Symmetries of scattering amplitudes in $\mathcal{N}=4$ SYM

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Based on work in collaboration with
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Scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory

- Extended spectrum of asymptotic on-shell states
  
  2 gluons with helicity $\pm 1$, 6 scalars with helicity 0, 8 gaugino with helicity $\pm \frac{1}{2}$

  all in the adjoint of the $SU(N_c)$ gauge group

- On-shell matrix elements of $S$-matrix:

  \[ A_n = \left. S \right|_{2}^{n} \]

  - Nontrivial functions of Mandelstam variables $s_{i...j}$ and 't Hooft coupling $a = g_{YM}^2 N_c$
  - Are independent on gauge choice
  - Probe (hidden) symmetries of gauge theory

Three questions in this talk:

- Do tree amplitudes in $\mathcal{N} = 4$ SYM have hidden symmetries?
- How powerful are these symmetries to completely determine the scattering amplitudes?
- What happens to these symmetries at loop level?
Color-ordered planar MHV, NMHV,... amplitudes

✔ Color-ordered planar partial amplitudes:

\[
A_n = \text{tr} \left[ T^{a_1} T^{a_2} \ldots T^{a_n} \right] A_n^{h_1, h_2, \ldots, h_n} (p_1, p_2, \ldots, p_n) + \text{[Bose symmetry]}
\]

✗ Quantum numbers: light-like momenta \((p_i^2 = 0)\), helicity \((h_i = 0, \pm \frac{1}{2}, \pm 1)\), color \((a_i)\)

✗ Amplitudes suffer from IR divergences \(\mapsto\) require regularization (dim.reg. with \(D = 4 - 2\epsilon\))

✔ The amplitudes are classified according to their total helicity

\[
h_{\text{tot}} = h_1 + \ldots + h_n = \{n, n-2, n-4, \ldots, -(n-2), -n\}
\]

✗ \(h_{\text{tot}} = \pm n, \pm (n-2)\): \(\mapsto\) amplitudes vanish to all loops due to supersymmetry

✗ \(h_{\text{tot}} = n-4\): \(\mapsto\) MHV amplitudes \(A^{--\ldots+}, A^{+-\ldots+}\)

\[
A_n^{\text{MHV}} = A_n^{\text{MHV(tree)}} (p_i, h_i) M_n^{\text{MHV}} (\{s_{ij}\}; a)
\]

All-loop corrections are described by a single scalar function! [Parke,Taylor]

✗ \(h_{\text{tot}} = n-4-2p\): \(\mapsto\) NP MHV amplitudes \(A^{--\ldots+}, A^{---\ldots+}\)

\[
A_n^{\text{NP MHV}} = \text{much more complicated structure compared with MHV amplitudes}
\]

Use supersymmetry to combine amplitudes into superamplitudes

Hidden Structures in Field Theory Amplitudes, 13 August 2009 - p. 3/25
From MHV amplitudes to MHV superamplitude in $\mathcal{N} = 4$ SYM

- On-shell helicity states in $\mathcal{N} = 4$ SYM:
  
  $G^\pm$ (gluons $h = \pm 1$), $\Gamma_A, \bar{\Gamma}^A$ (gluinos $h = \pm \frac{1}{2}$), $S_{AB}$ (scalars $h = 0$)

- Can be combined into a single on-shell superstate

\[
\Phi(p, \eta) = G^+(p) + \eta^A \Gamma_A(p) + \frac{1}{2} \eta^A \eta^B S_{AB}(p)
\]

\[
+ \frac{1}{3!} \eta^A \eta^B \eta^C \epsilon_{ABCD} \bar{\Gamma}^D(p) + \frac{1}{4!} \eta^A \eta^B \eta^C \eta^D \epsilon_{ABCD} G^-(p)
\]

- Combine all MHV amplitudes into a single MHV superamplitude

\[
A_n^{\text{MHV}} = (\eta_1)^4 (\eta_2)^4 \times A \left(G_1^- G_2^- G_3^+ \ldots G_n^+\right)
\]

\[
+ (\eta_1)^4 (\eta_2)^2 (\eta_3)^2 \times A \left(G_1^- \bar{S}_2 S_3 \ldots G_n^+\right) + \ldots
\]

- Spinor helicity formalism:

\[
p_i^2 = 0 \quad \Leftrightarrow \quad p_i^{\alpha\dot{\alpha}} \equiv p_i^\mu (\sigma_\mu)^{\alpha\dot{\alpha}} = \lambda_i^\alpha \bar{\lambda}_i^{\dot{\alpha}} \equiv |i\rangle [i]
\]

- Superamplitudes are functions of $\{\lambda_i, \bar{\lambda}_i, \eta_i\}$

\[
A_n (\Phi_1, \Phi_2, \ldots, \Phi_n) = A_n (\lambda_1, \bar{\lambda}_1, \eta_1; \ldots \lambda_n, \bar{\lambda}_n, \eta_n)
\]
Tree MHV superamplitude

