$\mathcal{N}$=8 Supergravity at Four Loops

John Joseph M. Carrasco

Hidden Structures in Field Theory Amplitudes, 2009

Neils Bohr International Academy

0905.2326 [hep-th]

Bern, JJMC, Dixon, Johansson, Roiban

Calculation
&
Structures.

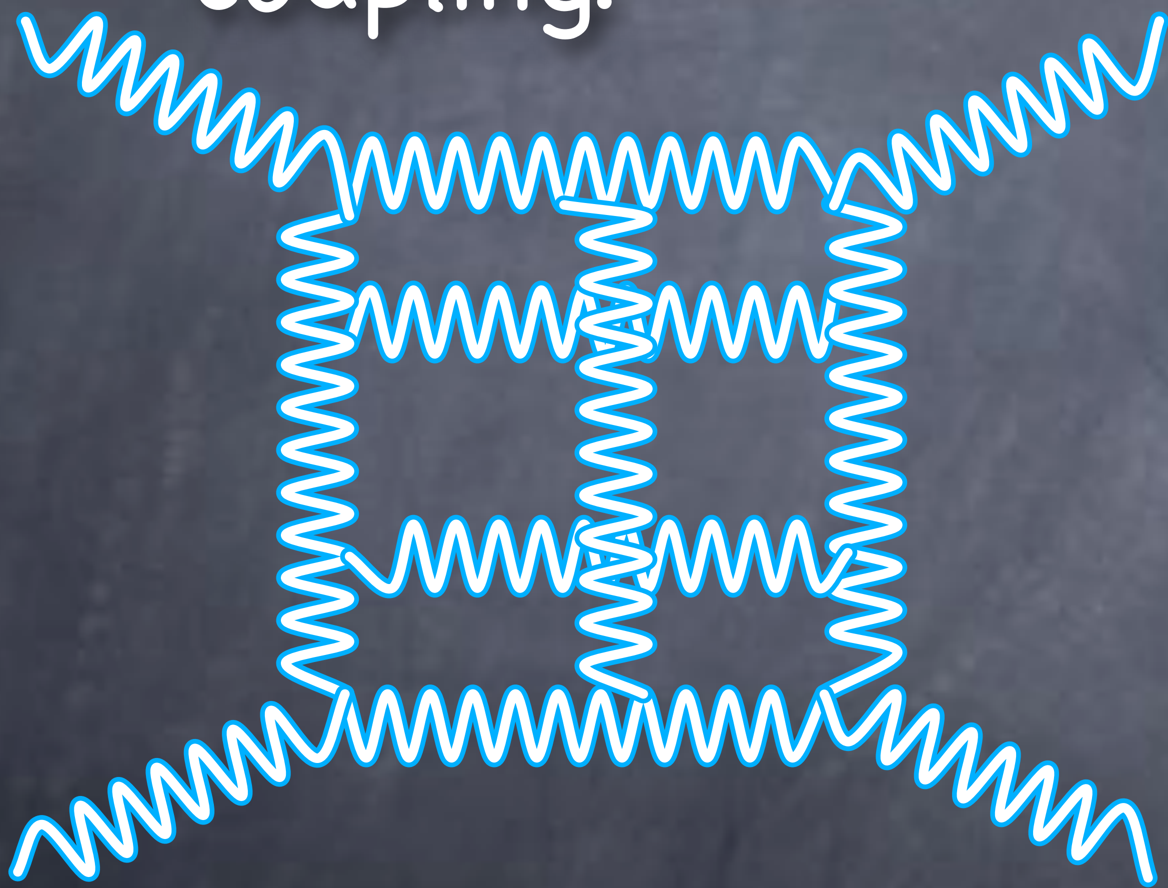
In this talk I'm going to present the calculation of the complete four-point four-loop N=8 supergravity amplitude and I will point out various structures that we discovered and/or exploited along the way.

Exciting Proposition:

Perturbatively finite QFT of gravity in 4D

Why surprising if possible:

Dimensionful coupling: $\kappa \sim m_{pl}^{-1}$ → non-renomalizable



No known **structure** to make up diff btw

$$\frac{(\kappa\ p^\mu p^\nu)\cdots}{\text{propagators}}$$ gravity

and

$$\frac{(g\ p^\mu)\cdots}{\text{propagators}}$$ gauge

Any responsible mechanism would fundamentally impact our understanding of gravity

Evidence of spectacular cancellations in $\mathcal{N}$=8!

D-dimensional calculations demonstrate certain cancellations to all loop orders, beyond known SUSY or string theory non-renormalization arguments. Bern, Dixon, Roiban (2007)

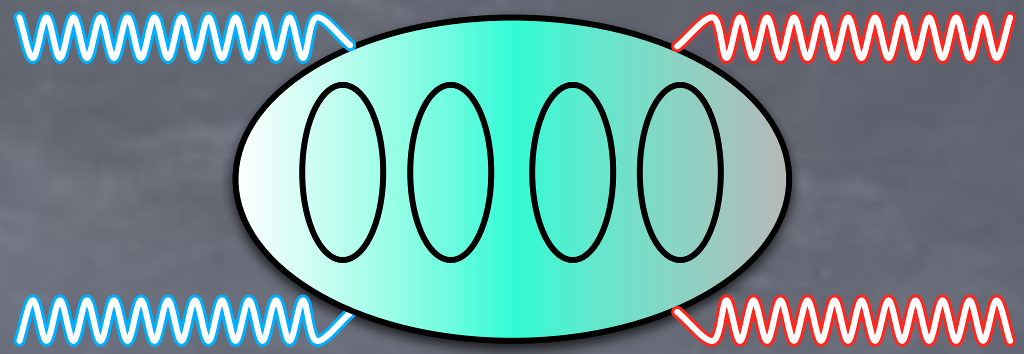
Dramatic 3-loop cancellations demonstrated. Bern, JJMC, Dixon, Johansson, Kosower, Roiban (2007,2008)

Evidence that responsible mechanism maybe somewhat generic to gravity theories. Bern, JJMC, Forde, Johansson, Ita (2007)

Suggestive hints that String dualities restrict form of effective action, possibly preventing divergences (issues of decoupling of towers of massive states cloud the situation) Chalmers [hep-th/0008162] (2000) ; Green, Vanhove, Russo (2007)

No proof of finiteness yet!

Why go after Four Loops?



1. To find structure responsible for cancellations, we want more data! (3-loops was only first chance to diverge from gauge-like powercounting.)

2. Direct challenge to a potential N = 6 superspace explanation suggested by Howe and Stelle. hep-th/0211279 (2003)

3. Bossard, Howe, Stelle predicted D = 5, L = 4 divergence from algebraic methods, avoiding superspace
0901.4661 [hep-th] (2009)

4. There is perhaps a fourth reason as well. Tools developed to probe higher loops can generalize to other more physical theories.

Bossard, Howe and Stelle 0901.4661 [hep-th]

Careful study considering new algebraic renormalization theorems exploiting cohomological methods as well as the full non-linear supersymmetry

"The algebraic formalism […] suggests that maximal supergravity is likely to diverge at four loops in $D = 5$ and at five loops in $D = 4$, unless other infinity suppression mechanisms not involving super-symmetry or gauge invariance are at work." Bossard, Howe & Stelle

Let's calculate!

What do we need?

Knowledge of Dark Lore

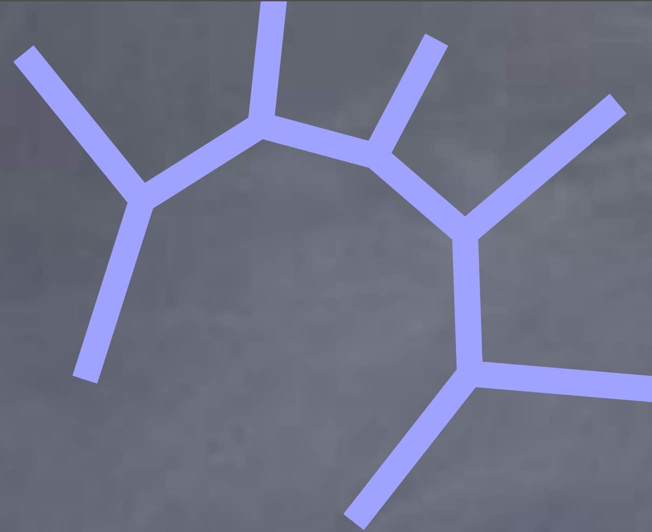
1. Generalized Unitarity Method — Bern, Dixon, Dunbar and Kosower (1994); Bern, Dixon and Kosower (1998, 2004, 2005); Bern, JJMC, Dixon, Johansson, Kosower, Roiban (2008)
   a. Method of maximal cuts — Bern, JJMC, Johanson and Kosower (2007); Britto, Cachazo, Feng; Buchbinder, Cachazo (2004); Cachazo and Skinner; Cachazo, Spradlin, Volovich (2008)
   b. Supersums — Bianchi, Freedman, Elvang, Kiermaier; Arkanki-Hamed, Cachazo, Kaplan; Brandhuber, Spence, Travaglini; Drummond, Korchemsky, Henn, Sokatchev; Bern, JJMC, Ita, Johansson, Roiban; Hall
2. KLT + "n-factors"

Kawai, Lewellen, and Tye (1986); Bern, Dixon, Dunbar, Perelstein, and Rozowsky (1998); Bern, JJMC, Johansson (2008)

See talks by: **Bern, Kosower, Roiban, Johansson, ...**

All this knowledge has been discussed in previous talks.

Pre-reqs.



Gross motor skills

1. Draw tree graphs (no cycles)
2. Add loops to diagrams
3. Identify matching graphs

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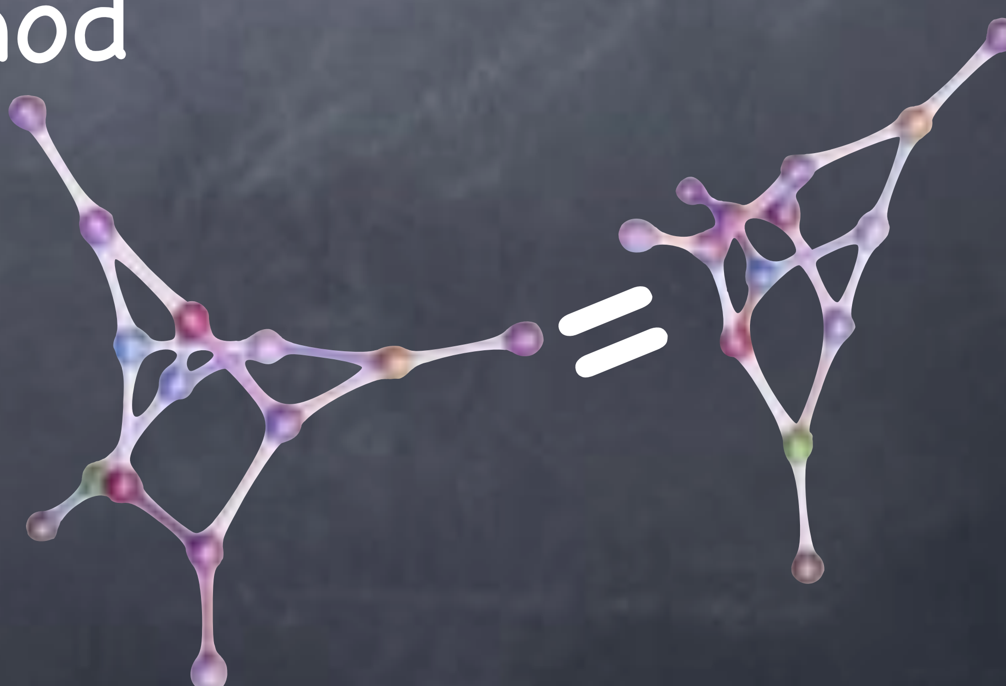

I’m going to take this opportunity to talk about some of the little-mentioned gross-motor skills required to use the Unitarity method, and KLT in the ways we’ve been pioneering for higher-loop calculation.

