

Supersymmetric gauge and gravity theories: from loop calculations to non-perturbative states

Radu Roiban
PSU

mainly based on work with Z. Bern, J.J. Carrasco, H. Ita, H. Johansson

Outline

- On supergravity perturbation theory
 - General considerations on UV behavior; goals; expectations
- YM and supergravity perturbative calculations and the relation between them
- Supersums
 - algebraic
 - index diagrams (more in Henrik's talk)
- bootstrapping low loop information: all order cancellations
- Some comments on effects of nonperturbative states of $\mathcal{N} = 8$

- $\mathcal{N} = 8$ supergravity

deWit, Freedman (1977)

Cremmer, Julia, +Scherk (1978, 1979)

- largest amount of supersymmetry and spins smaller than 2
- classically unique; smallest representation of the symmetry algebra contains the graviton
- spectrum: $2^8 = 256$ massless fields

helicity	-2	-3/2	-1	-1/2	0	+1/2	+1	+3/2	+2
number of fields	1	8	28	56	70	56	58	8	1
names	h^-	ψ_i^-	v_{ij}^-	χ_{ijk}^-	s_{ijkl}	χ_{ijk}^+	v_{ij}^+	ψ_i^+	h^+

- 32 supercharges, $SU(8)$ R-symmetry, $E_{7(7)}$ duality symmetry, ⟨add your name here⟩ symmetry

- spectrum = tensor product of 2 $\mathcal{N} = 4$ abelian vector multiplets

helicity	-1	-1/2	0	+1/2	+1
number of fields	1	4	6	4	1

- compactification of 11d supergravity on a torus

More on the spectrum of scalars:

133 scalars (in adjoint of $E_{7(7)}$) with local $SU(8)$ symmetry

→ $E_{7(7)}/SU(8)$ sigma model; classically $E_{7(7)}(\mathbb{R})$ symmetry of $\mathcal{N} = 8$

- $E_{7(7)}/SU(8)$ appears smooth
- $E_{7(7)}$ transformations can map $\langle \Phi_i \rangle = 0$ to $\langle \Phi_i \rangle \neq 0$ points; classically, coefficients are real
- Perturbation theory is usually carried out about the $\langle \Phi_i \rangle = 0$ point
- Solitons describing 1/2-BPS black holes
 - related to compactified solutions of 11d SUGRA
 - carry nontrivial electric/magnetic charges
 - nonzero asymptotic values of scalar fields \mapsto away from origin
 - can be massless

e.g. $ds^2 = -\left(1 - \frac{D}{r^2}\right)^{-1/2} dt^2 + \left(1 - \frac{D}{r^2}\right)^{+1/2} dx^2$

■ Should $\mathcal{N} = 8$ be a decoupling limit of string theory? **Probably not:** BPS supergravity solitons should have some string theory interpretation

■ $\mathcal{N} = 8$ supergravity is not a decoupling limit of **nonperturbative** string theory

Green, Ooguri, Schwarz

essentially because of the **Dirac quantization condition**

Distler

$$M_e M_m = n M_{\text{Pl}}^2 = \text{fixed} \quad \longrightarrow \quad \text{only one can be infinite}$$

■ **Contrast with SYM:** open and closed string physics are governed by different coupling constants – g_s and G_N ; **decoupling** $\leftrightarrow G_N \rightarrow 0$

Perturbation theory around $\langle \Phi \rangle = 0$: expansion around flat space

- $[G_N] < 0 \rightarrow$ nonrenormalizable by powercounting
 - **covariance**: on-shell counterterms are built out of invariants constructed from tensors, up to using the equations of motion
 - **dimensional analysis**: $[G_N] = -[R] = -2$
 $\rightarrow \#_{R-s} + \#_{D-s} = 1 + \# \text{loops}$
 - 1-loop: **pure** gravity and supergravity are finite 't Hooft, Veltman
 - 2-loops: pure gravity diverges Goroff, Sangotti; van de Ven
no susy extension \rightarrow 2-loop pure sugra is finite Grisar; Tomboulis
 - susy completion of R^4
Deser, Kay, Stelle; Kallosh, Howe, Stelle, Townsend; Gross, Witten
- expect infinitely many **potential** counterterms to all loops

- **supersymmetry**: non-renormalization of F -terms
dimensional analysis \rightarrow max loop order for susy finiteness

\mapsto order by order UV finiteness of supergravity perturbation theory appears to require the existence of either some infinite-dimensional symmetry or of some nontrivial dynamical mechanism

\mapsto understanding their origin and full implications will advance our understanding of perturbative quantum gravity

\mapsto nonperturbative questions are separate issues

- ◇ resummation of perturbation theory

- ◇ unitarity of S-matrix and black hole production

- ◇ what about perturbation theory away from $\langle \Phi \rangle = 0$

- some comments later on

YM and supergravity higher-loop calculations

Main tool: generalized unitarity

Bern, Dixon, Dunbar, Kosower

Bern, Dixon, Kosower

recent improvements by Britto, Cachazo, Feng; Bern, Carrasco, Johansson, Kosower

Buchbinder, Cachazo; Cachazo, Skinner



- **method is recursive:** trees \mapsto loops; low loops \mapsto higher loops
- **symmetries (dual conformal, susy)** Drummond, Henn, Smirnov, Sokatchev
Bern, Czakon, Dixon, Kosower, Smirnov
- **effective rules:** rung, box substitution, twist, etc
Bern, Rozowsky, Yan; Bern, Carrasco, Johansson, Kosower; Bern, Carrasco, Johansson
- (relatively) recently: leading singularity method
Buchbinder, Cachazo; Cachazo; Arkani-Hamed, Cachazo, Kaplan
- ◇ **Assumes states propagating in loops are same as external states**

Presentations of tree amplitudes for supergravity:

- KLT relations Kawai, Lewellen, Tye
- BCFW Cachazo, Svrcek, Brandhuber, Spence, Travaglini
Benincasa, Boucher-Veronneau, Cachazo, Arkani-Hamed, Kaplan; Hall
- “square numerators” Bern, Carrasco, Johansson
- manifestly symmetric ($D=4$) Nguyen, Spradlin, Volovich, Wen

KLT relations: recycle YM amplitudes into gravity amplitudes
hold in D -dimensions

Derived from string theory tree amplitudes and the observation that closed string states are created by bilinears in operators creating open string states; additional factors due to string zero modes

★ manifest reflection of $[\mathcal{N} = 8] = [\mathcal{N} = 4] \times [\mathcal{N} = 4]$

$$M_4^{\text{tr}}(1, 2, 3, 4) = -is_{12}A_4^{\text{tr}}(1, 2, 3, 4)A_4^{\text{tr}}(1, 2, 4, 3)$$

$$M_5^{\text{tr}}(1, 2, 3, 4, 5) = is_{12}s_{34}A_5^{\text{tr}}(1, 2, 3, 4, 5)A_5^{\text{tr}}(2, 1, 4, 3, 5) + (2 \leftrightarrow 3)$$

$$M_6^{\text{tr}} = 12 \text{ terms of the type } s^3 A_6 A_6$$

★ hold for all states

★ clean low energy limit and dim. red. (eqv. to vacuum stability)

◇ KLT+unitarity

$$\sum_{\text{sugra sts}} M_{n_1}(\dots l_1, l_2, l_3 \dots) M_{n_2}(\dots l_1, l_2, l_3 \dots) = \sum_{\text{KLT terms}} (\text{s}_{ij} \text{ factors})$$

$$\left[\sum_{\text{SYM sts}} A_{n_1}(\dots l_1, l_2, l_3 \dots) A_{n_2}(\dots l_1, l_2, l_3 \dots) \right] \left[\sum_{\text{SYM sts}} A'_{n_1}(\dots l_1, l_2, l_3 \dots) A'_{n_2}(\dots l_1, l_2, l_3 \dots) \right]$$

• Construction of sugra cuts is a 2-step process

1. construct all color-stripped SYM cuts
2. KLT them into supergravity cuts

Some questions:

- when two SYM theories $\xrightarrow{\text{KLT}}$ supergravity theory?
- what structures of SYM theories impact UV behavior of Sugra?

e.g. β -deformed $\mathcal{N} = 4$ SYM: spectrum and symmetries of $\mathcal{N} = 4$ SYM **except** only $\mathcal{N} = 1$ susy. **1-loop nonplanar amplitudes contain triangles**

