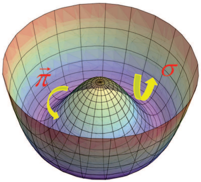
2. Chiral Perturbation Theory

- **Goldstone Theorem**
- **Chiral Symmetry**
- **Effective Goldstone Theory**
- **Chiral Symmetry Breaking**
- **Phenomenology**

Sigma Model

$\mathbf{\Phi}^{\mathsf{T}} \equiv (\sigma, \vec{\pi})$



$$\mathcal{L}_\sigma = \frac{1}{2}\,\partial_\mu \mathbf{\Phi}^{\mathsf{T}} \partial^\mu \mathbf{\Phi} \;-\; \frac{\lambda}{4}\,\left(\mathbf{\Phi}^{\mathsf{T}}\mathbf{\Phi} - v^2\right)^2$$

Global Symmetry: $\mathrm{O(4)} \sim \mathrm{SU(2)} \otimes \mathrm{SU(2)}$

- $\mathbf{v^2 < 0}$**:** $m_\Phi^2 = -\lambda\, v^2$
- $\mathbf{v^2 > 0}$**:** $\langle 0|\sigma|0\rangle = v \quad , \quad \langle 0|\vec{\pi}|0\rangle = 0$

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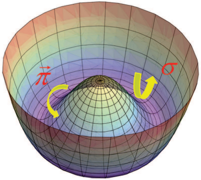

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SSB: $\quad O(4) \to O(3) \qquad [\frac{4\times 3}{2} - \frac{3\times 2}{2} = 3$ broken generators$]$

$$\mathcal{L}_\sigma = \frac{1}{2}\left\{\partial_\mu\hat{\sigma}\,\partial^\mu\hat{\sigma} + \partial_\mu\vec{\pi}\,\partial^\mu\vec{\pi} - M^2\hat{\sigma}^2\right\} - \frac{M^2}{2v}\,\hat{\sigma}\,\left(\hat{\sigma}^2 + \vec{\pi}^2\right) - \frac{M^2}{8v^2}\,\left(\hat{\sigma}^2 + \vec{\pi}^2\right)^2$$

$$\hat{\sigma} \equiv \sigma - v \qquad ; \qquad M^2 = 2\,\lambda\, v^2$$

3 Massless Goldstone Bosons

1) $\mathbf{\Sigma}(x) \equiv \sigma(x)\, \mathbf{I}_2 + i\, \vec{\tau}\, \vec{\pi}(x)$; $\langle \mathbf{A} \rangle \equiv \mathrm{Tr}\,(\mathbf{A})$

$$\mathcal{L}_\sigma = \frac{1}{4}\, \langle \partial_\mu \mathbf{\Sigma}^\dagger\, \partial^\mu \mathbf{\Sigma} \rangle - \frac{\lambda}{16}\, \Big(\langle \mathbf{\Sigma}^\dagger \mathbf{\Sigma} \rangle - 2\, v^2 \Big)^2$$

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Derivative Golstone Couplings

3) $E \ll M \sim v$:

$$\boxed{\mathcal{L}_\sigma \approx \frac{v^2}{4}\,\langle \partial_\mu \mathbf{U}^\dagger\,\partial^\mu \mathbf{U} \rangle}$$

Symmetry Realizations

Symmetry **G** $\{T_a\}$ ⟷ **Conserved charges** $\mathcal{Q}_a$

Noether Theorem: $\partial_\mu j_a^\mu = 0 \quad ; \quad \mathcal{Q}_a = \int d^3x \, j_a^0(x) \quad ; \quad \frac{d}{dt}\,\mathcal{Q}_a = 0$

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Wigner–Weyl	**Nambu–Goldstone**
$\mathcal{Q}_a\,\vert\,0\,\rangle = 0$	$\mathcal{Q}_a\,\vert\,0\,\rangle \neq 0$
• Exact Symmetry	• Spontaneously Broken Symmetry
• Degenerate Multiplets	• Massless Goldstone Bosons
• Linear Representation	• Non-Linear Representation

Chiral Symmetry $\quad \mathbf{m_q = 0}$ (Chiral Limit)

$$\mathcal{L}^0_{QCD} = -\frac{1}{4} G^{\mu\nu}_a G^a_{\mu\nu} + \bar{\mathbf{q}}_L \, i\gamma^\mu D_\mu \, \mathbf{q}_L + \bar{\mathbf{q}}_R \, i\gamma^\mu D_\mu \, \mathbf{q}_R$$

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$$q = \left(\frac{1-\gamma_5}{2}\right) q + \left(\frac{1+\gamma_5}{2}\right) q \equiv q_L + q_R$$

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- **The 0^- octet is nearly massless:** $m_\pi \approx 0$

