\langle r|x_{rs}x_{st}|t-1\rangle = \langle r|x_{rs}x_{st}|t\rangle = \langle r|x_{rt}x_{ts}|s-1\rangle = \langle r|x_{rt}x_{ts}|s\rangle = 0$$

- ✔ 'Master identity': all spurious poles of R_{rst} cancel in the linear combination of invariants:

$$R_{rst} + (R_{str} + R_{trs} - R_{s-1\,tr} - R_{t-1\,rs})$$

  - ✗ $n=8$ NMHV: general expression consistent with all symmetries

$$R_8^{\mathrm{NMHV};0} = \alpha R_{147} + \beta R_{148} + \gamma R_{157} + \delta R_{158} + \varepsilon R_{168} + \text{cyclic}$$

  Cancellation of spurious poles leads to

$$\alpha - \beta = \alpha + \gamma - \delta = 2\alpha - \gamma = \delta - \varepsilon = \beta + \gamma - \delta = \beta + \gamma - \varepsilon = 0$$

  This system is overdetermined but it has a unique solution

$$\beta = \alpha\,, \qquad \gamma = 2\alpha\,, \qquad \delta = \epsilon = 3\alpha$$

- ✔ The same relations (with $\alpha = \frac{1}{8}$) ensure the correct behavior in the collinear limit $p_i || p_{i+1}$

$$R_n(\ldots, i, i+1, \ldots) \stackrel{i||i+1}{\longrightarrow} R_{n-1}(\ldots, \ell, \ldots)$$

Tree amplitudes are uniquely fixed by symmetries + analyticity condition [GK,Sokatchev]

Do symmetries survive loop corrections?

- ✔ Loop corrections to the amplitudes necessarily induce infrared divergences
- ✔ The scattering amplitudes are well-defined in $D = 4 - 2\epsilon_{\rm IR}$ dimensions only
- ✔ *All-loop planar* (super)amplitudes can be split into a IR divergent and a finite part

$$\mathcal{A}_n^{(\rm all-loop)} = \text{Div}(1/\epsilon_{\rm IR})\ [\text{Fin} + O(\epsilon_{\rm IR})]$$

  - ✗ IR divergences (poles in $\epsilon_{\rm IR}$) exponentiate (in any gauge theory!) [Mueller],[Sen],[Collins],[Sterman],...

$$\text{Div}(1/\epsilon_{\rm IR}) = \exp\left\{-\frac{1}{2}\sum_{l=1}^{\infty} a^l \left(\frac{\Gamma_{\rm cusp}^{(l)}}{(l\epsilon_{\rm IR})^2} + \frac{G^{(l)}}{l\epsilon_{\rm IR}}\right)\sum_{i=1}^{n}(-s_{i,i+1})^{l\epsilon_{\rm IR}}\right\}$$

  - ✗ *IR divergences* are in the one-to-one correspondence with *UV divergences* of Wilson loops [Ivanov,GK,Radyushkin]

$\Gamma_{\rm cusp}(a) = \sum_l a^l \Gamma_{\rm cusp}^{(l)} =$ *cusp anomalous dimension of Wilson loops*

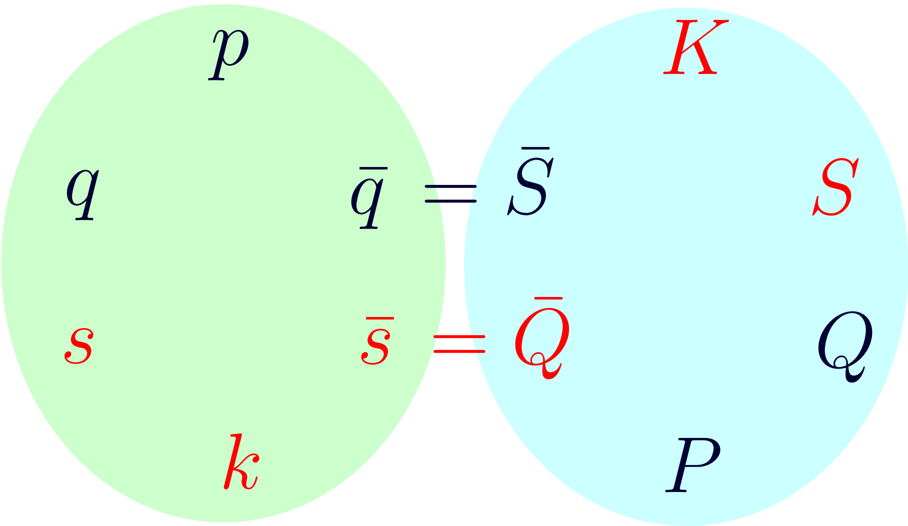
$G(a) = \sum_l a^l G_{\rm cusp}^{(l)} =$ *collinear anomalous dimension*

