$\mathcal{R}^4$-*counterterm and* $E_{7(7)}$-*symmetry in maximal supergravity*

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Hidden Structures in Field theory Amplitudes
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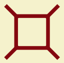
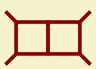
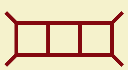
Counterterms in Theories of Gravity

Pure gravity

- is nonrenormalizable by power counting in four dimensions, because the coupling (Newtons constant) is dimensionful $G_{\text{Newton}} = \frac{1}{M^2_{\text{Planck}}}$.
-  does not suffer from divergences at the one loop level ($R_{\mu\nu} = 0$ onshell, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \rightarrow$ total divergence) [’t Hooft Veltman]
-  exhibits a two loop divergence related to the possible counterterm $R_{\mu\nu}{}^{\lambda\rho}\, R_{\lambda\rho}{}^{\sigma\tau}\, R_{\sigma\tau}{}^{\mu\nu}$ [Goroff Sagnotti][van de Ven]

$\mathcal{N} = 8$ supergravity

-  one loop: no divergence in pure gravity, so no divergence in supergravity
-  possible candidate $R_{\mu\nu}{}^{\lambda\rho}\, R_{\lambda\rho}{}^{\sigma\tau}\, R_{\sigma\tau}{}^{\mu\nu}$ from pure gravity produces amplitudes forbidden by supersymmetry: $\langle - + + + \cdots + \rangle$ [Grisaru 1977][Tomboulis 1977]
-  *admissible* counterterm $\mathcal{R}^4$ exists, which is the supersymmetric extension of

$$R^4 = t^{\mu_1\nu_1\ldots\mu_4\nu_4} t^{\rho_1\sigma_1\ldots\rho_4\sigma_4} R_{\mu_1\nu_1\rho_1\sigma_1} R_{\mu_2\nu_2\rho_2\sigma_2} R_{\mu_3\nu_3\rho_3\sigma_3} R_{\mu_4\nu_4\rho_4\sigma_4}$$

[Deser, Kay Stelle][Howe, Stelle Townsend][Kallosh 1981]

$\mathcal{N} = 8$ supergravity

Maximal supersymmetric gravity theory in four dimensions [DeWit Freedman][Cremmer, Julia Scherk][DeWit Nicolai]

- (selected) symmetries
  - supersymmetry with operators $Q_A, \tilde{Q}^A$, $A = 1 \ldots 8$
  - $E_{7(7)} \to \begin{cases} SU(8) & 63 \text{ compact generators} \\ \frac{E_{7(7)}}{SU(8)} & 70 \text{ noncompact generators} \end{cases}$
- Particle spectrum:

1	8	28	56	70	56	28	8	1
-2	-$\frac{3}{2}$	-1	-$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
B^-	F_A^-	B_{AB}^-	Λ_{ABC}^-	X_{ABCD}	Λ_+^{ABC}	B_+^{AB}	F_+^A	B_+

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Maximal supersymmetric gravity theory in four dimensions [DeWit Freedman][Cremmer, Julia Scherk][DeWit Nicolai]

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unitary gauge

- Particle spectrum:

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B^-	F_A^-	B_{AB}^-	Λ_{ABC}^-	X_{ABCD}	Λ_+^{ABC}	B_+^{AB}	F_+^A	B_+

- noncompact part of $E_{7(7)}$ connects different, physically indistinguishable vacua of the theory, which are labelled by the expectation values of the 70 scalars. (analogy to the Goldstone situation)
- in order to maintain unitary gauge, any noncompact E_7-rotation has to be accompanied by a compensating $SU(8)$ rotation

$\mathcal{R}^4$ counterterm

R: pure quantity, $\mathcal{R}$: supersymmetrized quantity

- suggested in 1981 as a counterterm for $\mathcal{N} = 8$ supergravity, transforms covariantly under supersymmetry and $SU(8)$ [Kallosh][Deser, Kay Stelle][Howe, Stelle Townsend]
- no explicit expression for supersymmetric extension of R^4 known, use of nonlinearized SUSY might be necessary for realization
- correponds to a divergence at three-loops, far beyond the results of recent considerations [Bossard, Howe Stelle][Green, Russo Vanhove]
- miraculously, the coefficient determined by renormalization vanishes [Bern, Dixon, Carrasco Johansson, Kosower]
- ***any symmetry not yet accounted for ?*** Candidate: $\frac{E_{7(7)}}{SU(8)}$
- usually: symmetry forbids the occurrence of the counterterm in the Lagrangian
  Here: $\frac{E_{7(7)}}{SU(8)}$ is not a symmetry of the lagrangian, but of the equations of motion.

