# Structure of gauge and gravity amplitudes in string and field theory

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Hidden Structures in Field Theory Amplitudes, August 14, 2009

based on work done with
N. Berkovits, N.E.J. Bjerrum-Bohr, P. Damgaard, M.B. Green, J. Russo

# Motivations

Recently we have experienced fantastic progress in the evaluation of on-shell gauge and gravity amplitudes in field theory.

Both on-shell S-matrix for $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA showed a rather remarkable simplicity compared to the Feynman graph approach

# Motivations

In $\mathcal{N} = 4$ SYM this simplicity is expected because of the extended supersymmetries and the structure of the gauge interaction of the theory

But the simplicity of the $\mathcal{N} = 8$ SUGRA S-matrix calls for an explanation:

- how much is due to supersymmetry?
- how much to gauge (diffeomorphism) invariance?
- Connection with string theory analysis

# Motivations

We are interested in theories with maximal supersymmetry in various dimensions $4 \leq D \leq 10$

Unfortunately the implementation of supersymmetry in perturbation in string theory and in field theory is not completely well understood and needs to be reconsidered

In this talk we will analyze the structure of the gauge and gravity amplitudes in string. At higher-loop we make use of Berkovits' pure spinor formulation of perturbative string to analyze the role of supersymmetry in the $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA amplitudes

# Outline

Part I

# Tree-level amplitudes

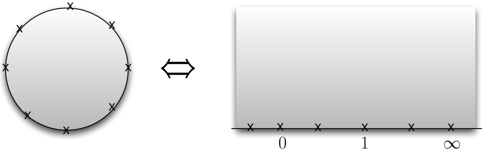
# Open string tree-level amplitudes

Color ordered tree-level SYM amplitude are obtained as the $\alpha' \to 0$ of the of open string amplitudes defined in the complex plane

$$A_{\rm SYM}(1,\ldots,n) = \lim_{\alpha'\to 0} \mathfrak{A}(1,\ldots,n)$$

$$\mathfrak{A}(1,\ldots,n) = \left\langle U^{(1)}(z_1)U^{(n-1)}(z_{n-1})U^{(n)}(z_n) \prod_{i=2}^{n-2} \int d^2 z_i V^{(i)} \right\rangle$$

# Cyclicity: $(n-1)!$ amplitudes




$PSL(2,\mathbb{R})$ invariance $z_1 = 0$, $z_{n-1} = 1$ and $z_n = +\infty$.
Ordered integral $\Re\mathrm{e}(z_1) < \cdots < \Re\mathrm{e}(z_n)$

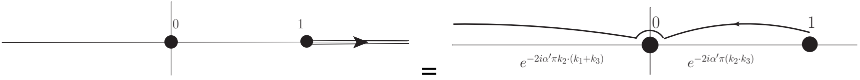
$$\mathfrak{A}(1,\dots,n) = \int_{\text{ordered}} \prod_{i=2}^{n-2} d^2 z_i \prod_{1\leq i<j\leq n} |z_i - z_j|^{2\alpha' k_i\cdot k_j} \sum_{(\zeta_j)\in\{0,1,z_i\}} L_k \prod_{i=2}^{n-2} \frac{1}{z_j-\zeta_j}$$

We define a set of $(n-3)!$ amplitudes by

$$\mathfrak{B}^\sigma = \mathfrak{A}(1, \underbrace{\sigma(2),\dots,\sigma(n-2)}_{\textit{permutation}}, n-1, n); \qquad \sigma \in \mathfrak{S}_{n-3}$$

# Monodromies: Step 1 $(n-2)!$ amplitudes

We deform the contour of integration [Bjerrum-bohr, Damgaard, Vanhove]



