Computing amplitudes using Wilson loops

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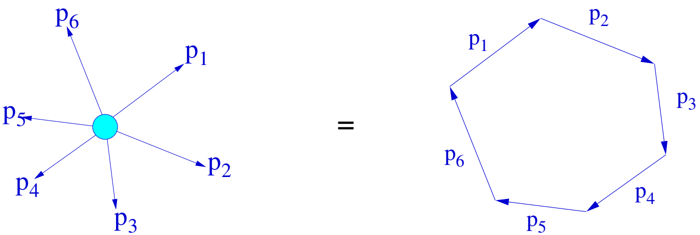
based on:
0707.1153 Brandhuber, Travaglini, P.H.
0902.2245 Anastasiou, Brandhuber, Khoze, Spence, Travaglini, P.H.

Work in progress

Brief introduction to amplitudes in $\mathcal{N} = 4$

- Duality between two objects in $\mathcal{N}$=4 Super Yang-Mills:



Gluon amplitudes = Wilson loops

- Vast simplification of the computation of amplitudes

Example

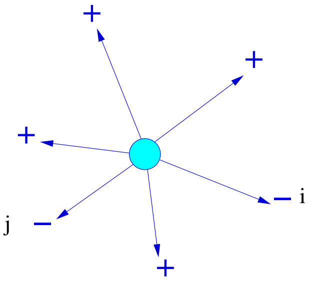
We compute all MHV 2-loop gluon scattering amplitudes (assuming the conjectured duality) for any n.

Motivation

- Theoretical
  - Hidden structures/symmetries (eg twistor string theory, no triangle hypothesis, dual conformal symmetry, integrability ...)
- Practical
  - Simpler/faster ways to compute amplitudes (recent advances include generalised unitarity/BCFW recursion relations etc.)
  - Background processes at LHC

MHV Amplitudes in $\mathcal{N} = 4$ SYM

- Colour-stripped, planar "Maximally Helicity Violating (MHV)" amplitudes



$$A_n = A_n^{\text{tree}} \mathcal{M}_n$$

- We will focus on $\mathcal{M}_n^{(L)}$

L-loop amplitude

The BDS conjecture [Anastasiou Bern Dixon Kosower 2003, Bern Dixon Smirnov 2005]

- IR divergences: dimensional regularisation $d = 4 - 2\epsilon$

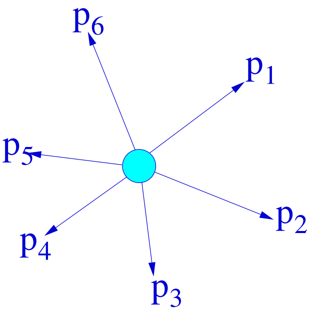
The BDS formula: an all-loop expression for any n

$$\log\left(\mathcal{M}_n(\epsilon)\right) = \sum_{L=1}^{\infty} a^L \left(f_{\mathcal{A}}^{(L)}(\epsilon)\mathcal{M}_n^{(1)}(L\epsilon) + C^{(L)}\right) + O(\epsilon)$$

- 'a' is the 't Hooft coupling constant
- Here $f_{\mathcal{A}}^{(L)}(\epsilon) = f_0^{(L)} + f_1^{(L)}\epsilon + f_2^{(L)}\epsilon^2$ where $f_i^{(L)}$ is a number.
- needs modification from $n = 6$ points...

Outline

Amplitude/Wilson loop duality



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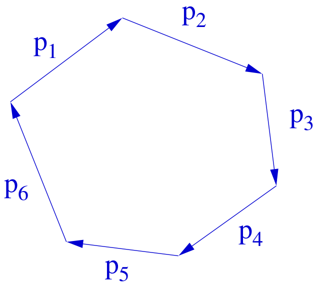


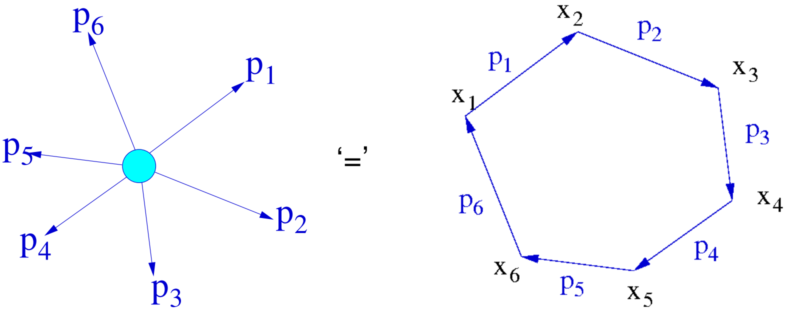
amplitude $\mathcal{M}_n$ '=' $\langle W[\mathcal{C}_n] \rangle$

$(d = 4 - 2\epsilon)$ $(d = 4 + 2\epsilon)$

- Wilson loop over the polygonal contour $\mathcal{C}_n$

$$W[\mathcal{C}] \; := \; \mathrm{Tr}\,\mathcal{P}\exp\left[ig \oint_{\mathcal{C}} d\tau \left(A_\mu(x(\tau))\dot{x}^\mu(\tau)\right)\right]$$