- ✔ *IR divergences preserve Poincaré supersymmetry but break conformal + dual conformal symmetry*

  IR divergences come from small momenta (=large distances) and, therefore, the conformal anomaly is not 'localized' (= difficult to control)

Anomalous symmetries at loop level

- Some symmetries ($p, q, \bar{q}, P, Q, \bar{S}, ...$) survive loop corrections while other ($s, \bar{s}, k, K, S, \bar{Q}, ...$) are broken



- But the anomalies are not independent: $[K, \bar{Q}] = S\,, \qquad [s, \bar{s}] = k$

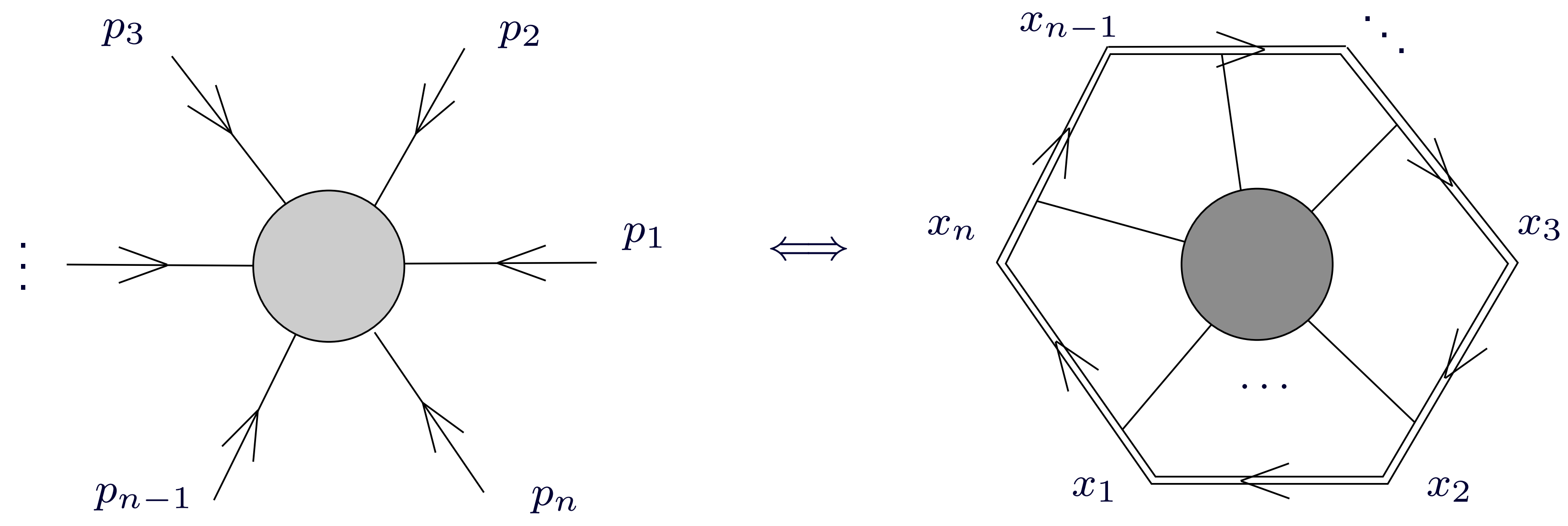
- Three independent anomalies are

$$s_{\alpha A} = \sum_i \frac{\partial^2}{\partial\lambda_i^\alpha \partial\eta_i^A}\,, \qquad \bar{Q}^A_{\dot\alpha} = \sum_i \eta_i^A \frac{\partial}{\partial\tilde{\lambda}_i^{\dot\alpha}}\,, \qquad K^{\alpha\dot\alpha} = \sum_i \left[x_i^{\alpha\dot\beta} x_i^{\beta\dot\alpha} \frac{\partial}{\partial x_i^{\beta\dot\beta}} + x_i^{\beta\dot\alpha}\theta_i^{\alpha B}\frac{\partial}{\partial\theta_i^{\beta B}} + \ldots\right]$$

- Dual conformal $K-$anomaly is *universal* for all superamplitudes (MHV, NMHV,...)