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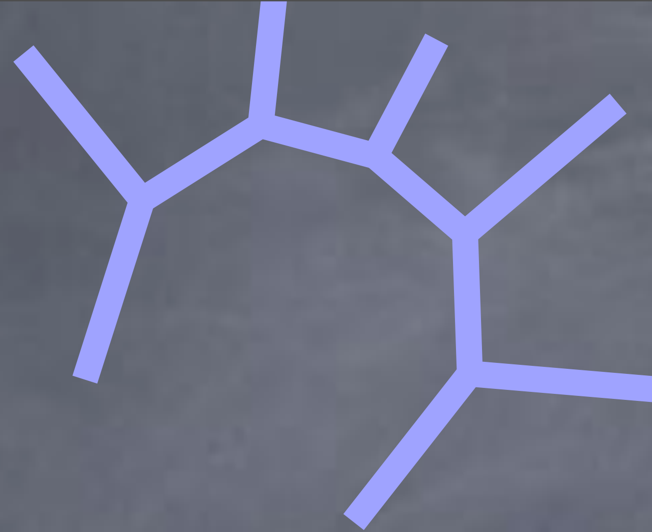
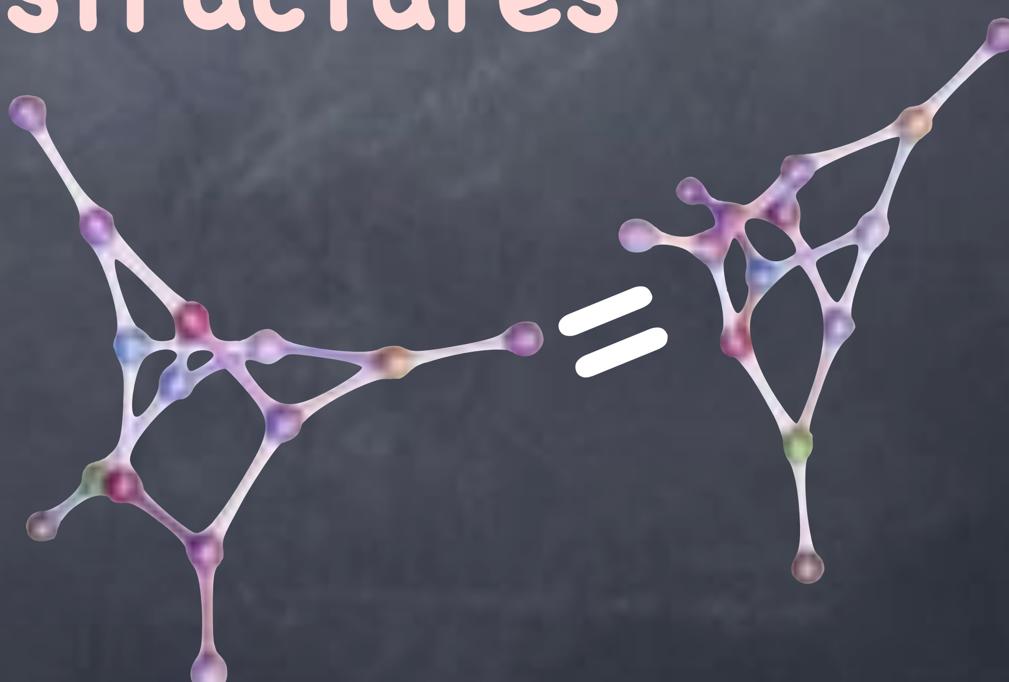


Gross motor skills

1. Draw tree graphs (no cycles)
2. Add loops to diagrams
3. Identify matching graphs

For this talk I'll discuss these skills. Don't worry, we'll find structures here too.



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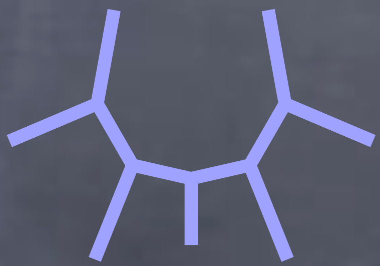


Why draw trees? To get value from cuts!

$$\sum_{\text{sYM states}} A^5_{tree}(k_1, k_2, l_3, l_2, l_1) \times A^5_{tree}(-l_1, -l_2, -l_3, k_3, k_4) =$$

$$\delta^{(8)}(\lambda_1^\alpha \eta_1^a + \lambda_2^\alpha \eta_2^a + \lambda_3^\alpha \eta_3^a + \lambda_4^\alpha \eta_4^a) \times \frac{1}{\langle 12\rangle\langle 2l_3\rangle\langle l_3 l_2\rangle\langle l_2 l_1\rangle\langle l_1 1\rangle}\Bigg[$$

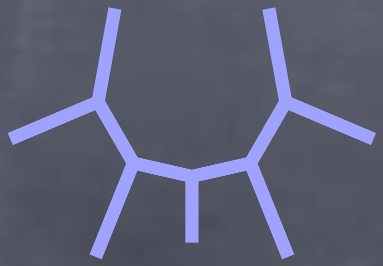
$$\frac{1}{\langle l_2 l_3\rangle\langle l_3 3\rangle\langle 3P_1^\flat\rangle\langle P_1^\flat l_2\rangle}\frac{1}{P_1^2}\frac{1}{\langle 4 l_1\rangle\langle l_1 P_1^\flat\rangle\langle P_1^\flat 4\rangle}\Big(\langle l_1 P_1^\flat\rangle\langle l_2 l_3\rangle\Big)^4$$

$$+\frac{1}{\langle P_2^\flat l_3\rangle\langle l_3 3\rangle\langle 3P_2^\flat\rangle}\frac{1}{P_2^2}\frac{1}{\langle l_2 P_2^\flat\rangle\langle P_2^\flat 4\rangle\langle 4 l_1\rangle\langle l_1 l_2\rangle}\Big(\langle l_3 P_2^\flat\rangle\langle l_1 l_2\rangle\Big)^4$$

$$+\frac{1}{\langle l_3 3\rangle\langle 34\rangle\langle 4P_3^\flat\rangle\langle P_3^\flat l_3\rangle}\frac{1}{P_3^2}\frac{1}{\langle P_3^\flat l_1\rangle\langle l_1 l_2\rangle\langle l_2 P_3^\flat\rangle}\Big(\langle l_3 P_3^\flat\rangle\langle l_1 l_2\rangle\Big)^4$$

$$+\frac{1}{\langle l_2 l_3\rangle\langle l_3 P_4^\flat\rangle\langle P_4^\flat l_2\rangle}\frac{1}{P_4^2}\frac{1}{\langle l_1 P_4^\flat\rangle\langle P_4^\flat 3\rangle\langle 34\rangle\langle 4 l_1\rangle}\Big(\langle l_1 P_4^\flat\rangle\langle l_2 l_3\rangle\Big)^4\Bigg]$$

Supersums on cuts are great. But individual cuts don't tell you what the full amplitude is. Need to be able to organize results into diagrams.



Drawing trees

$$\delta^{(8)}(\lambda_1^\alpha\eta_1^a+\lambda_2^\alpha\eta_2^a+\lambda_3^\alpha\eta_3^a+\lambda_4^\alpha\eta_4^a)\times\frac{1}{\langle 12\rangle\langle 2l_3\rangle\langle l_3l_2\rangle\langle l_2l_1\rangle\langle l_11\rangle}\Bigg[\frac{1}{\langle l_2l_3\rangle\langle l_33\rangle\langle 3P_1^\flat\rangle\langle P_1^\flat l_2\rangle}\frac{1}{P_1^2}\frac{1}{\langle 4l_1\rangle\langle l_1P_1^\flat\rangle\langle P_1^\flat 4\rangle}\Big(\langle l_1P_1^\flat\rangle\langle l_2l_3\rangle\Big)^4$$
$$+\frac{1}{\langle P_2^\flat l_3\rangle\langle l_33\rangle\langle 3P_2^\flat\rangle}\frac{1}{P_2^2}\frac{1}{\langle l_2P_2^\flat\rangle\langle P_2^\flat 4\rangle\langle 4l_1\rangle\langle l_1l_2\rangle}\Big(\langle l_3P_2^\flat\rangle\langle l_1l_2\rangle\Big)^4$$
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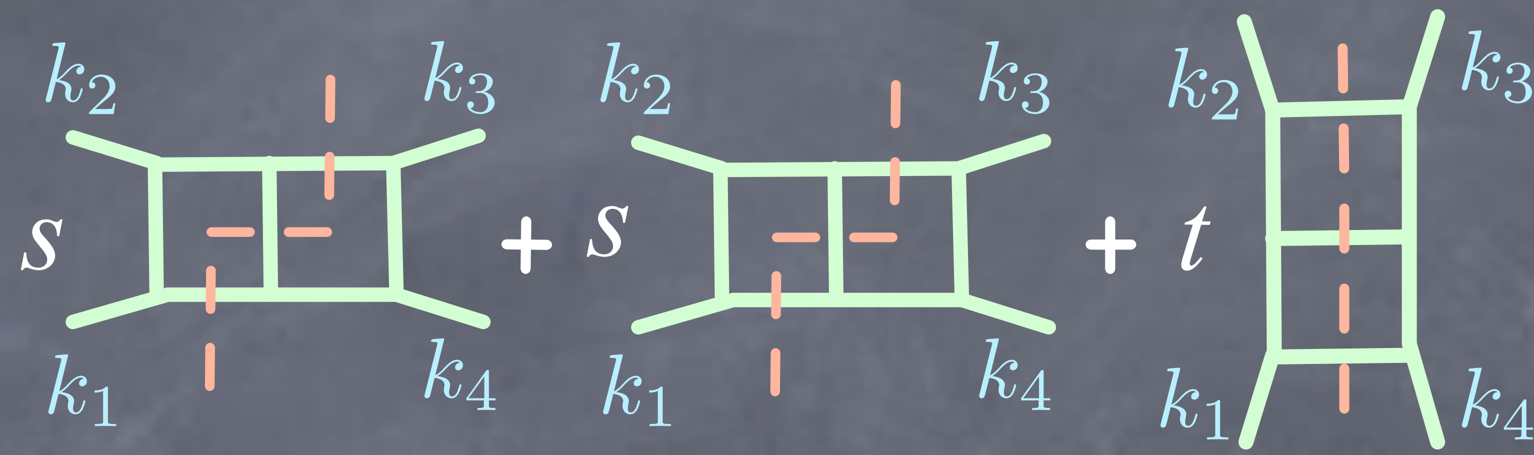
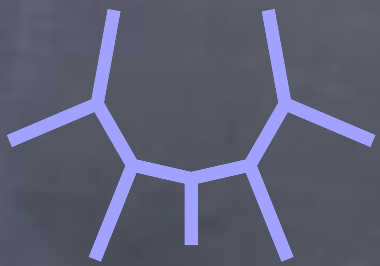
=