- All MHV amplitudes are combined into a single superamplitude (spinor notations $\langle ij \rangle = \lambda^\alpha_i \lambda_{j\alpha}$)

$$A_{n}^{\text{MHV}} = i \frac{\delta^{(4)}(\sum_{i=1}^{n} p_i) \delta^{(8)}(\sum_{i=1}^{n} \lambda^\alpha_i \eta^A_i)}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle}$$

- On-shell $\mathcal{N} = 4$ supersymmetry:

$$q_A^\alpha = \sum_i \lambda_{i,\alpha} \eta^A_i, \quad \bar{q}_A \dot{\alpha} = \sum_i \tilde{\lambda}_{i,\dot{\alpha}} \frac{\partial}{\partial \bar{\eta}^A_i} \implies q_A^\alpha A_{n}^{\text{MHV}} = \bar{q}_A \dot{\alpha} A_{n}^{\text{MHV}} = 0$$

- (Super)conformal invariance

$$k_{\alpha\dot{\alpha}} = \sum_i \frac{\partial^2}{\partial \lambda^\alpha_i \partial \tilde{\lambda}^\dot{\alpha}_i} \implies k_{\alpha\dot{\alpha}} A_{n}^{\text{MHV}} = 0$$

Much less trivial to verify for NMHV amplitudes

- In fact, (super)conformal symmetry is almost exact (due to holomorphic anomaly)

$$\bar{s} A_{n}^{\text{MHV}} \sim \sum_i \left( \eta_i \tilde{\lambda}_{i+1} - \eta_{i+1} \tilde{\lambda}_i \right) \delta^{(2)}(\lambda_i, \lambda_{i+1}) A_{n-1}^{\text{MHV}}$$

$\bar{s} A_{n}^{\text{MHV}}$ is localized at collinear configurations $p_i \parallel p_{i+1}$

[Nair] [Witten] [Bargheer, Beisert, Galleas, Loebbert, McLoughlin]
The $\mathcal{N} = 4$ superamplitudes possess a much bigger, dual superconformal symmetry

[Drummond, Henn, GK, Sokatchev]

Chiral dual superspace $(x_{\alpha \dot{\alpha}}, \theta^{A}_{\alpha}, \lambda_{\alpha})$:

\[ p = \sum_{i=1}^{n} p_i = 0 \quad \rightarrow \quad p_i = x_i - x_{i+1} \]

\[ q = \sum_{i=1}^{n} \lambda_i \eta_i = 0 \quad \rightarrow \quad \lambda_i \eta_i^A = (\theta_i - \theta_{i+1})^A_{\alpha} \]

The MHV superamplitude in the dual superspace

\[ A_{\text{MHV}}^n = i(2\pi)^4 \frac{\delta^{(4)}(x_1 - x_{n+1}) \delta^{(8)}(\theta_1 - \theta_{n+1})}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle} \]

$\mathcal{N} = 4$ supersymmetry in the dual superspace:

\[ Q^A_{\alpha} = \sum_{i=1}^{n} \frac{\partial}{\partial \theta^{A}_{\alpha}} , \quad \bar{Q}^A_{\dot{\alpha}} = \sum_{i=1}^{n} \theta^{A}_{\alpha} \frac{\partial}{\partial x^{\dot{\alpha}}_{\alpha}} , \quad P_{\alpha \dot{\alpha}} = \sum_{i=1}^{n} \frac{\partial}{\partial x^{\dot{\alpha}}_{\alpha}} \]

Dual supersymmetry

\[ Q^A_{\alpha} A_{\text{MHV}}^n = \bar{Q}^A_{\dot{\alpha}} A_{\text{MHV}}^n = P_{\alpha \dot{\alpha}} A_{\text{MHV}}^n = 0 \]
Dual $\mathcal{N} = 4$ superconformal symmetry II

✔ Super-Poincaré + inversion = conformal supersymmetry:

✗ Inversions in the dual superspace

$$ I[\chi_i^\alpha] = (x_i^{-1})^{\dot{\alpha}\beta} \chi_i^\beta, \quad I[\theta_i^\alpha A] = (x_i^{-1})^{\dot{\alpha}\beta} \theta_i^A $$

✗ Neighbouring contractions are dual conformal covariant

$$ I[\langle ii + 1 \rangle] = (x_i^2)^{-1} \langle ii + 1 \rangle $$

✗ Impose cyclicity, $x_{n+1} = x_1, \theta_{n+1} = \theta_1$, through delta functions. Then, only in $\mathcal{N} = 4$,

$$ I[\delta^4(x_1 - x_{n+1})] = x_1^8 \delta^4(x_1 - x_{n+1}) $$
$$ I[\delta^8(\theta_1 - \theta_{n+1})] = x_1^{-8} \delta^8(\theta_1 - \theta_{n+1}) $$