Amplitude/Wilson loop duality



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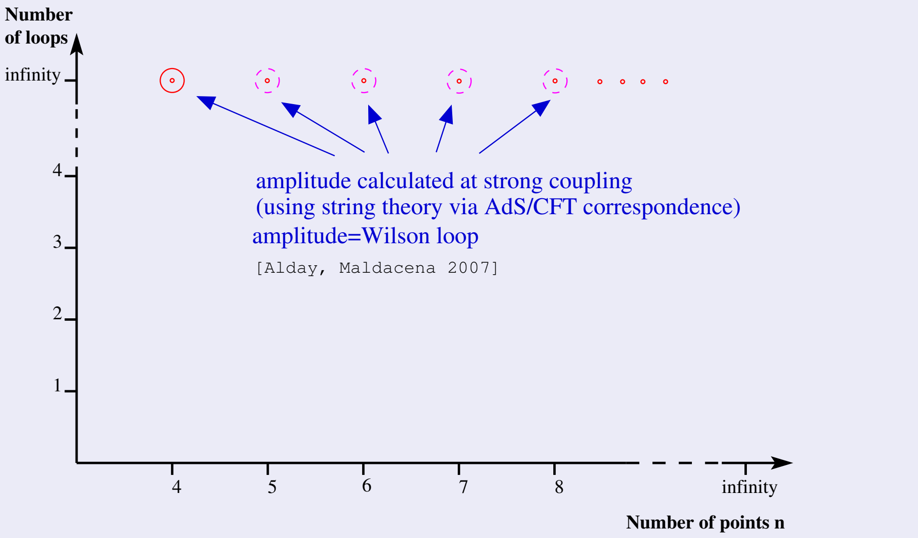
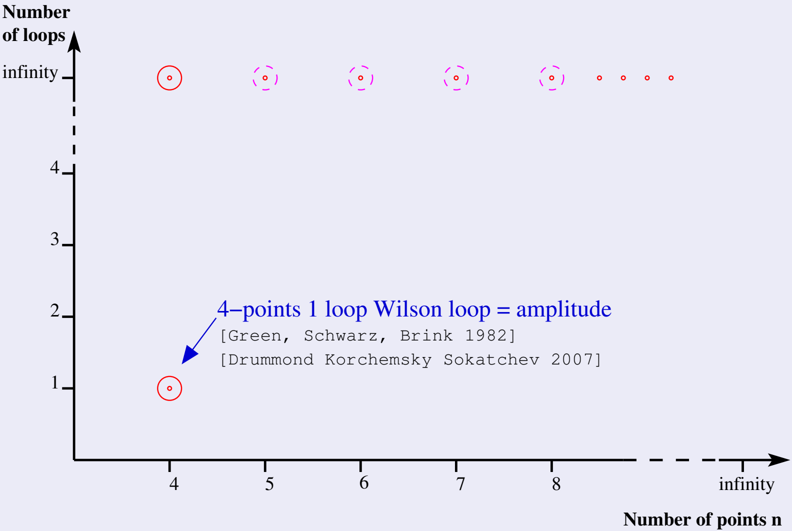
Evidence so far...



Evidence so far...



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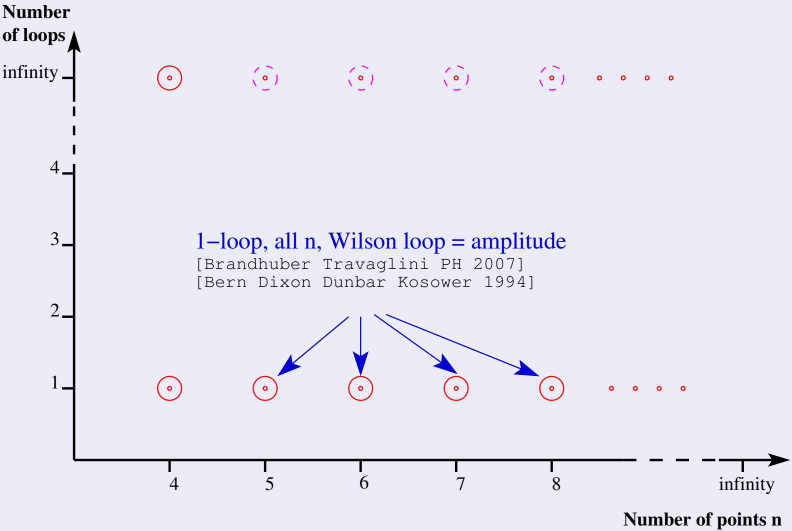

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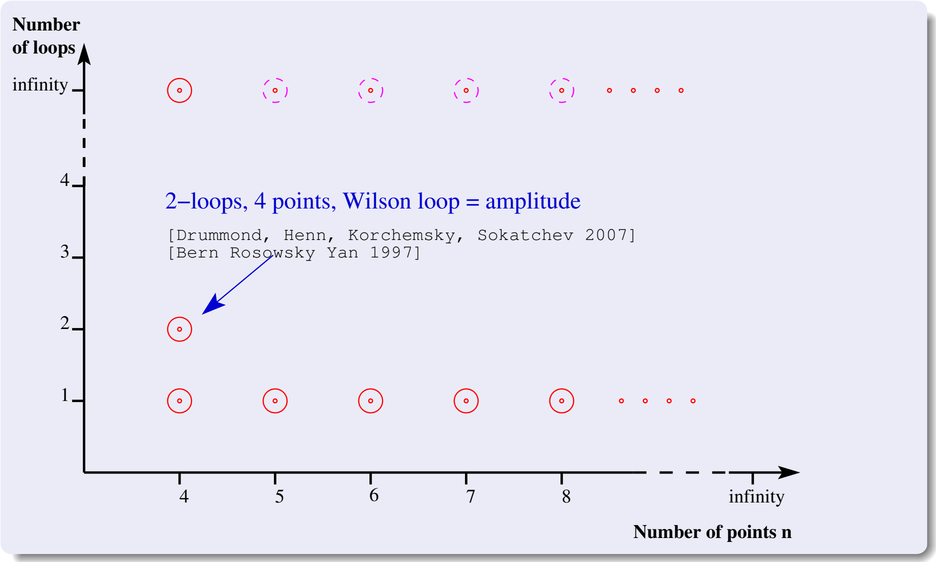