Commutation relations between $SU(8)$ generators T and $\frac{E_{7(7)}}{SU(8)}$-generators X:

$$[T,T] \sim T, \quad [X,T] \sim X, \quad [X,X] \sim T$$

Double soft scalar limit:

$$\mathcal{M}_{n+2}(X_1, X_2, \ldots) \xrightarrow[p_1,p_2 \to 0]{} \frac{1}{2}\sum_{i=3}^{n+2} \frac{p_i \cdot (p_2 - p_1)}{p_i \cdot (p_1 + p_2)} R(\eta_i) \mathcal{M}_n(3,4,\ldots)$$

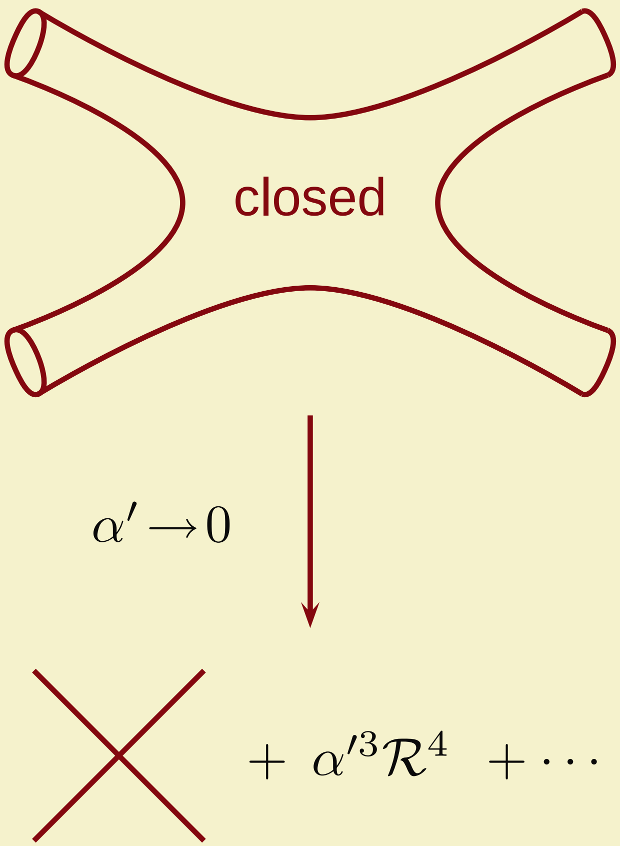
$R \sim [X_1, X_2]$

- relation proven for pure $\mathcal{N} = 8$ supergravity [Arkani-Hamed Cachazo, Kaplan]
- as expected: compensating $SU(8)$-rotation after noncompact $E_{7(7)}$-rotation

> Main idea: test the validity of the above relation for amplitudes derived from an action of the form $\int \mathcal{R} + c\mathcal{R}^4$

- How to obtain the corresponding amplitudes? Direct calculation in $\mathcal{N} = 8$ supergravity seems impossible.

String theory provides access to $\mathcal{R}^4$-corrected amplitudes:



- closed string theory reproduces $\mathcal{N} = 8$ supergravity after compactification to 4 dimensions and taking the point particle limit $\alpha' = 0$. [Green Schwarz]
- first correction to the effective action: $\alpha'^3\mathcal{R}^4$ [Gross Witten]
- expanding string amplitudes around $\alpha' = 0$ and keeping terms up to $\mathcal{O}(\alpha'^3)$ generates the same amplitudes as would be obtained from an action of the form $\int \mathcal{R} + \alpha'^3\mathcal{R}^4$

However ...

... not any string theory amplitude is ready available or easy to calculate.

$$\mathcal{M}_6(X_1^{ABCD}, X_{2\,ABCE}, 3, 4, 5, 6) \xrightarrow[p_1, p_2 \to 0]{} \frac{1}{2} \sum_{i=3}^{6} \frac{p_i \cdot (p_2 - p_1)}{p_i \cdot (p_1 + p_2)} R^D_E(\eta_i) \mathcal{M}_4{}^E{}_D(3, 4, 5, 6)$$

where $R\left([X^{I_1 \ldots I_4}, X_{I_5 \ldots I_8}]\right)^D_E = \varepsilon^{I_1 I_2 I_3 I_4 E}_{I_5 I_6 I_7 I_8 D} \times \eta_{iE} \partial_{\eta_{iD}}$.

Constraints

- amplitude has to be six-point at least (rhs nontrivial)
- amplitude needs NMHV signature: $\langle - - - + ++ \rangle$ (rhs at least MHV)
- scalars have to agree in three indices (from $E_{7(7)}$ commutation relations / proof of double soft scalar equality)
- remaining particles need to have two open indices

Additional tools required to calculate the six-point amplitude:

- supersymmetric Ward identities (SWI) and Kawai-Lewellen-Tye relations (KLT)
- Strategy: available *open* string amplitudes $\xrightarrow[\text{SWI}]{}$ certain class of open string amplitudes $\xrightarrow[\text{KLT}]{}$ closed string amplitude amplitude

$$0 = \langle [Q, \beta_1\beta_2 \cdots \beta_n] \rangle = \langle 0 | \, [Q, \beta_1\beta_2 \cdots \beta_n] \, | 0 \rangle = \sum_{i=1}^{n} \langle \beta_1\beta_2 \cdots [Q, \beta_i] \cdots \beta_n \rangle$$

$\mathcal{N} = 1$ supersymmetry with particles g^+, g^-, λ^+, λ^- and supersymmetry relations:

$$\left[Q(\eta), g^+(p)\right] = [p\eta] \, \lambda^+(p), \qquad \left[Q(\eta), \lambda^+(p)\right] = -\langle p\eta \rangle g^+(p),$$
$$\left[Q(\eta), g^-(p)\right] = \langle p\eta \rangle \lambda^-(p), \qquad \left[Q(\eta), \lambda^-(p)\right] = - \, [p\eta] \, g^-(p),$$