✔ The tree-level MHV superamplitude is covariant under dual conformal inversions

$$ I \left[ A_n^{\text{MHV}} \right] = (x_1^2 x_2^2 \ldots x_n^2) \times A_n^{\text{MHV}} $$

✔ Dual superconformal covariance is a property of all tree-level superamplitudes (MHV, NMHV, $N^2$ MHV,...) in $\mathcal{N} = 4$ SYM theory

[Drummond, Henn, GK, Sokatchev], [Brandhuber, Heslop, Travaglini]
Symmetries at tree amplitudes

✔ The relationship between conventional and dual superconformal $su(2, 2|4)$ symmetries:

$\begin{align*}
q & \quad \bar{q} \\
\bar{s} & \quad \bar{S} \\
k & \quad P
\end{align*}$

[Drummond, Henn, G.K., Sokatchev]

✔ The same symmetries appear at strong coupling from invariance of $AdS_5 \times S^5$ sigma model under bosonic [Kallosh, Tseytlin] + fermionic T-duality [Berkovits, Maldacena], [Beisert, Ricci, Tseytlin, Wolf]

✔ (Infinite-dimensional) closure of two symmetries has Yangian structure [Drummond, Henn, Plefka]

✔ All tree-level amplitudes are known [Drummond, Henn] from the supersymmetric generalization of the BCFW recursion relations [Brandhuber, Heslop, Travaglini], [Bianchi, Elvang, Freeman], [Arkani-Hamed, Cachazo, Kaplan]

Are tree level amplitudes completely determined by the symmetries?
Invariants of both symmetries

✓ The ‘ratio’ of two tree superamplitudes

\[ A_n = A_n^{\text{MHV}} \quad R_n = A_n^{\text{MHV}} \left[ R_n^{\text{MHV}} + R_n^{\text{NMHV}} + \ldots \right] \]

✓ The ratio \( R_n \)–functions are invariants of both conventional \((g)\) and dual \((G)\) symmetries:

\[
g \cdot R_n^{\text{NP MHV}} = G \cdot R_n^{\text{NP MHV}} = 0,
\]

\[
R_n^{\text{NP MHV}} = \text{Polynomial in } \theta \text{'s of degree } 4p
\]

General solution is unknown, classification of \( \text{NP MHV} \) superinvariants remains to be done

✗ MHV superinvariants \((p = 0)\) are trivial: \( R_n^{\text{MHV}} = \text{const} \)

✗ NMHV superinvariants \((p = 1)\) are nontrivial:

\[
R_{rst}(x, \lambda, \theta) = \frac{\langle s - 1s \rangle \langle t - 1t \rangle \delta^{(4)} (\langle r| x_{rs} x_{st} | \theta_{tr} \rangle + \langle r| x_{rt} x_{ts} | \theta_{sr} \rangle)}{x_{st}^2 \langle r| x_{rs} x_{st} | t - 1 \rangle \langle r| x_{rs} x_{st} | t \rangle \langle r| x_{rt} x_{ts} | s - 1 \rangle \langle r| x_{rt} x_{ts} | s \rangle}
\]

Supersymmetric extension of three-mass box coefficients

[Drummond,Henn,GK,Sokatchev]

[Bern,Dixon,Kosower]
How powerful are the symmetries?

General expression for the NMHV ratio function dictated by the symmetries

\[ g \cdot R_{n}^{\text{NMHV}} = G \cdot R_{n}^{\text{NMHV}} = 0 \quad \implies \quad R_{n}^{\text{NMHV}} = \sum_{r,s,t} c_{rst} R_{rst} \quad \text{(with } c_{rst} \text{ arbitrary!)} \]

✔ The combined action of conventional and dual superconformal symmetries is not sufficient to fix all the freedom in the tree-level amplitudes

✔ Additional information needed comes from the analytic properties of tree amplitudes: [GK, Sokatchev]

✗ ‘Physical’ poles in multi-particle invariant masses \((p_s + \ldots + p_{t-1})^2 = x_{st}^2 = 0\)

✗ Free from spurious singularities

✗ Correct collinear factorization \(p_i \parallel p_{i+1}\)

Analytical properties of \(R_{rst}^{-}\)invariants:

\[ R_{rst} \sim \left( \frac{x_{st}^2}{\langle r|x_{rs}x_{st}|t-1\rangle \langle r|x_{rs}x_{st}|t\rangle \langle r|x_{rt}x_{ts}|s-1\rangle \langle r|x_{rt}x_{ts}|s\rangle} \right)^{-1} \]

physical pole

spurious poles

Kinematical configuration corresponding to spurious pole at \(\langle r|x_{rs}x_{st}|t\rangle = 0\):

