Efficient Perturbative QFT from Integrability

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Based on arXiv:1411.4758 with D. Volin

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Overview

1. Planar $\mathcal{N} = 4$ SYM and the spectral problem
2. Harmonic oscillator
3. Quantum Spectral Curve for planar $\mathcal{N} = 4$ SYM
4. Weak coupling

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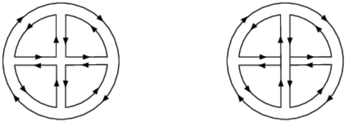

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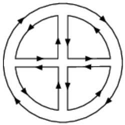

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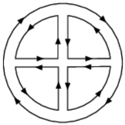
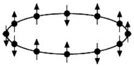
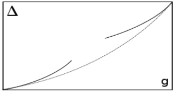


- Integrable: map to discrete spin chain model!





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"CFT data"

- 2-point functions

$$\langle O_i(x_1) O_j(x_2) \rangle \simeq \frac{\delta_{ij}}{|x_1 - x_2|^{2\Delta_i}}$$

- 3-point functions

$$\langle O_i(x_1) O_j(x_2) O_k(x_3) \rangle = \frac{C_{ijk}}{|x_1 - x_2|^{\Delta_j + \Delta_k - \Delta_i} |x_1 - x_3|^{\Delta_i + \Delta_k - \Delta_j} |x_2 - x_3|^{\Delta_i + \Delta_j - \Delta_k}}$$

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Which operators?

- Single traces of fundamental fields

$$\mathrm{Tr}(XY\nabla_+ Z\lambda A \ldots)$$

- Forms subsectors, e.g. $\mathfrak{sl}(2)$

$$\mathrm{Tr}(Z\nabla_+^S Z^{L-1} + \ldots)$$

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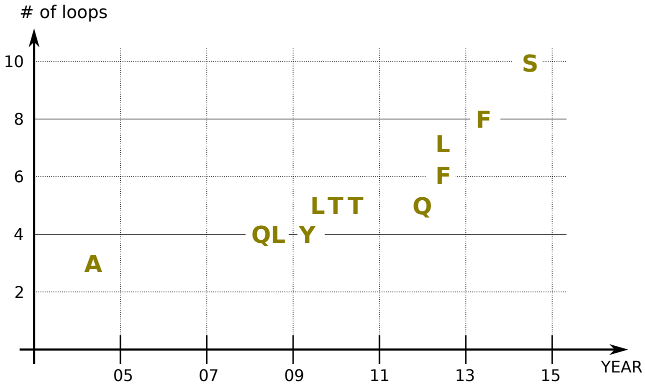

Spectral curve for 1-d harmonic oscillator ($V = \frac{1}{2}m\omega^2 x^2$)

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Classical spectral curve:

$$E \gg \hbar \quad \rightarrow \quad p \simeq \sqrt{2m(E - V)}$$

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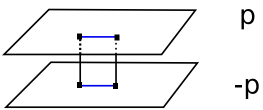

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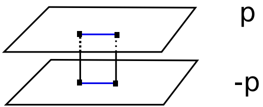

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p has N simple poles at zeros of ψ with residues $\frac{\hbar}{i}$

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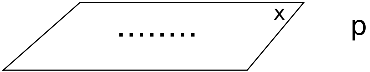

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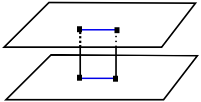

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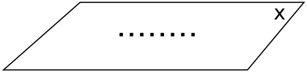

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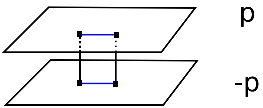

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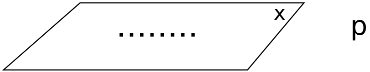
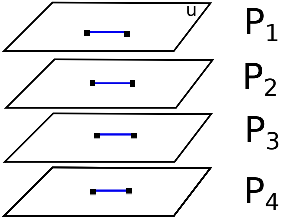


$$p = im\omega x + \frac{\hbar}{i}\sum_{i=1}^{N}\frac{1}{x - x_i} \qquad \rightarrow \qquad E = \hbar\omega\left(N + \frac{1}{2}\right)$$