- $K-$anomaly can be determined to all loops from Wilson loop/MHV amplitude duality, whereas the $s-$ and $\bar{Q}-$anomalies are hard to control

MHV scattering amplitudes/Wilson loop duality



MHV amplitudes are dual to light-like Wilson loops

$$\ln \mathcal{A}_n^{(\mathrm{MHV})} \sim \ln W(C_n) + O(1/N_c^2)\,, \qquad C_n = \text{light-like } n\text{−(poly)gon}$$

- ✔ At strong coupling, agrees with the BDS ansatz [Alday,Maldacena]
- ✔ At weak coupling, the duality was verified against BDS ansatz to two loops for $n \geq 4$
  [Drummond,Henn,GK,Sokatchev], [Anastasiou,Brandhuber,Heslop,Khoze,Spence,Travaglini]

  Wilson loops match the BDS ansatz for $n = 4, 5$ *but not for* $n \geq 6$

- ✔ Scattering amplitude/Wilson loop duality also holds in QCD but in the Regge limit only [GK]

Dual conformal $K-$anomaly

Dual conformal symmetry of the amplitudes $\Leftrightarrow$ Conformal symmetry of Wilson loops

Dual conformal anomaly $\Leftrightarrow$ Conformal anomaly of Wilson loops

✔ How could Wilson loops have conformal anomaly in $\mathcal{N}=4$ SYM?

- ✗ Were the Wilson loop well-defined (=finite) in $D=4$ dimensions it would be conformal invariant

$$W(C_n){=}W(C_n')$$

- ✗ ... but $W(C_n)$ has cusp UV singularities $\mapsto$ dim.reg. breaks conformal invariance

$$W(C_n) = W(C_n') \times \text{[cusp anomaly]}$$

✔ *All-loop* anomalous conformal Ward identities for the *finite part* of the Wilson loop

$$\ln W(C_n) = F_n^{(WL)} + \text{[UV divergencies]} + O(\epsilon)$$

Under special conformal transformations (boosts), to all orders, [Drummond,Henn,GK,Sokatchev]

$$K^\mu\, F_n \equiv \sum_{i=1}^{n} \left[2x_i^\mu (x_i\cdot\partial_{x_i}) - x_i^2\partial_{x_i}^\mu\right] F_n = \frac{1}{2}\Gamma_{\rm cusp}(a)\sum_{i=1}^{n} x^\mu_{i,i+1}\ln\left(\frac{x^2_{i,i+2}}{x^2_{i-1,i+1}}\right)$$

The same relations also hold at strong coupling [Alday,Maldacena],[Komargodski]

Dual conformal anomaly at work

Consequences of the conformal Ward identity for the finite part of the Wilson loop W_n:

✔ $n = 4, 5$ are special: there are no conformal invariants (too few distances due to $x^2_{i,i+1} = 0$)

$\Longrightarrow$ the Ward identity has a *unique all-loop solution* (up to an additive constant)

$$F_4 = \frac{1}{4}\Gamma_{\text{cusp}}(a) \ln^2\left(\frac{x^2_{13}}{x^2_{24}}\right) + \text{ const },$$

$$F_5 = -\frac{1}{8}\Gamma_{\text{cusp}}(a) \sum_{i=1}^{5} \ln\left(\frac{x^2_{i,i+2}}{x^2_{i,i+3}}\right) \ln\left(\frac{x^2_{i+1,i+3}}{x^2_{i+2,i+4}}\right) + \text{ const}$$

Exactly the BDS ansatz for the 4- and 5-point MHV amplitudes!

✔ Starting from $n = 6$ there are conformal invariants in the form of cross-ratios $u_{ijkl} = \frac{x^2_{il}x^2_{jk}}{x^2_{ik}x^2_{jl}}$

General solution to the Ward identity contains an arbitrary function of the conformal cross-ratios.

✔ Crucial test - go to six points at two loops where the answer is not determined by conformal symmetry [Drummond,Henn,GK,Sokatchev] [Bern,Dixon,Kosower,Roiban,Spradlin,Vergu,Volovich]

$$F_6^{(\text{WL})} = F_6^{(\text{MHV})} \neq F_6^{(\text{BDS})}$$

The Wilson loop/MHV amplitude duality holds at $n = 6$ to two loops!