\(-x_{rt}^2 x_{r+1,t}^2 x_{t+1,s}^2 + x_{r+1,t}^2 x_{rs} x_{t+1,s}^2 - x_{r+1,t+1}^2 x_{rs} x_{ts}^2 + x_{r,t+1}^2 x_{r+1,t}^2 x_{ts}^2 = 0\)

Spurious poles should cancel inside \(R_{n}^{\text{NMHV}}\)!
Cancellation of spurious poles

- Each invariant $R_{rst}$ has four sets of spurious poles
  \[ \langle r|x_{rs}x_{st}|t-1 \rangle = \langle r|x_{rs}x_{st}|t \rangle = \langle r|x_{rt}x_{ts}|s-1 \rangle = \langle r|x_{rt}x_{ts}|s \rangle = 0 \]

- 'Master identity': all spurious poles of $R_{rst}$ cancel in the linear combination of invariants:
  \[ R_{rst} + (R_{str} + R_{trs} - R_{s-1}tr - R_{t-1}rs) \]

- $n = 8$ NMHV: general expression consistent with all symmetries
  \[ R_{8}^{NMHV;0} = \alpha R_{147} + \beta R_{148} + \gamma R_{157} + \delta R_{158} + \varepsilon R_{168} + \text{cyclic} \]

  Cancellation of spurious poles leads to
  \[ \alpha - \beta = \alpha + \gamma - \delta = 2\alpha - \gamma = \delta - \varepsilon = \beta + \gamma - \delta = \beta + \gamma - \varepsilon = 0 \]

  This system is overdetermined but it has a unique solution
  \[ \beta = \alpha, \quad \gamma = 2\alpha, \quad \delta = \varepsilon = 3\alpha \]

- The same relations (with $\alpha = \frac{1}{8}$) ensure the correct behavior in the collinear limit $p_i\parallel p_{i+1}$
  \[ R_n(\ldots,i,i+1,\ldots) \xrightarrow{i\parallel i+1} R_{n-1}(\ldots,\ell,\ldots) \]

*Tree amplitudes are uniquely fixed by symmetries + analyticity condition* [GK,Sokatchev]
Do symmetries survive loop corrections?

✔ Loop corrections to the amplitudes necessarily induce infrared divergences

✔ The scattering amplitudes are well-defined in $D = 4 - 2\epsilon_{\text{IR}}$ dimensions only

✔ All-loop planar (super)amplitudes can be split into a IR divergent and a finite part

$$A_n^{(\text{all-loop})} = \text{Div}(1/\epsilon_{\text{IR}}) \left[ \text{Fin} + O(\epsilon_{\text{IR}}) \right]$$

✗ IR divergences (poles in $\epsilon_{\text{IR}}$) exponentiate (in any gauge theory!)

$$\text{Div}(1/\epsilon_{\text{IR}}) = \exp \left\{ -\frac{1}{2} \sum_{l=1}^{\infty} a^l \left( \frac{\Gamma_{\text{cusp}}^{(l)}}{(l\epsilon_{\text{IR}})^2} + \frac{G^{(l)}}{l\epsilon_{\text{IR}}} \right) \sum_{i=1}^{n} (-s_{i,i+1})^l \epsilon_{\text{IR}} \right\}$$

✗ IR divergences are in the one-to-one correspondence with UV divergences of Wilson loops

$[\text{Ivanov, GK, Radyushkin}]$

$$\Gamma_{\text{cusp}}(a) = \sum_{l} a^l \Gamma_{\text{cusp}}^{(l)} = \text{cusp anomalous dimension of Wilson loops}$$

$$G(a) = \sum_{l} a^l G_{\text{cusp}}^{(l)} = \text{collinear anomalous dimension}$$

✔ IR divergences preserve Poincaré supersymmetry but break conformal + dual conformal symmetry

IR divergences come from small momenta (=large distances) and, therefore, the conformal anomaly is not ‘localized’ (= difficult to control)
Anomalous symmetries at loop level

- Some symmetries \( (p, q, \bar{q}, P, Q, \bar{S}, \ldots) \) survive loop corrections while other \( (s, \bar{s}, k, K, S, \bar{Q}, \ldots) \) are broken.

- But the anomalies are not independent: \([K, \bar{Q}] = S\), \([s, \bar{s}] = k\)

- Three independent anomalies are

\[
\begin{align*}
 s\alpha A &= \sum_i \frac{\partial^2}{\partial \lambda_i^\alpha \partial \eta_i^A}, \\
 Q^{\alpha \dot{\alpha}} &= \sum_i \eta_i^A \frac{\partial}{\partial \tilde{\lambda}_i^\dot{\alpha}}, \\
 K^{\alpha \dot{\alpha}} &= \sum_i \left[ x_i^{\alpha \beta} x_i^{\beta \dot{\alpha}} \frac{\partial}{\partial x_i^{\beta \dot{\beta}}} + x_i^{\beta \dot{\alpha}} \theta_i^B \theta_i^\beta \frac{\partial}{\partial \theta_i^\beta} + \ldots \right]
\end{align*}
\]

- Dual conformal \( K \)–anomaly is **universal** for all superamplitudes (MHV, NMHV,...)