Jin, RR

Supersums

- chiral on-shell superspaces
- $\overline{\text{MHV}}$ amplitudes in MHV superspace
- MHV superrules
- structure of supersums
- algebraic form
- index diagrams
- lower susy

On-shell $D = 4$ superspace – various presentations

- Dual presentation of states of $\mathcal{N} = 4$ SYM:

$$\{g_{abcd}^+, f_{abc}^+, s_{ab}, f_a^-, g^-\} \xleftrightarrow{\epsilon^{abcd}} \{g_+, f_+^a, s^{ab}, f_{-}^{abc}, g_{-}^{abcd}\}$$

- Assemble in superfield

$$\Phi(\theta) = g^- + f_a^- \eta^a + s_{ab} \eta^a \eta^b + f_{abc}^+ \eta^a \eta^b \eta^c + g_{abcd}^+ \eta^a \eta^b \eta^c \eta^d$$

- Scattering of superfields \mapsto Superamplitude – $\mathcal{A}(\eta_1, \dots, \eta_n)$;

extract component amplitudes by Grassmann integration/differentiation

Nair; Elvang, Freedman, Kiermaier

- MHV superamplitude

$$\mathcal{A}_n^{\text{MHV}}(1, 2, \dots, n) \equiv \frac{i}{\prod_{j=1}^n \langle j \ (j+1) \rangle} \delta^{(8)} \left(\sum_{j=1}^n \lambda_\alpha^j \eta_j^a \right)$$

16 supercharges: $Q^{\alpha a} = \sum_i \lambda_i^\alpha \eta_i^a = \sum_u q_i^{\alpha a} \quad \tilde{Q}_a^{\dot{\alpha}} = \sum_i \tilde{\lambda}_i^{\dot{\alpha}} \tilde{\eta}_{ia} = \sum_i \tilde{q}_{ia}^{\dot{\alpha}}$

Only half can be manifest: \mapsto

- MHV (“chiral”) superspace: overall factor of $\delta^{(8)}(Q^{\alpha a})$
- $\overline{\text{MHV}}$ (“anti-chiral”) superspace: overall factor of $\delta^{(8)}(\tilde{Q}_a^{\dot{\alpha}})$
 - ◊ related by Fourier transform \hat{F}

Useful to have $\overline{\text{MHV}}$ in MHV superspace and vice-versa

V1: $\hat{F} \mathcal{A}^{\overline{\text{MHV}}}(1,2,\dots,n) = \frac{i(-1)^n}{\prod_{j=1}^n [j (j+1)]} \prod_{a=1}^4 \int d^2 \omega^a \prod_{i=1}^n \delta(\eta_i^a - \tilde{\lambda}_i^{\dot{\alpha}} \omega_{\dot{\alpha}}^a)$

◊ still proportional to $\delta^{(8)}(Q^{\alpha a})$ Drummond, Henn, Korchemsky, Sokatchev

V2: $\hat{F} \mathcal{A}^{\overline{\text{MHV}}}(1,2,\dots,n) = \frac{i(-1)^n}{\prod_{i=1}^n [i (i+1)]} \prod_{a=1}^4 \sum_{i < j} [i j] \partial_{\eta_i^a} \partial_{\eta_j^a} \eta_1^a \eta_2^a \cdots \eta_n^a$ Elvang, Freedman

◊ effective rule; treats $[i|, |j]$ as anticommuting

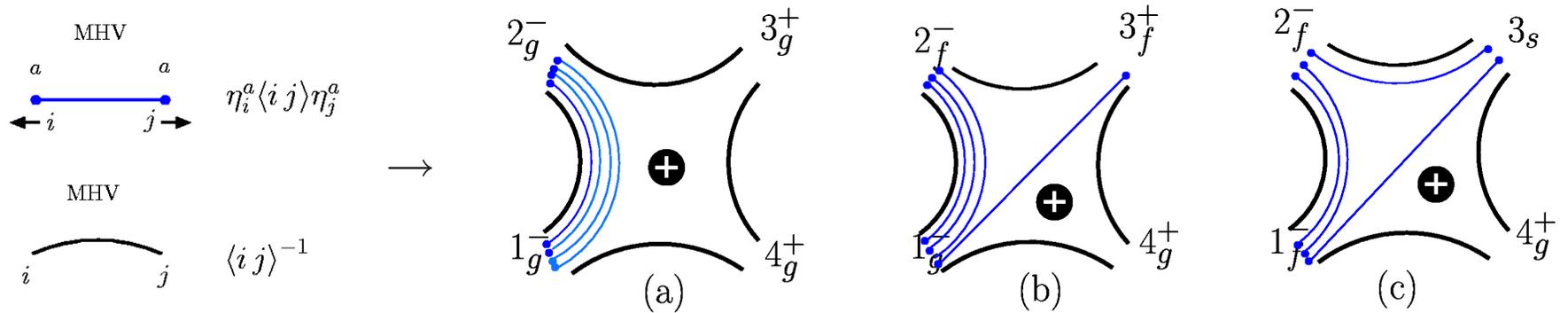
$$[i j] \tilde{\eta}_{ia} \tilde{\eta}_{ja} \xrightarrow{\hat{F}} \eta_1^a \cdots \eta_{i-1}^a [i| \eta_{i+1}^a \cdots \eta_{j-1}^a |j] \eta_{j+1}^a \cdots \eta_n^a$$

Index diagrams – pictorial representation of R-charge flow

Bern, Carrasco, Ita, Johansson, RR

$$\delta^{(8)}\left(\sum_{i=1}^n \lambda_{\alpha}^i \eta_i^{\alpha}\right) = \prod_{a=1}^4 \sum_{i < j} \langle i j \rangle \eta_i^a \eta_j^a \longrightarrow \text{basic factor is } \langle q_i^a q_j^a \rangle = \eta_i^a \langle i j \rangle \eta_j^a$$

- represent each factor diagrammatically



$$i \frac{\prod_{a=1}^4 \langle q_1^a q_2^a \rangle}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 1 \rangle} ; i \frac{\langle q_1^a q_2^a \rangle \langle q_1^b q_2^b \rangle \langle q_1^c q_2^c \rangle \langle q_1^d q_3^d \rangle}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 1 \rangle} ; i \frac{\langle q_1^a q_2^a \rangle \langle q_1^b q_2^b \rangle \langle q_1^c q_3^c \rangle \langle q_2^d q_3^d \rangle}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 1 \rangle}$$

- analogous representation for $\overline{\text{MHV}}$ amplitudes in $\overline{\text{MHV}}$ superspace

Fourier-transform “rule”: $[i j] \tilde{\eta}_{ia} \tilde{\eta}_{ja} \xrightarrow{\hat{F}} \eta_1^a \cdots \eta_{i-1}^a [i | \eta_{i+1}^a \cdots \eta_{j-1}^a | j] \eta_{j+1}^a \cdots \eta_n^a$

MHV superrules

Kiermaier, Elvang, Freedman; Boels, Mason, Skinner; Feng, Huang

- repackage of MHV rules for component amp's – just add “super”

For superamplitudes containing N^m MHV n -gluon amplitude –

- draw all tree graphs with $(m + 1)$ vertices and n external legs
- to each vertex associate the appropriate MHV superamplitude
- Off-shell extension – same as for bosonic amplitudes

$$\lambda_{P\alpha} \equiv P_{\alpha\dot{\alpha}}\zeta^{\dot{\alpha}} \text{ or } P^b = P - \frac{P^2}{2\zeta \cdot P}\zeta$$

- a propagator for each internal line (depends on superfields)
 - identifies the fermions of the internal line
 - integrates over common value

$$\int d^4\eta F(\eta)G(\eta) = \text{assume 4 different } \eta\text{-s}$$

$$F(\eta)\partial_\eta^4 G(\eta) + 4\partial_\eta F(\eta)\partial_\eta^3 G(\eta) + 6\partial_\eta^2 F(\eta)\partial_\eta^2 G(\eta) + 4\partial_\eta^3 F(\eta)\partial_\eta G(\eta) + \partial_\eta^4 F(\eta)G(\eta)$$

↳ implements sum over states

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 - integrates over common value – implements sum over states

$$\mathcal{A}_n^{N^m \text{MHV}} = i^m \sum_{\text{all graphs}} \int \left[\prod_{j=1}^m \frac{d^4 \eta_j}{P_j^2} \right] \mathcal{A}_{(1)}^{\text{MHV}} \mathcal{A}_{(2)}^{\text{MHV}} \cdots \mathcal{A}_{(m)}^{\text{MHV}} \mathcal{A}_{(m+1)}^{\text{MHV}}$$

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For superamplitudes containing N^m MHV n -gluon amplitude –

1. draw all tree graphs with $(m + 1)$ vertices and n external legs
2. to each vertex associate the appropriate MHV superamplitude
3. Off-shell extension – same as for bosonic amplitudes

$$\lambda_{P\alpha} \equiv P_{\alpha\dot{\alpha}}\zeta^{\dot{\alpha}} \text{ or } P^b = P - \frac{P^2}{2\zeta \cdot P}\zeta$$

4. a propagator for each internal line (depends on superfields)
 - identifies the fermions of the internal line
 - integrates over the common value – implements sum over states

Obs: on-shell condition may be imposed either by MHV vertex expansion or by a cut condition → sewing of general amplitudes by integration over common value of η evaluates the supersum.