MHV:
$$\begin{aligned}\langle [Q(\eta), g^- \, g^- \, \lambda^+ \, g^+ \, g^+ \, g^+] \rangle =& \langle 1\eta \rangle \langle \lambda^- \, g^- \, \lambda^+ \, g^+ \, g^+ \, g^+ \rangle + \langle 2\eta \rangle \langle g^- \, \lambda^- \, \lambda^+ \, g^+ \, g^+ \, g^+ \rangle \\ & - \langle 3\eta \rangle \langle g^- \, g^- \, g^+ \, g^+ \, g^+ \, g^+ \rangle\end{aligned}$$

NMHV:
$$\begin{aligned}\langle [Q(\eta), g^- \, g^- \, g^- \, \lambda^+ \, g^+ \, g^+] \rangle =& \langle 1\eta \rangle \langle \lambda^- \, g^- \, g^- \, \lambda^+ \, g^+ \, g^+ \rangle + \langle 2\eta \rangle \langle g^- \, \lambda^- \, g^- \, \lambda^+ \, g^+ \, g^+ \rangle \\ & + \langle 3\eta \rangle \langle g^- \, g^- \, \lambda^- \, \lambda^+ \, g^+ \, g^+ \rangle - \langle 4\eta \rangle \langle g^- \, g^- \, g^- \, g^+ \, g^+ \, g^+ \rangle\end{aligned}$$

Conventions:	$p_\mu p^\mu = \det(p^{\alpha\dot\alpha})$, $p^{\alpha\dot\alpha} = p_\mu(\sigma^{\alpha\dot\alpha})^\mu$, $p^{\alpha\dot\alpha} = \pi^\alpha \tilde\pi^{\dot\alpha}$,	$Q\vert 0\rangle = 0$
	$\langle ij \rangle = \pi_i^\alpha \pi_{j\alpha}$, $[ij] = \tilde\pi_{i\dot\alpha}\tilde\pi_j^{\dot\alpha}$, $s_{ij} = \alpha' \langle ij \rangle [ij] = 2\alpha' k_i \cdot k_j$	$Q(\eta) = Q^A \eta_A$

$$0 = \langle [Q, \beta_1\beta_2 \cdots \beta_n] \rangle = \langle 0| \, [Q, \beta_1\beta_2 \cdots \beta_n] \, |0\rangle = \sum_{i=1}^{n} \langle \beta_1\beta_2 \cdots [Q, \beta_i] \cdots \beta_n \rangle$$

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MHV: $\langle [Q(\eta), g^- \, g^- \, \lambda^+ \, g^+ \, g^+ \, g^+] \rangle = \langle 11\rangle \langle \lambda^- \, g^- \, \lambda^+ \, g^+ \, g^+ \, g^+\rangle + \langle 21\rangle \langle g^- \, \lambda^- \, \lambda^+ \, g^+ \, g^+ \, g^+\rangle$
$- \langle 31\rangle \langle g^- \, g^- \, g^+ \, g^+ \, g^+ \, g^+\rangle = 0$

NMHV: $\langle [Q(\eta), g^- \, g^- \, g^- \, \lambda^+ \, g^+ \, g^+] \rangle = \langle 11\rangle \langle \lambda^- \, g^- \, g^- \, \lambda^+ \, g^+ \, g^+\rangle + \langle 2\eta\rangle \langle g^- \, \lambda^- \, g^- \, \lambda^+ \, g^+ \, g^+\rangle$
$+ \langle 3\eta\rangle \langle g^- g^- \lambda^- \lambda^+ g^+ g^+\rangle - \langle 4\eta\rangle \langle g^- g^- g^- g^+ g^+ g^+\rangle = 0$

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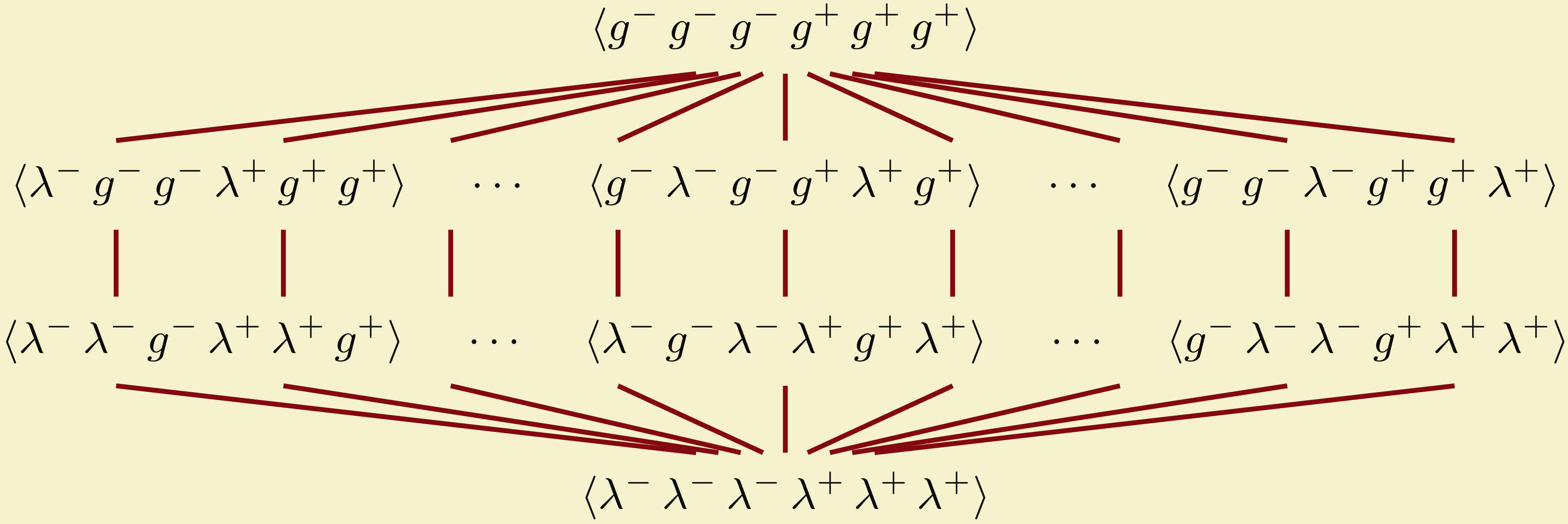
$\langle ij\rangle = \pi_i^\alpha \pi_{j\alpha}$, $[ij] = \tilde\pi_{i\dot\alpha} \tilde\pi_j^{\dot\alpha}$, $s_{ij} = \alpha' \langle ij\rangle [ij] = 2\alpha' k_i \cdot k_j$