- \( K \)–anomaly can be determined to all loops from Wilson loop/MHV amplitude duality, whereas the \( s \)– and \( \bar{Q} \)–anomalies are hard to control.
MHV amplitudes are dual to light-like Wilson loops

\[ \ln A_n^{(MHV)} \sim \ln W(C_n) + O(1/N_c^2), \quad C_n = \text{light-like } n-(\text{poly})gon \]

✔ At strong coupling, agrees with the BDS ansatz \[\text{[Alday,Maldacena]}\]

✔ At weak coupling, the duality was verified against BDS ansatz to two loops for \( n \geq 4 \)

\[\text{[Drummond,Henn,GK,Sokatchev, Anastasiou,Brandhuber,Heslop,Khoze,Spence,Travaglini]}\]

**Wilson loops match the BDS ansatz for** \( n = 4,5 \) **but not for** \( n \geq 6 \)

✔ Scattering amplitude/Wilson loop duality also holds in QCD but in the Regge limit only \[\text{[GK]}\]
**Dual conformal \( K \)–anomaly**

Dual conformal symmetry of the amplitudes ⇔ Conformal symmetry of Wilson loops

Dual conformal anomaly ⇔ Conformal anomaly of Wilson loops

✔ How could Wilson loops have conformal anomaly in \( \mathcal{N} = 4 \) SYM?

✗ Were the Wilson loop well-defined (=finite) in \( D = 4 \) dimensions it would be conformal invariant

\[ W(C_n) = W(C'_n) \]

✗ ... but \( W(C_n) \) has cusp UV singularities \( \mapsto \) dim.reg. breaks conformal invariance

\[ W(C_n) = W(C'_n) \times \text{[cusp anomaly]} \]

✔ All-loop anomalous conformal Ward identities for the finite part of the Wilson loop

\[
\ln W(C_n) = F_n^{(W_L)} + \text{[UV divergencies]} + O(\epsilon)
\]

Under special conformal transformations (boosts), to all orders, \[\text{[Drummond,Henn,GK,Sokatchev]}\]

\[
K^\mu F_n \equiv \sum_{i=1}^{n} \left[ 2x_i^\mu (x_i \cdot \partial x_i) - x_i^2 \partial x_i^\mu \right] F_n = \frac{1}{2} \Gamma_{\text{cusp}}(a) \sum_{i=1}^{n} x_{i,i+1}^\mu \ln \left( \frac{x_{i,i+2}^2}{x_{i-1,i+1}^2} \right)
\]

The same relations also hold at strong coupling \[\text{[Alday,Maldacena],[Komargodski]}\]
Dual conformal anomaly at work

Consequences of the conformal Ward identity for the finite part of the Wilson loop $W_n$:

- $n = 4, 5$ are special: there are no conformal invariants (too few distances due to $x_{i,i+1}^2 = 0$)

  $\implies$ the Ward identity has a unique all-loop solution (up to an additive constant)

$$F_4 = \frac{1}{4} \Gamma_{\text{cusp}}(a) \ln^2 \left( \frac{x_{13}^2}{x_{24}^2} \right) + \text{const},$$

$$F_5 = -\frac{1}{8} \Gamma_{\text{cusp}}(a) \sum_{i=1}^{5} \ln \left( \frac{x_{i,i+2}^2}{x_{i,i+3}^2} \right) \ln \left( \frac{x_{i+1,i+3}^2}{x_{i+2,i+4}^2} \right) + \text{const}$$

Exactly the BDS ansatz for the 4- and 5-point MHV amplitudes!

- Starting from $n = 6$ there are conformal invariants in the form of cross-ratios $u_{ijkl} = \frac{x_{i,l}^2 x_{j,k}^2}{x_{i,k}^2 x_{j,l}^2}$

General solution to the Ward identity contains an arbitrary function of the conformal cross-ratios.