Quantum Spectral Curve for planar $\mathcal{N} = 4$ SYM

[Gromov, Kazakov, Leurent, Volin, '13+'14]



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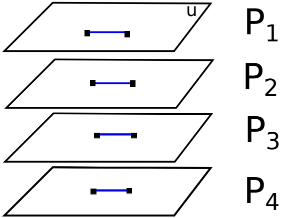

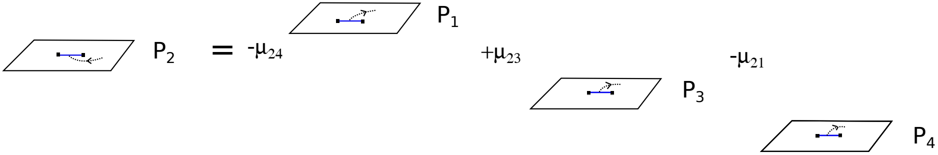

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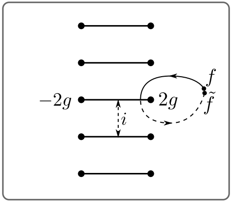
$\mathbf{P}_i(u)$ and $\mu_{ij}(u)$ (i=1,2,3,4) with $\sqrt{\ }$ branch points at $\pm 2g$ and $\pm 2g + i\mathbb{Z}$



$$
\begin{aligned}
\mu_{ij} - \tilde{\mu}_{ij} &= \tilde{\mathbf{P}}_i \mathbf{P}_j - \tilde{\mathbf{P}}_j \mathbf{P}_i \\
\tilde{\mathbf{P}}_i &= (\mu\chi)_i^{\ j} \mathbf{P}_j \\
\tilde{\mu}_{ij}(u) &= \mu_{ij}(u+i)
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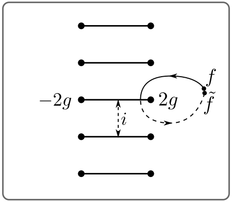

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Asymptotics at $u \to \infty$:

$$\mathbf{P}_1 \simeq A_1 u^{-\frac{L+2}{2}}, \quad \mathbf{P}_2 \simeq A_2 u^{-\frac{L}{2}}, \quad \mathbf{P}_3 \simeq A_3 u^{\frac{L-2}{2}}, \quad \mathbf{P}_4 \simeq A_4 u^{\frac{L}{2}}$$

where

$$A_1 A_4 = \frac{[(L+S-2)^2 - \Delta^2][(L-S)^2 - \Delta^2]}{16 i L (L-1)}, \quad A_2 A_3 = \frac{[(L-S+2)^2 - \Delta^2][(L+S)^2 - \Delta^2]}{16 i L (L+1)}$$

Quantum Spectral Curve at weak coupling

- $\mathrm{Tr}(Z\nabla_+^2 Z)$ $\quad (L = 2,\ S = 2)$

$$
\begin{array}{llll}
\mathbf{P}_1 = ? & \tilde{\mathbf{P}}_1 = ? & \mu_1 = ? & \Delta = 4 + ? \\
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$$\Delta = L + S + \mathcal{O}(g^2)$$

$$\Rightarrow \qquad A_1 A_4 = \mathcal{O}(g^2) \qquad A_2 A_3 = -i\,S\,\frac{L+S-1}{L-1} + \mathcal{O}(g^2)$$

$$\text{choose} \quad A_1 = g^2, \quad A_2 = 1$$

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- $\mathrm{Tr}(Z\nabla_+^2 Z)$ $\quad (L=2,\ S=2)$

$$\begin{array}{llll} \mathbf{P}_1 = 0 & \tilde{\mathbf{P}}_1 = ? & \mu_1 = ? & \Delta = 4+? \\ \mathbf{P}_2 = \frac{1}{u} & \tilde{\mathbf{P}}_2 = ? & \mu_2 = ? & \\ \mathbf{P}_3 = ? & & \mu_3 = ? & \\ \mathbf{P}_4 = ? & & \mu_4 = ? & \\ & & \mu_5 = ? & \end{array}$$