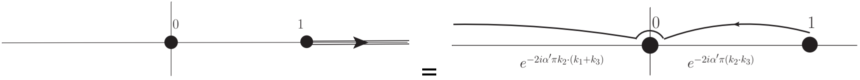

The real part of the monodromy relation leads to the stringy version of the Kleiss-Kuijf relations

$$\mathfrak{A}_n(\beta_1,\dots,\beta_r,1,\alpha_1,\dots,\alpha_s,n) = (-1)^r \times \Re\mathrm{e}\Big[\prod_{1\leq i<j\leq r} e^{2i\pi\alpha'(k_{\beta_i}\cdot k_{\beta_j})} \sum_{\sigma\subset \mathrm{OP}\{\alpha\}\cup\{\beta^T\}} \prod_{i=1}^{r}\prod_{j=1}^{s} e^{(\alpha_i,\beta_j)} \mathfrak{A}_n(1,\{\sigma\},n)\Big],$$

$\exp(\alpha,\beta) = \exp(2i\pi\alpha' k_\alpha\cdot k_\beta)$ if $\Re\mathrm{e}(z_\beta - z_\alpha) > 0$ or 1 otherwise

# Monodromies: Step 2 $(n-3)!$ amplitudes

We deform the contour of integration [Bjerrum-bohr, Damgaard, Vanhove]




The imaginary part of the monodromy relation

$$0=\Im\mathrm{m}\Big[\prod_{1\le i<j\le r} e^{2i\pi\alpha'(k_{\beta_i}\cdot k_{\beta_j})}\sum_{\sigma\subset \mathrm{OP}\{\alpha\}\cup\{\beta^T\}}\prod_{i=1}^{r}\prod_{j=1}^{s} e^{(\alpha_i,\beta_j)}\mathfrak{A}_n(1,\{\sigma\},n)\Big].$$

Proves the [Bern,Carrasco, Johanson,'08] relations in the field theory limit

# Minimal Basis for tree-level amplitudes

This implies that all ordered amplitudes can be expanded in the *minimal* basis $\mathfrak{B}$

$$\mathfrak{B}^{\sigma} = \mathfrak{A}(1, \underbrace{\sigma(2), \ldots, \sigma(n-2)}_{permutation}, n-1, n); \qquad \sigma \in \mathfrak{S}_{n-3}$$

$$\mathfrak{A}_n^{\text{ordered}} = \sum_{\sigma \in \mathfrak{S}_{n-3}} c_{\sigma}\, \mathfrak{B}^{\sigma}$$

$c_J$ are rational functions of degree$(c_{\sigma}) = 0$ in the $\sin(2\alpha' \pi\, p \cdot q)$
For gravity using that $V_{\text{grav}} = V_{\text{open}} \otimes \widetilde{V}_{\text{open}}$ gives

$$\mathfrak{M}_n^{\text{closed}} = \frac{\kappa^{n-2}}{(\alpha')^{n-3}} \sum_{\sigma, \tilde{\sigma} \in \mathfrak{S}_{n-3}} g_{\sigma\tilde{\sigma}}\, \mathfrak{B}^{\sigma} \widetilde{\mathfrak{B}}^{\tilde{\sigma}}$$

where the matrix $g_{\sigma\tilde{\sigma}}$ is symmetric and degree$(g_{IJ}) = n - 3$.

Part II

# One-loop amplitudes

# Basis of one-loop scalar integral functions

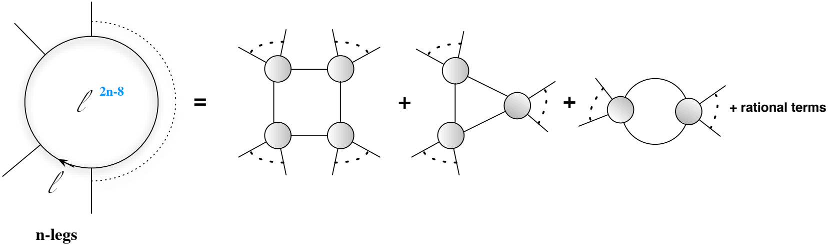
In $D = 4 - 2\epsilon$ one expands the amplitudes on a basis of scalar integral functions with massive external legs

$$\mathfrak{M}_{n;1}^{(4-2\epsilon)} = \sum_i bo_i\, I_{\square}^{(i)} + \sum_i t_i\, I_{\triangleright}^{(i)} + \sum_i bu_i\, I_{\circ}^{(i)} + c_{\text{rational pieces}}$$