$$= \sum_{\text{sYM states}} A^5_{tree}(k_1,k_2,l_3,l_2,l_1)\times A^5_{tree}(-l_1,-l_2,-l_3,k_3,k_4)$$

$$s\equiv(k_1+k_2)^2$$
$$t\equiv(k_1+k_4)^2$$

In order to organize the results into diagrams, you need to be able to draw all tree diagrams that contribute, and glue them together.



Drawing trees

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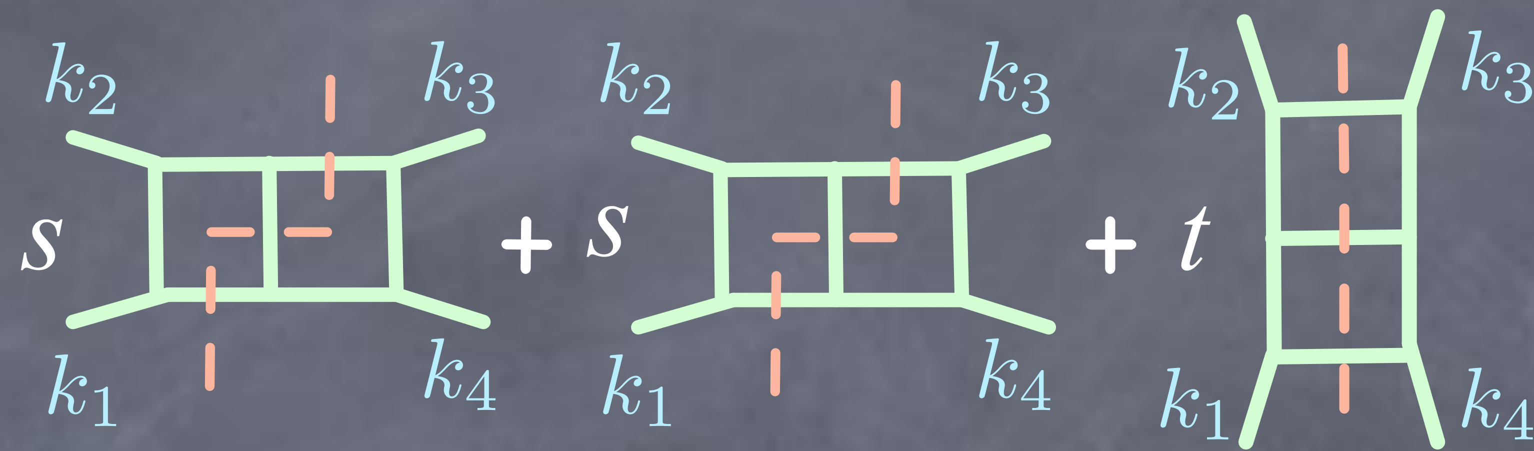

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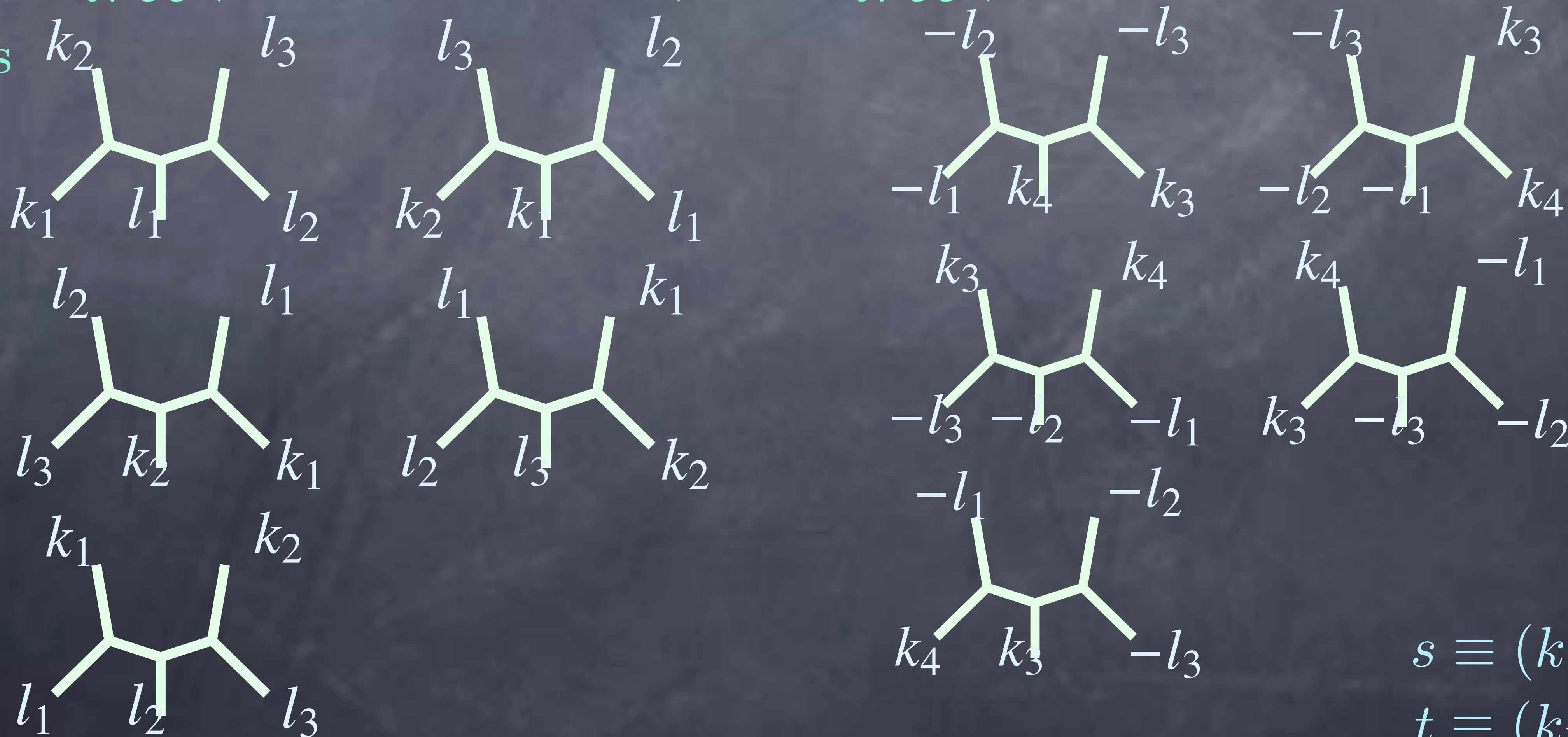

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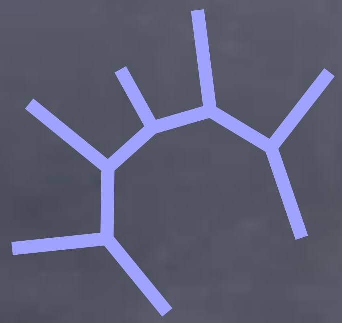


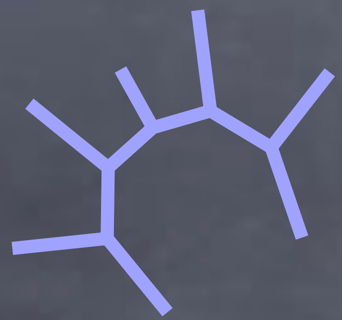
COLOR ORDERED
Drawing ^ Trees 101

Easy right?

- 9 point tree (good for a particular four-loop cut) has 429 color-ordered diagrams.

- If you’re good with your book-keeping you only need to do each n-point once

- By hand or by computer you’ll want to minimize isomorphism operations.



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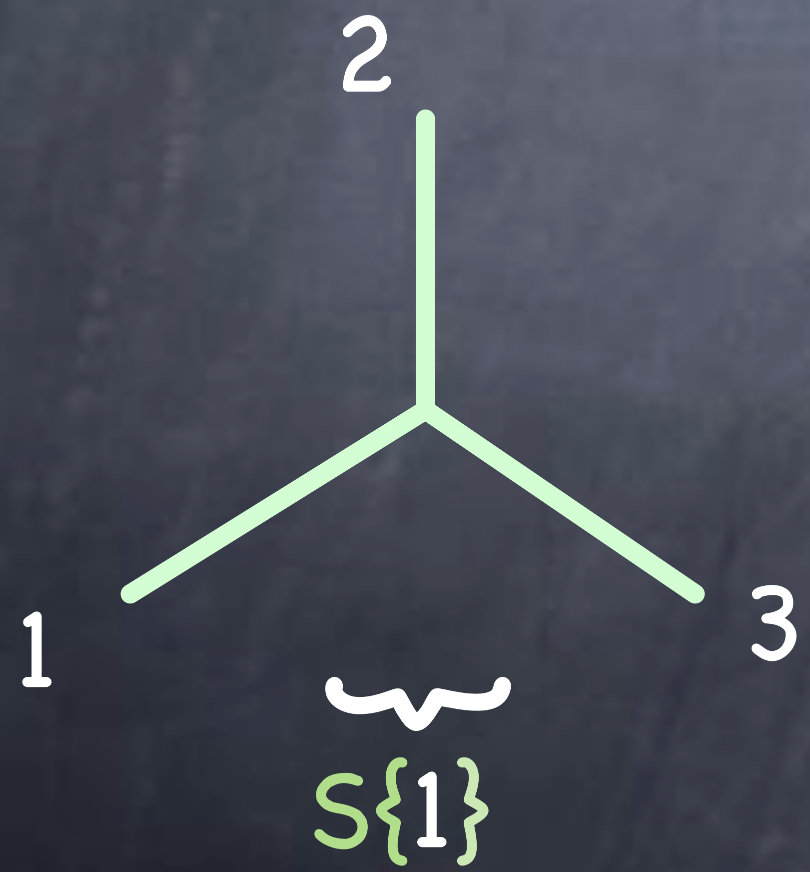
Drawing ^ Trees 101