General structure of a supercut

Bern, Carrasco, Ita, Johansson, RR

- **Supercut** – the collection of cuts in which any multi-particle state of SYM can cross the (generalized) cut; **organize in superspace**

$$\mathcal{C} = \int \left[\prod_{i=1}^k d^4 \eta_i \right] \mathcal{A}_{(1)}^{\text{tree}} \mathcal{A}_{(2)}^{\text{tree}} \mathcal{A}_{(3)}^{\text{tree}} \cdots \mathcal{A}_{(m)}^{\text{tree}}$$

- Generating function of all tree amplitudes w/ fixed # of legs

$$\mathcal{A}^{\text{tree}} = \mathcal{A}^{\text{MHV}} + \mathcal{A}^{\text{NMHV}} + \mathcal{A}^{\text{N}^2\text{MHV}} + \cdots + \mathcal{A}^{\text{N}^{(n-4)}\text{MHV}}$$

- **MHV superrules** \rightarrow suffices to consider supercuts in which $\mathcal{A}^{\text{tree}}$ contains only MHV and $\overline{\text{MHV}}$ components

\mapsto each supercut breaks up into pieces with each tree factor either MHV or $\overline{\text{MHV}}$ – “**holomorphicity configuration**”

$$\int \left[\prod_{i=1}^k d^4 \eta_i \right] \mathcal{A}_{(1)}^{\text{MHV}} \cdots \mathcal{A}_{(m')}^{\text{MHV}} \hat{\mathcal{A}}_{(m'+1)}^{\overline{\text{MHV}}} \cdots \hat{\mathcal{A}}_{(m)}^{\overline{\text{MHV}}}$$

◇ these “basic” components exhibit simple properties

1. each such term is an exact fourth power

- If $m' = m$

$$\int \left[\prod_i d^4 \eta_i \right] \prod_I \left(\prod_{a=1}^4 \delta^{(2)}(Q_I^a) \right) = \prod_{a=1}^4 \left(\int \left[\prod_i d\eta_i^a \right] \prod_I \delta^{(2)}(Q_I^a) \right),$$

- If $m' < m$: use Fourier transform

$$\prod_{a=1}^4 \int \left[\prod_i^n d\tilde{\eta}_{ia} e^{\eta_i^a \tilde{\eta}_{ia}} \right] \delta^{(2)} \left(\sum_{i=1}^n \tilde{\lambda}_\alpha^i \tilde{\eta}_i^a \right).$$

2. each supercut is manifestly proportional to $\delta^{(8)}(Q)$

$$\begin{aligned} \prod_{I=1}^r \delta^{(8)}(Q_I^a) &= \delta^{(8)} \left(\sum_{I=1}^r Q_I^a \right) \prod_{I=2}^r \delta^{(8)}(Q_I^a) \\ &= \delta^{(8)} \left(\sum_{i \in \mathcal{E}} \lambda_\alpha^i \eta_i^a \right) \prod_{I=2}^r \delta^{(8)}(Q_I^a) \end{aligned}$$

use sign rule $p \mapsto -p \Leftrightarrow \lambda \mapsto -\lambda$

◇ all supercuts and superamplitudes $\propto \delta^{(8)} \left(\sum_{i \in \mathcal{E}} \lambda_\alpha^i \eta_i^a \right)$

A consequence: UV finiteness of $\mathcal{N} = 4$ SYM

Renormalizable QFT w/ no more than 1 derivative per vertex

$$d_s = 4 - E + (D - 4)L - p$$

number of loop momenta coming out as
external momenta; p is half the number
of fermionic delta functions



- Here $p = 4 \mapsto d_s = -E + (D - 4)L$ i.e. finiteness in $D=4$
- Naively, critical dim. for SYM 4-point amplitudes:

$$D_c = 4 + \frac{4}{L} < 4 + \frac{6}{L} \quad \begin{array}{l} \swarrow \text{consequence of rung rule} \\ \searrow \text{harmonic superspace analysis} \end{array}$$

◇ 1/2 manifest susy captures most but not all susy cancellations

Supersums by solving linear equations

- **Goal:** carry out the η integrals sewing tree amplitudes
- **MHV superrules** \rightarrow η -dependence of any tree amplitude comes in the form of a product of δ functions: $N^m \text{MHV} \longleftrightarrow \prod_{i=1}^{m+2} \delta^{(4)}(f_i^a(p, \eta))$
with f_i linear functions of η
- In general, more δ functions than integration variables;
fewer δ functions \longleftrightarrow vanishing cut (SWI)
- ◇ **the algebraic supersum calculation is a 3(4) step process:**
 1. reconstruct overall supermomentum conservation
 2. for k cut lines, choose $4k$ of them and solve for integration η -s
 3. evaluate remaining δ functions; multiply by Jacobian
 - (4). restore symmetries; repeat 1,2,3 on other of δ -fct's

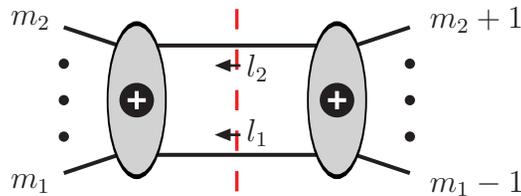
Useful tip: pick constraints with coef's of η independent of loop momenta

Supersums by solving linear equations

1. Cuts with MHV and MHV-expanded trees

- All δ functions are of the type $\delta^{(8)}(\sum_j \lambda_j^\alpha \eta_j^a)$

Example: supercut of 1-loop MHV superamplitude



$$\mathcal{C} = \int d^4 \eta_{l_1} \int d^4 \eta_{l_2}$$

$$\mathcal{A}^{\text{MHV}}(-l_1, m_1, \dots, m_2, -l_2) \mathcal{A}^{\text{MHV}}(l_2, m_2+1, m_1-1, l_1)$$

$$\delta^{(8)}\left(-\lambda_{l_1}^\alpha \eta_{l_1}^a - \lambda_{l_2}^\alpha \eta_{l_2}^a + \sum_{i=m_1}^{m_2} \lambda_i^\alpha \eta_i^a\right) \delta^{(8)}\left(\lambda_{l_1}^\alpha \eta_{l_1}^a + \lambda_{l_2}^\alpha \eta_{l_2}^a + \sum_{i=m_2+1}^{m_1-1} \lambda_i^\alpha \eta_i^a\right)$$

$$\downarrow$$

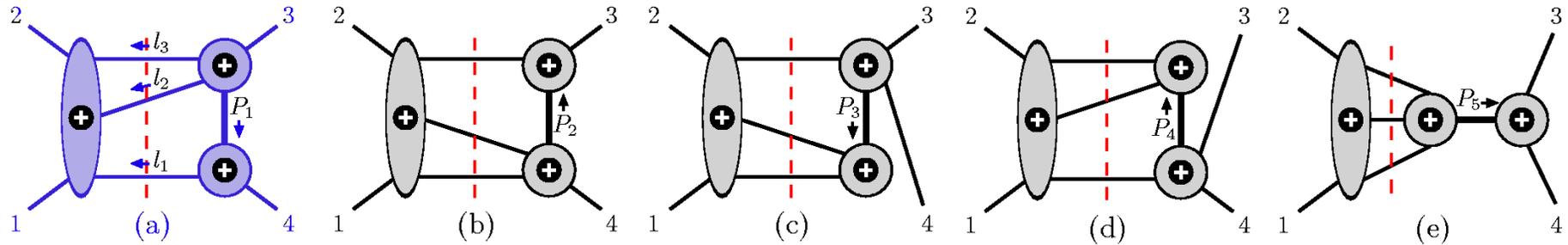
$$\delta^{(8)}\left(-\lambda_{l_1}^\alpha \eta_{l_1}^a - \lambda_{l_2}^\alpha \eta_{l_2}^a + \sum_{i=m_1}^{m_2} \lambda_i^\alpha \eta_i^a\right) \delta^{(8)}\left(\sum_{i=m_2+1}^{m_1-1} \lambda_i^\alpha \eta_i^a + \sum_{i=m_1}^{m_2} \lambda_i^\alpha \eta_i^a\right)$$