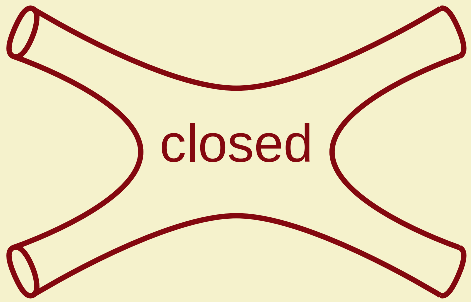
$Q|0\rangle = 0$
$Q(\eta) = Q^A \eta_A$

Supersymmetric Ward identities (SWI) II



- linear system of equations: 20 amplitudes and $30 = 6 + 18 + 6$ source terms equations $\rightarrow$ rank 18
- two amplitudes have to be known, in order to determine all others
- imagine the same consideration for $\mathcal{N} = 8$ supergravity ... algebraically not feasible. Second technique necessary: KLT

Kawai-Lewellen-Tye-relation (KLT)



$\alpha' \to 0$ ↓

$+ \ \alpha'^3 \mathcal{R}^4 \ + \cdots$

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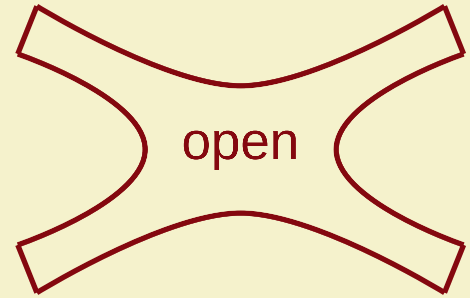
-2	-$\frac{3}{2}$	-1	-$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2

”=“

$$\left[\ \cdots \ \right]^2$$



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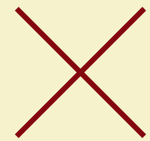


$+\alpha'^2 \mathcal{F}^4 + \alpha'^3 \cdots$

1	4	6	4	1
-1	-$\frac{1}{2}$	0	$\frac{1}{2}$	1

$$[\mathcal{N}=8] \quad = \quad [\mathcal{N}=4] \otimes [\mathcal{N}=4]$$

[Kawai, Tye Levellen]

$$\langle X^{1234}\, X_{1235}\, F^{5+} F_4^{-}\, B^{+}\, B^{-} \rangle \quad \Leftrightarrow \quad \langle g^{-}\, \lambda^{4+}\, g^{+}\, \lambda_4^{-}\, g^{+}\, g^{-} \rangle_L \times \langle g^{+}\, \lambda_5^{-}\, \lambda^{5+}\, g^{-}\, g^{+}\, g^{-} \rangle_R$$

- unique decomposition of any $\mathcal{N}=8$ state into two $\mathcal{N}=4$ states [Bianchi, Elvang Freedman]

Matching orders of α'

- String theory relation:

$$M_5(1,2,3,4,5) = i\pi \sin(\pi s_{12}) \sin(\pi s_{34})\, A_5(1,2,3,4,5)\, A_5(2,1,4,3,5) + \mathcal{P}(2,3)$$

- α'-expansion

$$\sin(\pi s_{12}) = \alpha'\pi\langle 12\rangle[12] - \frac{(\alpha'\pi\langle 12\rangle[12])^3}{3!} + \cdots$$
$$A_5(12345) = A_5^{\mathrm{SYM}}(12345) + A_5^{\mathcal{O}(\alpha'^2)}(12345) + A_5^{\mathcal{O}(\alpha'^3)}(12345) + \cdots\,,$$

