# Loops, Legs and Twistors

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# Twistors: Tree-Level Yang-Mills

## A Review of Twistor String Theory

In the beginning, there was Twistor String Theory [Witten (2003)].

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$$\mathcal{A}^{\mathrm{MHV}}(\lambda_i, \widetilde{\lambda}_i) = \int d^4x \exp\left( i x_{a\dot{a}} \lambda_i^a \widetilde{\lambda}_i^{\dot{a}} \right) \prod_{i=1}^{n} \frac{1}{\langle i\, i+1 \rangle}$$

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Then, recognizing $d^4x$ as the measure on the moduli space of lines in $\mathbb{P}^3$, Witten (2003) checked via difficult calculation several cases of his conjecture that:

The N$^k$MHV superamplitude in super Yang-Mills is supported on curves in $\mathbb{P}^{3|4}$ of degree $k+1$.

## The 'Connected Prescription'

According to twistor string theory, the N$^k$MHV superamplitude is given explicitly by [Roiban, MS, Volovich (2004)]:

$$\mathcal{A}^{\mathrm{N}^k\mathrm{MHV}}(\mathcal{Z}_i) = \int [\mathcal{D}\mathcal{C}_{k+1}] \, \frac{d^n z}{(z_1 - z_2)\cdots(z_n - z_1)} \prod_{i=1}^{n} \delta^{3|4}(\mathcal{Z}_i - \mathcal{C}_d(z_i))$$

- $\mathcal{Z}_i = (\lambda_i, \mu_i)$ where $\mu_i$ is related to $\widetilde{\lambda}_i$ by "Fourier transform".
- $\mathcal{C}_d(z)$ denotes a degree $d$ curve in $\mathbb{P}^{3|4}$.
- $[\mathcal{D}\mathcal{C}_d]$ denotes the measure on the moduli space of such curves
- $\prod \frac{1}{z_i - z_{i+1}}$ is the WZW current algebra correlator—here arising from vertex operators of open strings ending on an instanton which wraps the curve $\mathcal{C}(z)$.
- The delta functions force the specified $\mathcal{Z}_i$ to lie on the curve!

# Great Features of the Connected Prescription

## Manifest Properties

- Conformal symmetry is manifest for all superamplitudes.
- Dihedral symmetry $i \to i+1, i \to n+1-i$ is manifest for all superamplitudes.

## Almost Manifest Properties

- Parity symmetry [Roiban, MS, Volovich (2004), Witten (2004)]:

$$\mathrm{N}^k\mathrm{MHV}(\lambda_i, \widetilde{\lambda}_i) = \mathrm{N}^{n-k-4}\mathrm{MHV}(\widetilde{\lambda}_i, \lambda_i)$$

- Possesses the correct collinear limits for all superamplitudes.
- Possesses the correct soft limits for all superamplitudes.

## Not So Great Features ...

- Absence of spurious singularities is not obvious...

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- Absence of spurious singularities is not obvious...
- For that matter, possession of the correct multiparticle singularities is far from obvious!

- A little processing reveals that the formula must be interpreted as a contour integral of the form

$$\int d^m z \frac{h(\vec{z})}{f_1(\vec{z}) \cdots f_m(\vec{z})} = \sum_{\vec{z}_* : f_1(\vec{z}_*) = \cdots = f_m(\vec{z}_*) = 0} h(\vec{z}_*) \left[ \det \left( \frac{\partial f_i}{\partial z_j} \right) \right]^{-1}_{\vec{z} = \vec{z}_*}$$

- So, calculating any superamplitude reduces to the problem of finding the roots of some polynomial equations.

The numbers of roots are Eulerian numbers:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| n=4 | | | | 1 | | | |
| n=5 | | | | 1 | 1 | | |
| n=6 | | | 1 | 4 | 1 | | |
| n=7 | | | 1 | 11 | 11 | 1 | |
| n=8 | | 1 | 26 | 66 | 26 | 1 | |
| n=9 | 1 | 57 | 302 | 302 | 57 | 1 | |
| n=10 | 1 | 120 | 1191 | 2416 | 1191 | 120 | 1 |

MHV NMHV NNMHV

The total number of roots is $(n-3)!$ — so the formula is conceptually beautiful, but sadly computationally useless!