$$J = \det^4 \begin{vmatrix} \lambda_{l_1}^1 & \lambda_{l_2}^1 \\ \lambda_{l_1}^2 & \lambda_{l_2}^2 \end{vmatrix} = \langle l_1 l_2 \rangle^4 \rightarrow \mathcal{C} = -\delta^{(8)}(Q_\mathcal{E}) \langle l_1 l_2 \rangle^4 \times (\text{usual } \langle \rangle \text{ stuff})$$

Supersums by solving linear equations

- MHV-expanded trees \rightarrow more complicated results

Example: $\text{MHV} \times \overline{\text{MHV}}$



$$\delta^{(2)}(\lambda_1^\alpha \eta_1^a + \lambda_2^\alpha \eta_2^a - \lambda_{l_1}^\alpha \eta_{l_1}^a - \lambda_{l_2}^\alpha \eta_{l_2}^a - \lambda_{l_3}^\alpha \eta_{l_3}^a) \delta^{(2)}(\lambda_3^\alpha \eta_3^a + \lambda_{P_1}^\alpha \eta_{P_1}^a + \lambda_{l_2}^\alpha \eta_{l_2}^a + \lambda_{l_3}^\alpha \eta_{l_3}^a) \delta^{(2)}(\lambda_4^\alpha \eta_4^a + \lambda_{l_1}^\alpha \eta_{l_1}^a - \lambda_{P_1}^\alpha \eta_{P_1}^a)$$

$$\begin{aligned} \mathcal{C} = & \delta^{(8)}(\lambda_1^\alpha \eta_1^a + \lambda_2^\alpha \eta_2^a + \lambda_3^\alpha \eta_3^a + \lambda_4^\alpha \eta_4^a) \frac{1}{\langle 12 \rangle \langle 2l_3 \rangle \langle l_3 l_2 \rangle \langle l_2 l_1 \rangle \langle l_1 1 \rangle} \\ & \times \left[\frac{1}{\langle l_2 l_3 \rangle \langle l_3 3 \rangle \langle 3 P_1^b \rangle \langle P_1^b l_2 \rangle} \frac{1}{P_1^2} \frac{1}{\langle 4 l_1 \rangle \langle l_1 P_1^b \rangle \langle P_1^b 4 \rangle} \left(\langle l_1 P_1^b \rangle \langle l_2 l_3 \rangle \right)^4 \right. \\ & + \frac{1}{\langle P_2^b l_3 \rangle \langle l_3 3 \rangle \langle 3 P_2^b \rangle} \frac{1}{P_2^2} \frac{1}{\langle l_2 P_2^b \rangle \langle P_2^b 4 \rangle \langle 4 l_1 \rangle \langle l_1 l_2 \rangle} \left(\langle l_3 P_2^b \rangle \langle l_1 l_2 \rangle \right)^4 \\ & + \frac{1}{\langle l_3 3 \rangle \langle 3 4 \rangle \langle 4 P_3^b \rangle \langle P_3^b l_3 \rangle} \frac{1}{P_3^2} \frac{1}{\langle P_3^b l_1 \rangle \langle l_1 l_2 \rangle \langle l_2 P_3^b \rangle} \left(\langle l_3 P_3^b \rangle \langle l_1 l_2 \rangle \right)^4 \\ & \left. + \frac{1}{\langle l_2 l_3 \rangle \langle l_3 P_4^b \rangle \langle P_4^b l_2 \rangle} \frac{1}{P_4^2} \frac{1}{\langle l_1 P_4^b \rangle \langle P_4^b 3 \rangle \langle 3 4 \rangle \langle 4 l_1 \rangle} \left(\langle l_1 P_4^b \rangle \langle l_2 l_3 \rangle \right)^4 \right] \end{aligned}$$

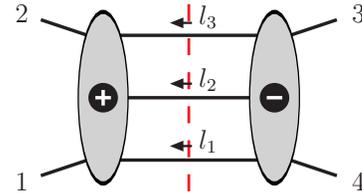
◇ preferable to avoid (MHV-)expanding trees

Supersums by solving linear equations

2. Cuts with MHV and $\overline{\text{MHV}}$ trees

- Use Fourier-transform of $\overline{\text{MHV}}$ amplitude to MHV superspace

- Treat ω on equal footing with η



$$\int d\eta_{l_1}^a d\eta_{l_2}^a d\eta_{l_3}^a d^2\omega^a \delta^{(2)}(\lambda_1^\alpha \eta_1^a + \lambda_2^\alpha \eta_2^a - \lambda_{l_1}^\alpha \eta_{l_1}^a - \lambda_{l_2}^\alpha \eta_{l_2}^a - \lambda_{l_3}^\alpha \eta_{l_3}^a) \\ \times \delta(\eta_{l_1}^a - \tilde{\lambda}_{l_1}^{\dot{\alpha}} \omega_{\dot{\alpha}}^a) \delta(\eta_{l_2}^a - \tilde{\lambda}_{l_2}^{\dot{\alpha}} \omega_{\dot{\alpha}}^a) \delta(\eta_{l_3}^a - \tilde{\lambda}_{l_3}^{\dot{\alpha}} \omega_{\dot{\alpha}}^a) \delta(\eta_3^a - \tilde{\lambda}_3^{\dot{\alpha}} \omega_{\dot{\alpha}}^a) \delta(\eta_4^a - \tilde{\lambda}_4^{\dot{\alpha}} \omega_{\dot{\alpha}}^a)$$



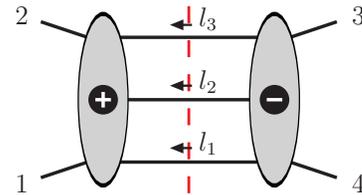
$$\delta^2(\lambda_1^\alpha \eta_1^a + \lambda_2^\alpha \eta_2^a + \lambda_3^\alpha \eta_3^a + \lambda_4^\alpha \eta_4^a) \\ \times \int d\eta_{l_1}^a d\eta_{l_2}^a d\eta_{l_3}^a d^2\omega^a \delta(\eta_{l_1}^a - \tilde{\lambda}_{l_1}^{\dot{\alpha}} \omega_{\dot{\alpha}}^a) \delta(\eta_{l_2}^a - \tilde{\lambda}_{l_2}^{\dot{\alpha}} \omega_{\dot{\alpha}}^a) \delta(\eta_{l_3}^a - \tilde{\lambda}_{l_3}^{\dot{\alpha}} \omega_{\dot{\alpha}}^a) \delta(\eta_3^a - \tilde{\lambda}_3^{\dot{\alpha}} \omega_{\dot{\alpha}}^a) \delta(\eta_4^a - \tilde{\lambda}_4^{\dot{\alpha}} \omega_{\dot{\alpha}}^a)$$

$$J = \det^4 \begin{pmatrix} 1 & 0 & 0 & -\tilde{\lambda}_{l_1}^1 & -\tilde{\lambda}_{l_1}^2 \\ 0 & 1 & 0 & -\tilde{\lambda}_{l_2}^1 & -\tilde{\lambda}_{l_2}^2 \\ 0 & 0 & 1 & -\tilde{\lambda}_{l_3}^1 & -\tilde{\lambda}_{l_3}^2 \\ 0 & 0 & 0 & -\tilde{\lambda}_3^1 & -\tilde{\lambda}_3^2 \\ 0 & 0 & 0 & -\tilde{\lambda}_4^1 & -\tilde{\lambda}_4^2 \end{pmatrix} \mapsto \frac{\delta^{(8)}\left(\sum_{i=1}^4 \lambda_i^\alpha \eta_i^a\right) [34]^4}{\langle 12 \rangle \langle 2l_3 \rangle \langle l_3 l_2 \rangle \langle l_2 l_1 \rangle \langle l_1 1 \rangle [34] [4l_1] [l_1 l_2] [l_2 l_3] [l_3 3]}$$