Dual conformal symmetry beyond MHV

One-loop NMHV superamplitudes

- ✔ $n-$gluon one-loop NMHV amplitudes are known [Bern,Dixon,Kosower]
- ✔ New result for one-loop NMHV superamplitude: [Henn,Drummond,GK,Sokatchev]

$$\mathcal{A}_n^{\rm NMHV;1} = \mathcal{A}_n^{\rm MHV;1} \times \Big[R_n^{\rm NMHV}(x_i, \lambda_i, \theta_i^A) + O(\epsilon) \Big]$$

IR divergences break symmetries of NMHV and MHV but they cancel inside the 'ratio function'

One-loop NMHV 'ratio function' $=$ ***sum of tree-level dual superconformal invariants:***

$$R_n^{\rm NMHV} = \sum_{p,q,r} R_{pqr}(\lambda, \tilde\lambda, \theta)\, V_{pqr}(x_{ij}^2)$$

- ✗ Helicity structure is invariant under both (conventional and dual) superconformal symmetries
- ✗ Loop corrections are described by scalar functions

$$V_{pqr} = 1 + a\, V^{(1)}(\{u_{pqr}\}) + O(a^2)\,, \qquad u_{pqr} = \text{dual conformal cross-ratios}$$

  they are dual conformal invariants made of IR finite combinations of 1-loop scalar box integrals [Henn,Drummond,GK,Sokatchev],[Brandhuber,Heslop,Travaglini],[Elvang,Freedman,Kiermaier]

- ✗ $R_n^{\rm NMHV}$ is free from spurious singularities and has correct collinear limit

The ratio function is dual conformal invariant but it is not superconformal invariant, why ?

Dual supersymmetry $\overline{Q}-$anomaly

✔ Main idea: Instead of $\bar{Q}\mathcal{A}_n$ let us compute its discontinuity $\mathrm{Disc}_s\left(\bar{Q}\mathcal{A}_n\right) = \bar{Q}(\mathrm{Disc}_s\mathcal{A}_n)$

$$\mathrm{Disc}_{s_{123}}\mathcal{A}_6^{\mathrm{MHV};1} = \mathcal{A}^{\mathrm{MHV};0}(-\ell_1,1,2,3,-\ell_2)\star\mathcal{A}^{\mathrm{MHV};0}(\ell_2,4,5,6,\ell_1)\,,$$

Two tree amplitudes are integrated over the phase space of on-shell states ℓ_1 and ℓ_2

$$\begin{aligned}\delta_{\bar{Q}}\mathrm{Disc}_{s_{123}}\mathcal{A}_6^{\mathrm{MHV};1} &= \delta_{\bar{Q}}\mathcal{A}^{\mathrm{MHV};0}(-\ell_1,1,2,3,-\ell_2)\star\mathcal{A}^{\mathrm{MHV};0}(\ell_2,4,5,6,\ell_1)\\ &+\mathcal{A}^{\mathrm{MHV};0}(-\ell_1,1,2,3,-\ell_2)\star\delta_{\bar{Q}}\mathcal{A}^{\mathrm{MHV};0}(\ell_2,4,5,6,\ell_1)\end{aligned}$$

✔ If tree supeamplitudes were exactly invariant, $\delta_{\bar{Q}}\mathcal{A}^{\mathrm{MHV};0} = 0$, then $\delta_{\bar{Q}}\mathrm{Disc}_{s_{123}}\mathcal{A}_6^{\mathrm{MHV};1} = 0$.
But they are not due to holomorphic anomaly! [Cachazo,Svrcek,Witten],[Bena,Bern,Kosower,Roiban]

✔ $\bar{Q}-$anomaly of one-loop NMHV ratio function

$$\begin{aligned}\bar{Q}^A_{\dot\alpha}\left(\mathrm{Disc}_{x_{14}^2}R_6^{\mathrm{NMHV};1}\right) &\sim (\eta_1[23]+\eta_2[31]+\eta_3[12])^A R_{146}\\ &\times\left(\frac{\tilde\lambda_{1\dot\alpha}[6|x_{63}|3\rangle}{x_{14}^2[61][12]}+\frac{\tilde\lambda_{3\dot\alpha}[4|x_{41}|1\rangle}{x_{14}^2[23][34]}\right)+(i\to i+3)\neq 0\end{aligned}$$

Holomorphic anomaly is responsible for the breakdown of $\bar{Q}=\bar{s}$ symmetry of the ratio function (but not of the dual conformal symmetry!) [GK,Sokatchev]

Analyticity constraints at loop level

All-loop superamplitude $\mathcal{A}_n$ should be free from spurious poles + have correct collinear behaviour:

- ✔ Spurious pole cancellation operates 'horizontally', i.e. it establishes a property of the $n-$particle amplitude, without reference to other amplitudes
- ✔ Collinear factorization operates 'vertically', i.e. it recursively relates the $n-$particle amplitude to the amplitudes with fewer particles
- ✔ Example: One loop correction to NMHV$_{n=7}$ ratio function:

$$R_6^{\mathrm{NMHV};1} = \frac{1}{2} R_{146} V_{146}(u_1, u_2, u_3) + \text{(cyclic)}$$

$$V_{146} = -\ln u_1 \ln u_3 + \frac{1}{2} \sum_{k=1}^{3} \left[\ln u_k \ln u_{k+1} + \mathrm{Li}_2(1-u_k)\right] - \frac{\pi^2}{6} \, ,$$

with u_i dual conformal cross-ratios, e.g. $u_1 = x_{15}^2 x_{24}^2/(x_{14}^2 x_{25}^2)$

- ✗ Spurious pole: $u_1 = 1$, $u_2, u_3 =$ arbitrary

$$V_{146}(1, u_2, u_3) - V_{146}(u_3, 1, u_2) = 0$$

- ✗ Collinear limit: $u_1 \to 0$, $u_3 \to 1 - u_2$

$$V_{146}(0, u_2, 1-u_2) + V_{146}(0, 1-u_2, u_2) = 0$$

The two approaches are equivalent at tree level but lead to *different* constraints at loop level!

Symmetry of all-loop superamplitudes

DHKS proposal for all-loop superamplitude in $\mathcal{N} = 4$ SYM:

$$\mathcal{A}_n(x_i, \lambda_i, \theta_i^A) = \mathcal{A}_n^{\rm MHV} + \mathcal{A}_n^{\rm NMHV} + \mathcal{A}_n^{\rm N^2MHV} + \ldots + \mathcal{A}_n^{\overline{\rm MHV}}$$

- ✔ At tree level, $\mathcal{A}_n$ is fixed by conventional and dual symmetries + analyticity conditions
- ✔ At loop level, both symmetries become anomalous due to IR divergences + holomorphic anomaly
- ✔ The dual superconformal symmetry is restored in the ratio of superamplitudes $\mathcal{A}_n$ and $\mathcal{A}_n^{\rm MHV}$

$$\mathcal{A}_n(x_i, \lambda_i, \theta_i^A) = \mathcal{A}_n^{\rm MHV} \times \Big[R_n(x_i, \lambda_i, \theta_i^A) + O(\epsilon)\Big]$$

The ratio function

$$R_n = 1 + R_n^{\rm NMHV} + R_n^{\rm N^2MHV} + \ldots$$

is *IR finite* and, most importantly, it is *dual conformal invariant* [Drummond,Henn,GK,Sokatchev]

$$K^{\alpha\dot\alpha} R_n^{(\text{all loops})} = 0$$

The conjecture was recently proven to one loop [Brandhuber,Heslop,Travaglini]

... but all-loop proof is still missing

Superamplitudes in twistor space

Twistor (half-Fourier) transform $(\lambda, \tilde\lambda, \eta) \mapsto (\lambda, \mu, \psi)$ in split signature $(++--)$ [Witten'03]

$$T[\mathcal{A}_n](\{\lambda,\mu,\psi\}) = \int \prod_1^n \frac{d^2\tilde\lambda_i}{(2\pi)^2}\, d^4\eta_i\, \mathrm{e}^{i([\mu_i \tilde\lambda_i] + \psi_{iA}\eta_i^A)} \mathcal{A}_n(\{\lambda, \tilde\lambda, \eta\})\,,$$

- ✔ The twistor transform of the MHV superamplitude

$$\mathcal{A}_n^{\mathrm{MHV}} = \frac{i}{\prod_1^n \langle i\, i+1\rangle} (2\pi)^4 \delta^{(4)}(\sum_1^n \lambda_i \tilde\lambda_i)\; \delta^{(8)}(\sum_1^n \lambda_i \eta_i)$$

$$T\left[\mathcal{A}_n^{\mathrm{MHV}}\right] = i \int d^4X d^8\Theta\, \frac{\prod_1^n \delta^{(2)}(\mu_i + \lambda_i X)\; \delta^{(4)}(\psi_i + \langle i\,\Theta\rangle)}{\langle 12\rangle \ldots \langle n-1\, n\rangle\langle n1\rangle}$$

  Is localized on the line in the twistor space

$$\mu_{i\dot\alpha} + \lambda_i^\alpha X_{\alpha\dot\alpha} = 0\,, \qquad \psi_{i\,A} + \langle i\,\Theta_A\rangle = 0\,,$$