## The Connected Prescription in Ambitwistor Space

Motivated by the work of Arkani-Hamed, Cachazo, Cheung & Kaplan (2009), let us consider "Fourier transforming" some of the twistor variables

$$\mathcal{Z}_i = (\lambda_i, \mu_i) \qquad \rightarrow \qquad \mathcal{W}_i = (\widetilde{\mu}_i, \widetilde{\lambda}_i)$$

via

$$\mathcal{A}(\mathcal{W}_a, \mathcal{Z}_J) = \int \exp\left( i \sum_a \mathcal{W}_a \cdot \mathcal{Z}_a \right) \mathcal{A}(\mathcal{Z}_i)$$

For the N$^k$MHV superamplitude, an astoundingly convenient choice is to leave precisely $k+2$ particles in the $\mathcal{Z}$ representation and transform all others to $\mathcal{W}$.

The integral over the moduli space of curves is then a triviality...

# The ‘Connected Link’ Formula

... leading to the formula

$$\mathcal{A}(\mathcal{W}_i, \mathcal{Z}_J) = \int d^{(n-k-2)\times(k+2)} c_{iJ}\, U(c_{iJ})\ \exp\left( i \sum_{i,J} c_{iJ} \mathcal{W}_i \cdot \mathcal{Z}_J \right)$$

with the ‘link representation’

$$U(c_{iJ}) = \int \frac{d^n z}{(z_1 - z_2)\cdots(z_n - z_1)} \frac{d^n c}{c_1 \cdots c_n} \prod_{i,J} \delta\left( c_{iJ} - \frac{c_i c_J}{z_i - z_J} \right)$$

Reminder: this is a contour integral, with the delta-functions indicating which singularities the contour is supposed to encircle.

## **An Example:** $A(1^+, 2^+, 3^+, 4^+, 5^-, 6^-, 7^-)$

The link representations are simple to work out on a case-by-case basis, for example

$$U^{++++---} = \frac{c_{25}c_{26}c_{36}c_{37}}{(c_{15}c_{26} - c_{16}c_{25})(c_{36}c_{47} - c_{37}c_{46})}\delta(S_{123:567})\delta(S_{234:567})$$

Generically, the N$^k$MHV superamplitude involves $k(n-k-4)$ delta-functions of 'sextics':

$$S_{ijk:lmn} = c_{im}c_{in}c_{jl}c_{kl}c_{km} \pm 5 \text{ permutations}$$

## Returning to Physical Space

Now the biggest benefit of 'link representations' is that going to physical space is trivial:

$$A(\lambda_i, \widetilde{\lambda}_i) = \int d^{(n-k-2)\times(k+2)} c_{iJ}\, U(c_{iJ}) \prod_i \delta^2(\lambda_i - c_{iJ}\lambda_J) \prod_J \delta^2(\widetilde{\lambda}_J + c_{iJ}\lambda_i)$$

The delta-functions here give $2n-4$ linear equations in terms of $(n-k-2)(k+2)$ variables, which can be solved in terms of $k(n-k-4)$ parameters (we'll call them $\tau$).

Returning to our example we now have

$$A^{++++---} = \int d\tau_1 d\tau_2 \frac{c_{25}c_{26}c_{36}c_{37}}{(c_{15}c_{26} - c_{16}c_{25})(c_{36}c_{47} - c_{37}c_{46})} \delta(S_{123:567})\delta(S_{234:567})$$

where each $c$ is linear in $\tau_1, \tau_2$ and the $S_{ijk:lmn}$ are quartic.