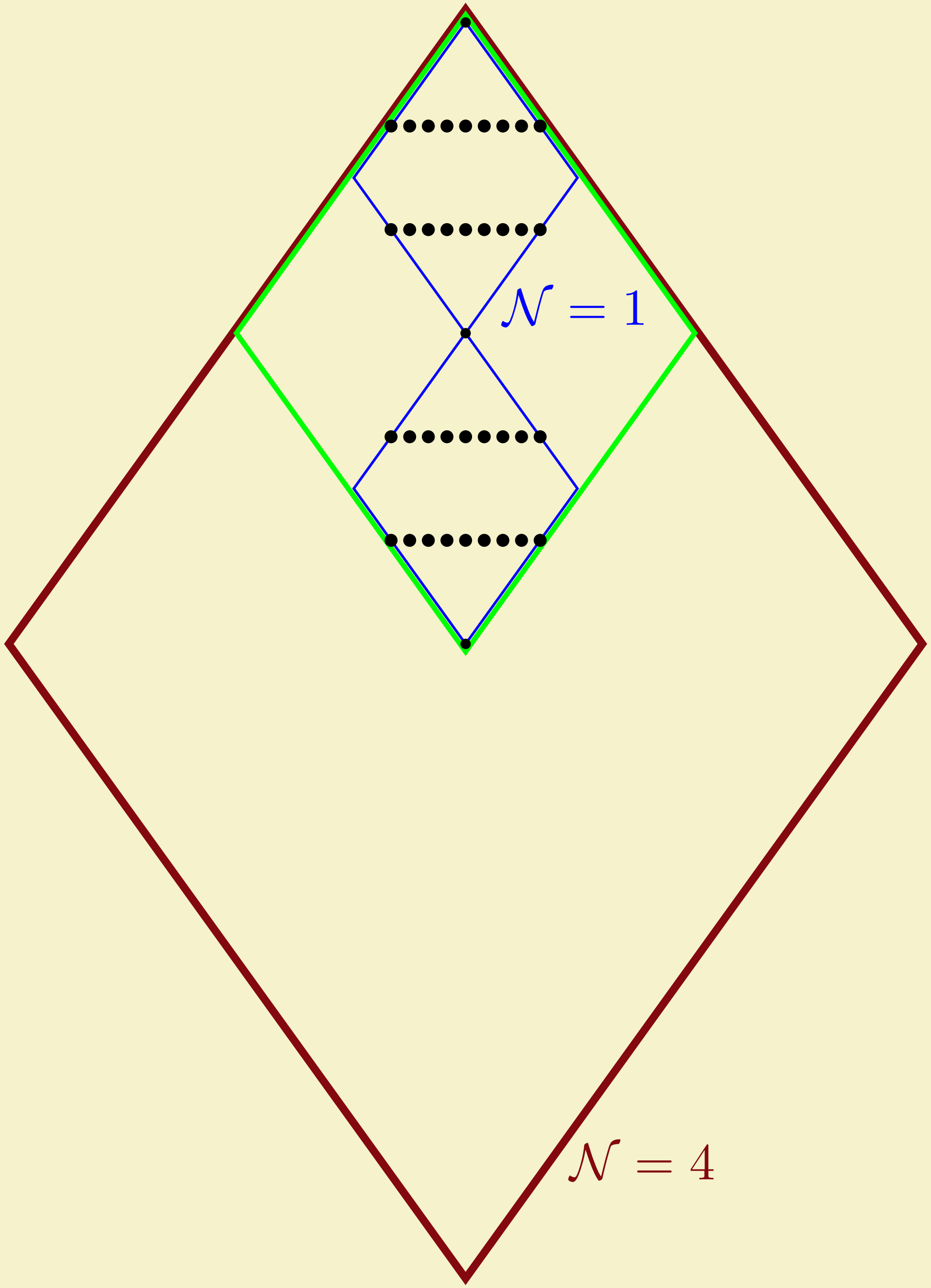
- any particular order in α' of M_5 can be determined by multiplying all suitable combinations of orders of the right hand side:

$$M_5^{\mathcal{O}(\alpha'^2)}(1,2,3,4,5) = i\pi^3 s_{12}s_{34}\Big[A_5^{\mathrm{SYM}}(12345)\, A_5^{\mathcal{O}(\alpha'^2)}(21435) + A_5^{\mathcal{O}(\alpha'^2)}(12345)\, A_5^{\mathrm{SYM}}(21435)$$
$$-\tfrac{\pi^2}{6}(s_{12}^2 + s_{34}^2) A_5^{\mathrm{SYM}}(12345)\, A_5^{\mathrm{SYM}}(21435)\Big] + \mathcal{P}(2,3) = 0\,.$$

How does everything combine?

- try to find a $\mathcal{N}=8$ amplitude satisfying all constraints, whose KLT-decomposition consists of $\mathcal{N}=1$ amplitudes only

$$\langle X^{1234}\, X_{1235}\, F^{5+} F_4^-\, B^+\, B^-\rangle = \text{KLT}\Big[\langle g^-\,\lambda^{4+}\,g^+\,\lambda_4^-\,g^+\,g^-\rangle_L \times\langle g^+\,\lambda_5^-\,\lambda^{5+}\,g^-\,g^+\,g^-\rangle_R\Big]$$