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- STEP 2:

$$
\begin{aligned}
\mu_1{}^{[2]} &= (1 - \mathbf{P}_2\,\mathbf{P}_3)\,\mu_1 + \mathbf{P}_2^2\,\mu_2 \\
\mu_2{}^{[2]} &= -\mathbf{P}_3^2\,\mu_1 + (1 + \mathbf{P}_2\,\mathbf{P}_3)\,\mu_2
\end{aligned}
\qquad f^{[n]}(u) \equiv f(u + \frac{in}{2})
$$

$$
\Rightarrow \quad \frac{1}{\mathbf{P}_2^2}\,\mu_1 - \left(\frac{\mathbf{P}_3}{\mathbf{P}_2} - \frac{\mathbf{P}_3^{[2]}}{\mathbf{P}_2^{[2]}} + \frac{1}{\mathbf{P}_2^2} + \frac{1}{\left(\mathbf{P}_2^{[2]}\right)^2} \right) \mu_1{}^{[2]} + \frac{1}{\left(\mathbf{P}_2^{[2]}\right)^2}\,\mu_1{}^{[4]} = 0
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Quantum Spectral Curve at weak coupling

- $\mathrm{Tr}(Z\nabla_+^2 Z) \quad (L = 2,\ S = 2)$

$$\mathbf{P}_1 = 0 \qquad \tilde{\mathbf{P}}_1 = u^2 \qquad \mu_1 = -\frac{i}{6} + \frac{u}{2} + \frac{iu^2}{2} \qquad \Delta = 4 + ?$$

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$$\mathbf{P}_3 = -6i \qquad \mu_3 = ?$$

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- STEP 4:

$$\frac{1}{\mathbf{P}_2^2}\mu_4 - \left(\frac{\mathbf{P}_3}{\mathbf{P}_2} - \frac{\mathbf{P}_3^{[2]}}{\mathbf{P}_2^{[2]}} + \frac{1}{\mathbf{P}_2^2} + \frac{1}{\left(\mathbf{P}_2^{[2]}\right)^2}\right)\mu_4^{[2]} + \frac{1}{\left(\mathbf{P}_2^{[2]}\right)^2}\mu_4^{[4]}$$
$$= \frac{\tilde{\mathbf{P}}_2\mathbf{P}_4}{\mathbf{P}_2^2} - \frac{\tilde{\mathbf{P}}_2^{[2]}\mathbf{P}_4^{[2]}}{\left(\mathbf{P}_2^{[2]}\right)^2} + \left(\frac{\mathbf{P}_4^{[2]}}{\mathbf{P}_2^{[2]}} - \frac{\mathbf{P}_4}{\mathbf{P}_2}\right)\mu_3^{[2]}$$

Construct solution using

$$\Psi(f) \equiv \sum_{n=0}^{\infty} f^{[2n]}, \qquad \text{s.t.} \quad \Psi(f - f^{[2]}) = f$$

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$$\mathbf{P}_4 = -12iu \qquad \mu_4 = -2u - 3iu^2 - 8u^3 - 15iu^4 + 6u^5$$
$$\mu_5 = 12iu^2 - 18u^3 + 6iu^4 - 18u^5 - 6iu^6$$

- STEP 4:

$$\frac{1}{\mathbf{P}_2^2}\mu_4 - \left(\frac{\mathbf{P}_3}{\mathbf{P}_2} - \frac{\mathbf{P}_3^{[2]}}{\mathbf{P}_2^{[2]}} + \frac{1}{\mathbf{P}_2^2} + \frac{1}{\left(\mathbf{P}_2^{[2]}\right)^2}\right)\mu_4{}^{[2]} + \frac{1}{\left(\mathbf{P}_2^{[2]}\right)^2}\mu_4{}^{[4]}$$
$$= \frac{\tilde{\mathbf{P}}_2\mathbf{P}_4}{\mathbf{P}_2^2} - \frac{\tilde{\mathbf{P}}_2^{[2]}\mathbf{P}_4^{[2]}}{\left(\mathbf{P}_2^{[2]}\right)^2} + \left(\frac{\mathbf{P}_4^{[2]}}{\mathbf{P}_2^{[2]}} - \frac{\mathbf{P}_4}{\mathbf{P}_2}\right)\mu_3^{[2]}$$