- ✔ All superamplitudes are conjectured to be supported on holomorphic curves of higher degree: connected/disconnected prescriptions [Nair],[Witten],[Roiban,Spradlin,Volovich],[Bern,Del Duca,Dixon,Kosower],...
- ✔ Simple form of BCFW recurrence relations [Mason,Skinner],[Arkani-Hamed et al]
- ✔ Simple action of conformal group: the line parameters X and Θ are transformed as the coordinates of *configuration superspace* [GK,Sokatchev]

Twistor transform of NMHV superamplitude I

Half-Fourier transform of the NMHV amplitude

$$\mathcal{A}_n^{\rm NMHV} = \sum_{3\leq a+1<b\leq n-1} \mathcal{A}_{nab}\,, \qquad \mathcal{A}_{nab} = \mathcal{A}_n^{\rm MHV}\, R_{nab}\,.$$

seems to be an impossible task, because of the very non-trivial dependence of R's on $\tilde\lambda$

$$R_{nab} = \frac{\langle a-1\,a\rangle\langle b-1\,b\rangle\;\delta^{(4)}(\sum_1^{a-1}\langle n|x_{nb}x_{ba}^{-1}|i\rangle\eta_i + \sum_1^{b-1}\langle n|x_{na}x_{ab}^{-1}|i\rangle\eta_i)}{x_{ab}^2\langle n|x_{nb}x_{ba}^{-1}|a-1\rangle\langle n|x_{nb}x_{ba}^{-1}|a\rangle\langle n|x_{na}x_{ab}^{-1}|b-1\rangle\langle n|x_{na}x_{ab}^{-1}|b\rangle}\,,$$

- ✔ First, we define the spinors $\langle\rho_0| + \langle\rho| + \langle\sigma| = 0$

$$\langle\rho_0| \equiv \langle n|\,, \qquad \langle\rho| = \langle n|x_{nb}x_{ba}^{-1}\,, \qquad \langle\sigma| = \langle n|x_{na}x_{ab}^{-1}\,,$$

- ✔ Second, use the Faddeev-Popov approach to introduce them into R's via delta function integrals

$$\begin{aligned} R(\langle n|x_{nb}x_{ba}^{-1}) &= \int d^2\rho\; R(\rho)\;\delta^{(2)}\left(\langle\rho| - \langle n|x_{nb}x_{ba}^{-1}\right) \\ &= |x_{ab}^2|\int d^2\rho\, d^2\tilde\rho\; R(\rho)\;\exp\{-i\,(\langle\rho|x_{ba}|\tilde\rho] + \langle n|x_{nb}|\tilde\rho])\}\,. \end{aligned}$$

- ✔ Incomplete cancellation $|x_{ab}^2|/x_{ab}^2 = \mathrm{sgn}(x_{ab}^2)$ leads to breakdown of conformal symmetry of tree NMHV amplitude

[Mason,Skinner],[Arkani-Hamed et al.]

Twistor transform of NMHV superamplitude II

Twistor transform of the partial NMHV amplitude $\mathcal{A}_{nab}$ [GK,Sokatchev]

$$T\left[\mathcal{A}_{nab}\right] = i\int d^4X \int d^2\rho\, d^2\tilde{\rho}\, \frac{(\prod)_{nab}}{\Delta_{nab}} \times \text{[Fermionic counterpart]}$$