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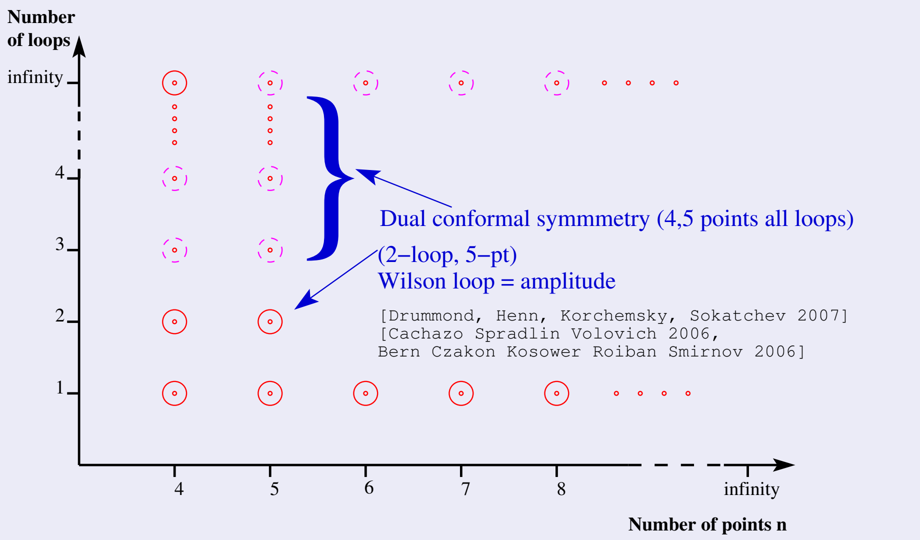
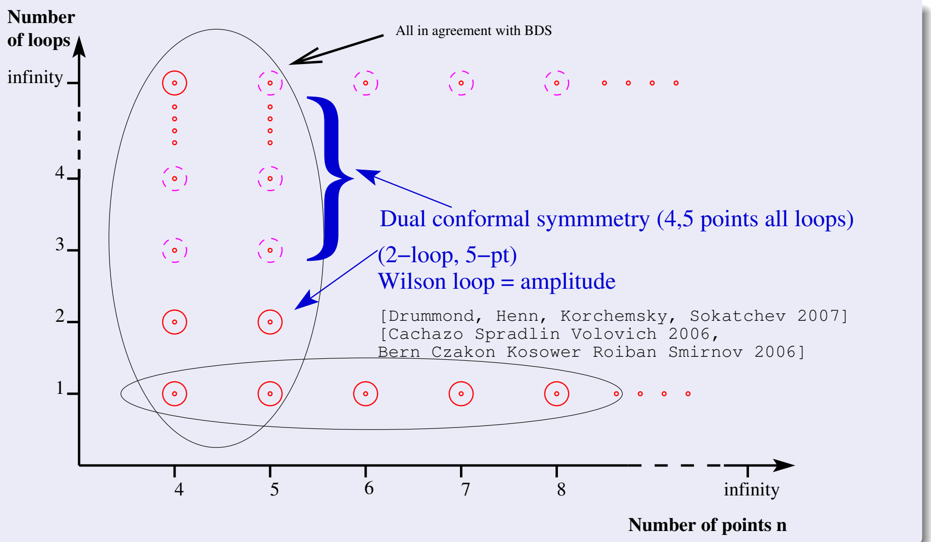


Evidence so far...



Remainder function

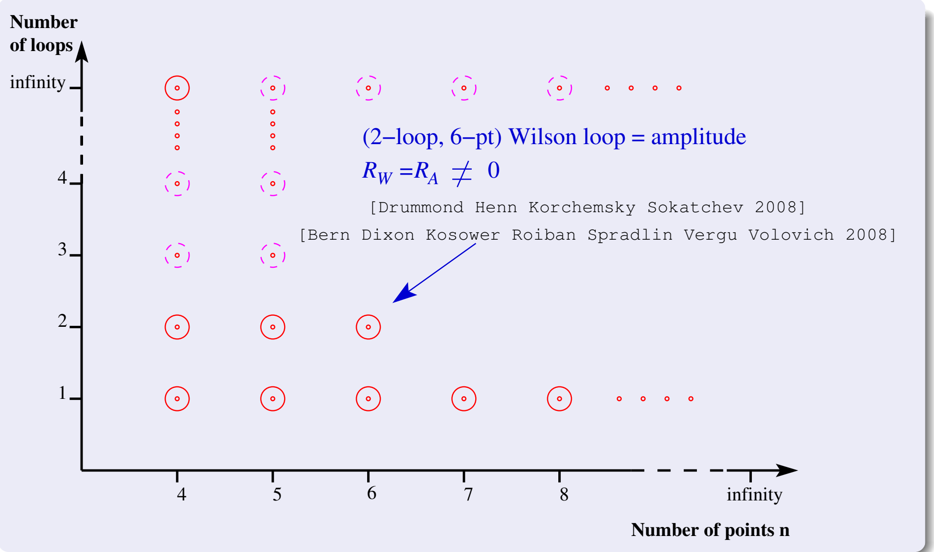
$n \geq 6$

$$\log\left(\mathcal{M}_n(\epsilon)\right) = \sum_{l=1}^{\infty} a^l f_{\mathcal{A}}^{(l)}(\epsilon)\mathcal{M}_n^{(1)}(l\epsilon) + C_{\mathcal{A}}(a) + \mathcal{R}_n^{\mathcal{A}}(p_i; a) + O(\epsilon)$$

$$\log\left(W_n(\epsilon)\right) = \sum_{l=1}^{\infty} a^l f_{W}^{(l)}(\epsilon)W_n^{(1)}(l\epsilon) + C_{w}(a) + \mathcal{R}_n^{W}(p_i; a) + O(\epsilon)$$