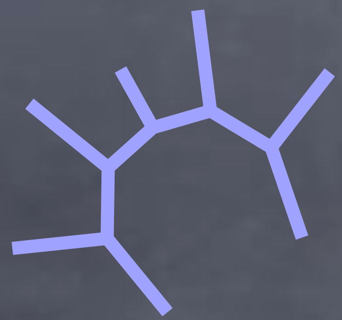
Simple algorithm to go from set of **n** leg tree graphs to **n+1**:

For all graphs **g** in S{n},

for each edge **e** between leg n and 1,

create a graph in S{n+1} with leg (n+1) connected to edge **e**





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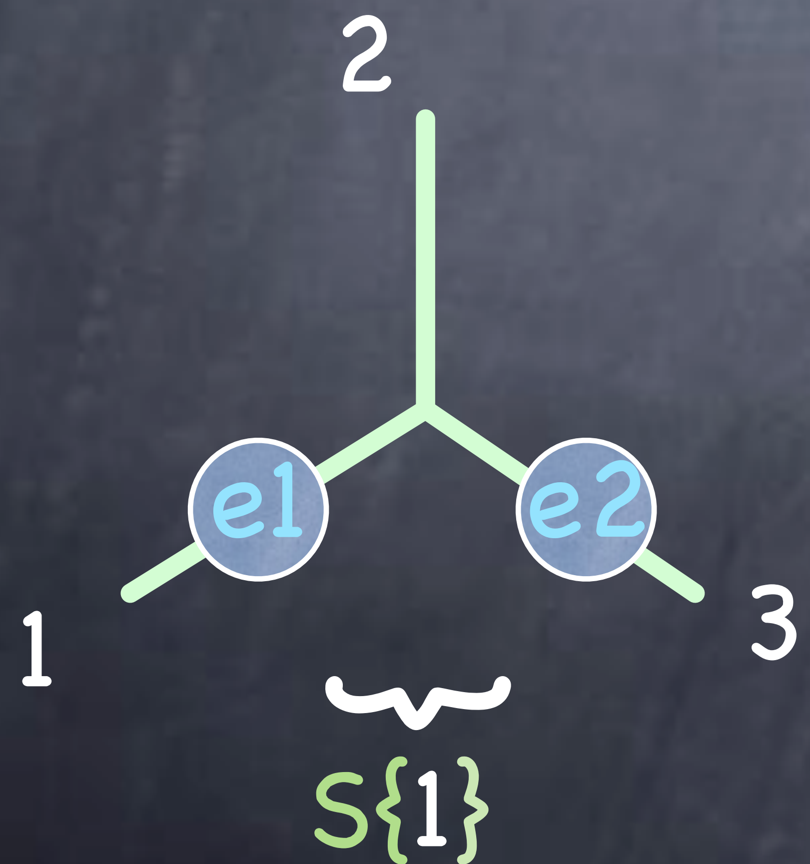

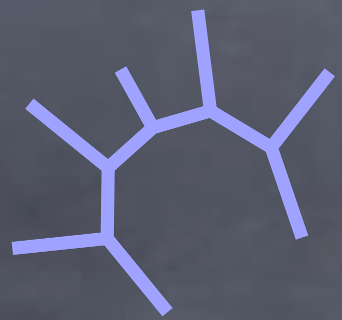

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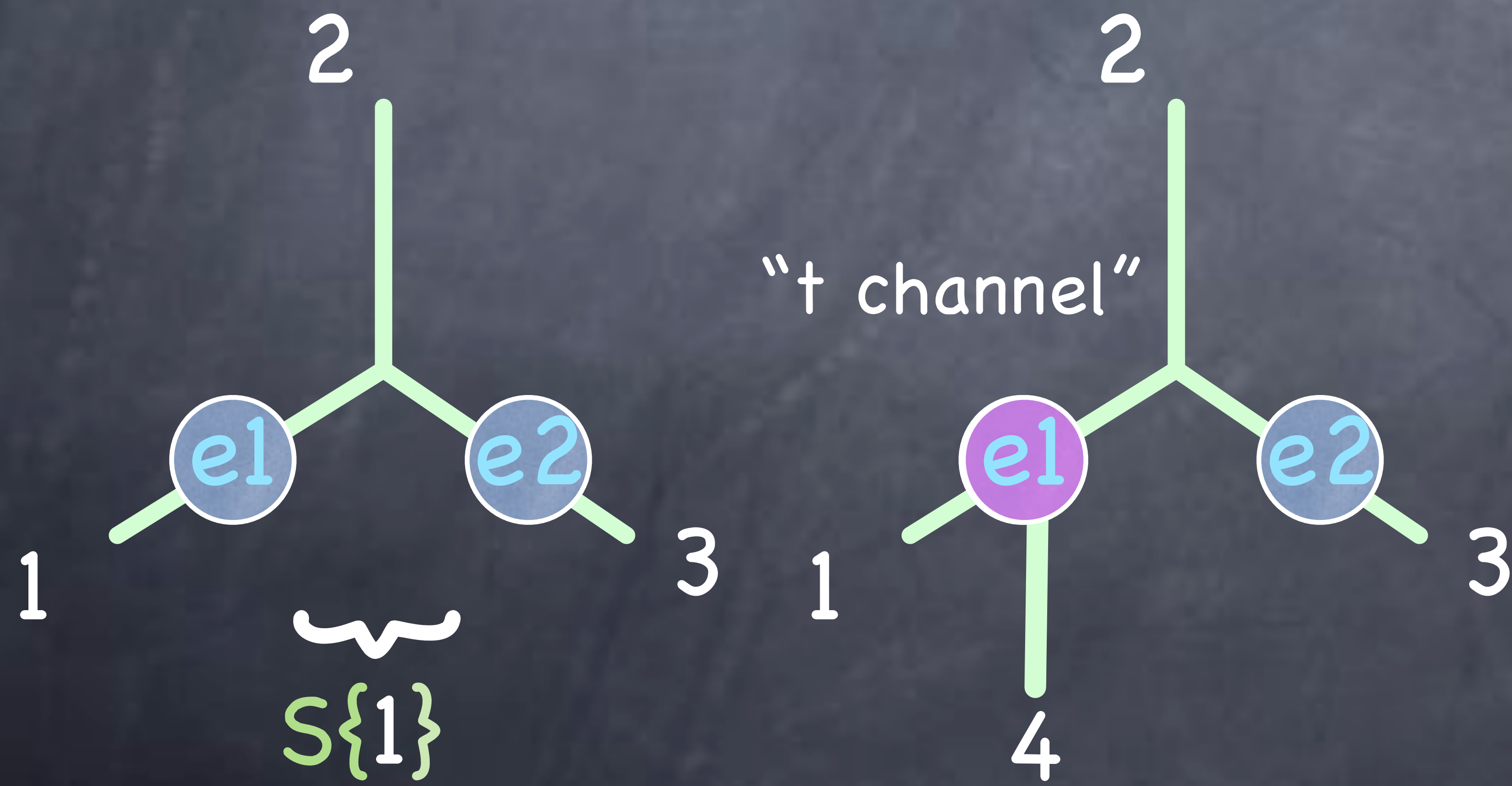