Supersums by solving linear equations

2. Cuts with MHV and $\overline{\text{MHV}}$ trees

- Use Fourier-transform of $\overline{\text{MHV}}$ amplitude to MHV superspace
- Treat ω on equal footing with η



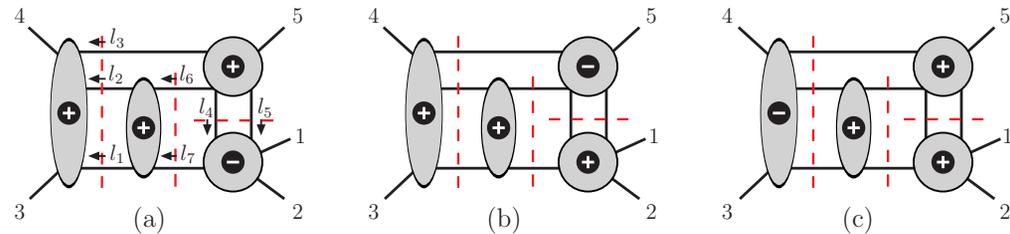
$$\int d\eta_{l_1}^a d\eta_{l_2}^a d\eta_{l_3}^a d^2\omega^a \delta^{(2)}(\lambda_1^\alpha \eta_1^a + \lambda_2^\alpha \eta_2^a - \lambda_{l_1}^\alpha \eta_{l_1}^a - \lambda_{l_2}^\alpha \eta_{l_2}^a - \lambda_{l_3}^\alpha \eta_{l_3}^a) \\ \times \delta(\eta_{l_1}^a - \tilde{\lambda}_{l_1}^{\dot{\alpha}} \omega_{\dot{\alpha}}^a) \delta(\eta_{l_2}^a - \tilde{\lambda}_{l_2}^{\dot{\alpha}} \omega_{\dot{\alpha}}^a) \delta(\eta_{l_3}^a - \tilde{\lambda}_{l_3}^{\dot{\alpha}} \omega_{\dot{\alpha}}^a) \delta(\eta_3^a - \tilde{\lambda}_3^{\dot{\alpha}} \omega_{\dot{\alpha}}^a) \delta(\eta_4^a - \tilde{\lambda}_4^{\dot{\alpha}} \omega_{\dot{\alpha}}^a)$$

$$J = \det^4 \begin{pmatrix} 1 & 0 & 0 & -\tilde{\lambda}_{l_1}^1 & -\tilde{\lambda}_{l_1}^2 \\ 0 & 1 & 0 & -\tilde{\lambda}_{l_2}^1 & -\tilde{\lambda}_{l_2}^2 \\ 0 & 0 & 1 & -\tilde{\lambda}_{l_3}^1 & -\tilde{\lambda}_{l_3}^2 \\ 0 & 0 & 0 & -\tilde{\lambda}_3^1 & -\tilde{\lambda}_3^2 \\ 0 & 0 & 0 & -\tilde{\lambda}_4^1 & -\tilde{\lambda}_4^2 \end{pmatrix} = [34]^4$$

- ◇ **General rule:** for every $\overline{\text{MHV}}$ factor with 2 external legs, p and k , the supercut gets a factor of $[pk]^4$ – improved powercounting?

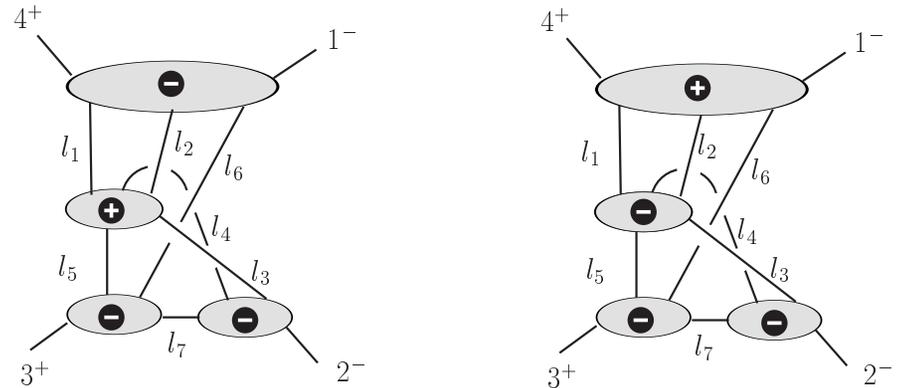
Some comments:

- not restricted to 4-points
- not restricted to low orders



$$(\langle l_1 l_2 \rangle \langle l_3 l_6 \rangle [12])^4 ; \langle l_1 l_2 \rangle^4 \langle l_7 | l_4 + l_5 | 5 \rangle^4 ; (\langle l_4 l_5 \rangle \langle l_6 l_7 \rangle [34])^4$$

- not restricted to planar cuts



$$(\langle 1 2 \rangle [l_7 2] [l_6 3] [1 4])^4 ; (\langle 1 2 \rangle [2 3] [l_3 l_4] [l_5 l_7])^4$$

◇ used in the construction of the 4-loop $\mathcal{N} = 4$ and $\mathcal{N} = 8$ 4-pt amplitudes

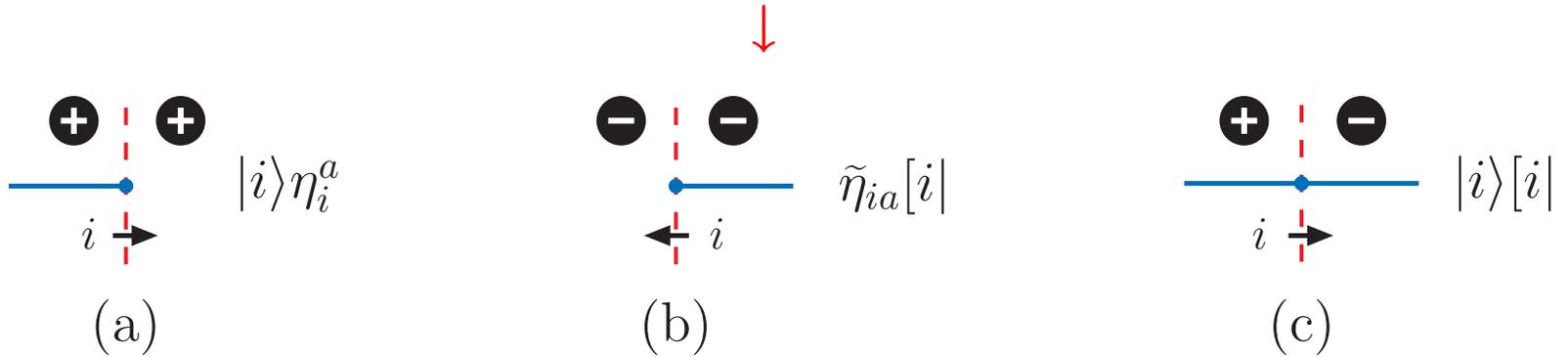
Bern, Carrasco, Dixon, Johansson, RR

- it accounts for all identities involving over-antisymmetrization
- restricted to $D = 4$
- cannot isolate contribution of a single state; restriction to $\mathcal{N} = 4$ solved by use of index diagrams