- Crucial test - go to six points at two loops where the answer is not determined by conformal symmetry

  [Drummond,Henn,GK,Sokatchev] [Bern,Dixon,Kosower,Roiban,Spradlin,Vergu,Volovich]

$$F_{6}^{(WL)} = F_{6}^{(MHV)} \neq F_{6}^{(BDS)}$$

The Wilson loop/MHV amplitude duality holds at $n = 6$ to two loops!
Dual conformal symmetry beyond MHV

One-loop NMHV superamplitudes

✔ $n$-gluon one-loop NMHV amplitudes are known

✔ New result for one-loop NMHV superamplitude:

$$\mathcal{A}_{n}^{\text{NMHV}}; 1 = \mathcal{A}_{n}^{\text{MHV}}; 1 \times \left[ R_{n}^{\text{NMHV}}(x_{i}, \lambda_{i}, \theta_{A}^{i}) + O(\epsilon) \right]$$

IR divergences break symmetries of NMHV and MHV but they cancel inside the ‘ratio function’

One-loop NMHV ‘ratio function’ = sum of tree-level dual superconformal invariants:

$$R_{n}^{\text{NMHV}} = \sum_{p,q,r} R_{pqr}(\lambda, \tilde{\lambda}, \theta) V_{pqr}(x_{ij}^{2})$$

✗ Helicity structure is invariant under both (conventional and dual) superconformal symmetries

✗ Loop corrections are described by scalar functions

$$V_{pqr} = 1 + a V^{(1)}(\{u_{pqr}\}) + O(a^{2}), \quad u_{pqr} = \text{dual conformal cross-ratios}$$

they are dual conformal invariants made of IR finite combinations of 1-loop scalar box integrals

✗ $R_{n}^{\text{NMHV}}$ is free from spurious singularities and has correct collinear limit

The ratio function is dual conformal invariant but it is not superconformal invariant, why?
Dual supersymmetry $\overline{Q}$–anomaly

- Main idea: Instead of $\overline{Q} A_n$ let us compute its discontinuity
  \[ \text{Disc}_s (\overline{Q} A_n) = \overline{Q} (\text{Disc}_s A_n) \]
  \[ \text{Disc}_{s123} A_{6 \text{MHV};1} = A_{\text{MHV};0} (-\ell_1, 1, 2, 3, -\ell_2) \star A_{\text{MHV};0} (\ell_2, 4, 5, 6, \ell_1) , \]

Two tree amplitudes are integrated over the phase space of on-shell states $\ell_1$ and $\ell_2$

\[ \delta \overline{Q} \text{Disc}_{s123} A_{6 \text{MHV};1} = \delta \overline{Q} A_{\text{MHV};0} (-\ell_1, 1, 2, 3, -\ell_2) \star A_{\text{MHV};0} (\ell_2, 4, 5, 6, \ell_1) + A_{\text{MHV};0} (-\ell_1, 1, 2, 3, -\ell_2) \star \delta \overline{Q} A_{\text{MHV};0} (\ell_2, 4, 5, 6, \ell_1) \]

- If tree superamplitudes were exactly invariant, $\delta \overline{Q} A_{\text{MHV};0} = 0$, then $\delta \overline{Q} \text{Disc}_{s123} A_{6 \text{MHV};1} = 0$. But they are not due to holomorphic anomaly!

- $\overline{Q}$–anomaly of one-loop NMHV ratio function

\[ \overline{Q}_\alpha^A \left( \text{Disc}_{x_{14}^2} R_{6 \text{NMHV};1} \right) \sim (\eta_1[23] + \eta_2[31] + \eta_3[12]) A R_{146} \]
\[ \times \left( \frac{\tilde{\lambda}_1 \alpha[6|x_{63}|3]}{x_{14}^2[61][12]} + \frac{\tilde{\lambda}_3 \alpha[4|x_{41}|1]}{x_{14}^2[23][34]} \right) + (i \rightarrow i + 3) \neq 0 \]

Holomorphic anomaly is responsible for the breakdown of $\overline{Q} = \overline{s}$ symmetry of the ratio function (but not of the dual conformal symmetry!)
Analyticity constraints at loop level

All-loop superamplitude $A_n$ should be free from spurious poles + have correct collinear behaviour:

- Spurious pole cancellation operates ‘horizontally’, i.e. it establishes a property of the $n$–particle amplitude, without reference to other amplitudes

- Collinear factorization operates ‘vertically’, i.e. it recursively relates the $n$–particle amplitude to the amplitudes with fewer particles

- Example: One loop correction to NMHV$_{n=7}$ ratio function:

$$R_{146}^{NMHV;1} = \frac{1}{2} R_{146} V_{146}(u_1, u_2, u_3) + \text{(cyclic)}$$

$$V_{146} = -\ln u_1 \ln u_3 + \frac{1}{2} \sum_{k=1}^{3} [\ln u_k \ln u_{k+1} + \text{Li}_2(1 - u_k)] - \frac{\pi^2}{6} ,$$

with $u_i$ dual conformal cross-ratios, e.g. $u_1 = \frac{x_{15}^2 x_{24}^2}{(x_{14}^2 x_{25}^2)}$

- Spurious pole: $u_1 = 1, u_2, u_3 = \text{arbitrary}$

$$V_{146}(1, u_2, u_3) - V_{146}(u_3, 1, u_2) = 0$$

- Collinear limit: $u_1 \to 0, u_3 \to 1 - u_2$

$$V_{146}(0, u_2, 1 - u_2) + V_{146}(0, 1 - u_2, u_2) = 0$$

The two approaches are equivalent at tree level but lead to different constraints at loop level!
Symmetry of all-loop superamplitudes