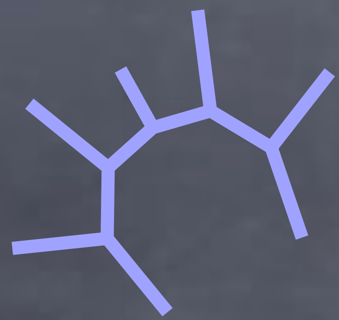

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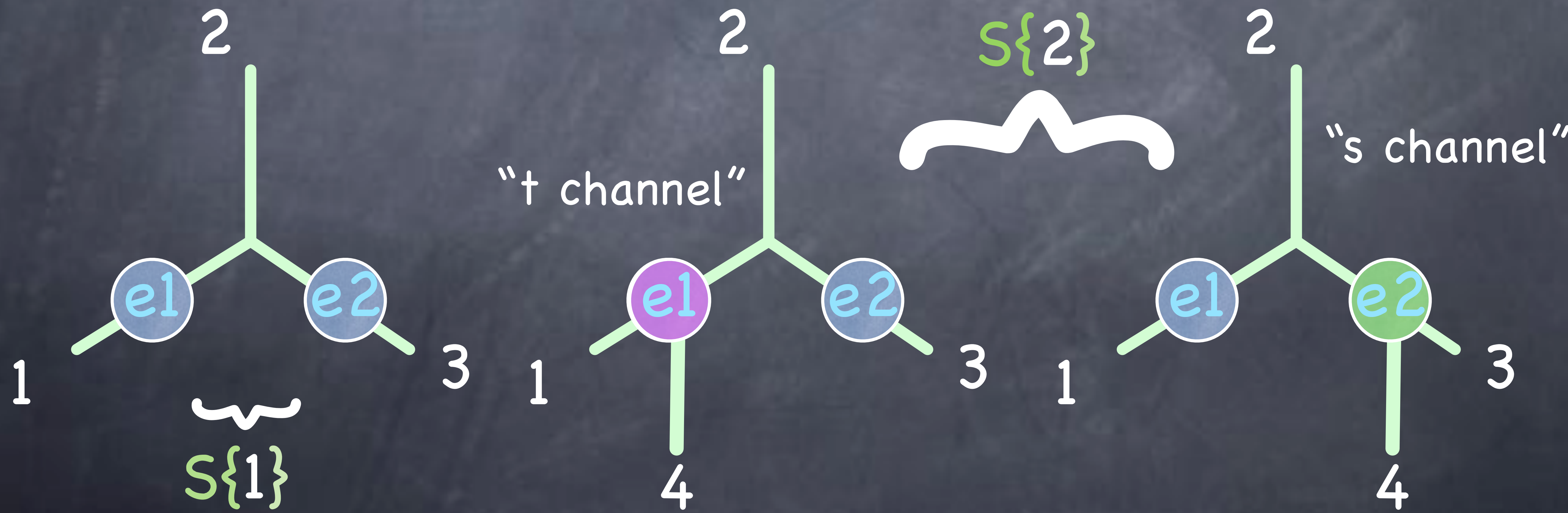
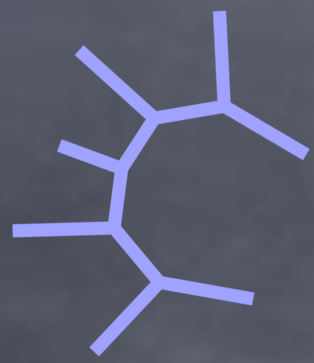
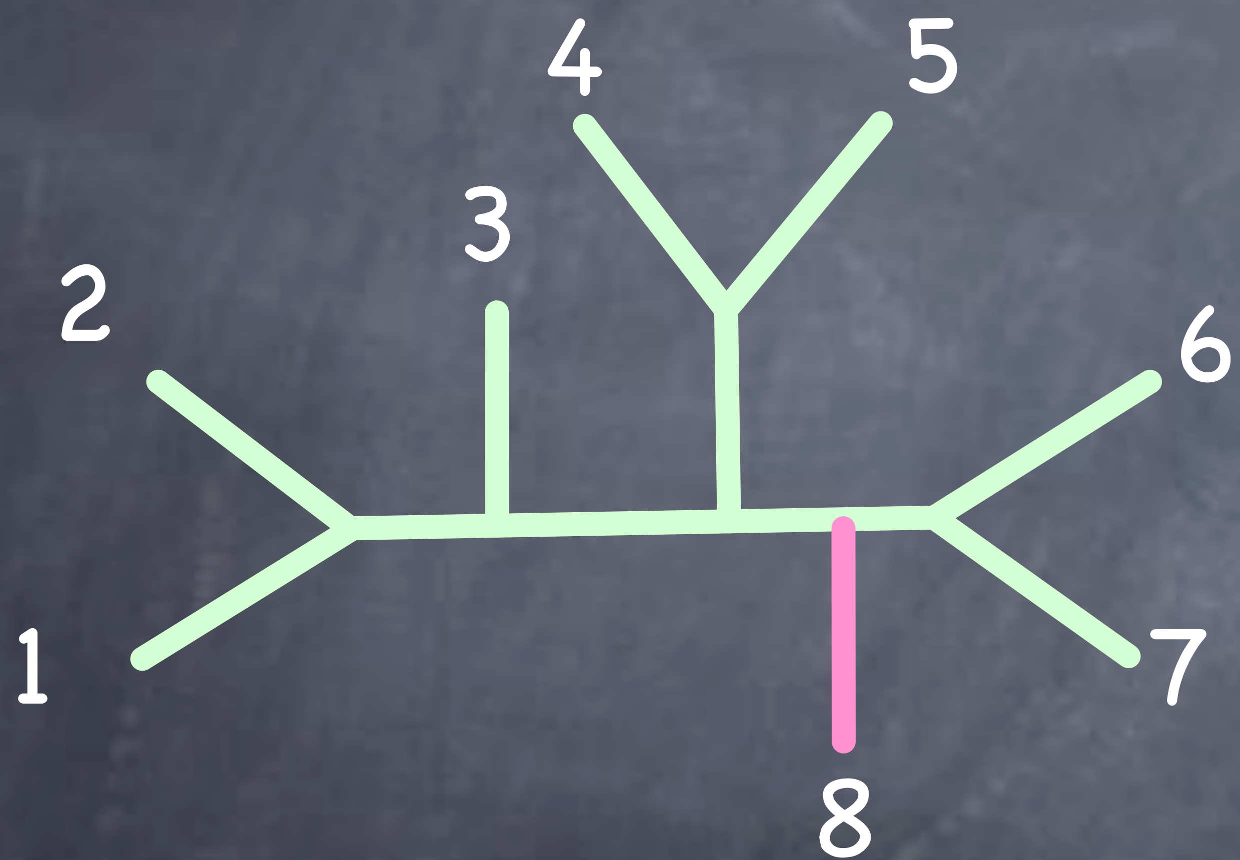


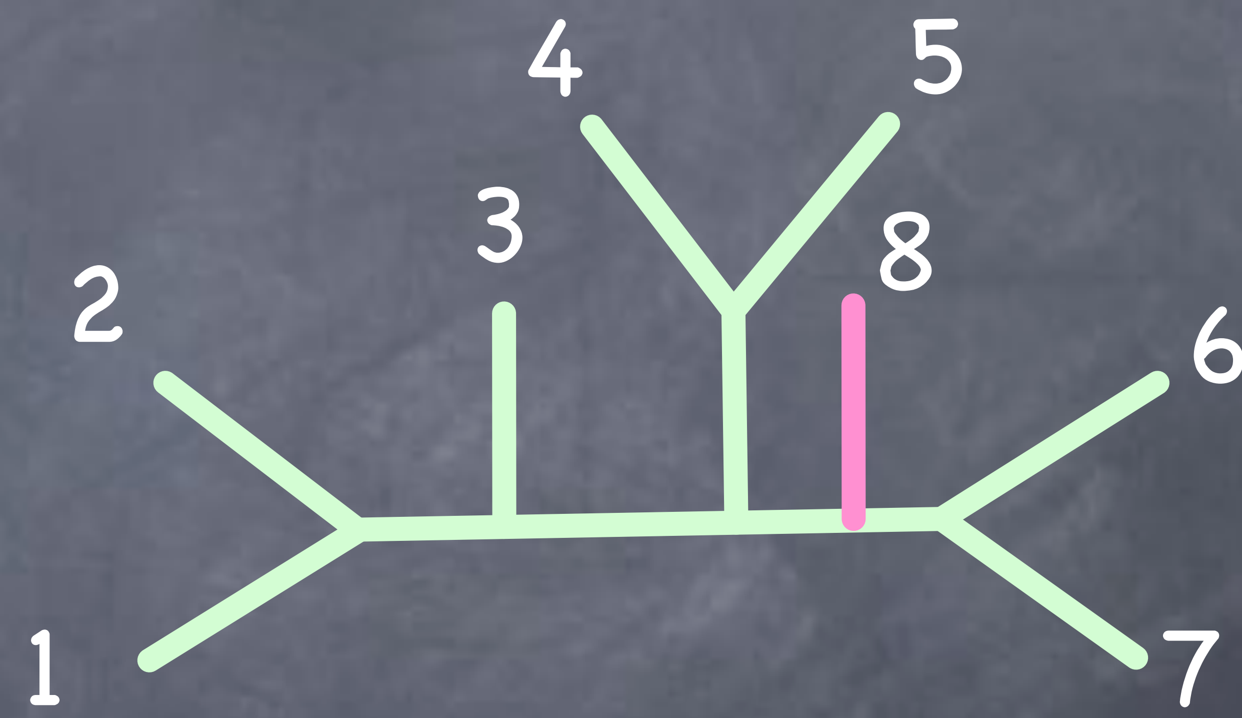


COLOR ORDERED
Drawing ^ Trees 101

Need to pay attention to orientation:



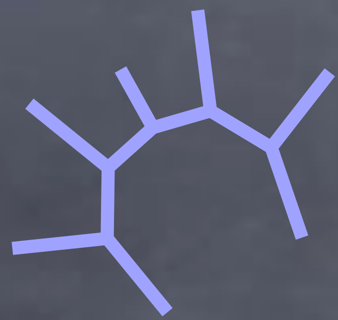
OK



Not OK

(you want representations that can encode the difference)

NON-COLOR ORDERED



Drawing ^ Trees 101

Even simpler algorithm to go from set of **n** leg non-color ordered tree graphs to **n+1**: **(No orientation to worry about.)**

For all graphs **g** in **S{n}**, for every edge **e**,
create a graph in **S{n+1}** with leg (n+1) connected to edge **e**.

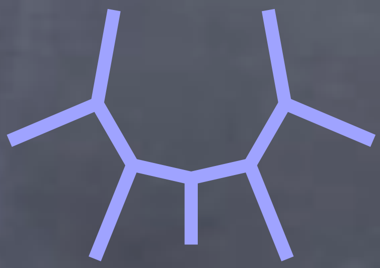
n leg cubic graph has

n ext + (n-3) internal edges = 2n-3 edges

|S{n+1}| = (2n-3) |S{n}| = (2n-3)((2n-5) |S{n-1}|) =

(2n-3) (2n-5) (2n-7) ... (3) = (2n-3)!!

|S{n}|=(2(n-1)-3)!!=(2n-5)!!



COLOR ORDERED Structure

Number of distinct color-ordered cubic n-trees:

$$C_{n-2} = \frac{2^{(n-2)}\,(2\,n-5)!!}{(n-1)\,!}$$

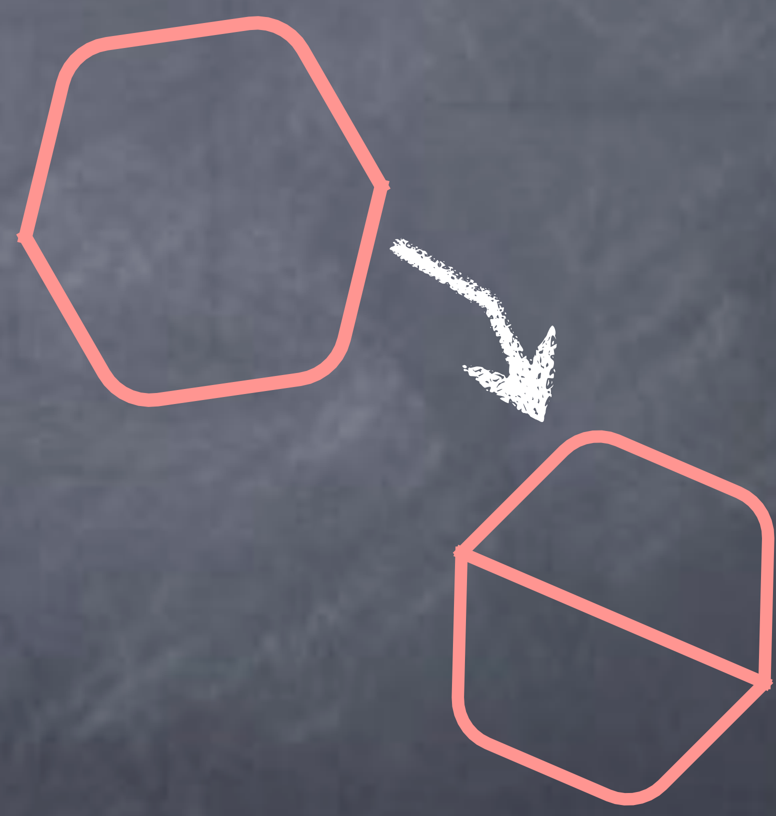
Number of ways of cutting a **n** convex polygon into **n-2** triangles with (non-intersecting) straight lines.

Euler's Polygon Division Problem

(dual-space rep of 6-trees)

How do we know what cuts span the amplitude?

