Emergent twistor geometry in scattering amplitudes

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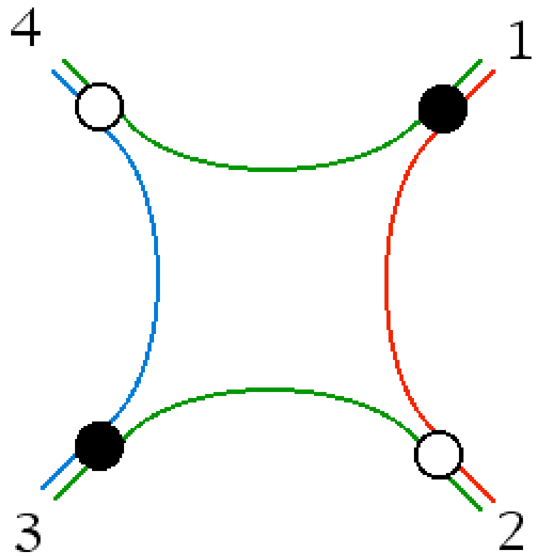
Origins

1970s: Roger Penrose's *twistor programme* suggests twistor space is the fundamental setting for:

- **QFT amplitudes (and dynamical theory) with twistor-geometric regularisation.**
- Particle spectrum.
- General relativity.

Conformal symmetry but also *breaking* of conformal symmetry is central.



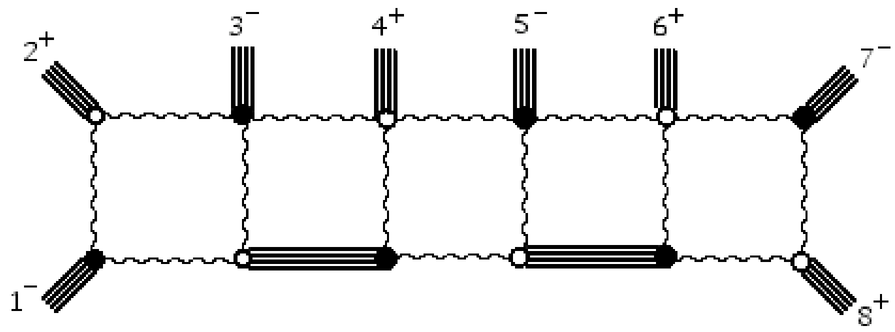
1988: The tree-level scattering amplitude for four gauge fields, expressed as a twistor diagram, shows string-like characteristics.

Impact of the BCFW recursion relation

2005: BCFW recursion is naturally expressed in the joining of twistor diagrams, leading to twistor diagrams for all tree-level amplitudes. **hep-th/0503060**.

Example: One of the 20 terms for A($1^-2^+3^-4^+5^-6^+7^-8^+$), as evaluated by Britto Cachazo and Feng in 2004:

$$\frac{\langle 13\rangle^4[68]^4\langle 5|(6+7+8)|4]^4}{S_{123}S_{678}\langle 12\rangle\langle 23\rangle[67][78]\langle 1|2+3|4][8|6+7|5\rangle[6|(7+8)(1+2+3)|4]\langle 5|(6+7+8)(1+2)|3\rangle}\,\delta(\sum_{i=1}^{8}p_i)$$



The lines in the twistor diagrams represent *quadruple poles* and *boundaries.*

Super-BCFW in super-twistor diagrams

2005: Extend twistors to N=4 super-twistors. Define diagram elements as *super-boundaries*, drawn as *arcs*, as in **hep-th/0512336**. The quadruple poles are eliminated.

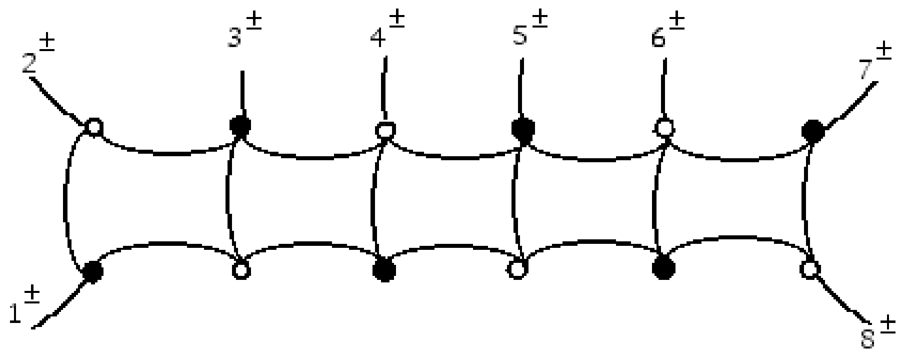
The BCFW restriction on helicities is eliminated.