DHKS proposal for all-loop superamplitude in $\mathcal{N} = 4$ SYM:

$$A_n(x_i, \lambda_i, \theta_i^A) = A_n^{MHV} + A_n^{NMHV} + A_n^{N^2MHV} + \ldots + A_n^{MHV}$$

- At tree level, $A_n$ is fixed by conventional and dual symmetries + analyticity conditions
- At loop level, both symmetries become anomalous due to IR divergences + holomorphic anomaly
- The dual superconformal symmetry is restored in the ratio of superamplitudes $A_n$ and $A_n^{MHV}$

$$A_n(x_i, \lambda_i, \theta_i^A) = A_n^{MHV} \times \left[ R_n(x_i, \lambda_i, \theta_i^A) + O(\epsilon) \right]$$

The ratio function

$$R_n = 1 + R_n^{NMHV} + R_n^{N^2MHV} + \ldots$$

is **IR finite** and, most importantly, it is **dual conformal invariant**

$$K^{\alpha\dot{\alpha}} R_n^{\text{all loops}} = 0$$

The conjecture was recently proven to one loop

... but all-loop proof is still missing
Superamplitudes in twistor space

Twistor (half-Fourier) transform \((\lambda, \tilde{\lambda}, \eta) \mapsto (\lambda, \mu, \psi)\) in split signature \((++--)\)  

\[
T[A_n](\{\lambda, \mu, \psi\}) = \int \prod_{1}^{n} \frac{d^2 \tilde{\lambda}_i}{(2\pi)^2} d^4 \eta_i \, e^{i(\mu_i \tilde{\lambda}_i + \psi_i A \eta_i^A)} A_n(\{\lambda, \tilde{\lambda}, \eta\}),
\]

✔ The twistor transform of the MHV superamplitude

\[
A_n^{\text{MHV}} = \frac{i}{\prod_{1}^{n} \langle i \, i + 1 \rangle (2\pi)^4 \delta^{(4)}(\sum_{1}^{n} \lambda_i \tilde{\lambda}_i) \delta^{(8)}(\sum_{1}^{n} \lambda_i \eta_i)}
\]

\[
T[A_n^{\text{MHV}}] = i \int d^4 X d^8 \Theta \prod_{1}^{n} \delta^{(2)}(\mu_i + \lambda_i X) \delta^{(4)}(\psi_i + \langle i \Theta \rangle) \frac{\langle 12 \rangle \ldots \langle n - 1 \, n \rangle \langle n1 \rangle}{\langle \rangle}
\]

Is localized on the line in the twistor space

\[
\mu_{i\dot{\alpha}} + \lambda_i^\alpha X_{\alpha \dot{\alpha}} = 0, \quad \psi_i A + \langle i \Theta A \rangle = 0,
\]

✔ All superamplitudes are conjectured to be supported on holomorphic curves of higher degree: connected/disconnected prescriptions  
[Nair],[Witten],[Roiban,Spradlin,Volovich],[Bern,Del Duca,Dixon,Kosower],...

✔ Simple form of BCFW recurrence relations  
[Mason,Skinner],[Arkani-Hamed et al]

✔ Simple action of conformal group: the line parameters \(X\) and \(\Theta\) are transformed as the coordinates of configuration superspace  
[GK,Sokatchev]
Twistor transform of NMHV superamplitude I

Half-Fourier transform of the NMHV amplitude

\[ A_n^{NMHV} = \sum_{3 \leq a+1 < b \leq n-1} A_{nab}, \quad A_{nab} = A_n^{MHV} R_{nab}. \]

seems to be an impossible task, because of the very non-trivial dependence of \( R \)'s on \( \tilde{\lambda} \)

\[ R_{nab} = \frac{\langle a - 1 b \rangle \langle b - 1 a \rangle \delta^{(4)} (\sum_{1}^{a-1} \langle n | x_{nb} x_{ba}^{-1} | i \rangle \eta_i + \sum_{1}^{b-1} \langle n | x_{na} x_{ab}^{-1} | i \rangle \eta_i) }{x_{ab}^2 \langle n | x_{nb} x_{ba}^{-1} | a - 1 \rangle \langle n | x_{nb} x_{ba}^{-1} | a \rangle \langle n | x_{na} x_{ab}^{-1} | b - 1 \rangle \langle n | x_{na} x_{ab}^{-1} | b \rangle}, \]

✔ First, we define the spinors \( \langle \rho_0 | + \langle \rho | + \langle \sigma | = 0 \)

\[ \langle \rho_0 | \equiv \langle n |, \quad \langle \rho | = \langle n | x_{nb} x_{ba}^{-1} |, \quad \langle \sigma | = \langle n | x_{na} x_{ab}^{-1} |, \]