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Evaluating this contour integral gives

$$\sum \frac{c_{25}c_{26}c_{36}c_{37}}{(c_{15}c_{26} - c_{16}c_{25})(c_{36}c_{47} - c_{37}c_{46})} \left[ \det \frac{\partial(S_{123:567}, S_{234:567})}{\partial(\tau_1, \tau_2)} \right]^{-1}$$

where the sum runs over all 11 roots of

$$S_{123:567}(\tau_1, \tau_2) = S_{234:567}(\tau_1, \tau_2) = 0.$$

and one can check numerical agreement with the amplitude.

This is just the familiar connected prescription, in new variables.

## An Amazing Object

But consider more generally the object

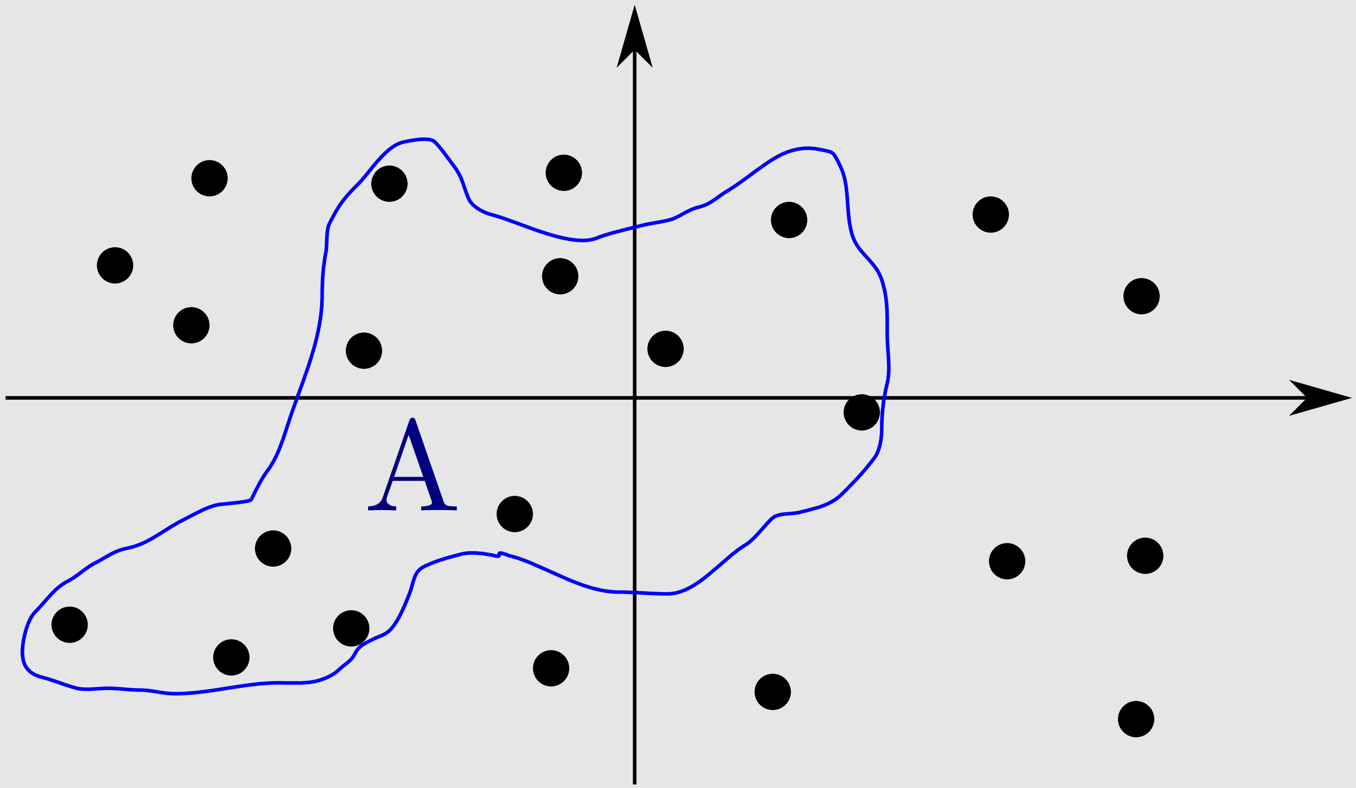
$$T^{++++---}(\tau_1,\tau_2) = \frac{c_{25}c_{26}c_{36}c_{37}}{(c_{15}c_{26}-c_{16}c_{25})(c_{36}c_{47}-c_{37}c_{46})}\,\frac{1}{S_{123:567}S_{234:567}}$$