Construct solution using

$$\Psi(f) \equiv \sum_{n=0}^{\infty} f^{[2n]}, \qquad \text{s.t.} \quad \Psi(f - f^{[2]}) = f$$

Quantum Spectral Curve at weak coupling

- $\mathrm{Tr}(Z\nabla_+^2 Z)$ $\quad (L = 2,\ S = 2)$

$$\mathbf{P}_1 = 0 \qquad \tilde{\mathbf{P}}_1 = u^2 \qquad \mu_1 = -\tfrac{i}{6} + \tfrac{u}{2} + \tfrac{iu^2}{2} \qquad \Delta = 4 + 12g^2$$

$$\mathbf{P}_2 = \tfrac{1}{u} \qquad \tilde{\mathbf{P}}_2 = u + 3u^3 \qquad \mu_2 = -u - 3iu^2 + 2u^3$$

$$\mathbf{P}_3 = -6i \qquad \mu_3 = -3u^2 - 6iu^3 + 3u^4$$

$$\mathbf{P}_4 = -12iu \qquad \mu_4 = -2u - 3iu^2 - 8u^3 - 15iu^4 + 6u^5$$

$$\mu_5 = 12iu^2 - 18u^3 + 6iu^4 - 18u^5 - 6iu^6$$

- HIGHER LOOPS:

$$\Psi(1 + 2u + 6u^2) = u + (3 + i)u^2 + 2iu^3$$

$$\Psi\left(\frac{1}{u^a}\right) = \sum_{n=0}^{\infty} \frac{1}{(u + in)^a} \equiv \eta_a$$

$$\Psi(\eta_5) = -i\eta_4 + (1 + iu)\eta_5$$

$$\eta_5 = \frac{1}{u^5} - i\zeta_5 + 5\zeta_6 u + 15i\zeta_7 u^2 + \ldots \qquad \zeta_a \equiv \sum_{n=1}^{\infty} \frac{1}{n^a}$$

Quantum Spectral Curve at weak coupling

$\Delta = 4 + 12g^2$

Quantum Spectral Curve at weak coupling

$\Delta = 4 + 12g^2 - 48g^4$

Quantum Spectral Curve at weak coupling

$\Delta = 4 + 12g^2 - 48g^4 + 336g^6$

Quantum Spectral Curve at weak coupling

$$\Delta = 4 + 12g^2 - 48g^4 + 336g^6 + g^8\left(-2496 + 576\,\zeta_3 - 1440\,\zeta_5\right)$$

Quantum Spectral Curve at weak coupling

$$\Delta = 4 + 12g^2 - 48g^4 + 336g^6 + g^8\big(-2496 + 576\,\zeta_3 - 1440\,\zeta_5\big) + g^{10}\big(15168 + 6912\,\zeta_3 - 5184\,\zeta_3^2 - 8640\,\zeta_5 + 30240\,\zeta_7\big)$$

Quantum Spectral Curve at weak coupling

$$\Delta = 4 + 12g^2 - 48g^4 + 336g^6 + g^8\left(-2496 + 576\,\zeta_3 - 1440\,\zeta_5\right) + g^{10}\left(15168 + 6912\,\zeta_3 - 5184\,\zeta_3^2 - 8640\,\zeta_5 + 30240\,\zeta_7\right) + g^{12}\left(-7680 - 262656\,\zeta_3 - 20736\,\zeta_3^2 + 112320\,\zeta_5 + 155520\,\zeta_3\,\zeta_5 + 75600\,\zeta_7 - 489888\,\zeta_9\right)$$