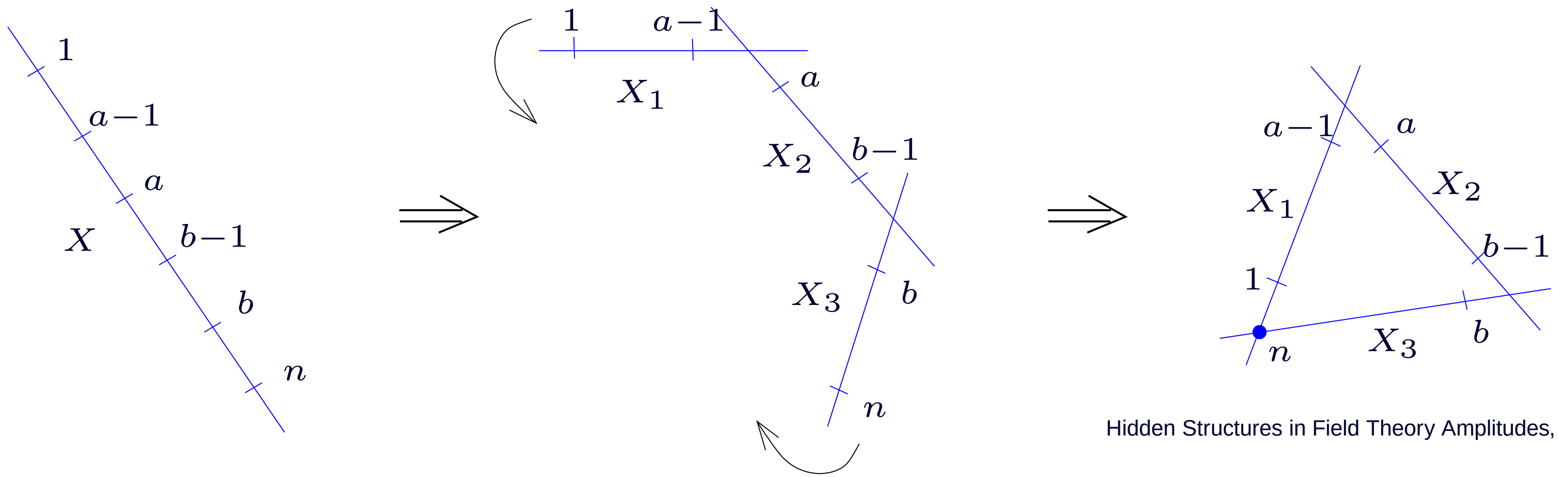
- ✔ Integrand = product of three MHV-like lines in twistor space

$$\frac{(\prod)_{nab}}{\Delta_{nab}} = \frac{\prod_1^{a-1}\delta^{(2)}(\mu_i + \langle i|X_1)}{\langle 12\rangle \ldots \langle a-1\,\rho\rangle} \times \frac{\prod_a^{b-1}\delta^{(2)}(\mu_i + \langle i|X_2)}{\langle \rho\, a\rangle \ldots \langle b-1\,\sigma\rangle} \times \frac{\prod_b^{n}\delta^{(2)}(\mu_i + \langle i|X_3)}{\langle \sigma\, b\rangle \ldots \langle n1\rangle}$$

- ✔ Line moduli: $X_1 = X - \rho_0\tilde{\rho}\,, \qquad X_2 = X + \sigma\tilde{\rho}\,, \qquad X_3 = X$
- ✔ Three vectors X_{12}, X_{23} and X_{31} are *lightlike* $\Longrightarrow$ three lines intersect pairwise

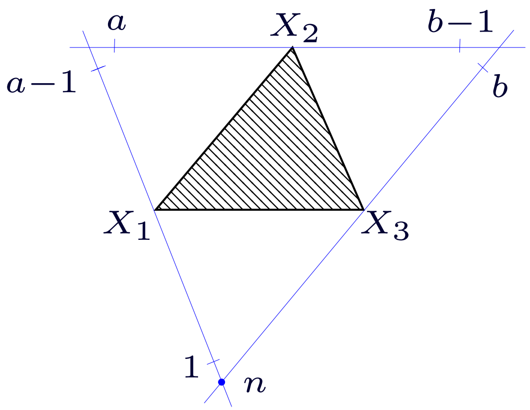
$$X_{12} = \rho\,\tilde{\rho}\,, \qquad X_{23} = \sigma\,\tilde{\rho}\,, \qquad X_{31} = \rho_0\,\tilde{\rho}\,, \qquad X_{12} + X_{23} + X_{31} = 0$$