Apparently we've learned that the contour integral

$$\oint d\tau_1 d\tau_2\, T^{++++---}(\tau_1,\tau_2)$$

over the contour which encircles the 11 poles of the second piece calculates the ‘connected prescription representation’ for the tree-level amplitude.

But $T^{++++---}$ has plenty of other poles in the $(\tau_1, \tau_2)$ plane (actually, 11 of them — a numerological coincidence). What do THOSE residues compute?



To explore these other contours we will use the...

Global Residue Theorem:

$$\sum_{\vec{z}_*:f_1(\vec{z}_*)=\cdots f_2(\vec{z}_*)=0} h(\vec{z}_*) \left[\det\left(\frac{\partial f_i}{\partial z_j}\right)\right]^{-1} = 0$$

Important: this is true as long as $h(z)$ has no poles!

In our example there are seven different global residue identities:

$$T^{++++---}(\tau_1,\tau_2) = \frac{c_{25}c_{26}c_{36}c_{37}}{(c_{15}c_{26}-c_{16}c_{25})(c_{36}c_{47}-c_{37}c_{46})}\frac{1}{S_{123:567}S_{234:567}}$$

$$(1): \quad f_1 = (c_{15}c_{26}-c_{16}c_{25}), \qquad f_2 = (c_{36}c_{47}-c_{37}c_{46})S_{123:567}S_{234:567}$$

$$(2): \quad f_1 = (c_{36}c_{47}-c_{37}c_{46}), \qquad f_2 = (c_{15}c_{26}-c_{16}c_{25})S_{123:567}S_{234:567}$$

$$(3): \quad f_1 = S_{123:567}, \qquad f_2 = (c_{15}c_{26}-c_{16}c_{25})(c_{36}c_{47}-c_{37}c_{46})S_{234:567}$$

$$(4): \quad f_1 = S_{234:567}, \qquad f_2 = (c_{15}c_{26}-c_{16}c_{25})(c_{36}c_{47}-c_{37}c_{46})S_{123:567}$$

$$(5): \quad f_1 = (c_{15}c_{26}-c_{16}c_{25})(c_{36}c_{47}-c_{37}c_{46}), \qquad f_2 = S_{123:567}S_{234:567}$$

$$(6): \quad f_1 = (c_{15}c_{26}-c_{16}c_{25})S_{234:567}, \qquad f_2 = (c_{36}c_{47}-c_{37}c_{46})S_{123:567}$$

$$(7): \quad f_1 = (c_{15}c_{26}-c_{16}c_{25})S_{123:567}, \qquad f_2 = (c_{36}c_{47}-c_{37}c_{46})S_{234:567}$$

I would not waste screen space writing them all out if they weren't all extremely important!

Let’s just look at one of them:

$$(6): \quad f_1 = (c_{15}c_{26} - c_{16}c_{25})S_{234:567}, \qquad f_2 = (c_{36}c_{47} - c_{37}c_{46})S_{123:567}$$

Clearly, all of the ‘connected prescription’ residues

$$S_{123:567} = S_{234:567} = 0$$

also contribute to the global residue identity number (6), but other residues contribute as well, namely solutions of:

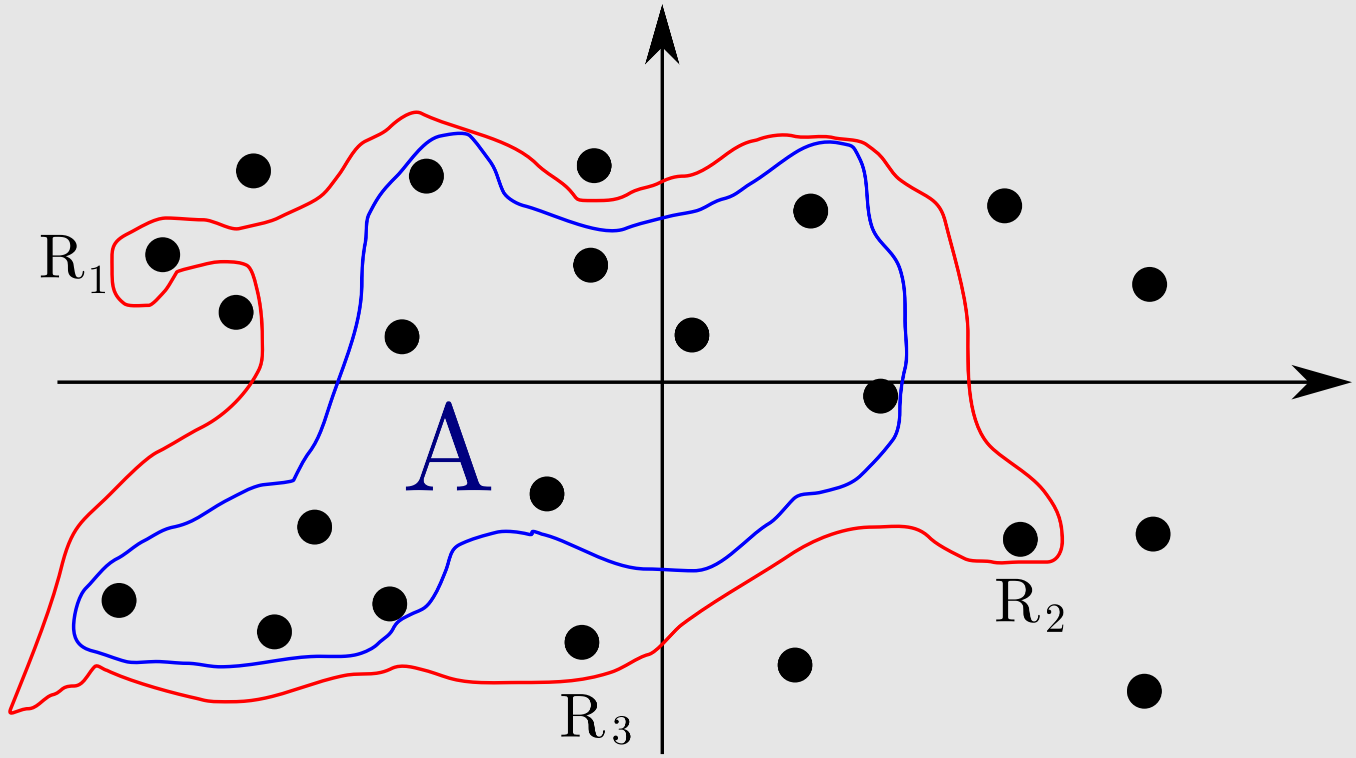
$$(c_{15}c_{26} - c_{16}c_{25}) = S_{123:567} = 0$$

$$S_{234:567} = (c_{36}c_{47} - c_{37}c_{46}) = 0$$

$$(c_{15}c_{26} - c_{16}c_{25}) = (c_{36}c_{47} - c_{37}c_{46}) = 0$$

Amazingly each of these three sets of equations has precisely one solution in the $(\tau_1, \tau_2)$ plane!

One way to see that this is not outrageous is to note that $S_{ijk:lmn}$ dramatically simplifies on the locus where $c_{il}c_{jm} - c_{im}c_{lj} = 0$.




So, the global residue theorem says the sum of the red residues is ZERO.

But the connected prescription tells us that the sum of the blue residues is the tree-level (++++—) amplitude.