- non-zero remainder function found for the two-loop six-point amplitude and the Wilson loop [Drummond Henn Korchemsky Sokatchev 2008, Bern Dixon Kosower Roiban Spradlin Vergu Volovich 2008]

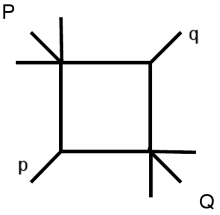
Evidence so far...



Wilson loop calculations, 1-loop

- the expression for the $n-point$ amplitude and for the WL are very closely related:

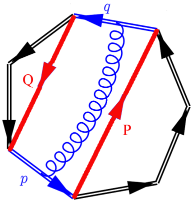
Amplitude $= \sum_{p,q}$



$+O(\epsilon)$

=

Wilson loop $= \sum_{p,q}$



$$P = \sum_{k=p+1}^{q-1} k \qquad Q = \sum_{k=q+1}^{p-1} k$$

▸ WL

Wilson loop calculations, 1-loop

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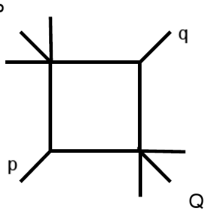
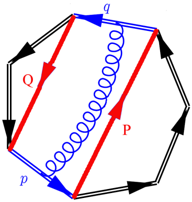
$$\text{Amplitude} = \sum_{p,q}$$



$$+O(\epsilon)$$

$=$

$$\text{Wilson loop} = \sum_{p,q}$$

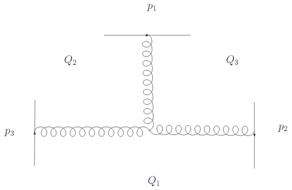


$$P = \sum_{k=p+1}^{q-1} k \qquad Q = \sum_{k=q+1}^{p-1} k$$

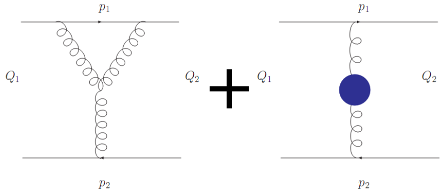
▸ WL

2-loop n-point Wilson loop (log of)

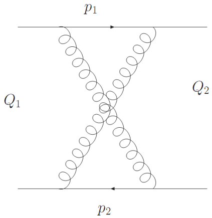
Only four new “master” integrals to be computed for all *n*



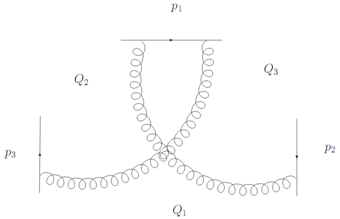
$f_H(p_1, p_2, p_3; Q_1, Q_2, Q_3)$



$f_Y(p_1, p_2; Q_1, Q_2)$

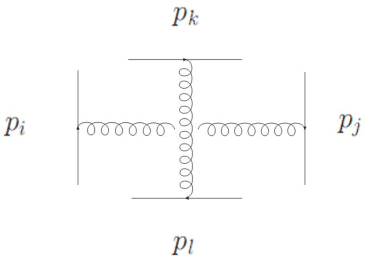


$f_X(p_1, p_2; Q_1, Q_2)$



$f_C(p_1, p_2, p_3; Q_1, Q_2, Q_3)$

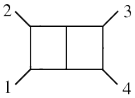
Also factorised cross diagram



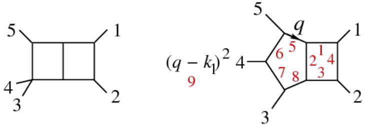
- This is given by the product of two one loop diagrams
- $-1/2 f_{\mathcal{P}}(p_i, p_j; Q_{ji}, Q_{ij}) f_{\mathcal{P}}(p_k, p_l; Q_{lk}, Q_{kl})$

(Compare with amplitude (parity even part))

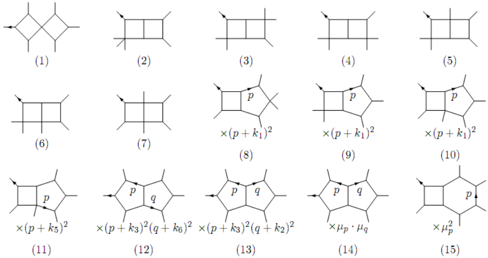
$n = 4$



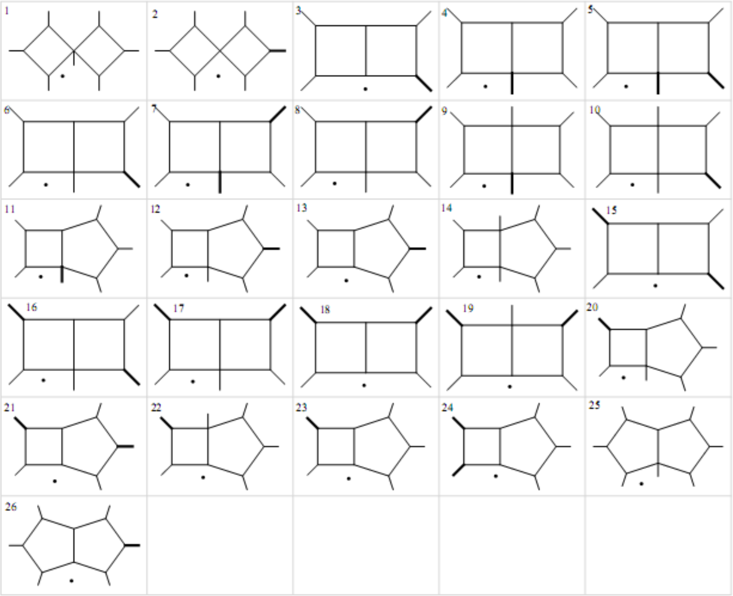
$n = 5$



$n = 6$



$n = 7$ [Vergu]