This basis of scalar integral functions captures the IR and UV divergences of the one-loop amplitudes

# The no triangle property in $\mathcal{N} = 8$

$\mathcal{N} = 8$ amplitudes $2n - 8$ powers of loop momenta should contains boxes, triangles, bubbles and rational terms



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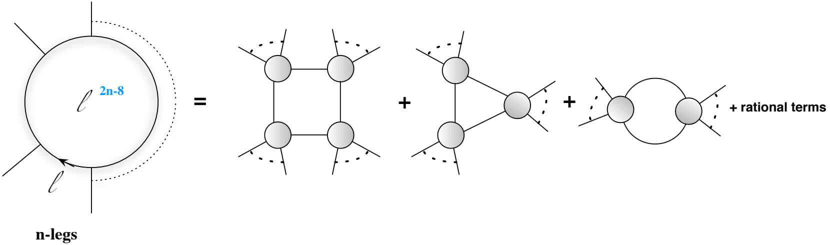



Explicit computations by [Bjerrum-Bohr et al., Bern et al.] showed that the amplitudes reduce to scalar box integral functions like for $\mathcal{N} = 4$ SYM.

For this one needs a reduction formula that takes into account *all* the cancellations occurring in Gravity [Bjerrum-Bohr, Vanhove]

# The no triangle property of $\mathcal{N} = 8$ amplitudes

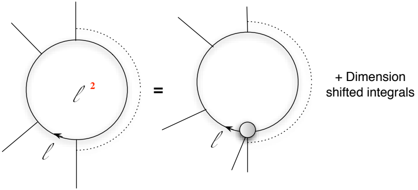
- Gravity does not have color factor
  - summation over all the permutations at one-loop
  - Sum over all the planar and non-planar diagrams at higher loop order
- Gauge invariance $\varepsilon_{\mu\nu} \to \varepsilon_{\mu\nu} + \partial_\mu v_\nu + \partial_\nu v_\mu$

Unordered amplitudes are more than just the sum over all orderings of color ordered amplitudes.
All the various orderings have the *same* tensorial structure

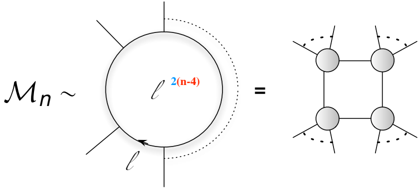
$$\mathfrak{M} = \sum_r t_r \int_0^\infty \frac{dT}{T} T^{\frac{2n-D}{2}} \int_0^1 \prod_{i=1}^{n-1} d\nu_i \, \mathcal{P}(\partial Q_n) \, e^{-T \sum_{r,s} (k_r \cdot k_s) \, G^{(1)}_{r,s}}$$

$Q_n = \sum_{i<j} (k_i \cdot k_j) \, [(\nu_i - \nu_j)^2 - |\nu_i - \nu_j|]$

# The no triangle property of $\mathcal{N} = 8$ amplitudes




New reduction formulas for unordered integrals where two powers of loop momenta are cancelled at each steps [Bjerrum-Bohr, Vanhove]




For $\mathcal{N} = 8$ sugra amplitude the no triangle property arises because the amplitude has $n - 4$ powers of $\ell^2$

Part III

# The UV behaviour $\mathcal{N} = 4$ super-Yang-Mills in various dimensions

# $\mathcal{N} = 4$ SYM in various dimensions

The coupling constant of $\mathcal{N} = 4$ SYM has dimension

$$[g^2_{\rm YM}] = (length)^{D-4}$$

Power counting gives that the 4-point $L$-loop amplitude $\mathfrak{A}_{4;L}$ has superficial UV divergence ($\Lambda$ UV cut-off)

$$[\mathfrak{A}^{(D)}_{4;L}] = \Lambda^{(D-4)L}$$

UV finite in $D < 4$

- could be logarithmically diverging in four dimensions

Supersymmetry improves this power counting

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$$[\mathfrak{A}^{(D)}_{4;L}] = \Lambda^{(D-4)L-4}\, t_8 F^4$$

$$\text{UV finite in } D < 4 + \frac{4}{L}$$

- Off-shell $\mathcal{N} = 2$ superspace is enough to assure finiteness in four dimensions by factorizing a $t_8 F^4$ term