✔ Second, use the Faddeev-Popov approach to introduce them into \( R \)'s via delta function integrals

\[ R(\langle n | x_{nb} x_{ba}^{-1} |) = \int d^2 \rho \ R(\rho) \ \delta^{(2)} \left( \langle \rho | - \langle n | x_{nb} x_{ba}^{-1} \right) \]

\[ = |x_{ab}^2| \int d^2 \rho \ d^2 \tilde{\rho} \ R(\rho) \ \exp \left\{ -i \left( \langle \rho | x_{ba} | \tilde{\rho} \rangle + \langle n | x_{nb} | \tilde{\rho} \rangle \right) \right\}. \]

✔ Incomplete cancellation \( |x_{ab}^2| / x_{ab}^2 = \text{sgn}(x_{ab}^2) \) leads to breakdown of conformal symmetry of tree NMHV amplitude

[Mason, Skinner], [Arkani-Hamed et al.]
Twistor transform of NMHV superamplitude II

Twistor transform of the partial NMHV amplitude $A_{nab}$

$$T[A_{nab}] = i \int d^4 X \int d^2 \rho d^2 \tilde{\rho} (\prod_{nab}^{(2)}) \times [\text{Fermionic counterpart}]$$

- Integrand = product of three MHV-like lines in twistor space

$$\frac{(\prod_{nab})}{\Delta_{nab}} = \prod_{1}^{a-1} \delta(2)(\mu_i + \langle i | X_1 \rangle) \times \prod_{a}^{b-1} \delta(2)(\mu_i + \langle i | X_2 \rangle) \times \prod_{n}^{b} \delta(2)(\mu_i + \langle i | X_3 \rangle)$$

- Line moduli:
  - $X_1 = X - \rho_0 \tilde{\rho}$, $X_2 = X + \sigma \tilde{\rho}$, $X_3 = X$

- Three vectors $X_{12}, X_{23}$ and $X_{31}$ are lightlike $\implies$ three lines intersect pairwise

$$X_{12} = \rho \tilde{\rho}, \quad X_{23} = \sigma \tilde{\rho}, \quad X_{31} = \rho_0 \tilde{\rho}, \quad X_{12} + X_{23} + X_{31} = 0$$

Transforming a single MHV line into three NMHV lines

\[
\begin{array}{c}
X \\
\downarrow \quad a-1 \\
\downarrow b-1 \\
X_1 \\
\downarrow a \\
\downarrow b \\
X_2 \\
\downarrow b-1 \\
\downarrow b \\
X_3 \\
\downarrow n \\
\downarrow n \\
n
\end{array}
\]
Geometric interpretation of dual superconformal invariants

NMHV superinvariant = Three intersecting lines = (Coplanar) lightlike triangle in moduli space

$\begin{align*}
X_3 \\
X_1 \\
X_2
\end{align*}$

MHV superinvariants = $(2k + 1)$ intersecting lines = Nonplanar triangulated surfaces in moduli space

$\begin{align*}
X_5 \\
X_1 \\
X_2 \\
X_3 \\
X_4
\end{align*}$

$\begin{align*}
a_1 \\
b_1 \\
b_2 \\
1 \\
n
\end{align*}$

$\begin{align*}
a \\
b \\
a-1
\end{align*}$

$\begin{align*}
1 \\
n
\end{align*}$
Conclusions and open questions

✔ Tree amplitudes in $\mathcal{N} = 4$ SYM respect conventional and dual superconformal symmetries but their combined action is not sufficient to fix the amplitudes. The additional information needed comes from analyticity properties of the amplitudes.

✔ At loop level, both symmetries are broken by IR divergences + holomorphic anomaly. The dual conformal anomaly is well understood but how to control the remaining anomalies?

We need the dual model for $\mathcal{N} = 4$ superamplitude:

Dual model for the MHV amplitude = light-like Wilson loop

Dual model for the MHV+NMHV+ . . . + $\overline{\text{MHV}}$ amplitude = ???

$\overline{Q}$—anomaly indicates that the dual model does not respect Poincaré supersymmetry.

How could it be?

✔ Weak/strong coupling paradox:

At weak coupling, the $\overline{Q}$—anomaly is present to all loops $\overline{Q} R_n = a f_1 + a^2 f_2 + \ldots$ but it is not seen at strong coupling !?

$$A_{n}^{\text{NP MHV}} \sim \exp \left( -\sqrt{a} S_{\text{min}} \right) \left[ 1 + O\left(1/\sqrt{a}\right) \right] \quad \Rightarrow \quad R_{n}^{\text{NP MHV}} \sim 1 + O\left(1/\sqrt{a}\right)$$

What is the meaning of holomorphic anomaly in string theory?