Therefore we learn that

$$A^{++++---} = -R_1 - R_2 - R_3$$

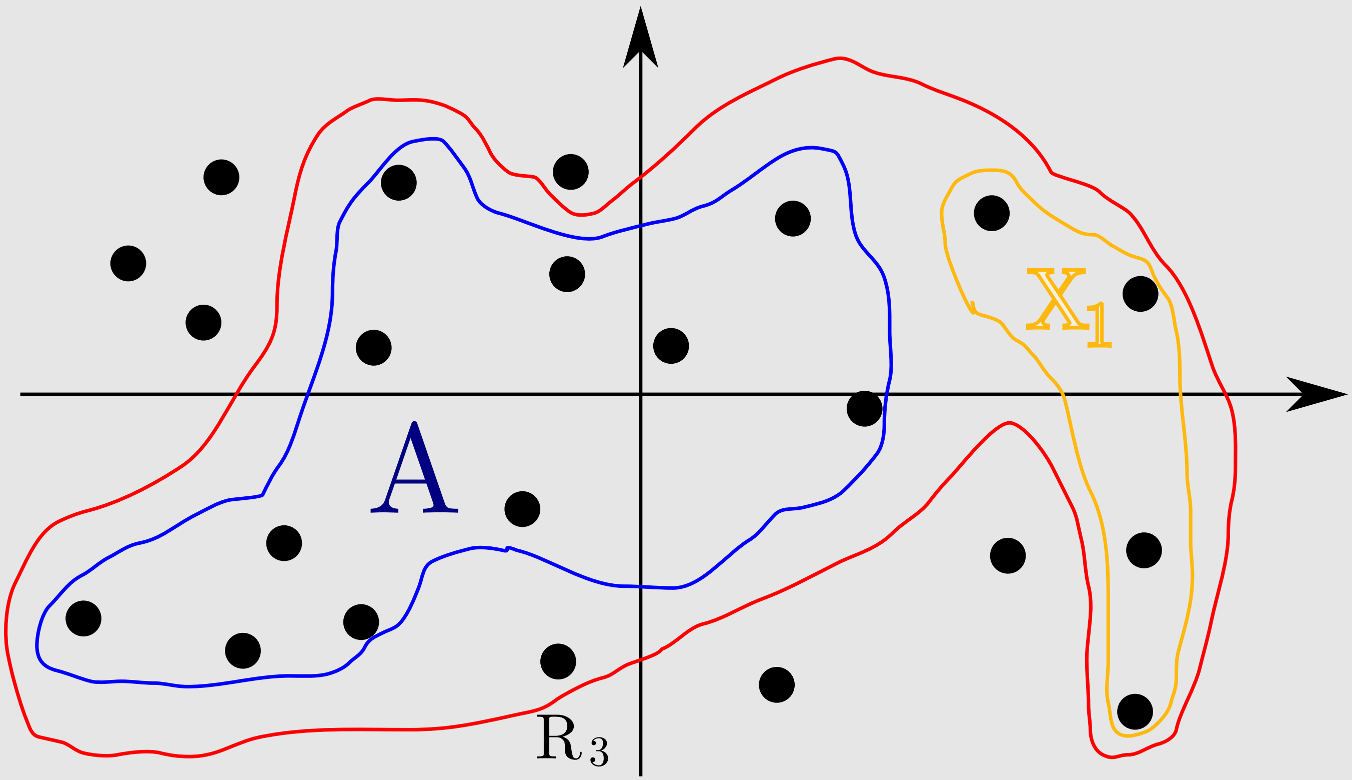
Now, I mentioned that the equations determining the locations of $R_1, R_2, R_3$ have one solution each; you can easily solve for the corresponding $(\tau_1, \tau_2)$.