Complete 2-loop Wilson loop

- The logarithm of the complete n-sided Wilson loop is given in terms of the four new master diagrams together with the one loop diagram $f_P(p_i, p_j; Q_{ji}, Q_{ij})$ as

$$\sum_{1\le i<j<k\le n} \Big[f_H(p_i, p_j, p_k; Q_{jk}, Q_{ki}, Q_{ij}) + f_C(p_i, p_j, p_k; Q_{jk}, Q_{ki}, Q_{ij}) + f_C(p_j, p_k, p_i; Q_{ki}, Q_{ij}, Q_{jk}) + f_C(p_k, p_i, p_j; Q_{ij}, Q_{jk}, Q_{ki}) \Big]$$
$$+ \sum_{1\le i<j\le n} \Big[f_X(p_i, p_j; Q_{ji}, Q_{ij}) + f_Y(p_i, p_j; Q_{ji}, Q_{ij}) + f_Y(p_j, p_i; Q_{ij}, Q_{ji}) \Big]$$
$$+ \sum_{1\le i<k<j<l\le n} (-1/2) f_P(p_i, p_j; Q_{ji}, Q_{ij}) f_P(p_k, p_l; Q_{lk}, Q_{kl})$$

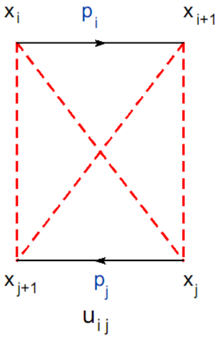
Outline

Computations at $n = 6, 7, 8\ldots$

- Using sector decomposition and the numerical techniques of [Anastasiou Beerli Daleo (2007,2008), Lazopoulos Melnikov Petriello (2007), Anastasiou Melnikov Petriello (2005)] we compute the 2-loop master integrals
- Computations of WL performed for $n = 4, 5, 6, 7, 8 \to$ considerable amount of data collected.
- Verified that the remainder function is conformally invariant
- Verified cyclic and parity (dihedral) symmetry
- Collinear limits

Conformal invariants: cross-ratios

- Number of independent cross-ratios is $n(n-5)/2$
- Basis:



$$u_{ij} = \frac{x^2_{ij+1} x^2_{ji+1}}{x^2_{ij} x^2_{i+1j+1}}$$

- This ignores the Gram determinant $n(n-5)/2 > 3n - 15$
- physical kinematics will form a $3n - 15$ dimensional slice of this space of cross-ratios

Hexagon computations

- 3 cross-ratios

$$u_{36} = \frac{x_{31}^2 x_{46}^2}{x_{36}^2 x_{41}^2} := u_1 \; , \; u_{14} = \frac{x_{15}^2 x_{24}^2}{x_{14}^2 x_{25}^2} := u_2 \; , \; u_{25} = \frac{x_{26}^2 x_{35}^2}{x_{25}^2 x_{36}^2} := u_3$$

- remainder function $\rightarrow \mathcal{R}(u_1, u_2, u_3)$

Hexagon Calculations

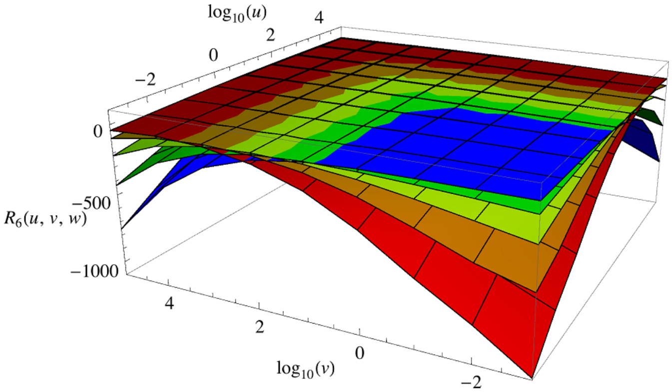
- Checks of conformal invariance of the Remainder (previously done by DHKS/BDKSVV):

(u_1, u_2, u_3)	$\mathcal{R}_6^{\rm WL}(A)$	$\mathcal{R}_6^{\rm WL}(B)$	$\mathcal{R}_6^{\rm WL}(C)$
$(1/9, 1/9, 1/9)$	5.18056	5.18096	5.18102
$(1/4, 1/4, 1/4)$	1.08916	1.08916	1.08919
$(1, 1, 1)$	-2.70814	-2.7066	-2.70657
$(100, 100, 100)$	-2.09134	-2.09204	-2.09228