  [Mandelstam; Howe, Stelle, West; Brink, Lindgren, Nilsson]

  - ☹ BUT: does not explain the non-renormalisation theorems for $F^4$ for $L \geq 2$
  - ☹ BUT: does not lead to the correct divergences structure at $L = 2$ in $D = 8$ and $L = 3$ in $D = 6$

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$$[\mathfrak{A}^{(D)}_{4;L}] = \Lambda^{(D-4)L-4-2\gamma_L}\, \partial^{2\gamma_L}\, t_8 F^4 \qquad L \geq 2$$

$$\text{UV finite in } D < 4 + \frac{4 + 2\gamma_L}{L}$$

- Perturbative computations by [Bern, Dixon, Dunbar, Perelstein, Rozowski] indicates a better behaviour with

$$\gamma_L = 1$$

- confirmed by $\mathcal{N} = 3$ superspace arguments in $D = 4$

[Howe, Stelle]

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  - Validity of this rule to all orders?
  - What about the color factors dependence?

# Superconformal multiplet of $\mathcal{N} = 4$ algebra

We consider the representation of the $\mathcal{N} = 4$ superconformal algebra.

There are TRUE BPS states which do not develop an anomalous dimension, and FAKE F-term developing an anomalous dimension that can be written as D-terms. [Drummond, Heslop, Howe, Kerstan]

For $D < 10$ one can construct the dimension 2 operator

$$\mathcal{K} = \mathrm{tr}(\varphi^i \varphi_i)$$

- $\frac{1}{2}$-BPS states: There are single-trace operators and multi-trace operators: $t_8\,\mathrm{tr}(F^4)$ and $t_8\,(\mathrm{tr}F^2)^2$, both TRUE BPS states.
- $\frac{1}{4}$-BPS states: TRUE F-term are at least double trace operators: $\partial^2\, t_8(\mathrm{tr}F^2)^2$. The single trace operator is a fake F-term since $\partial^2\, t_8\mathrm{tr}(F^4) \sim \int d^{16}\theta\, \mathcal{K}$
- $\frac{1}{8}$-BPS states: True F-term are at least triple traces, and $\partial^4\, t_8(\mathrm{tr}F^2)^2 \sim \int d^{16}\theta \mathcal{K}^2$ is a fake F-term.

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# Zero mode saturation in Pure Spinor amplitudes

We consider the four points open string amplitudes within the (non-minimal) pure spinor formalism of [Berkovits]

An $L$-loop open string amplitude with $H$-handles and $B$-boundaries has $3(L-1)$ real moduli with $L = B + 2H - 1$

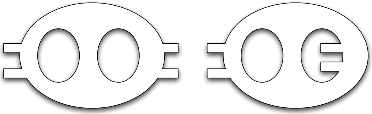
$$\mathfrak{A}_L^4 \sim g_s^{L-1} \int d^{3L-3}\tau \, \langle \prod_{i=1}^{3L-3} (\mu_i | b) \, V_1 \cdots V_4 \rangle$$

SYM vertex operator

$$V = \int d^2 z \, :(\partial x^m A_m + \partial\theta^\alpha A_\alpha + d_\alpha W^\alpha + F_{mn}(\lambda\gamma^{mn} w)) \, e^{ik\cdot X}:$$

the dimension 1 gaugino superfield

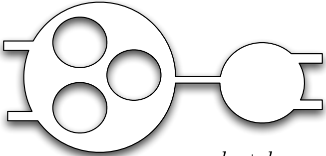
$$W^\alpha \sim \chi^\alpha + \theta_\beta F^{\alpha\beta} + \cdots$$