Plugging them in leads to the formula:

$$A^{++++---} = \frac{\langle 5|6+7|1]^3}{s_{671}\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle[6\,7][7\,1]\langle 2|1+7|6]}$$
$$+\frac{\langle 7|1+2|3+4|5\rangle^3}{s_{712}s_{345}\langle 7\,1\rangle\langle 1\,2\rangle\langle 3\,4\rangle\langle 4\,5\rangle\langle 2|1+7|6]}$$
$$-\frac{\langle 7|6+5|4]^3}{\langle 3|4+5|6]\langle 7\,1\rangle\langle 1\,2\rangle\langle 2\,1\rangle[4\,5][5\,6]}$$

The twistor string arrives at the BCFW party, fashionably late!

But that was only one of the seven GRT identities!



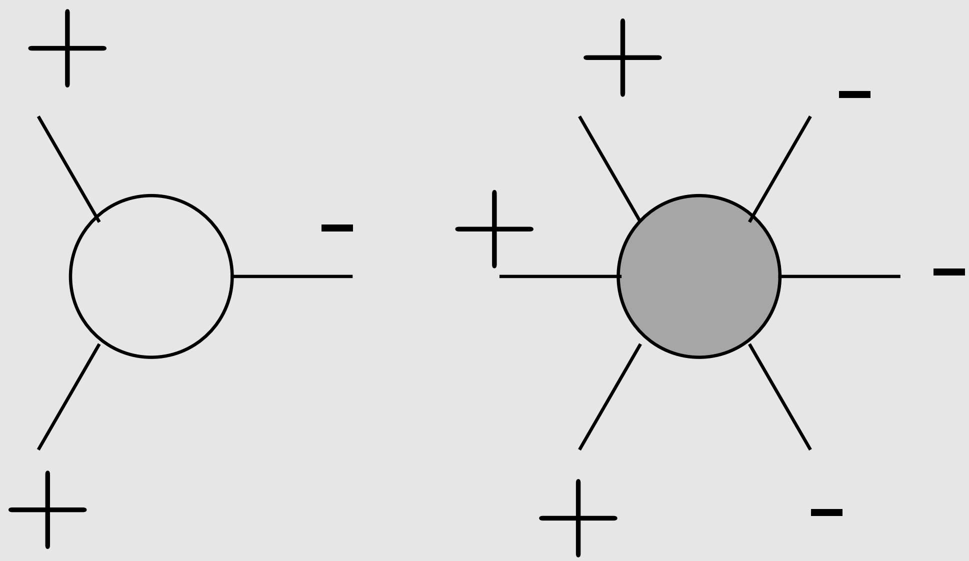

Another identity expresses the amplitude as

$$A = -R_3 - X_1$$

where $X_1$ is a sum of four residues whose locations are specified by the roots of a quartic polynomial.

This is an 'intermediate prescription' [Bena, Bern, Kosower (2004)]

That means that we apply BCFW once to express the amplitude




but now, instead of applying BCFW again on the term on the right, we use the connected formula which expresses the $(+++---)$ amplitude as a sum over the roots of a quartic polynomial.

Other GRT identities give another 'intermediate' decomposition, and consistency conditions between various representations.

## Summary: The Twistor String Amplitude

The twistor string amplitude is

$$\mathcal{T}(\mathcal{Z}_i) = \int [\mathcal{D}\mathcal{C}_{k+1}] \frac{d^n z}{(z_1 - z_2)(z_2 - z_3)\cdots(z_n - z_1)} \prod_{i=1}^{n} \delta^{3|4}(\mathcal{Z}_i - \mathcal{C}(z_i))$$

This object is to be understood as the integrand of a contour integral.

Various different choices of contour compute various apparently different but actually equivalent representations for all tree amplitudes in SYM.

The connected prescription of Roiban, MS, Volovich is related to BCFW by a change of contour!!!