The diagrams give super-amplitudes. An external super-wave-function is integrated over a super-volume.

2008: The analogous formalism in (+ + − −) signature is given by Arkani-Hamed et al., Mason and Skinner.

Example: computing all 8-field NNMHV tree amplitudes. The amplitude is still the sum of 20 diagrams, but each can be evaluated efficiently for all helicity choices at once (70 non-trivial cases).

Typical super-diagram:



$$\frac{F(H)^4}{S_{123}S_{678}\langle 12\rangle\langle 23\rangle[67][78]\langle 1|2+3|4][8|6+7|5\rangle[6|(7+8)(1+2+3)|4]\langle 5|(6+7+8)(1+2)|3\rangle}\,\delta\Big(\sum_{i=1}^{8} p_i\Big)$$

where the 70 helicity factors F(H) are tabulated in **hep-th/0603101**

Contours and regularisation

Twistor diagrams for space-time are not yet defined as precisely as they are in $(+\,+\,-\,-)$ signature. The contours are not specified systematically.

Space-time integrals are over one copy of Minkowski space, i.e. with boundaries at the light-cone at infinity. The corresponding twistor contours have boundaries defined by the infinity twistor, thus breaking conformal invariance explicitly.

Different contours are needed for each channel, or kinematic region.

Dimensional regularisation is not relevant. But twistor space has the natural extra dimension of scale in *non-projective* twistor space. It allows the changes:

$$W_\alpha Z^\alpha \to W_\alpha Z^\alpha - k$$

$$I_{\alpha\beta} X^\alpha Z^\beta \to I_{\alpha\beta} X^\alpha Z^\beta - m, \qquad I^{\alpha\beta} W_\alpha Y_\beta \to I^{\alpha\beta} W_\alpha Y_\beta - m$$

and then integration over 4-dimensional twistor spaces, instead of 3-dimensional projective twistor spaces.

This allows the correct expression of infra-red singularities at least in the 4-field case (already known in 1990 period).

Possibility that the sign factors in $(+\,+\,-\,-)$ theory will help make contour specification systematic.

String-like properties

Emergent geometrical structure: the diagrams resemble open strings.

The diagrams are non-unique, having many equivalence relations, so must be considered as representatives of a deeper structure.

For other than MHV, it is only a sum of a set of diagrams that has physical significance. Can this sum be described in a more geometrical way? Twistor diagrams are (super-)homology definitions, suggesting deeper contour relations to be found.

MHV diagrams can be described as polygons. Then the equivalence relations amount to equivalence under any triangulation. (Arkani-Hamed, Cachazo, et al.).

For MHV, the Kleiss-Kuijf relations can also be represented entirely through rules for attaching and subdividing triangles.

BCFW recursion implies the Kleiss-Kuijf relations for ALL tree amplitudes.
(Simple. Unpublished, but well known to Arkani-Hamed et al.).

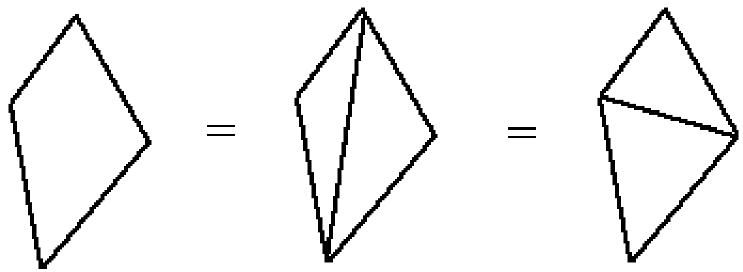
This also suggests that a unified sum of digrams should have deeper string-like properties.

Connection with soft limit, KLT and gravitational amplitude questions.

Dual conformal invariance

2009: Apart from MHV and anti-MHV, all amplitudes appear non-uniquely as sums of BCF terms with cancelling spurious poles, and a hidden dihedral symmetry.

Express dual conformal invariance by defining a dual 'momentum-twistor space'. The NMHV amplitudes then appear as (super-)volumes of 4-polytopes with manifest dihedral symmetry, and no spurious poles. **0905.1473v1(hep-th)**.



In two dimensions, a four-term relation between triangular areas.
In three dimensions, a five-term relation between tetrahedral volumes.
In four dimensions, a six-term relation between simplicial 4-volumes. This is the essential identity needed in NMHV expressions.