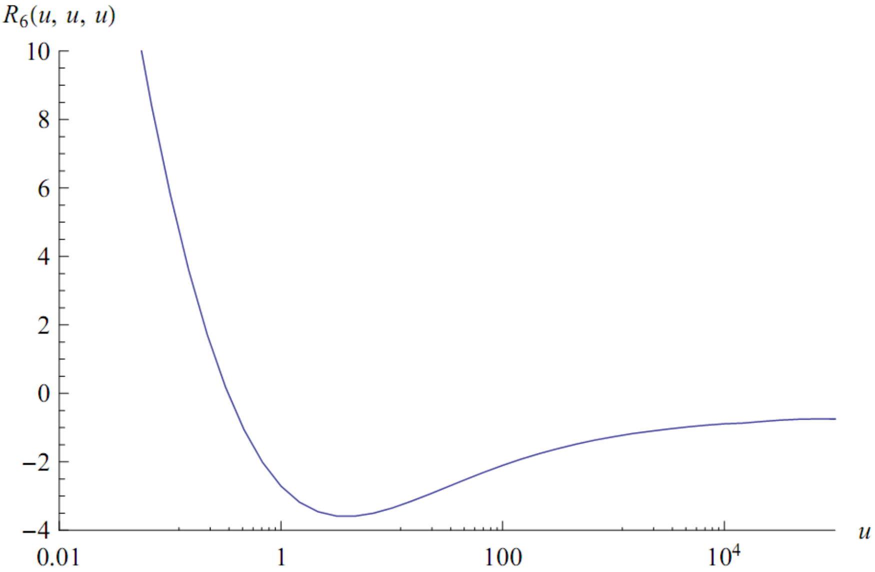
(A), (B), (C) are three different but conformally equivalent kinematics.

6-pnt Wilson loop

- $\mathcal{R}_6^W$ with $u_1 = u,\ u_2 = v,\ u_3 = w$
- $w = 1$ blue, $w = 10$ green, $w = 100$ yellow, $w = 1000$ orange, $w = 10000$ red



Plot of $\mathcal{R}_6(u, u, u)$



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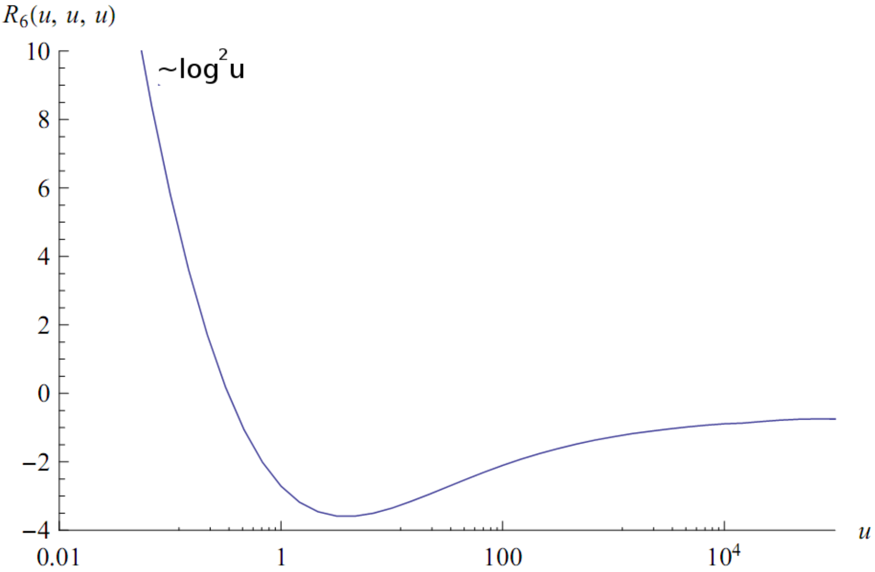
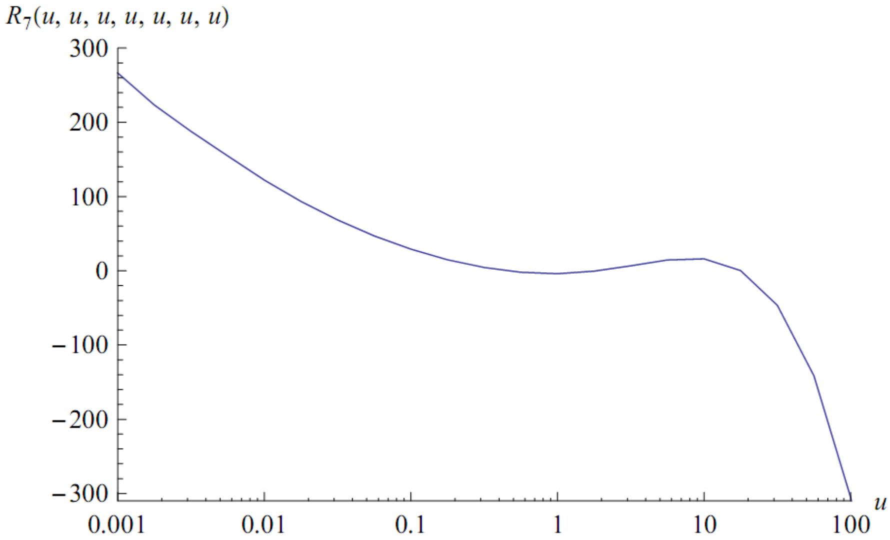
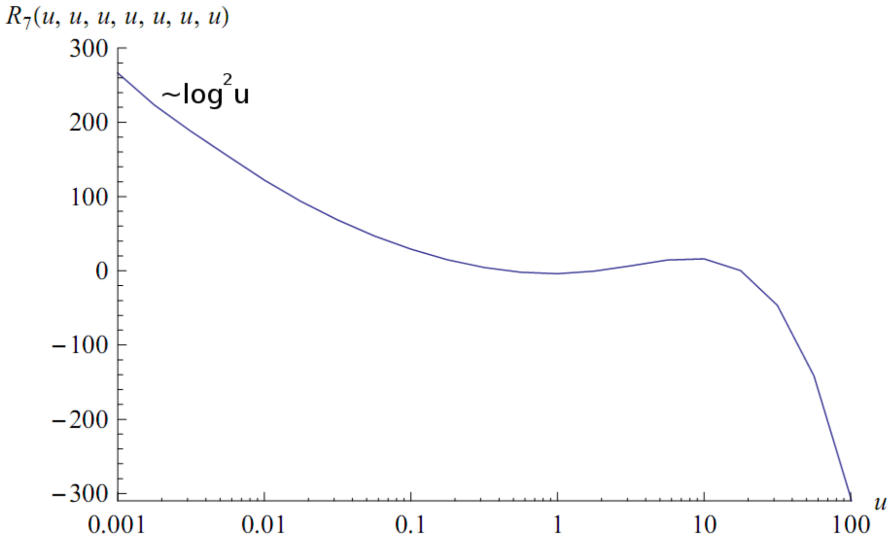