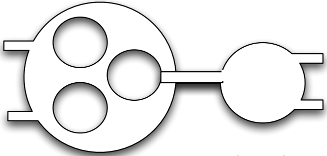
Zero mode saturation gives for $L \leq 6$ at a generic point in the moduli space

$$\mathfrak{A}_L^4 \sim g_s^{L-1} \int d^{3L-3}\tau \, (\mu_\tau|\partial x)^{L-2} \int d^{16}\theta\theta^{12-2L} \, W^4$$

But contributions from the boundary of the moduli space can generate reduction of number of derivatives



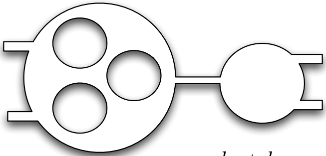
$$\mathrm{Tr}(T_1T_2T_3T_4)\,\frac{k_1+k_2}{k_1\cdot k_2}$$



$$\mathrm{Tr}([T_1,T_2])\mathrm{Tr}([T_3,T_4])\frac{k_1+k_2}{k_1\cdot k_2}=0$$

- Single trace operator: there are $L-2$ open string poles

$$\mathfrak{A}_L^4 \sim g_s^{L-1}\, N^L\, k^2\, t_8 \mathrm{tr} F^4$$



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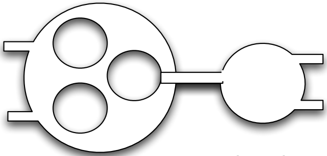

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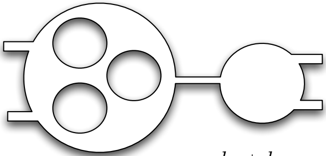
The $\partial^2\,\mathrm{tr}(F^4)$ operator gets all loop order correction from $L \geq 2$



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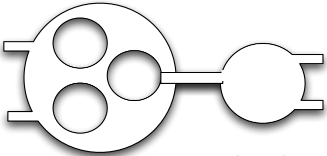

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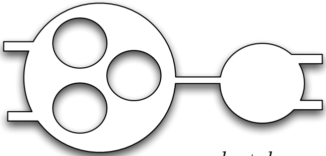

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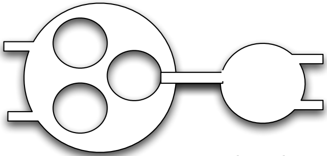

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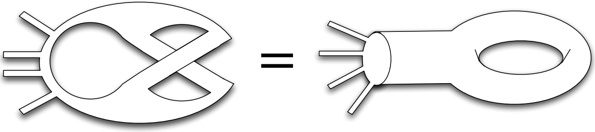
- $\partial^2 t_8(\mathrm{tr}F^2)^2$ operator is not renormalized above $L=2$.
- $\partial^4\, t_8(\mathrm{tr}F^2)^2$ operator gets perturbative contributions from $L\geq 3$.

# Gravitational corrections

Adding an handle suppresses two boundaries and lead of $1/N^2$ corrections



The leading low-energy limit $\ell_s \to 0$ of the $L$-loop open string amplitude is given by

$$\mathfrak{A}_L^4 \sim g_s^{L-1}\, N^L\, (1 + \frac{c_1}{N^2} + \frac{c_2}{N^4} + \cdots)\, k^2\, t_8 \mathrm{tr}(F^4)$$

$c_i$ is given by pure SYM and mixed SUGRA+SYM contributions

# $\mathcal{N} = 4$ SYM in various dimensions

[Berkovits, Green, Russo, Vanhove]

$$[\mathfrak{A}_{4;L}^{(D)}] = \Lambda^{(D-4)L-4-2\gamma_L}\, \partial^{2\gamma_L}\, t_8 F^4 \qquad L \geq 2$$

- The leading UV divergence for $\mathcal{N} = 4$ SYM 4-point amplitudes is given by the the single trace operator $\partial^2\, t_8 \mathrm{tr} F^4$

$$D < D_c = 4 + \frac{6}{L}$$

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$$D < D_c = 4 + \frac{6}{L}$$