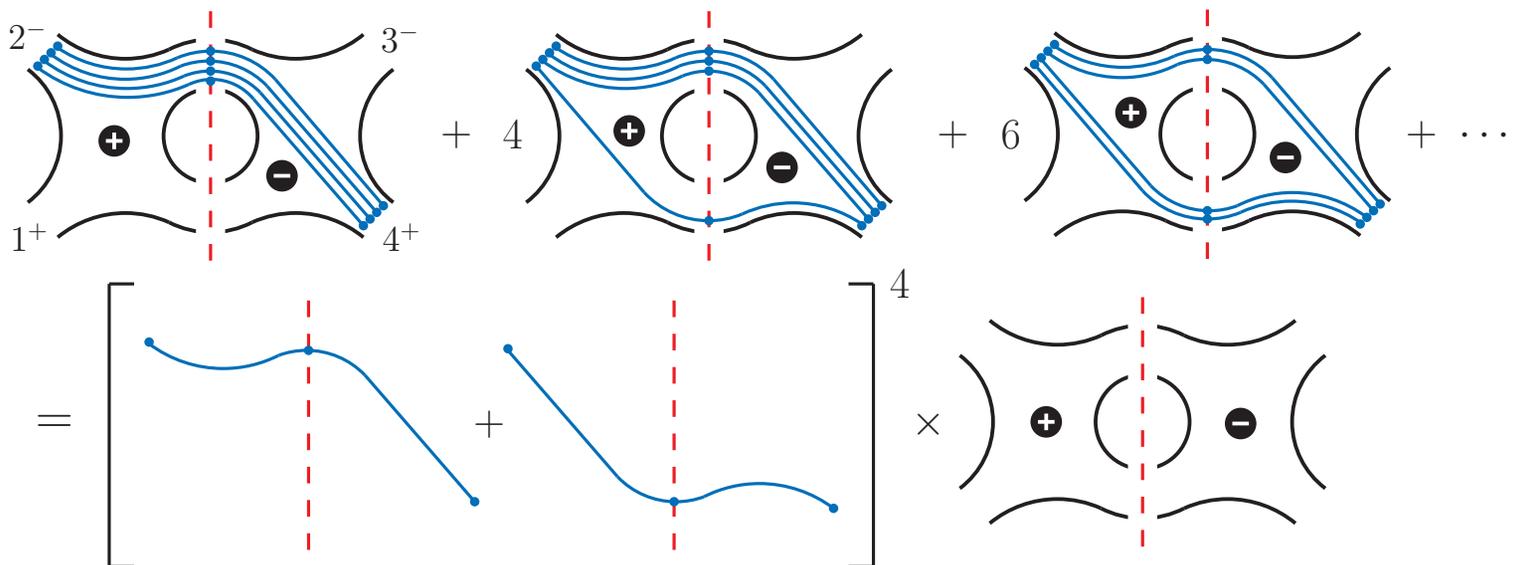
Sewing of index diagrams

- sewing operators in various superspaces

$$\int d\eta \mathcal{A}_L^{\text{MHV}}(\eta) \mathcal{A}_R^{\text{MHV}}(\eta) \quad \int d\tilde{\eta} \mathcal{A}_L^{\overline{\text{MHV}}}(\tilde{\eta}) \mathcal{A}_R^{\overline{\text{MHV}}}(-\tilde{\eta}) \quad \int d\eta d\tilde{\eta} e^{\eta\tilde{\eta}} \mathcal{A}_L^{\text{MHV}}(\eta) \mathcal{A}_R^{\overline{\text{MHV}}}(\tilde{\eta})$$



Example: $A_4 \times A_4$ in $\text{MHV} \times \overline{\text{MHV}}$ superspace



Diagrams \mapsto spinor expressions

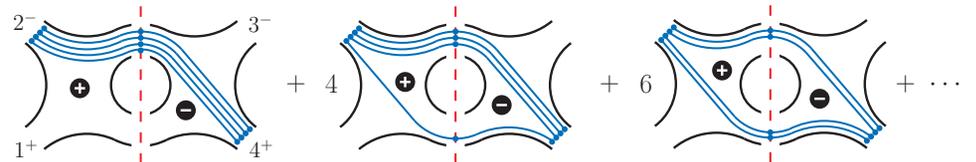
- Index diagrams identify the terms that survive η integration; only signs are needed

V1 Transform all to MHV superspace:

$$[\tilde{q}_{ia} \tilde{q}_{ja}] \xrightarrow{\widehat{F}} \eta_1^a \cdots \eta_{i-1}^a [i | \eta_{i+1}^a \cdots \eta_{j-1}^a | j] \eta_{j+1}^a \cdots \eta_n^a$$

Extract sign as $\eta_{i_1}^a \cdots \eta_{i_n}^a \mapsto \sigma[i_1, \dots, i_n]$

Example:



$$\left((\langle q_2 q_{l_1} \rangle) ([\tilde{q}_{l_1} \tilde{q}_4]) + (\langle q_2 q_{l_2} \rangle) ([\tilde{q}_{l_2} \tilde{q}_4]) \right)^4 = \left[\text{diagram 1} + \text{diagram 2} \right]^4 \times \text{diagram 3}$$

$$\left((\eta_2 \langle 2 l_1 \rangle \eta_{l_1}) (\eta_{l_2} [l_1 | \eta_3 | 4]) + \eta_2 \langle 2 l_2 \rangle \eta_{l_2} ([l_2 | \eta_{l_1} \eta_3 | 4]) \right)^4 = (\langle 23 \rangle [34])^4$$

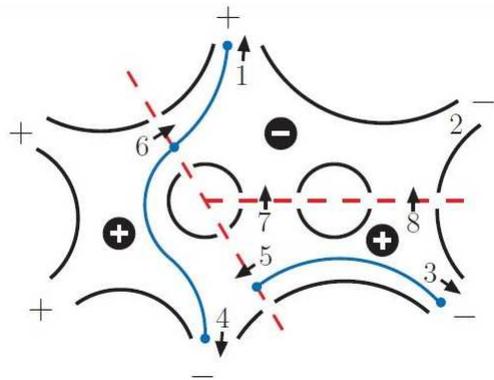
Diagrams \mapsto spinor expressions

- Index diagrams identify the terms that survive η integration; only signs are needed

V2 use mixed superspace

- for each unbroken index line write spinor string
- attach Grassmann parameters at the ends of strings
- $\eta_{i_1} \dots \eta_{i_l} \tilde{\eta}_{j_1} \dots \tilde{\eta}_{j_m} \mapsto \sigma[i_1 \dots i_l] \sigma[j_1 \dots j_m]$
- treat η and $\tilde{\eta}$ as commuting

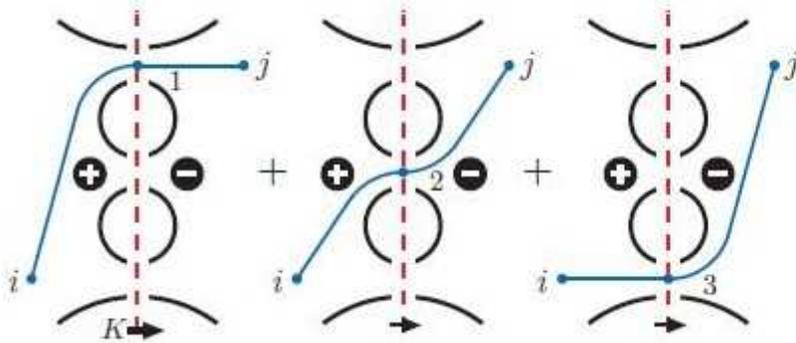
Example:



$$(\tilde{\eta}_1 [1|6|4\rangle \eta_4) (\eta_3 \langle 3\ 5\rangle \eta_5) \mapsto -[1|6|4\rangle \langle 3\ 5\rangle$$

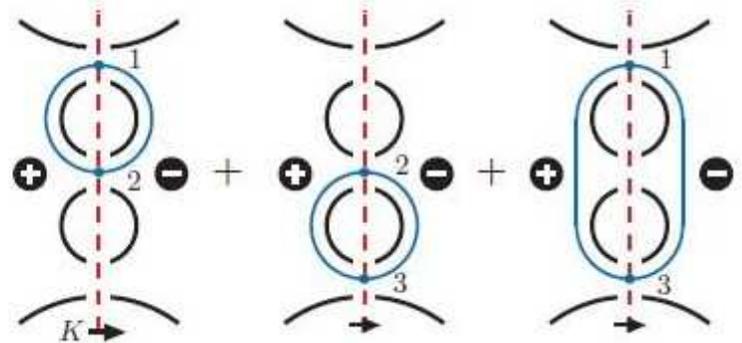
Simplify [%]

- momentum conservation



$$\eta_i \langle i | l_1 | j \rangle \tilde{\eta}_j + \eta_i \langle i | l_2 | j \rangle \tilde{\eta}_j + \eta_i \langle i | l_3 | j \rangle \tilde{\eta}_j$$