Quantum Spectral Curve at weak coupling

$$\Delta = 4 + 12g^2 - 48g^4 + 336g^6 + g^8\big(-2496 + 576\,\zeta_3 - 1440\,\zeta_5\big) + g^{10}\big(15168 + 6912\,\zeta_3 - 5184\,\zeta_3^2 - 8640\,\zeta_5 + 30240\,\zeta_7\big) + g^{12}\big(-7680 - 262656\,\zeta_3 - 20736\,\zeta_3^2 + 112320\,\zeta_5 + 155520\,\zeta_3\,\zeta_5 + 75600\,\zeta_7 - 489888\,\zeta_9\big) + g^{14}\big(-2135040 + 5230080\,\zeta_3 - 421632\,\zeta_3^2 + 124416\,\zeta_3^3 - 229248\,\zeta_5 + 411264\,\zeta_3\,\zeta_5 - 993600\,\zeta_5^2 - 1254960\,\zeta_7 - 1935360\,\zeta_3\,\zeta_7 - 835488\,\zeta_9 + 7318080\,\zeta_{11}\big)$$

Quantum Spectral Curve at weak coupling

$$\Delta = 4 + 12g^2 - 48g^4 + 336g^6 + g^8\left(-2496 + 576\,\zeta_3 - 1440\,\zeta_5\right) + g^{10}\left(15168 + 6912\,\zeta_3 - 5184\,\zeta_3^2 - 8640\,\zeta_5 + 30240\,\zeta_7\right) + g^{12}\left(-7680 - 262656\,\zeta_3 - 20736\,\zeta_3^2 + 112320\,\zeta_5 + 155520\,\zeta_3\,\zeta_5 + 75600\,\zeta_7 - 489888\,\zeta_9\right) + g^{14}\left(-2135040 + 5230080\,\zeta_3 - 421632\,\zeta_3^2 + 124416\,\zeta_3^3 - 229248\,\zeta_5 + 411264\,\zeta_3\,\zeta_5 - 993600\,\zeta_5^2 - 1254960\,\zeta_7 - 1935360\,\zeta_3\,\zeta_7 - 835488\,\zeta_9 + 7318080\,\zeta_{11}\right) + g^{16}\Big(54408192 - 83496960\,\zeta_3 + 7934976\,\zeta_3^2 + 1990656\,\zeta_3^3 - 19678464\,\zeta_5 - 4354560\,\zeta_3\,\zeta_5 - 3255552\,\zeta_3^2\,\zeta_5 + 2384640\,\zeta_5^2 + 21868704\,\zeta_7 - 6229440\,\zeta_3\,\zeta_7 + 22256640\,\zeta_5\,\zeta_7 + 9327744\,\zeta_9 + 23224320\,\zeta_3\,\zeta_9 + \frac{65929248}{5}\,\zeta_{11} - 106007616\,\zeta_{13} - \frac{684288}{5}\,Z_{11}^{(2)}\Big)$$

Quantum Spectral Curve at weak coupling

$$\Delta = 4 + 12g^2 - 48g^4 + 336g^6 + g^8\big(-2496 + 576\,\zeta_3 - 1440\,\zeta_5\big) + g^{10}\big(15168 + 6912\,\zeta_3 - 5184\,\zeta_3^2 - 8640\,\zeta_5 + 30240\,\zeta_7\big) + g^{12}\big(-7680 - 262656\,\zeta_3 - 20736\,\zeta_3^2 + 112320\,\zeta_5 + 155520\,\zeta_3\,\zeta_5 + 75600\,\zeta_7 - 489888\,\zeta_9\big) + g^{14}\big(-2135040 + 5230080\,\zeta_3 - 421632\,\zeta_3^2 + 124416\,\zeta_3^3 - 229248\,\zeta_5 + 411264\,\zeta_3\,\zeta_5 - 993600\,\zeta_5^2 - 1254960\,\zeta_7 - 1935360\,\zeta_3\,\zeta_7 - 835488\,\zeta_9 + 7318080\,\zeta_{11}\big) + g^{16}\Big(54408192 - 83496960\,\zeta_3 + 7934976\,\zeta_3^2 + 1990656\,\zeta_3^3 - 19678464\,\zeta_5 - 4354560\,\zeta_3\,\zeta_5 - 3255552\,\zeta_3^2\,\zeta_5 + 2384640\,\zeta_5^2 + 21868704\,\zeta_7 - 6229440\,\zeta_3\,\zeta_7 + 22256640\,\zeta_5\,\zeta_7 + 9327744\,\zeta_9 + 23224320\,\zeta_3\,\zeta_9 + \frac{65929248}{5}\,\zeta_{11} - 106007616\,\zeta_{13} - \frac{684288}{5}\,Z_{11}^{(2)}\Big) + g^{18}\Big(-1014549504 + 1140922368\,\zeta_3 - 51259392\,\zeta_3^2 - 20155392\,\zeta_3^3 + 575354880\,\zeta_5 - 14294016\,\zeta_3\,\zeta_5 - 26044416\,\zeta_3^2\,\zeta_5 + 55296000\,\zeta_5^2 + 15759360\,\zeta_3\,\zeta_5^2 - 223122816\,\zeta_7 + 34020864\,\zeta_3\,\zeta_7 + 22063104\,\zeta_3^2\,\zeta_7 - 92539584\,\zeta_5\,\zeta_7 - 113690304\,\zeta_7^2 - 247093632\,\zeta_9 + 119470464\,\zeta_3\,\zeta_9 - 245099520\,\zeta_5\,\zeta_9 - \frac{186204096}{5}\,\zeta_{11} - 278505216\,\zeta_3\,\zeta_{11} - 253865664\,\zeta_{13} + 1517836320\,\zeta_{15} + \frac{15676416}{5}\,Z_{11}^{(2)} - 1306368\,Z_{13}^{(2)} + 1306368\,Z_{13}^{(3)}\Big)$$