- The double trace $\partial^{\beta_L}\, t_8 (\mathrm{tr} F^2)^2$ the critical dimension for UV divergences

$$\begin{aligned} D_c &= 4 + \frac{4+2[L/2]}{L}, & \beta_L &= L & \text{for } L \leq 4 \\ D_c &= 4 + \frac{8}{L}, & \beta_L &= 4 & \text{for } L \geq 3 \end{aligned}$$

Results in agreement with computations by [Bern, Dixon, Roiban, Carrasco, Johansson, to appear]

# Critical dimensions

[Berkovits, Green, Russo, Vanhove]

| | L=1 | L=2 | L=3 | L=4 | L=5 |
|---|---|---|---|---|---|
| $s^{\gamma_L}\, t_8 \mathrm{tr}(F^4)$ | $D_c = 8$ <br> $\gamma_1 = 0$ | $D_c = 7$ <br> $\gamma_2 = 1$ | $D_c = 6$ <br> $\gamma_3 = 1$ | $D_c = \frac{11}{2}$ <br> $\gamma_4 = 1$ | $D_c = \frac{26}{5}$ <br> $\gamma_5 = 1$ |
| $s^{\beta_L}\, t_8 (\mathrm{tr} F^2)^2$ | $D_c = 8$ <br> $\beta_1 = 0$ | $D_c = 7$ <br> $\beta_2 = 1$ | $D_c = \frac{20}{3}$ <br> $\beta_3 = 2$ | $D_c = 6$ <br> $\beta_4 = 2$ | $D_c = \frac{28}{5}$ <br> $\beta_5 = 2$ |

For $L \geq 4$ the UV divergence is dominated by the single trace term

$$\Lambda^{(D-4)L-6}\, \partial^2 t_8 \mathrm{tr}(F^4) \qquad L \geq 2$$

the double-trace term is subleading

$$\Lambda^{(D-4)L-8}\, \partial^4 t_8 (\mathrm{tr} F^2)^2 \qquad L \geq 3$$

Part IV

# The UV behaviour $\mathcal{N} = 8$ supergravity in various dimensions

# Constraints on $\mathcal{N} = 8$ supergravity amplitudes

Gravity has a dimensional coupling constant

$$[1/\kappa_{(D)}^2] = \text{mass}^{D-2}$$

An $L$-loop $n$-point gravity amplitude in $D$-dimensions has the dimension

$$[\mathfrak{M}_L^{(D)}] = \text{mass}^{(D-2)L+2}$$

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# Constraints on $\mathcal{N} = 8$ supergravity amplitudes

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# Superspace counting and UV behaviour

- Critical dimension for UV divergence is

$$D \geq D_c = 2 + \frac{6 + 2\beta_L}{L}$$

- Depending on the various implementations of supersymmetry

$$6 \leq 6 + 2\beta_L \leq 18$$

- With a first possible divergence in $D = 4$ at
  - $L \geq 3$ [Howe, Lindstrom, Stelle '81]
  - $L \geq 5$ [Howe, Stelle '06; Bossard, Howe, Stelle '09]
  - $L \geq 8$ [Kallosh '81]
  - $L \geq 9$: $\beta_L \leq 6$ [Green,Russo,Vanhove '06]
  - $L = \infty$: $\beta_L = L$ [Green,Russo,Vanhove '06]

# Supersymmetry in $\mathcal{N} = 8$ amplitudes

[Berkovits] pure spinor formalism for closed string amplitudes gives

- Zero-mode saturation in 4-graviton amplitudes

$$\mathfrak{M}_L \sim \int d^{16}\theta d^{16}\bar{\theta}\, \theta^{12-2L}\bar{\theta}^{12-2L}\, W^4\, I_L + \cdots \sim D^{2L} R^4\, I_L + \cdots$$

$$W_{\alpha\beta,a_1 a_2} = F_{\alpha\beta,a_1 a_2} + \cdots + \theta^{\gamma}_{a_1}\, \bar{\theta}^{\delta}_{a_2}\, R_{\alpha\gamma\beta\delta} + \cdots$$