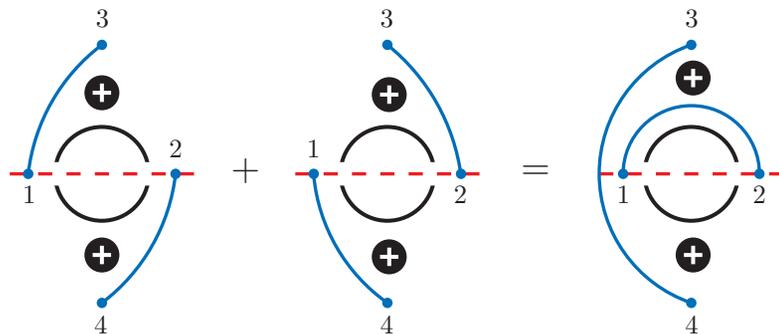
$$= \eta_i \langle i | l_1 + l_2 + l_3 | j \rangle \tilde{\eta}_j$$



$$-\langle l_1 l_2 \rangle [l_2 l_1] - \langle l_2 l_3 \rangle [l_3 l_2] - \langle l_1 l_3 \rangle [l_3 l_1]$$

$$= -(l_1 + l_2 + l_3)^2$$

- Schouten's identity



$$\langle -q_1 q_3 \rangle \langle -q_2 q_4 \rangle + \langle q_1 q_4 \rangle \langle q_2 q_3 \rangle$$

$$= \langle -q_1 q_2 \rangle \langle q_3 q_4 \rangle$$

Reduced supersymmetry

$$\begin{aligned} \text{supersum} &= (A_1 + A_2 + A_3 + \dots)^4 \equiv (A_1 + A_2 + A_3 + \dots)^{\mathcal{N}} \\ &= \sum_i A_i^4 + 4 \sum_{i \neq j} A_i A_j^3 + \dots \end{aligned}$$

gluon states \uparrow
2-fermion states \uparrow } fix the $\mathcal{N} = 4$ supersum

- gluons – 4 index lines routed the same way
- 2-fermion – 3 index lines routed the same way

Require that some set of index lines are always routed together \Leftrightarrow
ignore fields carrying subsets of the corresponding indices

E.g. $a = 3, 4$ together \mapsto no contributions from f_+^3 , s^{a3} ($a \neq 4$), etc

- Two possible field contents:

$$g_+, f_+^a, s^{ab}, s^{34}, f_-^{b34}, g_-^{ab34}, a, b = 1, 2$$

$$g_+, f_+^a, f_-^{234}, g_-^{a234}$$

$$\text{supersum} = (A_1 + A_2 + A_3 + \dots)^{\mathcal{N}} (A_1^{4-\mathcal{N}} + A_2^{4-\mathcal{N}} + A_3^{4-\mathcal{N}} + \dots)$$

- Corresponding MHV supervertex

$$\mathcal{A}_n^{\text{MHV}}(1, 2, \dots, n) = \frac{i}{\prod_{j=1}^n \langle j \ (j+1) \rangle} \left(\prod_{a=1}^{\mathcal{N}} \delta^{(2)}(Q^a) \right) \times \left(\sum_{i < j}^n \langle i \ j \rangle^{4-\mathcal{N}} \prod_{a=\mathcal{N}+1}^4 \eta_i^a \eta_j^a \right)$$

bookeeping ↑

◇ Overall $\delta^{(\mathcal{N})}(Q_{\mathcal{E}})$ is guaranteed

SYM supersums \rightarrow Supergravity supersums

- chiral superspace w/ 16 supercharges $\delta^{(8)}(Q) \mapsto \delta^{(16)}(Q)$
 same construction applies, albeit with twice as many equations
- KLT: constructs supergravity supercuts from SYM supersums

$$\begin{aligned}
 M_n^{L\text{-loop}} \Big|_{\text{cut}} &= \sum_{\mathcal{N}=8} M_{n_1}^{\text{tree}} M_{n_2}^{\text{tree}} \\
 &= \sum_{\mathcal{N}=8} \left(\sum_{i,j} g_{ij} A_{n_1}^{(i)} A_{n_1}^{(j)} \right) \left(\sum_{k,l} g_{kl} A_{n_2}^{(k)} A_{n_2}^{(l)} \right) \\
 &= \sum_{i,j,k,l} g_{ij} g_{kl} \left(\sum_{\mathcal{N}=4} A_{n_1}^{(i)} A_{n_2}^{(k)} \right) \left(\sum_{\mathcal{N}=4} A_{n_1}^{(j)} A_{n_2}^{(l)} \right)
 \end{aligned}$$

regular SYM supersums



- sugra supercuts do not hide additional cancellation mechanisms beyond those existing in 2 copies of SYM supercuts

Bootstrapping low orders to all orders

- Chiral on-shell superspace constraints

- $\mathcal{N} = 4$ SYM: finite but not all cancellations accounted for
- $\mathcal{N} = 8$ SUGRA: improved superficial d.o.d. vanishes at $L = 3$

$$d_s = (D-2)L+2 \quad \mapsto \quad \widehat{d}_s = d_s - 8 = (D-2)L - 6 < 0 \text{ in } D = 4 \text{ for } L < 3$$

$$D_c = 2 + \frac{6}{L}$$

★ naive app. of rung rule \oplus mild assumptions: $\widehat{\widehat{d}}_s = \widehat{d}_s - 4 \rightarrow L < 5$

$$D_c = 2 + \frac{10}{L}$$

Bootstrapping low orders to all orders

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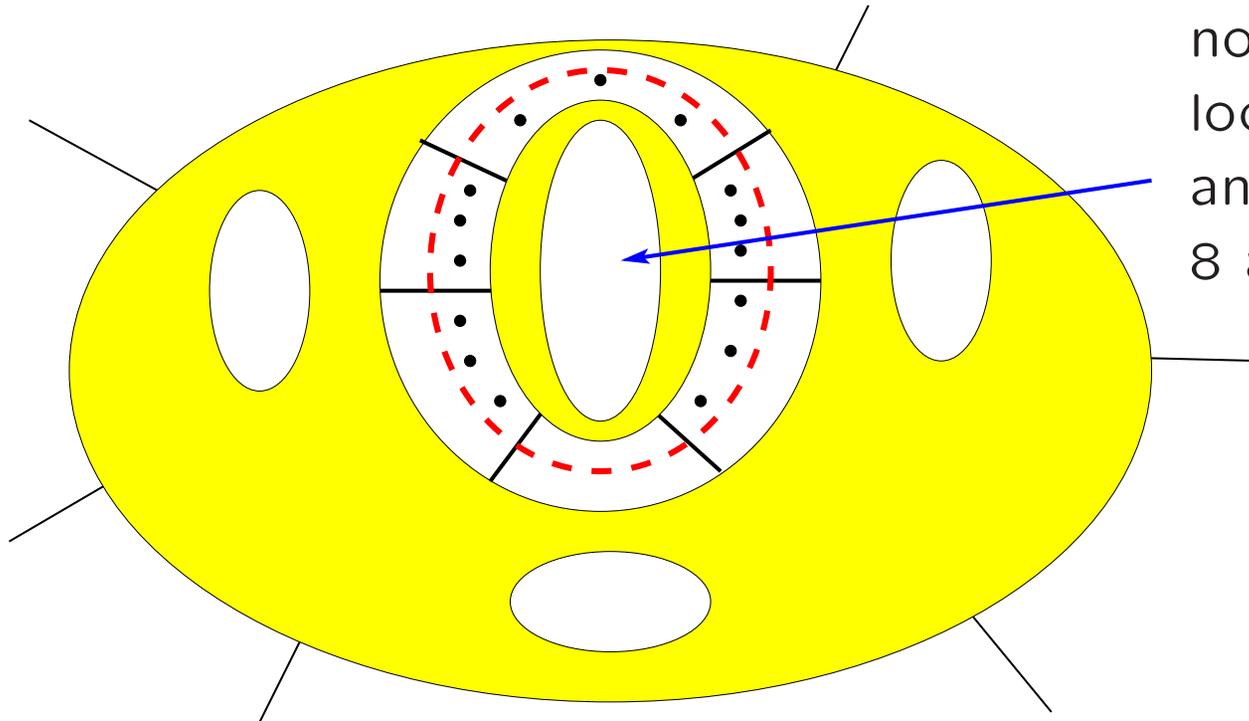
- ★ naive app. of rung rule \oplus mild assumptions: $\widehat{\widehat{d}}_s = \widehat{d}_s - 4 \rightarrow L < 5$