1) Draw all (trivalent) vacuum diagrams
2) Dress them with external legs in all (distinct) ways



Gross motor skill #2:
Add loops to graphs

Finding all trivalent vacuum diagrams

Start with:

Choose all unique pairs of edges (including {self, self})

Edge pair:

{self,self}

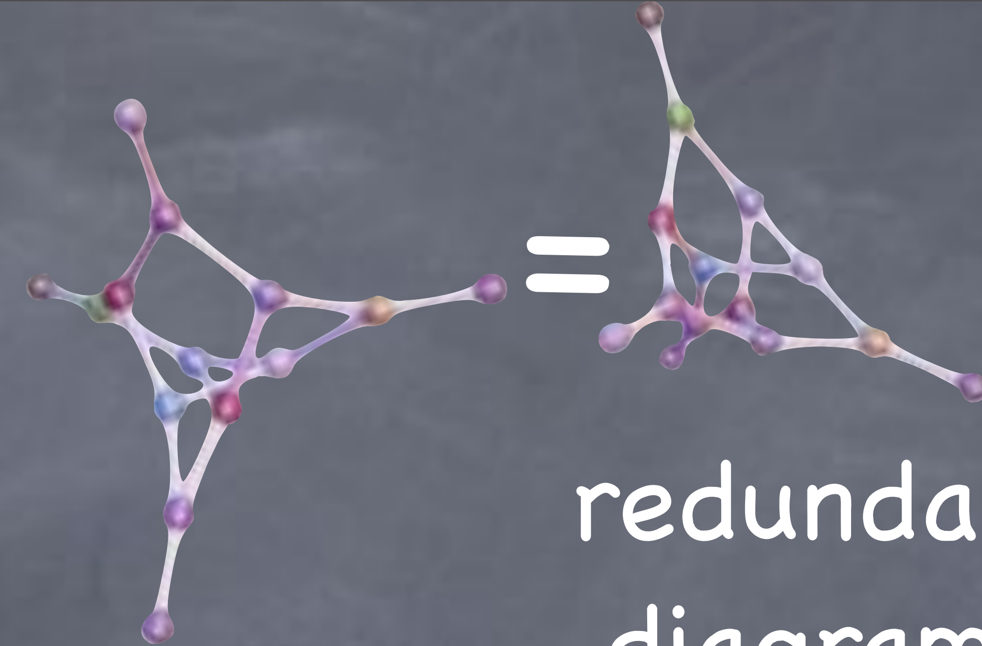
distinct: {A,B}

A

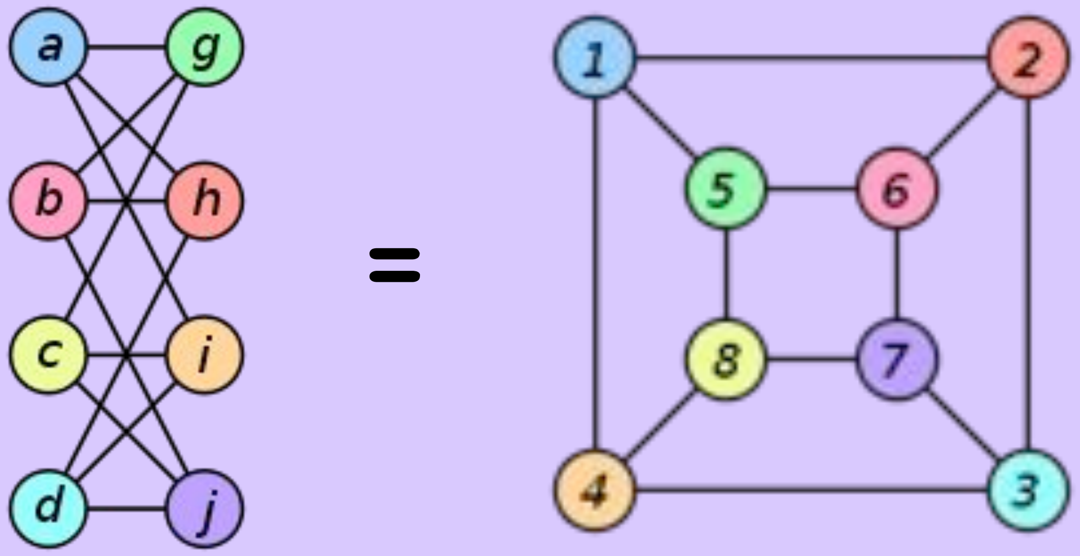
B

Discard: redundant diagrams, 1 particle reducible

Gross motor skill #3:
Identify matching graphs



redundant diagrams



via

$f(a) = 1$
$f(b) = 6$
$f(c) = 8$
$f(d) = 3$
$f(g) = 5$
$f(h) = 2$
$f(i) = 4$
$f(j) = 7$

taken from wikipedia

automorphism special case useful for symmetrization and determining symmetry factors

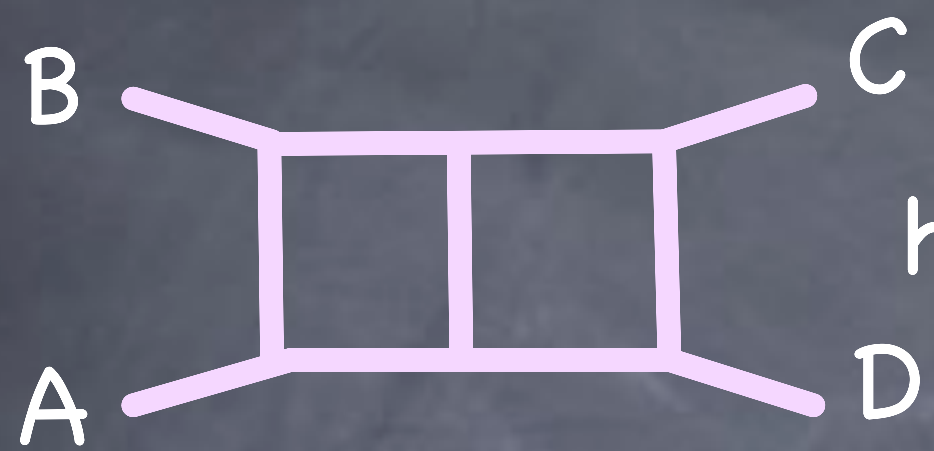
Fascinating and extensive literature:
graph isomorphism problem

Many implementations sufficiently speedy for graphs this small. c.f. Mathematica's Combinitorica, or Brendon D. McKay's Nauty

(http://cs.anu.edu.au/~bdm/nauty/)

Certainly possible to just build your own for these purposes: 3 loop 4-particle cut was done with home-rolled isomorphism prior to consulting literature. Turns out a lot of very clever people have thought about this and come up with some appreciable optimizations.

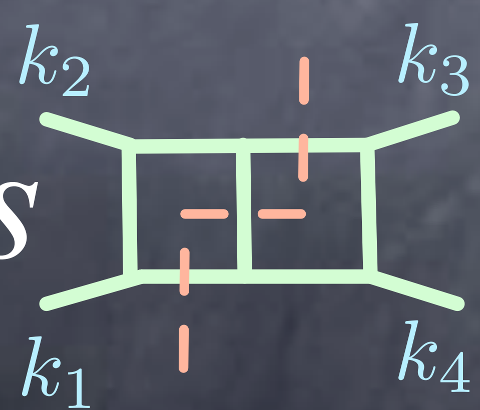
Given knowledge of an integral, e.g.

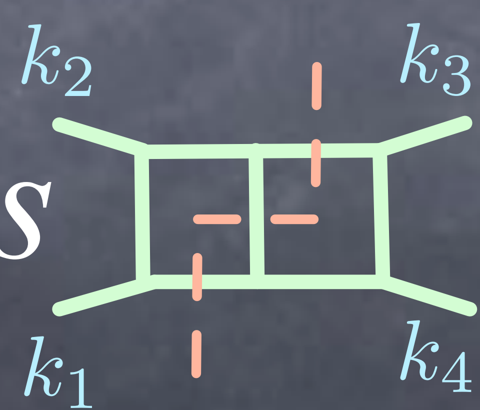


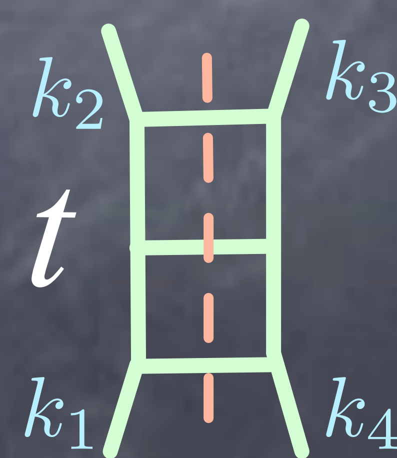
has numerator $(A+B)^2$

graph isomorphism necessary to dress cuts, e.g.

$$\sum_{\text{sYM states}} A^5_{tree}(k_1,k_2,l_3,l_2,l_1)\times A^5_{tree}(-l_1,-l_2,-l_3,k_3,k_4)$$

$= \; s$ 

$+ \; s$ 

$+ \; t$ 

How KLT is used in practice!

$$s \equiv (k_1+k_2)^2$$
$$t \equiv (k_1+k_4)^2$$

How KLT is used in practice!

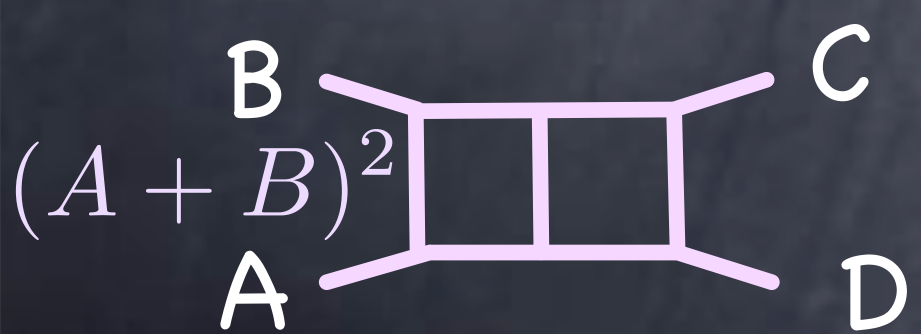
$$\sum_{N=8 \text{ states}} M_5^{\text{tree}}(1,2,\ell_3,\ell_2,\ell_1) M_5^{\text{tree}}(3,4,-\ell_1,-\ell_2,-\ell_3)$$