Transforming a single MHV line into three NMHV lines

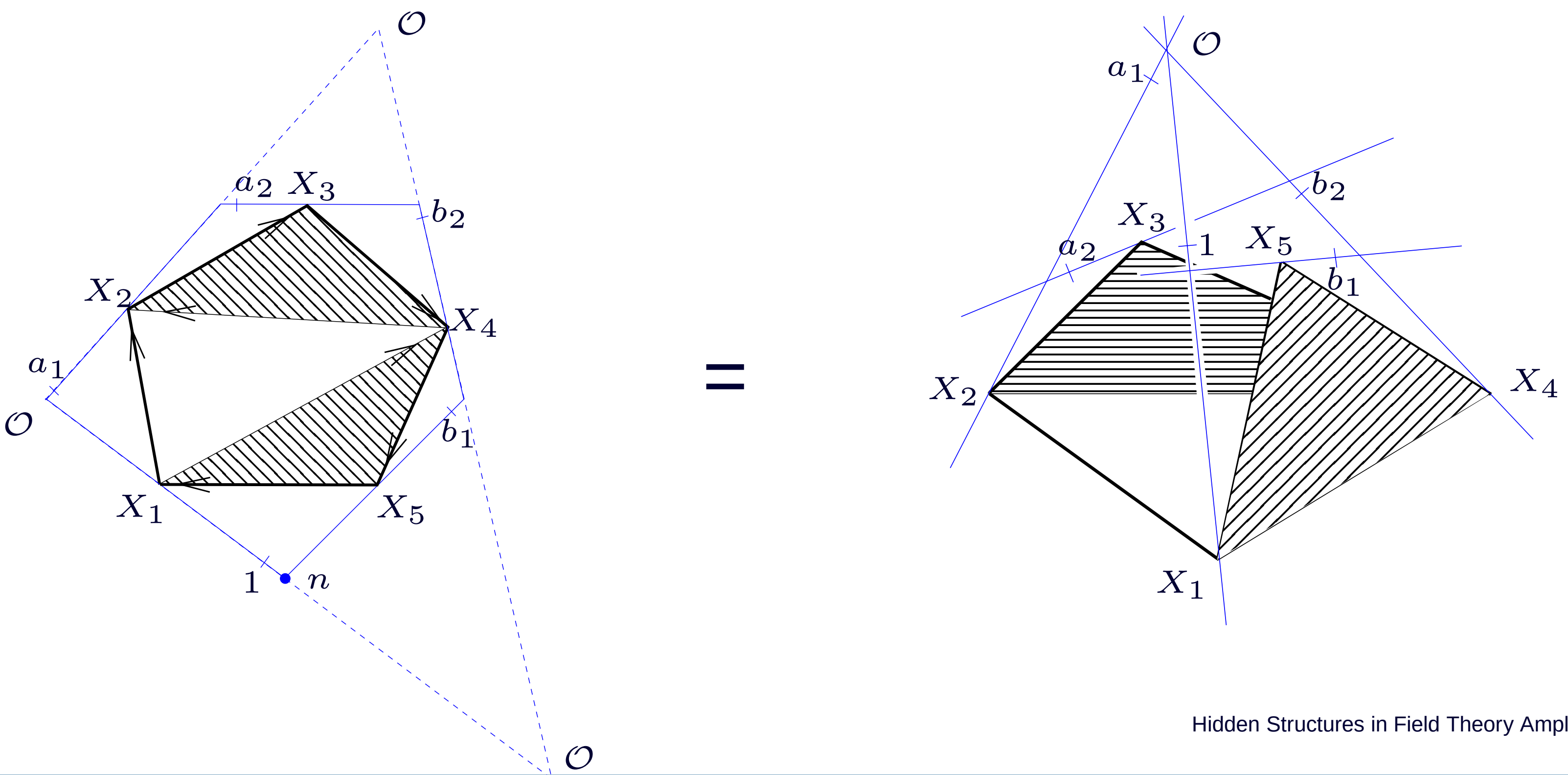


Geometric interpretation of dual superconformal invariants

NMHV superinvariant = Three intersecting lines = (Coplanar) lightlike triangle in moduli space



N^{k}MHV superinvariants = $(2k+1)$ intersecting lines = Nonplanar triangulated surfaces in moduli space



Conclusions and open questions

- ✔ Tree amplitudes in $\mathcal{N}=4$ SYM respect conventional and dual superconformal symmetries but their combined action is not sufficient to fix the amplitudes. The additional information needed comes from analyticity properties of the amplitudes.

- ✔ At loop level, both symmetries are broken by IR divergences + holomorphic anomaly. The dual conformal anomaly is well understood but how to control the remaining anomalies?

  We need the dual model for $\mathcal{N}=4$ superamplitude:

  Dual model for the MHV amplitude = light-like Wilson loop

  Dual model for the MHV+NMHV+ $\ldots +$ $\overline{\text{MHV}}$ amplitude = ???

  $\bar{Q}-$anomaly indicates that the dual model does not respect Poincaré supersymmetry.

  How could it be?

- ✔ Weak/strong coupling paradox:

  At weak coupling, the $\bar{Q}-$anomaly is present to all loops $\bar{Q}R_n = af_1 + a^2 f_2 + \ldots$ but it is not seen at strong coupling !?

$$\mathcal{A}_n^{\mathrm{N^pMHV}} \sim \exp\left(-\sqrt{a}S_{\min}\right)\left[1+O(1/\sqrt{a})\right] \qquad \Rightarrow \qquad R_n^{\mathrm{N^pMHV}} \sim 1 + O(1/\sqrt{a})$$

  What is the meaning of holomorphic anomaly in string theory?