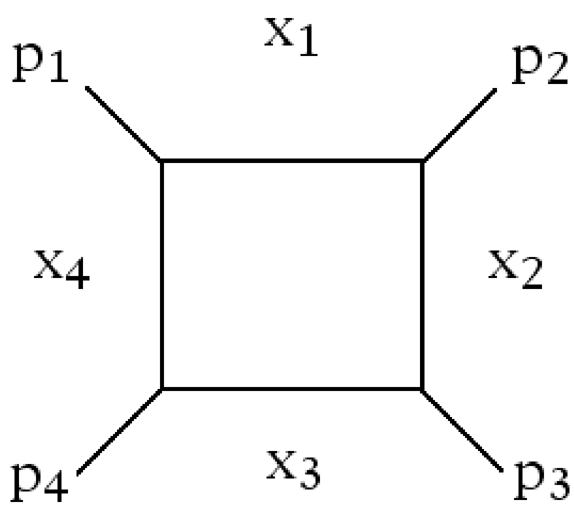
For n gauge fields, the polytope has
$n(n-3)/2$ vertices (corresponding to the physical poles), $n(n-3)$ edges, $n(n-1)/2$ faces and n hyperfaces.

The different BCF expressions correspond to different ways of partitioning these polytopes into simplexes.

Extension beyond NMHV will involve more than simple volume integrals, but seems to involve only projective twistor integration.

Puzzle: how to combine the insights from dual conformal symmetry, which depend on fixing a colour-ordering, with the simplicities of the Kleiss-Kuijf relations, which relate different colour orderings.

Loop integrals in momentum-twistor space



Use region space and momentum-twistors.

Scalar 4-mass box-function is fundamental case.

$$\int \frac{1}{(k-x_1)^2(k-x_2)^2(k-x_3)^2(k-x_4)^2}\ d^4k$$

$$\lim_{\mu\to 0}\int \frac{1}{((k-x_1)^2-\mu^2)((k-x_2)^2-\mu^2)((k-x_3)^2-\mu^2)((k-x_4)^2-\mu^2)}\ d^4k$$

This transforms naturally into a six-dimensional projective twistor integral:

$$\lim \oint \frac{DZ_1 \wedge DZ_2}{(P_{\alpha\beta} Z_1^\alpha Z_2^\beta) \quad (Q_{\alpha\beta} Z_1^\alpha Z_2^\beta) \quad (R_{\alpha\beta} Z_1^\alpha Z_2^\beta) \quad (S_{\alpha\beta} Z_1^\alpha Z_2^\beta)}$$

Here P, Q, R, S are skew two-index twistors, which may be considered as elements of a CP^5.

When $\mu = 0$, these are skew and *simple*; when $\mu \neq 0$ they are skew and not simple.

For $\mu \neq 0$ there is a compact contour integration, giving the correct dilogarithmic answer when taking the limit $\mu = 0$ in the correct (Feynman) way.

Conformal breaking shows up in this limiting process, although the amplitude is conformally invariant within each kinematical region.

The geometry of transversals

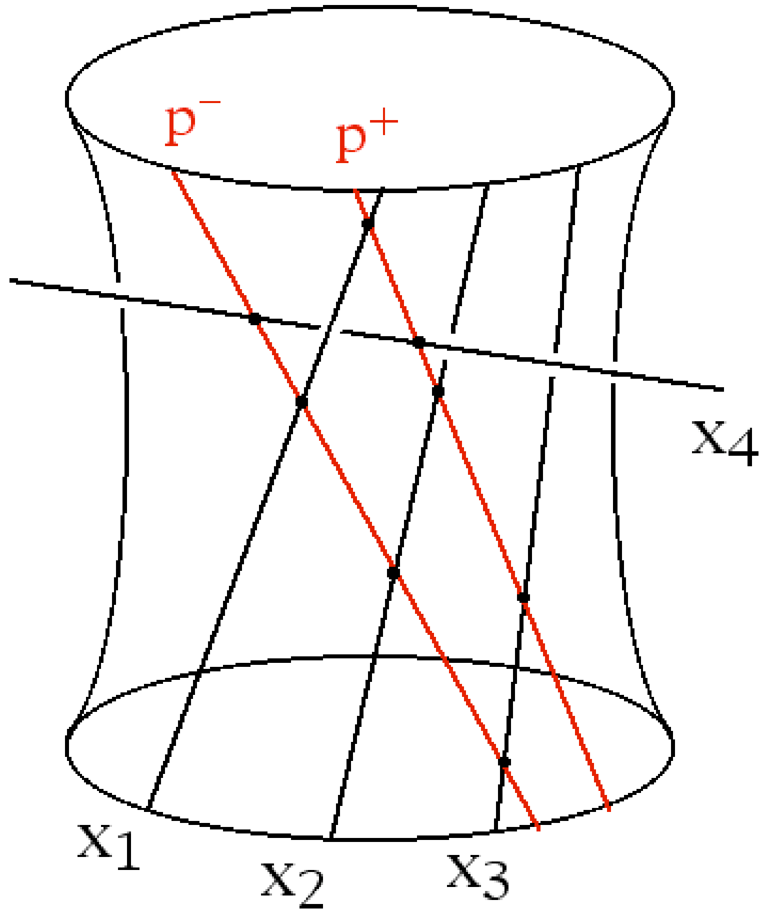
For the 4-mass ampitude, the region momenta are not null-separated, and the corresponding lines in twistor space are all skew.