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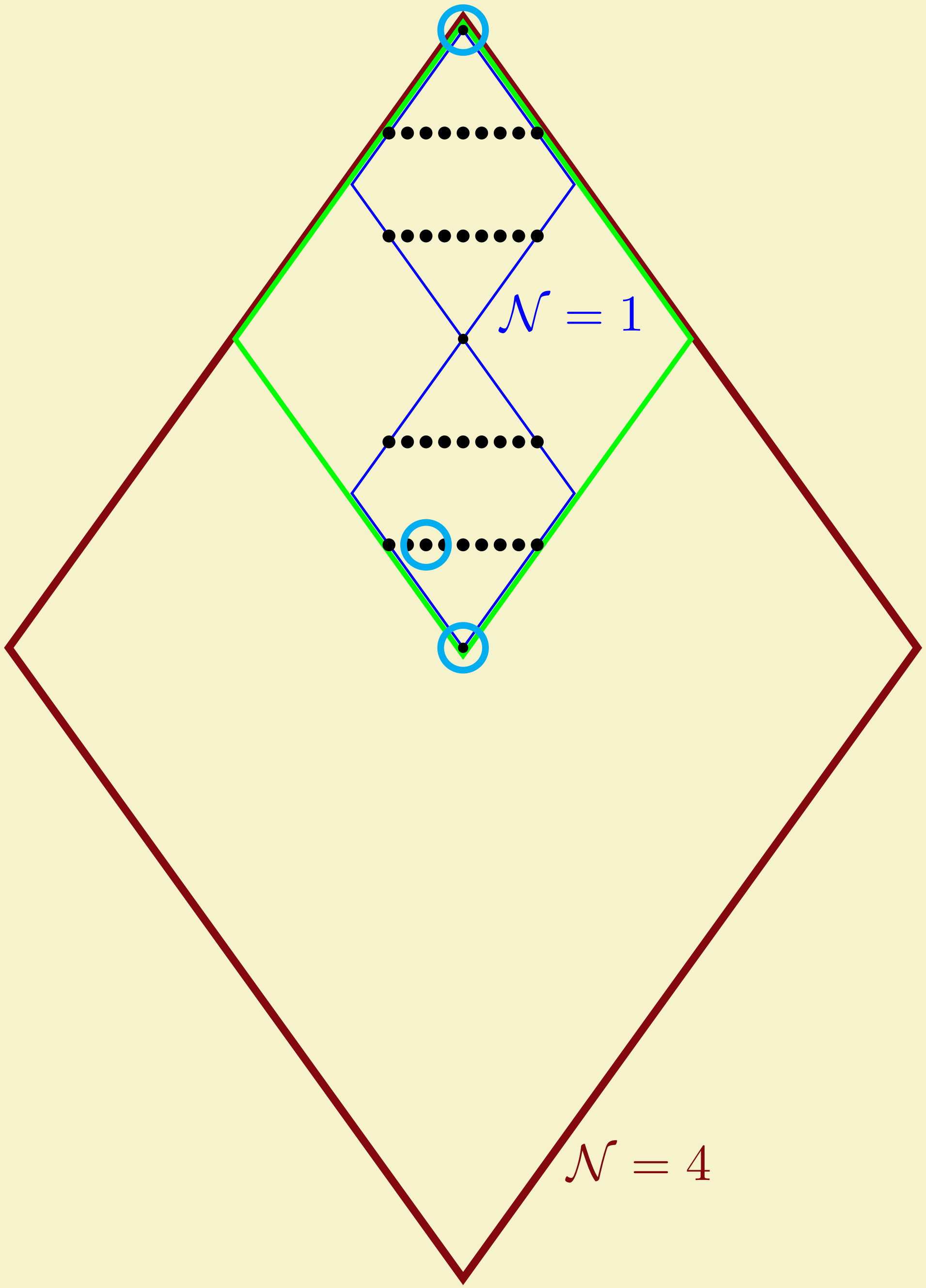
$$\langle X^{1234}\, X_{1235}\, F^{5+} F_4^-\, B^+\, B^- \rangle = \text{KLT}\Big[\langle g^-\, \lambda^{4+}\, g^+\, \lambda_4^-\, g^+\, g^- \rangle_L \times \langle g^+\, \lambda_5^-\, \lambda^{5+}\, g^-\, g^+\, g^- \rangle_R\Big]$$

- consider the available open string calculations $\langle g^-\, g^-\, g^-\, g^+\, g^+\, g^+ \rangle$, $\langle \phi^-, \phi^-, \phi^-, \phi^+, \phi^+, \phi^+ \rangle$, $\langle \phi^-, \phi^-, \lambda^-, \lambda^+, \phi^+, \phi^+ \rangle$ [Stieberger Taylor]