Quantum Spectral Curve at weak coupling

$$\begin{aligned}
\Delta = \ldots \\
+g^{20}\Big(&16445313024 - 13069615104\,\zeta_3 - 1509027840\,\zeta_3^2 + 578949120\,\zeta_3^3 - 14929920\,\zeta_3^4 - 11247547392\,\zeta_5 \\
&+1213581312\,\zeta_3\,\zeta_5 + 1234206720\,\zeta_3^2\,\zeta_5 - 70170624\,\zeta_3^3\,\zeta_5 - 1390279680\,\zeta_5^2 - 654842880\,\zeta_3\,\zeta_5^2 + \frac{6966252288}{175}\,\zeta_5^3 \\
&+377212032\,\zeta_7 - 1610841600\,\zeta_3\,\zeta_7 + 154680192\,\zeta_3^2\,\zeta_7 + 222341760\,\zeta_5\,\zeta_7 + 133788672\,\zeta_3\,\zeta_5\,\zeta_7 + 868662144\,\zeta_7^2 \\
&+4915257984\,\zeta_9 - 332646912\,\zeta_3\,\zeta_9 - 91072512\,\zeta_3^2\,\zeta_9 + 1099699200\,\zeta_5\,\zeta_9 + 2275620480\,\zeta_7\,\zeta_9 + \frac{9793211904}{5}\,\zeta_{11} \\
&-2334572928\,\zeta_3\,\zeta_{11} + 2713772160\,\zeta_5\,\zeta_{11} - \frac{787483944}{175}\,\zeta_{13} + 3372969600\,\zeta_3\,\zeta_{13} - \frac{4308536566944}{875}\,\zeta_{15} \\
&-21661960320\,\zeta_{17} + \frac{752219136}{5}\,Z_{11}^{(2)} - \frac{5070791808}{175}\,Z_{13}^{(2)} - \frac{7159104}{7}\,Z_{13}^{(3)} + \frac{2716063488}{175}\,Z_{15}^{(2)} - \frac{17895168}{25}\,Z_{15}^{(3)} \\
&+11943936\,\zeta_3\,Z_{11}^{(2)}\Big) + O(g^{22})
\end{aligned}$$

where e.g.

$$Z_{13}^{(3)} = -\zeta_{3,7,3} + \zeta_3\,\zeta_{3,7} + 12\,\zeta_5\,\zeta_{3,5} + 6\,\zeta_5\,\zeta_8 \qquad \zeta_{a,b,c} = \sum_{1\le n_1<n_2<n_3}^{\infty} \frac{1}{n_1^a n_2^b n_3^c}$$

[Broadhurst, Kreimer '95]

Outlook

Future directions

- Other sectors, twists
- AdS_4/CFT_3 and AdS_3/CFT_2
- Direct derivation of Quantum Spectral Curve