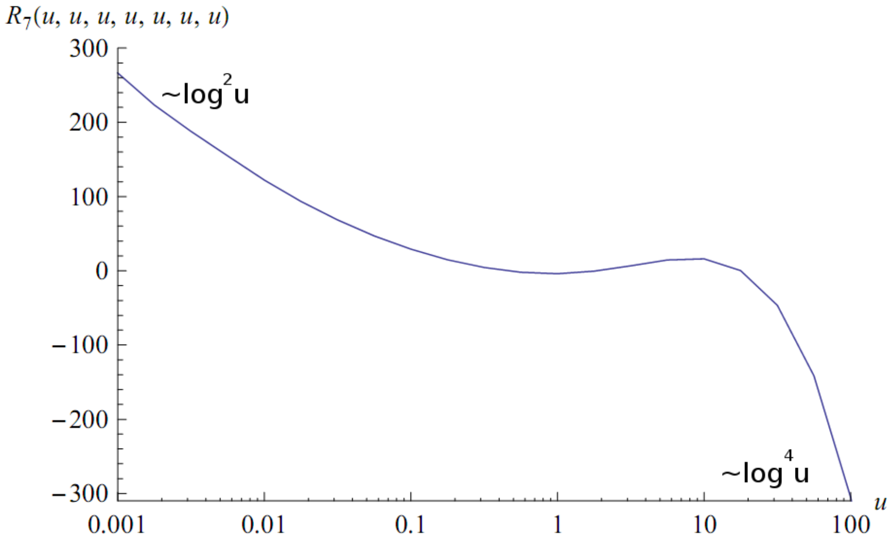
Plot of $\mathcal{R}_7(u, u, u, u, u, u, u)$



Plot of $\mathcal{R}_7(u, u, u, u, u, u, u)$



Plot of $\mathcal{R}_7(u, u, u, u, u, u, u)$



8 points

- Conformal invariance
- Cyclicity and parity (also checked at $6, 7$ points)

Collinear limits

- $\mathcal{R}_n(u)$ should have trivial simple collinear limits

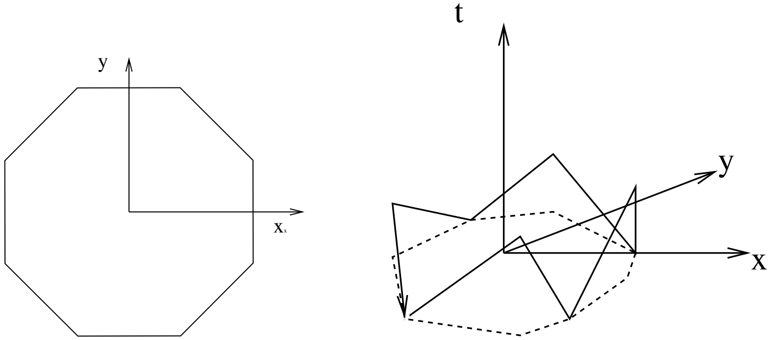
$$\mathcal{R}_n \longrightarrow \mathcal{R}_{n-1}$$

- We verify this for $n = 6, 7, 8$ (with no constant shifts)

Outline

Higher n

- We can compute for arbitrarily large n
- Alday and Maldacena recently considered special n-point amplitude kinematics at strong coupling via string theory



- Momenta in $2+1$ dimensions in notation (t, z)

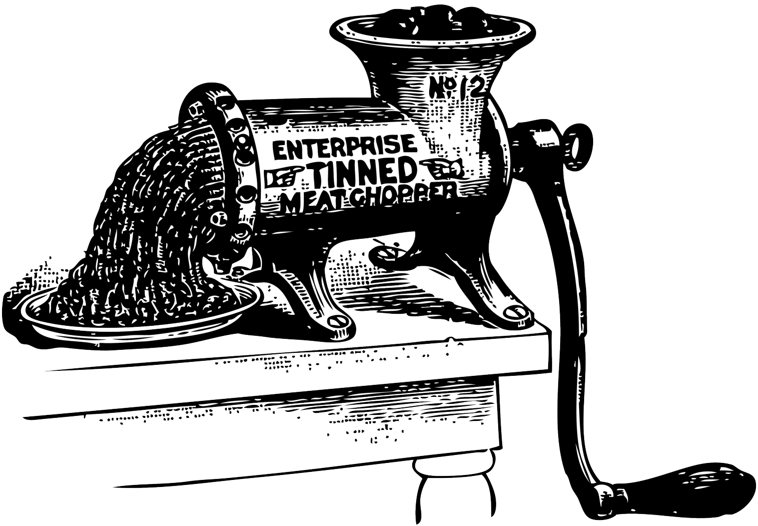
$$x_{2k} \;=\; \left(2\sin\frac{\pi}{2n},\; e^{i\pi\frac{2k+1}{n}}\right)\,, \qquad x_{2k+1} \;=\; \left(0,\; e^{i\pi\frac{2k}{n}}\right)$$