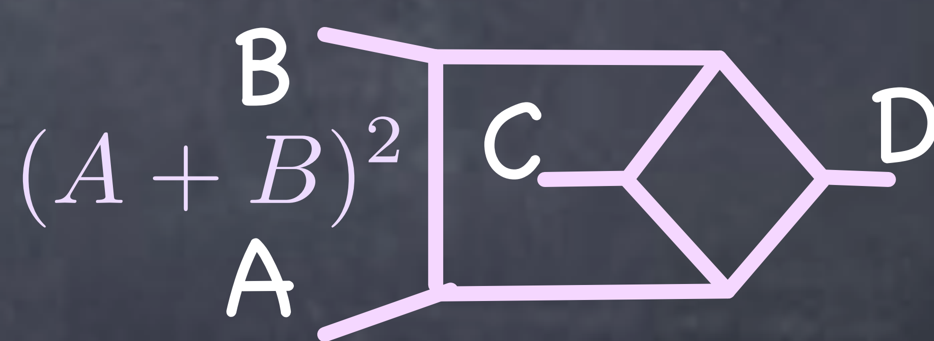
$$= -(\ell_1+k_1)^2(\ell_3+k_2)^2(\ell_3-k_3)^2(\ell_1-k_4)^2$$

$$\times \left[\sum_{N=4 \text{ states}} A_5^{\text{tree}}(\ell_1,1,2,\ell_3,\ell_2) A_5^{\text{tree}}(-\ell_3,3,4,-\ell_1,-\ell_2) \right]$$

$$\times \left[\sum_{N=4 \text{ states}} A_5^{\text{tree}}(1,\ell_1,\ell_3,2,\ell_2) A_5^{\text{tree}}(3,-\ell_3,-\ell_1,4,-\ell_2) \right]$$

$$+\{1 \leftrightarrow 2\} + \{3 \leftrightarrow 4\} + \{1 \leftrightarrow 2,\, 3 \leftrightarrow 4\}$$



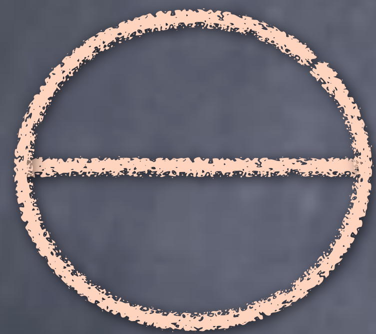


Because we can: (a) Draw trees, and (b) Match graphs. Knowing YM amplitudes, we can find their contributions to various cuts. Rather then having to perform the susysum time and time again, we just draw pictures, match graphs, and dress appropriately.

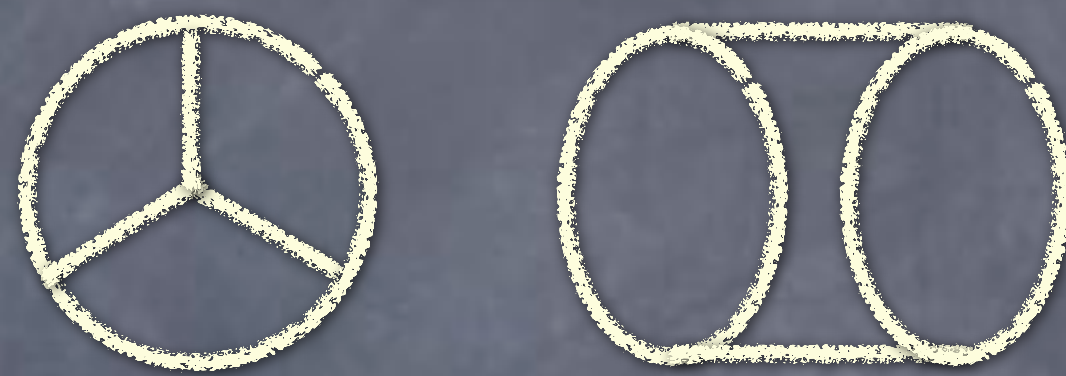
Of course we've now got all sorts of algebra to do on the gravity side to assign to the appropriate gravity graphs on the cut, but here too, since we can draw trees and match graphs, we haven't had to do any expensive gravity SUSY-summing.

With all tools in place we can find all contributing vacuums

2 loops



3 loops



4 loops

Four-Loop Amplitude Construction

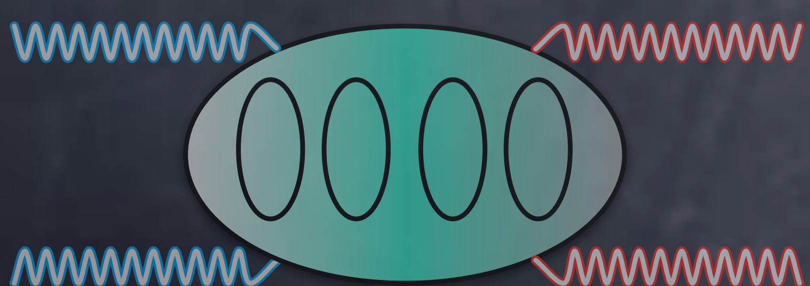
4-loop trivalent
1PI vacuum graphs

Attach 4 external legs.

Remove all diagrams with 2, 3-point sub-graphs.

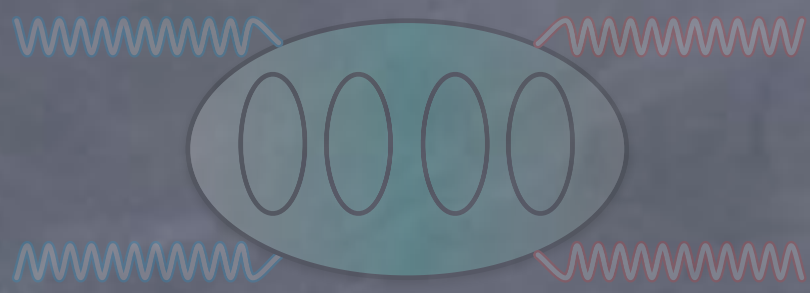
left with 50 diagram topologies or integrals

$$M_4^{\text{4-loop}} = \left(\frac{\kappa}{2}\right)^{10} stu M_4^{\text{tree}} \sum_{S_4} \sum_{i=1}^{50} c_i I_i$$



ext. leg perms

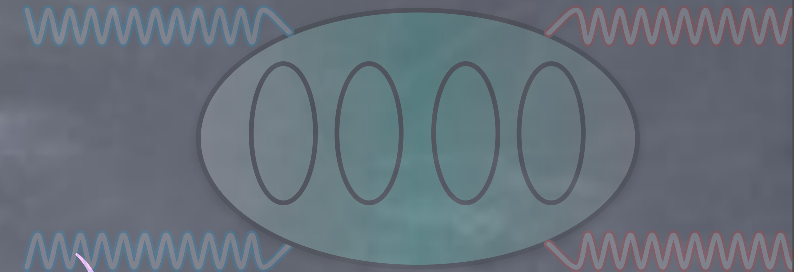
symmetry factor



Want to see the integrals?

(follow download instructions at the end of the talk!)

Four-Loop Cuts



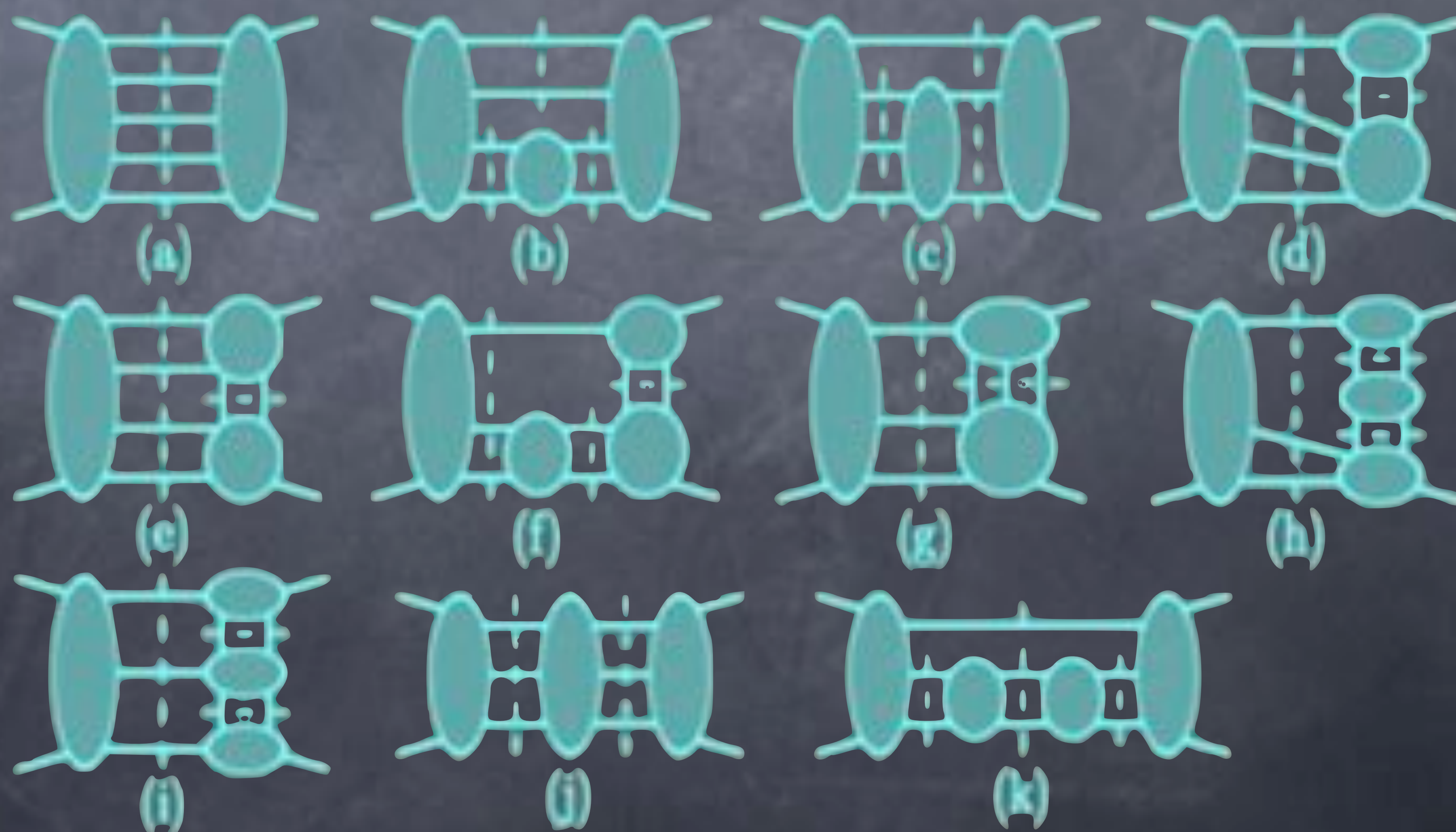
$$I_i = \int \left[\prod_{p=1}^{4} \frac{d^D l_{n_p}}{(2\pi)^D} \right] \frac{N_i(l_j, k_j)}{l_1 l_2 \ldots l_{13}}$$

Numerators determined from 2906 maximal and near maximal cuts

YM diags thru KLT used as truth.