- All-loop consequences of low-loop calculations

generalized unitarity method \longrightarrow properties of subamplitudes

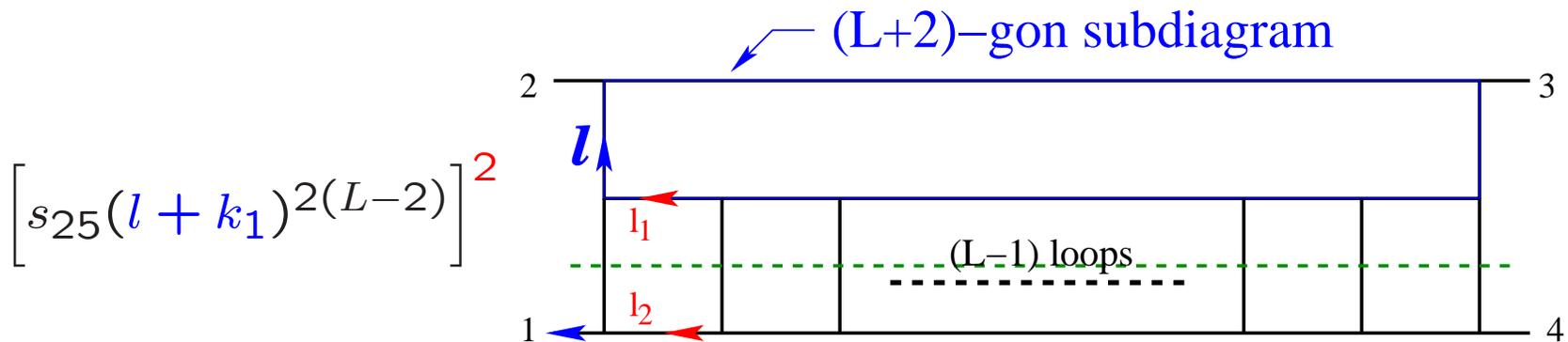
- 1-loop no-triangle behavior Bern, Dixon, Dunbar, Perelstein, Rozowsky
Bern, Bjerum-Bohr, Dunbar; Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager
for a proof: Bjerrum-Bohr, Vanhove; Arkani-Hamed, Cachazo, Kaplan
- 2-, 3-, 4-loop 4-point amplitudes Bern, Dixon, Dunbar, Perelstein, Rozowsky
Bern, Carrasco, Dixon, Johanson, Kosower, RR
Bern, Carrasco, Dixon, Johanson, RR

- no-triangle behavior

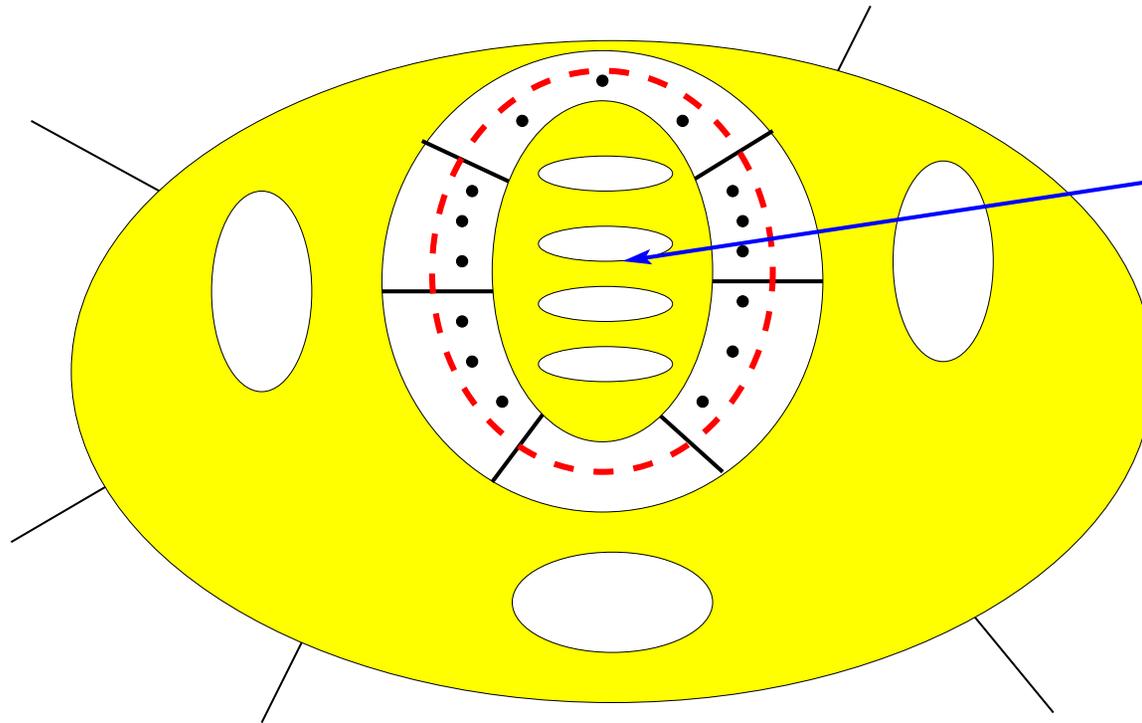


no triangles in any 1-loop subamplitude of any loop n -point $\mathcal{N} = 8$ amplitude

- Explicit test at 3-, 4-loops



- same for isolating 2-, 3-, 4-loop subamplitudes (only global UV)



any 2-, 3-, 4-loop subamplitude should be the same as the corresponding 2-, 3-, 4-loop amplitude

$n \geq 5$ only UV behavior may be inferred

- ★ add extra leg does not change overall powercounting:
2 derivatives per vertex vs. additional propagator

Assume perturbative finiteness; what next?

Asymptoticity, summability, vacuum stability, and such

- In generic QFT, perturbation theory is asymptotic. Planar $\mathcal{N} = 4$ is expected to have finite radius of convergence $\lambda_c \leq \pi^2$, but not nonplanar $\mathcal{N} = 4$.

- (Borel) summability? Not clear

However – many classical solutions with the same energy

→ potential tunneling → may be related to nonsumability

(cf. double-well anharmonic oscillator)

⇒ Resummation of perturbation theory requires additional input
accounting for $\exp(-\frac{S_{\text{cl}}}{G_N})$ corrections

★ Similar properties are shared by perturbative string theory

Gross, Periwal

- Unitarity vs. states not propagating in loops

An analogy: sine-Gordon theory $L = \frac{1}{2}(\partial\phi)^2 + \frac{m_0^2}{\beta^2} \cos(\beta\phi)$

- weak coupling expansion of the exact S-matrix agrees with perturbative calculations, order by order in perturbation theory

Arefeva, Korepin

- spectrum of L consists of a soliton, an anti-soliton and soliton-anti-soliton bound states

Dashen, Hasslacher, Neveu
Zamolodchikov²

$$m_{\text{sol}} = \frac{8m}{\gamma} \quad m_k = 2m_{\text{sol}} \sin \frac{k\gamma}{16} \quad \gamma = \frac{\beta^2}{1 - \frac{\beta^2}{8\pi}} \quad k = 1, 2, \dots, \frac{8\pi}{\gamma}$$

while solitons are visible as poles in the exact S-matrix of fundamental particles, they *do not* propagate in loops

- For $8\pi \geq \beta^2 \geq \frac{8\pi}{1+\pi}$ soliton is lightest state $0 \leq m_{\text{sol}} \leq m$

⇒ appropriate description is in terms of solitons: m . Thirring model

A possibility:

1. large fraction of information on the other states of supergravity is already captured by perturbation theory around origin
2. additional information for these states is necessary to resum perturbation theory
3. the set of perturbative states may change as one moves around $E_{7(7)}/SU(8)$
4. the set of perturbative expansions around all points of $E_{7(7)}/SU(8)$ would allow resummation of the S-matrix

Summary

- Spectrum and symmetries of $\mathcal{N} = 8$ supergravity; relation to string theory; relation between SYM and SUGRA perturbation theories; some consequences of finiteness; nonperturbative questions
- Supersums are an integral part of calculations within the generalized unitarity method; efficient computation; simple results; diagrammatic technology; extension to reduced susy
- UV consequences of supersums; all-loop consequences of finite-order calculations
- Is it possible that, even though not propagating in loops, all states of the theory make some appearance in perturbation theory around origin?