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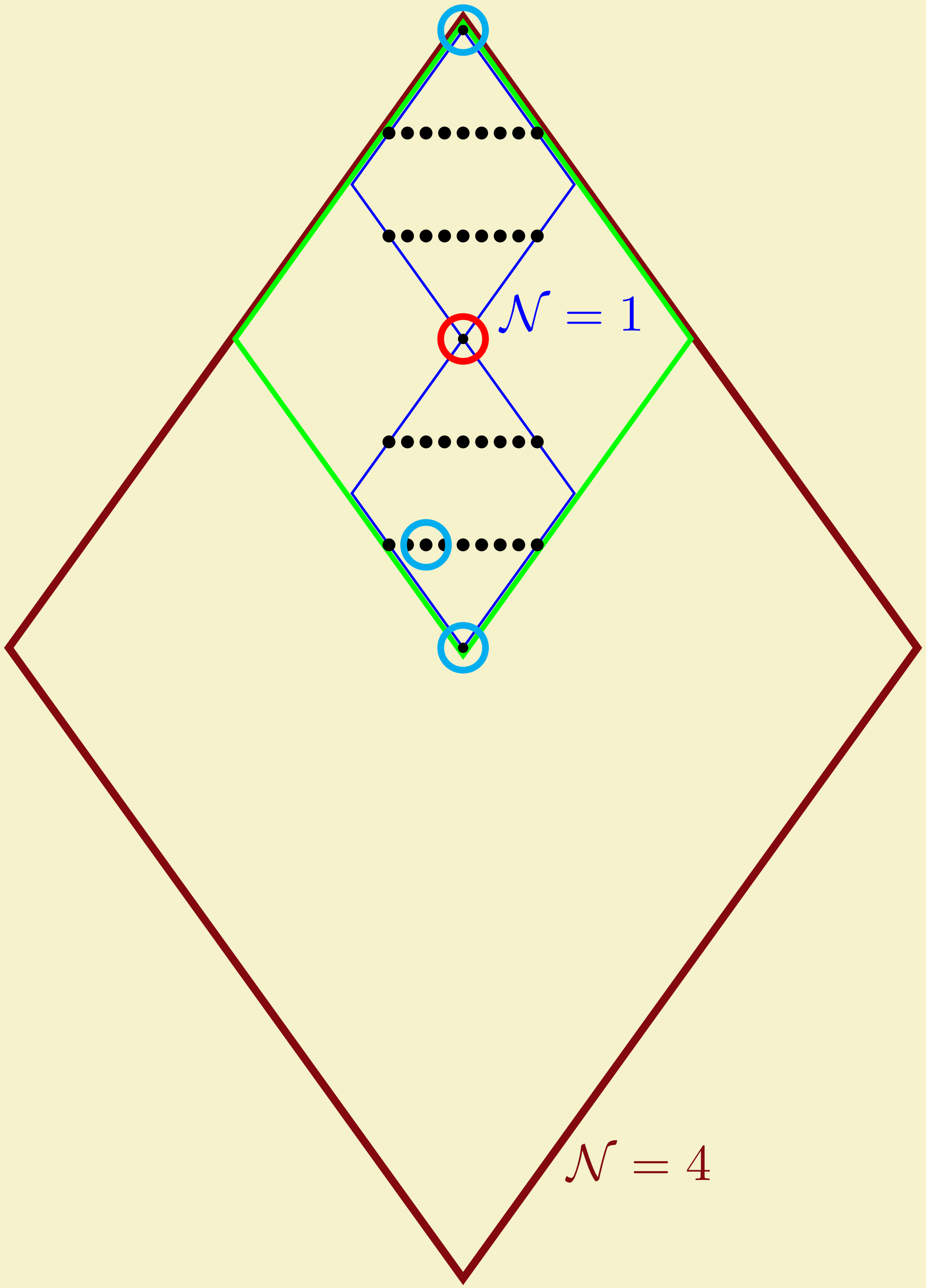
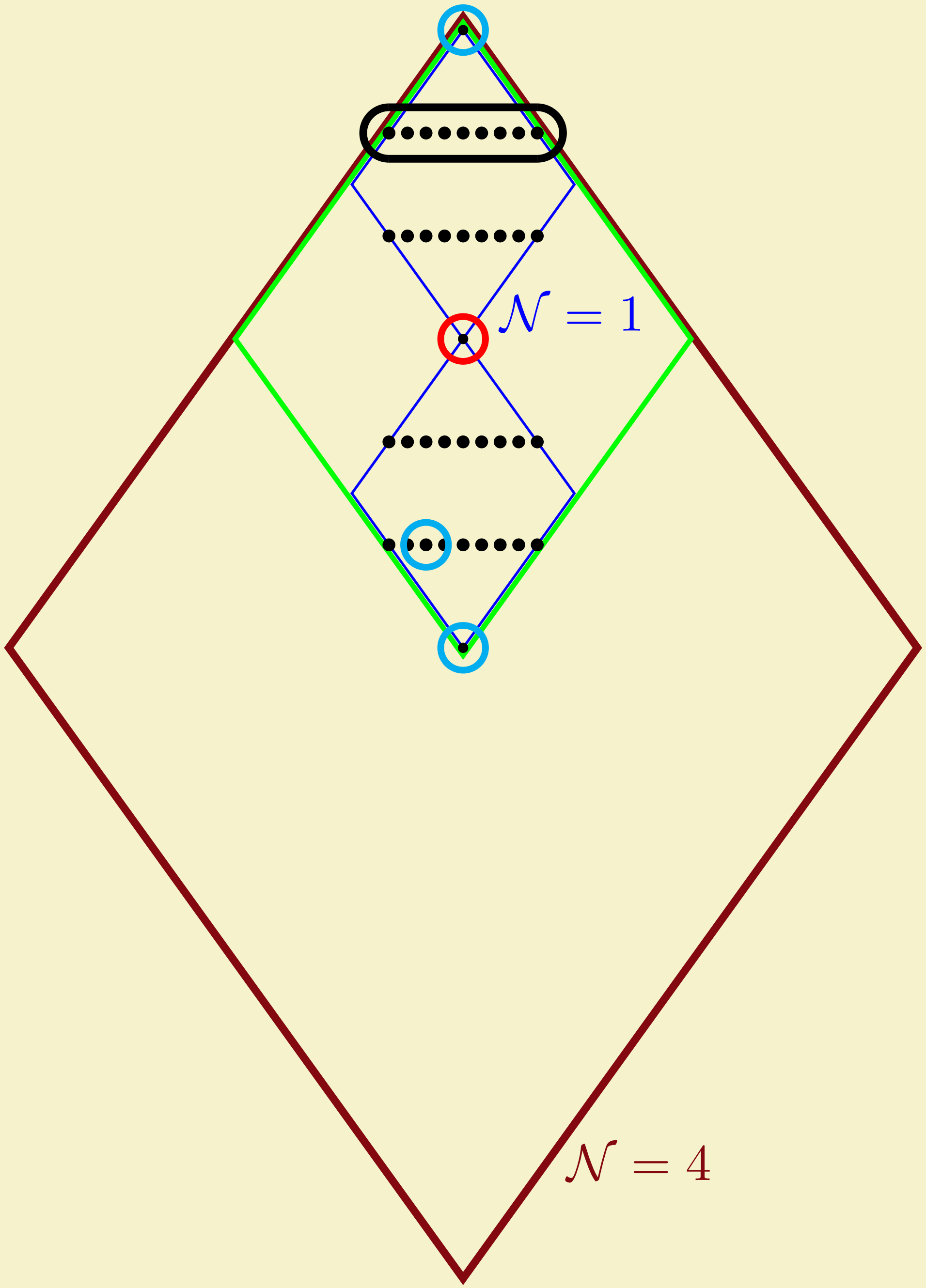
- determine in a first step the amplitude $\langle \lambda^-\, \lambda^-\, \lambda^-\, \lambda^+\, \lambda^+\, \lambda^+ \rangle$