The loop amplitude is dominated by the on-shell internal momentum, corresponding to the two solutions of:

$(p-x_1)^2 = (p-x_2)^2 = (p-x_3)^2 = (p-x_4)^2 = 0$

These are only quadratic equations but the solution is still awkward to express in space-time.

In twistor geometry, these are simply the two transversals of the four skew lines.



Degeneration to the 3-mass case occurs when two neighbouring regions become null-separated, i.e. lines meet in twistor space.

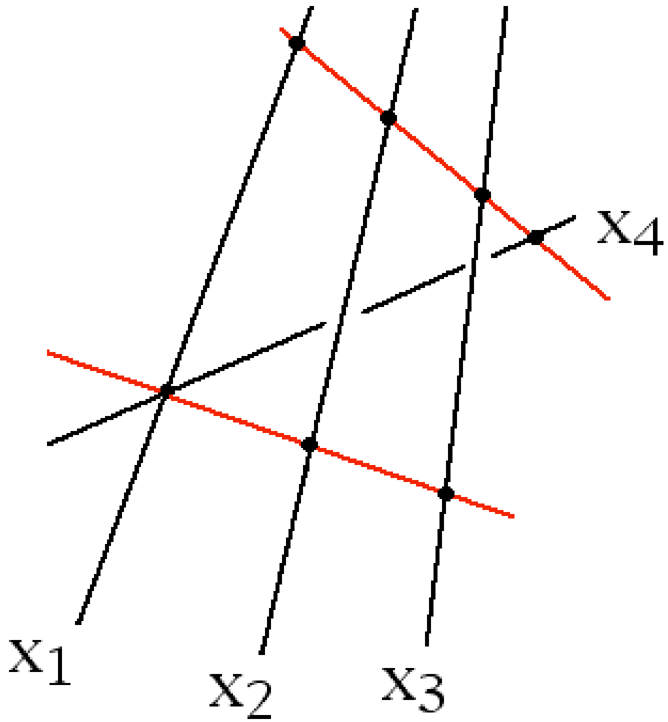
The picture of the two transversals then changes.

If lines x_1 and x_4 meet, they define a point and a plane. One transversal passes through the point, and one lies in the plane.

This shows the bifurcation of the 3-mass integral into two helicity types.

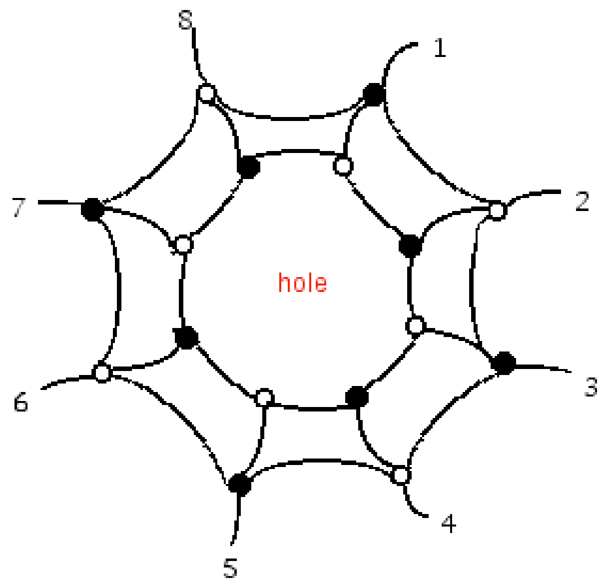
This pattern continues down to the 0-mass box.

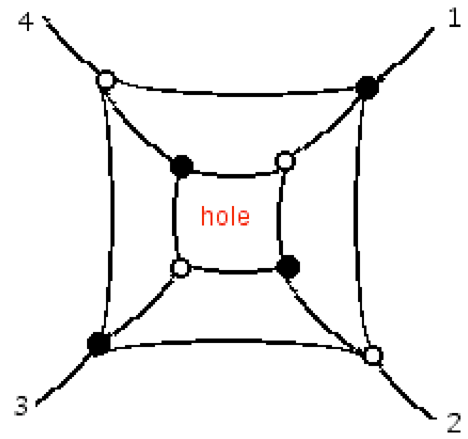
The mass μ has a substantial role in regularisation.



Twistor diagrams for loop integrals

Conjecture: one-loop N=4 gauge theory diagrams for eight fields and four fields:





Work in progress: defining contours for integrating these.