- space projection $=$ regular polygon, zig-zags in time

- this kinematics leads to the cross-ratios

$$u_{ij} = 1 \, , \qquad\qquad i - j = \text{odd} \, ,$$
$$u_{ij} = 1 - \left(\frac{\sin \frac{\pi}{n}}{\sin \frac{\pi a}{n}} \right)^2 \, , \qquad i - j = 2a \, ,$$

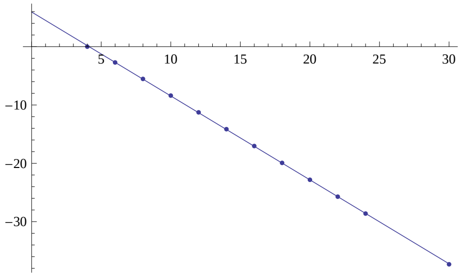
- At strong coupling: $A_n = \pi \left(\frac{3}{8} n - 2 + \frac{2}{n} \right)$ [Alday Maldacena]
- does the weak coupling result share any features with this?
- eg naive counting of two loop diagrams $\Rightarrow$ n^4 growth
- put the above kinematics in our program ...


No 12
ENTERPRISE
TINNED
MEAT CHOPPER

- Results:

no.points	$\mathcal{R}_n$
$n = 6$	-2.708
$n = 8$	-5.528
$n = 10$	-8.386
$n = 12$	-11.261
$n = 14$	-14.145
$n = 16$	-17.034
$n = 18$	-19.926
$n = 20$	-22.820
$n = 22$	-25.716
$n = 24$	-28.614
$n = 30$	-37.311

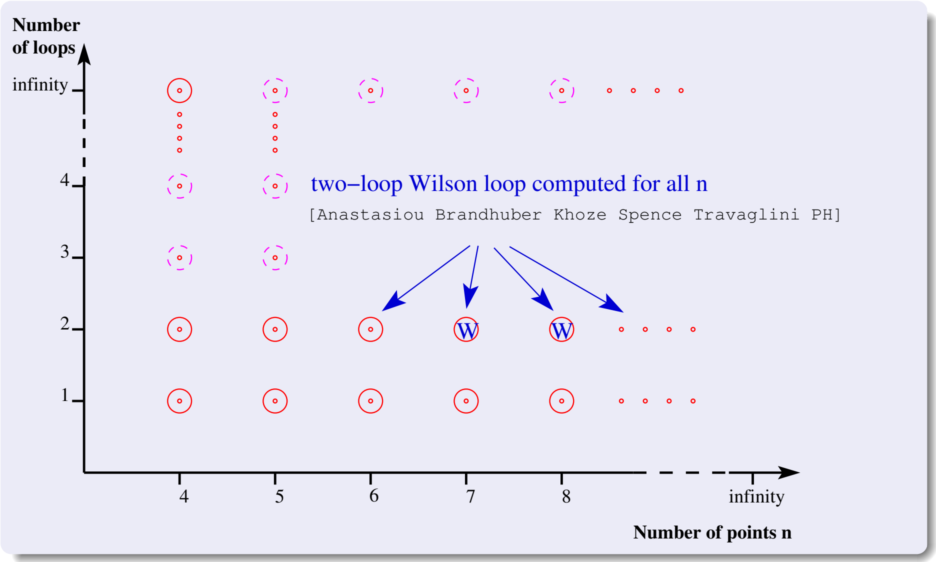
Plot of two-loop data versus linear fit



- Best linear fit: $\mathcal{R}_n \approx 5.94061 - 1.43878n$
  Error ~ 0.1
- $+1/n$ term: $\mathcal{R}_n \approx 6.3689 - 1.4538n - 2.1928/n$
  Error ~ 0.01
- Including $1/n^2$ term
  $\mathcal{R}_n \approx -1.45128n + 6.26917 - 1.13934/n - 2.8661/n^2$
  Error ~ 0.0005 = numerical error

Summary of results

- Summary: the number of distinct integrals for the 2-loop n-gon WL is independent of n
- We compute all n-sided polygonal light-like Wilson loops at two loops (eg recent computation of an $n = 30$ WL)
- no additional complexity as n increases: the number of diagrams increases but the type of integral is n-independent
- Assuming the amplitude/Wilson loop duality we compute two-loop planar MHV amplitudes for *any* number of points


Number
of loops
infinity
4
3
2
1
two–loop Wilson loop computed for all n
[Anastasiou Brandhuber Khoze Spence Travaglini PH]
W
W
4
5
6
7
8
infinity
Number of points n

Future directions

- amplitude calculation at $n \geq 7$-points needed! [Vergu]
- analytic determination of $\geq$6-pnt amplitude/Wilson loop
- Proof of WL/amplitude duality
- Generalisations of WL to NMHV amplitudes etc. [Berkovits Maldacena]
- Generalisations to other theories
- Understanding the role of standard (super)conformal symetry $\Rightarrow$ Yangian, infinite new symmetries (integrability) [Beisert Ricci Tseytlin Wolf, Berkovits Maldacena, Drummond Henn Plefka, Bargheer Beisert Galleas Loebbert McLoughlin]