- **The vacuum is not invariant (SSB):** $\langle 0 | \, (\bar{q}_L q_R + \bar{q}_R q_L) \, | 0 \rangle \neq 0$

8 Massless 0^- Goldstone Bosons

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Energy Scale	Fields	Effective Theory
M_W	W, Z, γ, g τ, μ, e, ν_i t, b, c, s, d, u	Standard Model
	↓ OPE	
$\lesssim m_c$	$\gamma, g\ ;\ \mu, e, \nu_i$ s, d, u	$\mathcal{L}_{\rm QCD}^{(n_f=3)}$, $\mathcal{L}_{\rm eff}^{\Delta S=1,2}$
	↓ $N_C \to \infty$	
M_K	$\gamma\ ;\ \mu, e, \nu_i$ π, K, η	χPT

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Parity $\longrightarrow$ even dimension ; $\mathbf{U}\,\mathbf{U}^\dagger = 1 \longrightarrow 2n \geq 2$

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$\longrightarrow$

$$\mathcal{L}_2 = \frac{f^2}{4} \langle \partial_\mu \mathbf{U}^\dagger \, \partial^\mu \mathbf{U} \rangle$$

Derivative Coupling

Goldstones become free at zero momenta

$$\mathcal{L}_2 = \frac{f^2}{4}\langle\partial_\mu \mathbf{U}^\dagger \partial^\mu \mathbf{U}\rangle = \partial_\mu\pi^-\partial^\mu\pi^+ + \frac{1}{2}\,\partial_\mu\pi^0\partial^\mu\pi^0 + \cdots$$

$$+ \frac{1}{6f^2}\left\{\left(\pi^+ \overleftrightarrow{\partial}_\mu \pi^-\right)\left(\pi^+ \overleftrightarrow{\partial}^\mu \pi^-\right) + 2\left(\pi^0 \overleftrightarrow{\partial}_\mu \pi^+\right)\left(\pi^- \overleftrightarrow{\partial}^\mu \pi^0\right) + \cdots\right\}$$

$$+ O\left(\pi^6/f^4\right)$$

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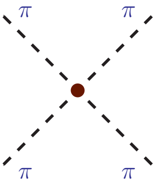
Chiral Symmetry Determines the Interaction:



$$T\left(\pi^+\pi^0 \to \pi^+\pi^0\right) = \frac{t}{f^2}$$

$t \equiv (p'_+ - p_+)^2$

Weinberg

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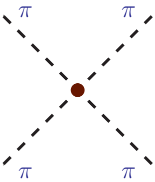

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Non-Linear Lagrangian: $2\pi \to 2\pi, 4\pi, \cdots$ related

Explicit Symmetry Breaking

$$\mathcal{L}_{QCD} \equiv \mathcal{L}^0_{QCD} + \bar{\mathbf{q}}\,(\not{\mathbf{v}} + \not{\mathbf{a}}\,\gamma_5)\,\mathbf{q} - \bar{\mathbf{q}}\,(\mathbf{s} - i\,\gamma_5\,\mathbf{p})\,\mathbf{q}$$

$$= \mathcal{L}^0_{QCD} + \bar{\mathbf{q}}_L\,\not{\mathbf{l}}\,\mathbf{q}_L + \bar{\mathbf{q}}_R\,\not{\mathbf{r}}\,\mathbf{q}_R - \bar{\mathbf{q}}_R\,(\mathbf{s} + i\,\mathbf{p})\,\mathbf{q}_L - \bar{\mathbf{q}}_L\,(\mathbf{s} - i\,\mathbf{p})\,\mathbf{q}_R$$

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$$\mathbf{l}_\mu \equiv \mathbf{v}_\mu - \mathbf{a}_\mu = e\,\mathcal{Q}\,A_\mu + \cdots$$

$$\mathbf{r}_\mu \equiv \mathbf{v}_\mu + \mathbf{a}_\mu = e\,\mathcal{Q}\,A_\mu + \cdots \qquad \mathcal{Q} \equiv \frac{1}{3}\,\mathrm{diag}(2,-1,-1)$$

$$\mathbf{s} = \mathcal{M} + \cdots \qquad ; \qquad \mathcal{M} \equiv \mathrm{diag}(m_u, m_d, m_s)$$

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Local $SU(3)_L \otimes SU(3)_R$ Symmetry:

$$\mathbf{q}_L \to g_L\,\mathbf{q}_L \qquad \mathbf{q}_R \to g_R\,\mathbf{q}_R$$

$$\mathbf{l}_\mu \to g_L\,\mathbf{l}_\mu\,g_L^\dagger + i\,g_L\,\partial_\mu g_L^\dagger$$
$$\mathbf{r}_\mu \to g_R\,\mathbf{r}_\mu\,g_R^\dagger + i\,g_R\,\partial_\mu g_R^\dagger$$
$$(\mathbf{s} + i\,\mathbf{p}) \to g_R\,(\mathbf{s} + i\,\mathbf{p})\,g_L^\dagger$$