## The Very Recent Work of Arkani-Hamed et. al.

$$\mathcal{T}(\mathcal{Z}_i) = \int [\mathcal{D}\mathcal{C}_{k+1}] \frac{d^n z}{(z_1 - z_2)(z_2 - z_3)\cdots(z_n - z_1)} \prod_{i=1}^{n} \delta^{3|4}(\mathcal{Z}_i - \mathcal{C}(z_i))$$

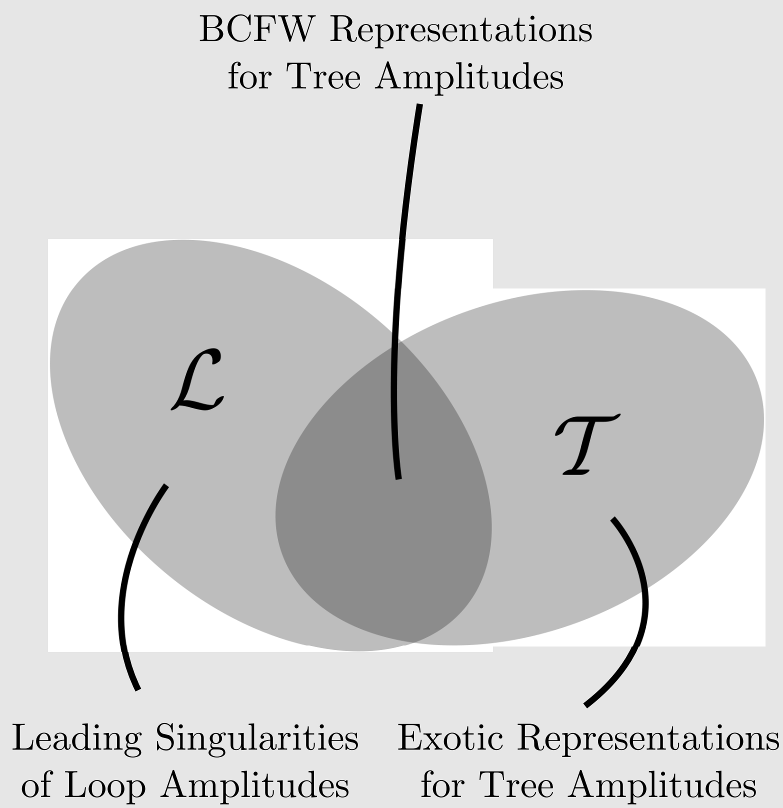
Arkani-Hamed, Cachazo, Cheung, Kaplan have recently written down a tantalizingly similar integrand

$$\mathcal{L}(\mathcal{Z}_i) = \int \frac{[dC]_{k\times n}}{(12\cdots k)(23\cdots(k+1))(n1\cdots(n-1))} \prod_{i=1}^{n} \delta^{4|4}(C_{\alpha i}\mathcal{Z}_i)$$

where $C$ is a $k \times n$ matrix and the denominator factors are its minors.

Here, different choices of contour compute BCFW representations of tree-level amplitudes as well as, conjecturally, leading singularities of loop amplitudes to all orders.

The domain of overlap of the contour integrals $\mathcal{L}, \mathcal{T}$:




This picture suggests the existence of an object $\mathcal{D}$ (for ‘dual’) which encapsulates everything; including ‘connected’ prescriptions for leading singularities of loop amplitudes!

Different gauge-fixings of $\mathcal{D}$ could lead to $\mathcal{L}, \mathcal{T}$.

## An Important Comment

The most immediate open problem in the work of Arkani-Hamed et. al. is that there is no known ‘dictionary’ telling you which contour computes which object from $\mathcal{L}$.

One contour might compute a tree amplitude, another a three-loop leading singularity, and it is not yet known how to tell a priori which contour computes what.

In contrast, our $\mathcal{T}$ carries with it the specification of a certain contour (the one implicit in the connected prescription) which we know calculates the tree-level amplitude; the global residue theorem gives a systematic procedure for generating other representations of the same tree amplitude.

## Many Open Questions...

- Understand better these sextics and their intersections.

- Can we make a more direct translation to the work of Arkani-Hamed et. al., specifically can we make a general statement about which contour they need to use to get the tree superamplitude?

- Can we write down a ‘twistor integrand’ which includes information about loop superamplitudes?

- Is there a ‘connected prescription’ for gravity superamplitudes?