How does everything combine?

- try to find a $\mathcal{N}=8$ amplitude satisfying all constraints, whose KLT-decomposition consists of $\mathcal{N}=1$ amplitudes only
$$\langle X^{1234}\, X_{1235}\, F^{5+} F_4^- \, B^+\, B^- \rangle = \text{KLT}\Big[\langle g^-\, \lambda^{4+}\, g^+\, \lambda_4^-\, g^+\, g^- \rangle_L \times \langle g^+\, \lambda_5^-\, \lambda^{5+}\, g^-\, g^+\, g^- \rangle_R \Big]$$
- consider the available open string calculations $\langle g^-\, g^-\, g^-\, g^+\, g^+\, g^+ \rangle$, $\langle \phi^-, \phi^-, \phi^-, \phi^+, \phi^+, \phi^+ \rangle$, $\langle \phi^-, \phi^-, \lambda^-, \lambda^+, \phi^+, \phi^+ \rangle$ [Stieberger Taylor]
- determine in a first step the amplitude $\langle \lambda^-\, \lambda^-\, \lambda^-\, \lambda^+\, \lambda^+\, \lambda^+ \rangle$
- Having now *two* amplitudes from the first $\mathcal{N}=1$ diamond on the disposal, calculate all permutations of amplitudes of type $\langle \lambda\lambda gggg \rangle$
- apply KLT-relations in order to determine the desired $\mathcal{N}=8$ result



Result and Outlook

$$\langle X^{1234} X_{1235} F^{5+} F_4^- B^+ B^- \rangle\Big|_{\mathcal{O}(\alpha'^3)} \xrightarrow[p_1,p_2\to 0]{} \frac{1}{2}\sum_{i=3}^{6} \frac{p_i\cdot(p_2-p_1)}{p_i\cdot(p_1+p_2)} R_5^4(\eta_i)\langle F^{5+} F_4^- B^+ B^- \rangle\Big|_{\mathcal{O}(\alpha'^3)}$$

$$= \frac{1}{2}\left[\frac{p_3\cdot(p_2-p_1)}{p_3\cdot(p_1+p_2)}\langle F^{4+} F_4^- B^+ B^- \rangle\Big|_{\mathcal{O}(\alpha'^3)} - \frac{p_4\cdot(p_2-p_1)}{p_4\cdot(p_1+p_2)}\langle F^{5+} F_5^- B^+ B^- \rangle\Big|_{\mathcal{O}(\alpha'^3)}\right]$$

- Double soft scalar limit relation does indeed hold for amplitudes in an $\alpha'^3\mathcal{R}^4$-corrected $\mathcal{N}=8$ supergravity $\rightarrow$ $\mathcal{R}^4$-term not restricted by onshell symmetry $E_{7(7)}/SU(8)$
- vanishing coefficient still awaits explanation

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$$= \frac{1}{2} \left[\frac{p_3 \cdot (p_2 - p_1)}{p_3 \cdot (p_1 + p_2)} \langle F^{4+} F_4^- B^+ B^- \rangle\Big|_{\mathcal{O}(\alpha'^3)} - \frac{p_4 \cdot (p_2 - p_1)}{p_4 \cdot (p_1 + p_2)} \langle F^{5+} F_5^- B^+ B^- \rangle\Big|_{\mathcal{O}(\alpha'^3)} \right]$$

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Thanks!