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- No reduction of derivatives from massless closed string poles
  [Berkovits,Green,Russo,Vanhove]

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- The first full superspace integral is met at six loops $L = 6$

$$\mathfrak{M}_6 \quad \sim \quad \int d^{16}\theta d^{16}\bar\theta\, W^4 \sim D^{12} R^4 + \text{susy completion}$$

This is candidate counterterm for the first possible UV divergence at 9-loop in D=4 [ Green, Russo, Vanhove, ’06]

Confirmed by the $1 \leq L \leq 4$ $\mathcal{N} = 8$ supergravity amplitudes computations of [Bern, Carrasco, Johansson, Roiban]

# Non-renormalisation theorems

The $\beta_L = L$ rule implies that $[\mathfrak{M}_L^{(D)}] = \text{mass}^{(D-4)L-6} D^{2L} R^4$

- 1-loop non-renormalisation of $R^4$: $\beta_L \geq 2$ for $L \geq 2$

$$\text{UV divergence for } L = 1: \quad D \geq 8$$
$$\text{First UV divergence in 4D:} \quad L \geq 3 + \beta_L \geq 5 \text{ loops}$$

- 2-loop non-renormalisation of $D^4 R^4$: $\beta_L \geq 3$ for $L \geq 3$
- 3-loop non-renormalisation of $D^6 R^4$: $\beta_L \geq 4$ for $L \geq 4$
- After 6-loop supersymmetric protection runs out: $\beta_L = 6$ for $L \geq 6$

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$$\text{UV divergence for } L = 2: \qquad D \geq 7$$
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- 3-loop non-renormalisation of $D^6 R^4$: $\beta_L \geq 4$ for $L \geq 4$
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$$\begin{aligned} \text{UV divergence for } L = 3: &\qquad D \geq 6 \\ \text{First UV divergence in 4D:} &\qquad L \geq 3 + \beta_L \geq 7 \text{ loops} \end{aligned}$$

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$$[\mathfrak{M}_L^{(D)}] = \mathrm{mass}^{(D-2)L-18}\, D^{12} R^4$$

leading to a 9-loop divergence in $D = 4$ with for counter-term

$$\int d^{32}\theta\, (W^{ij}_{\alpha\beta})^4 = D^{12} R^4 + \textit{susy completion}$$

# Critical dimensions for UV divergences

$\mathcal{N}=8$ supergravity has critical dimension [Green, Russo, Vanhove]

$$[\mathfrak{M}_L^{(D)}] = \mathrm{mass}^{(D-4)L-6}\, D^{2L} R^4$$

$$D < D_c = 4 + \frac{6}{L}; \quad \text{for} \quad L \leq 6$$

- $\mathcal{N}=4$ SYM UV behaviour is dominated by the single trace term

$$[\mathfrak{A}_L] = \mathrm{mass}^{(D-4)L-6}\, D^2\, t_8 \mathrm{tr} F^4$$

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- The double-trace term in $\mathcal{N}=4$ SYM has critical dimension

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$$D < D_c = 4 + \frac{8}{L} \quad \text{for} \quad L \geq 3$$

[Berkovits, Green, Russo, Vanhove]

# Summary & Outlook

We have showed how maximally supersymmetry is realised in gauge theory amplitude in various dimensions

- In $\mathcal{N} = 4$ SYM the double-trace operator are more protected than the single trace operators
  - The pure spinor string analysis is in perfect agreement with the field theory expectations
- Both in $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA supersymmetric protection runs out of steam at some loop order
- Extension of these cancellations beyond 6 loops needs an other mechanism
  - New cancellations? Need a very good control of the tree amplitudes where some remarkable cancellations are already seen [Bjerrum-Bohr, Vanhove], [Bjerrum-Bohr, Damgaard, Vanhove]
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