See Henrik's talk for N=4 integrals

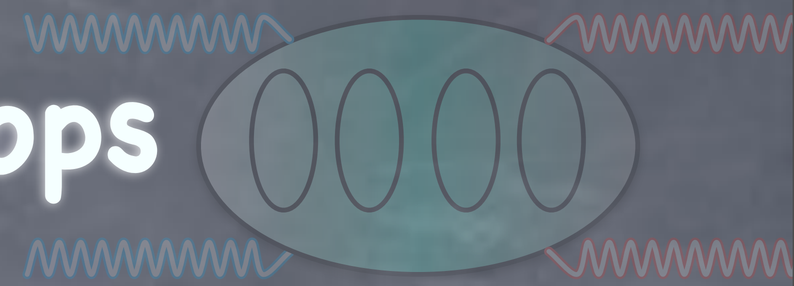
Completeness of ansatz verified on 26 generalized cuts



Following the method of maximal cuts (c.f. 0705.1864, 0808.4112 [hep-th]):
 * we first fix those coefficients of the N_i that contribute when the number of cut propagators is maximal (13)
 * we then consider cuts with 12 cut lines, fixing the coefficients assoc. w/ single inverse propagators l_n^2 (contact terms).
 * We continue this procedure down to nine cut lines, considering, in total, 2906 distinct cuts.

At this point, the resulting expression is complete. Can verify with only 26 cuts, sufficient to completely determine any four-loop four-point amplitude in any massless theory. The 11 cuts that cannot be straightforwardly verified using lower-loop four-point amplitudes in two-particle cuts are shown above.

UV Divergence at Four Loops



$$I_i = \int \left[\prod_{p=1}^{4} \frac{d^D l_{n_p}}{(2\pi)^D} \right] \frac{N_i(l_j, k_j)}{l_1 l_2 \ldots l_{13}}$$

Leading numerators $N_i \sim O(k^4 l^8)$

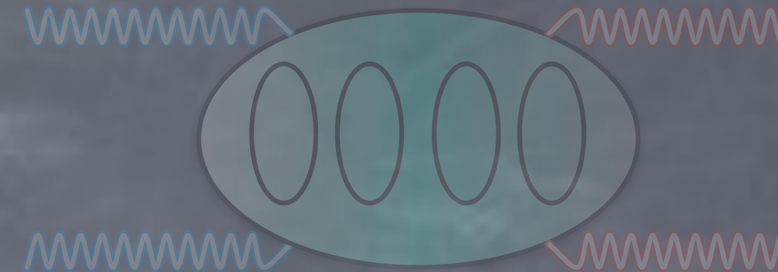
would have D = 4.5 divergence

Represented by integrals which cancel in the full amplitude

Sub-leading divergence: $O(k^5 l^7)$

trivially vanishes under integration by Lorentz invariance

UV Divergence at Four Loops



$N_i \sim O(k^6 l^6)$ corresponding to D = 5 div.

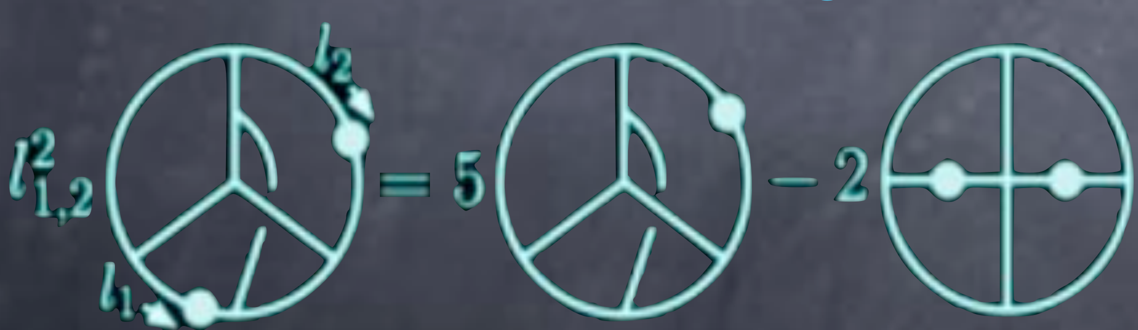
Expand the integrands about small external momenta:

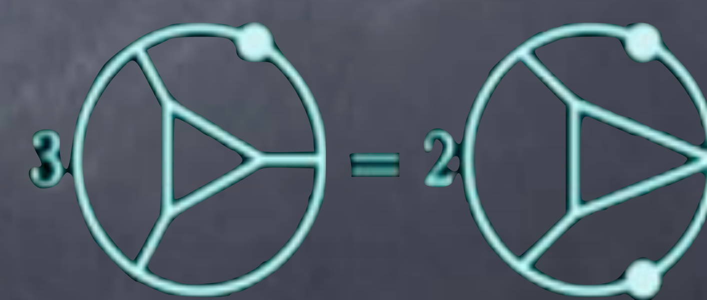
$$N_i^{(6)} + N_i^{(7)} \frac{K_n \cdot l_j}{l_j^2} + N_i^{(8)} \left(\frac{K_n^2}{l_j^2} + \frac{K_n \cdot l_j \; K_q \cdot l_p}{l_j^2 l_p^2} \right)$$

(K_i annotates sums over external momenta)

Marcus & Sagnotti UV extraction method

cancels after using D = 5 integral identities like:

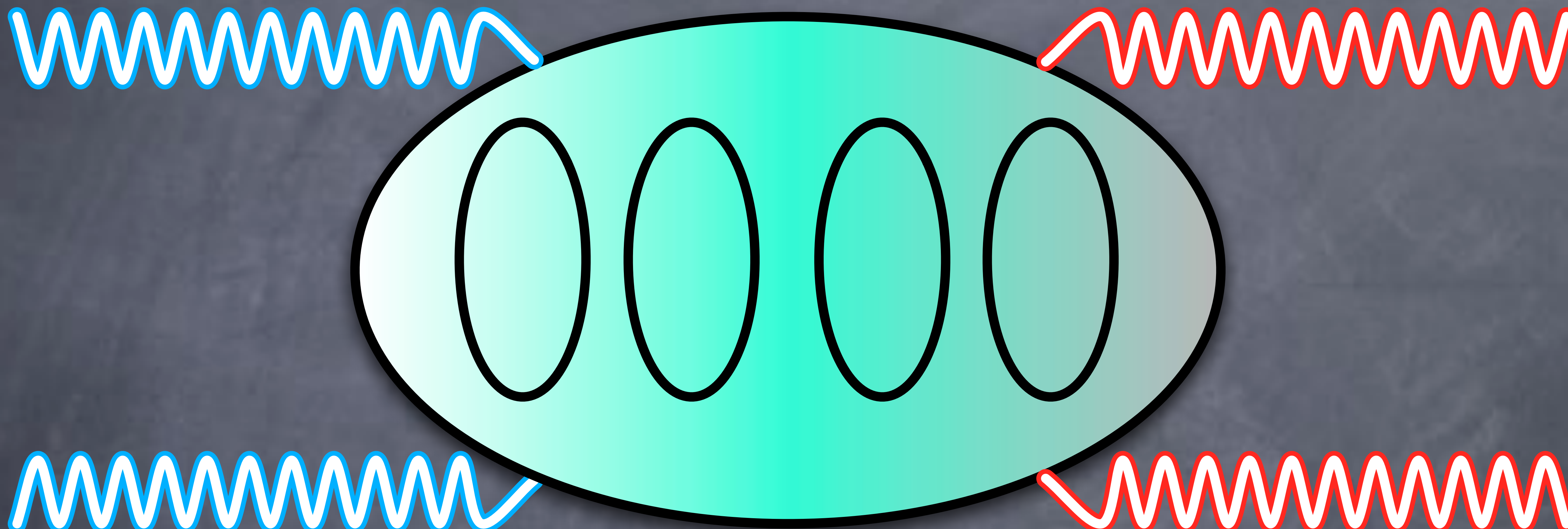




Many ways of expanding the contributing integrals I_i in terms of independent momenta. Each must be equivalent order by order in small external momenta. Equating expansions is sufficient to produce all required integral identities to demonstrate the cancellations of D=5 divergences.

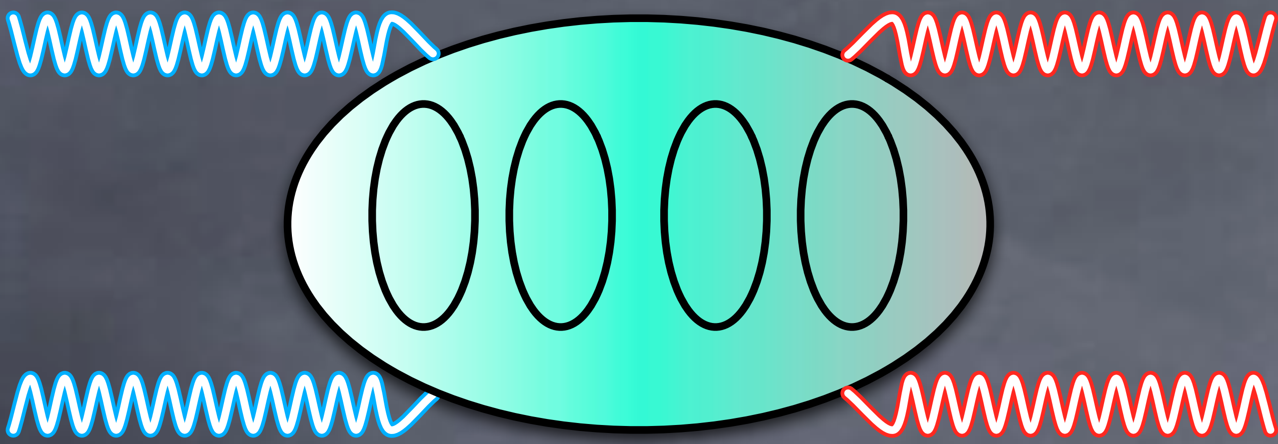
Verified by explicit analytical integration in $D=5-2\epsilon$

Four Loop $\mathcal{N}$=8 SUGRA



is finite in D=5!

actually finite for D < 5.5



actually finite for D < 5.5

Verified at 4 loops the all-loop order D-dimensional cancellations predicted by Bern, Dixon, Roiban

Developed, extended, and refined higher-loop calculation methods exposing surprising relations:

Box Cut, "twist rule", Jacobi-like relations, only (n-3)! indep. color-ordered Amps, n-factor KLT, supersum structure, applications to theories w/ less SUSY

Story's not over: there exists structure yet to be found.

Open Data available at:

EPAPS Document No. E-PRLTAO-103-025932

http://ftp.aip.org/epaps/phys_rev_lett/E-PRLTAO-103-025932/

http://www.aip.org/pubservs/epaps.html.