Lowest-Order Effective Lagrangian:

$$\mathcal{L} = \frac{f^2}{4} \langle D_\mu \mathbf{U} \, D^\mu \mathbf{U}^\dagger + \chi \, \mathbf{U}^\dagger + \mathbf{U} \, \chi^\dagger \rangle$$

$$D_\mu \mathbf{U} = \partial_\mu \mathbf{U} - i \, \mathbf{r}_\mu \, \mathbf{U} + i \, \mathbf{U} \, \mathbf{l}_\mu$$

$$\chi \equiv 2 \, B_0 \, (\mathbf{s} + i \, \mathbf{p})$$

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Currents:

$$\mathbf{J}^\mu_L = \frac{\partial}{\partial\, \mathbf{l}_\mu} \mathcal{L}_2 = \frac{i}{2} f^2\, D^\mu \mathbf{U}^\dagger\, \mathbf{U} = \frac{f}{\sqrt{2}}\, D^\mu \mathbf{\Phi} + \cdots$$

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$$\langle 0 | \, (J^\mu_A)_{12} \, | \pi^+(p) \rangle = i \sqrt{2} \, f \; p^\mu \quad \longrightarrow \quad f = f_\pi \approx 92.4 \text{ MeV}$$

$(\pi^+ \to \mu^+ \nu_\mu)$

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$$\bar{\mathbf{q}}^j_L \mathbf{q}^i_R = -\frac{\partial \mathcal{L}_2}{\partial (\mathbf{s} - i\, \mathbf{p})^{ji}} = -\frac{f^2}{2} B_0 \mathbf{U}^{ij} \quad \longrightarrow$$

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Quark Masses:

$$\frac{f^2}{4}\,\langle \chi\,\mathbf{U}^\dagger + \mathbf{U}\,\chi^\dagger\rangle \quad\rightarrow\quad \mathcal{L}_m = -B_0\,\langle \mathcal{M}\,\Phi^2\rangle$$

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Isospin limit: $\quad \mathbf{m_u = m_d = \hat{m}}$

$$\frac{M_\pi^2}{2\,\hat{m}} = \frac{M_K^2}{\hat{m} + m_s} = \frac{3\, M_\eta^2}{2\,\hat{m} + 4\, m_s} = B_0$$

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- **Gell-Mann–Okubo:** $4\,M_K^2 = M_\pi^2 + 3\,M_\eta^2$

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Quark Mass Ratios:

Dashen Theorem

$$\left(M^2_{K^0} - M^2_{K^\pm}\right)_{\mathrm{em}} = \left(M^2_{\pi^0} - M^2_{\pi^\pm}\right)_{\mathrm{em}} + \mathcal{O}(e^2 p^2)$$

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Proof: $e^2 \langle \mathcal{Q}_{\mathrm{R}} \mathrm{U} \mathcal{Q}_{\mathrm{L}} \mathrm{U}^\dagger \rangle = -\frac{2e^2}{f^2} \left(\pi^+\pi^- + K^+K^-\right) + \mathcal{O}(\phi^4) \quad ; \quad \mathcal{Q}_X \to g_X \, \mathcal{Q}_X \, g_X^\dagger$ □

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→ $$\mathbf{m_u : m_d : m_s = 0.55 : 1 : 20.3}$$ Weinberg

$$\frac{f^2}{4}\,\langle \chi\,\mathbf{U}^\dagger + \mathbf{U}\,\chi^\dagger \rangle \;=\; -B_0\,\langle\,\mathcal{M}\,\Phi^2\rangle \;+\; \frac{B_0}{6\,f^2}\,\langle\,\mathcal{M}\,\Phi^4\rangle \;+\;\cdots$$

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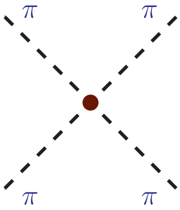

$$T\left(\pi^+\pi^0\to\pi^+\pi^0\right) \;=\; \frac{t - M_\pi^2}{f_\pi^2}$$

$t \equiv (p'_+ - p_+)^2$

Weinberg

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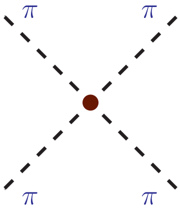

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$\mathcal{L}_2$ ⟷ Current Algebra 60's

Chiral Power Counting

$\mathbf{U}$	$\mathcal{O}(p^0)$
$D_\mu \mathbf{U}\, ,\ \mathbf{l}_\mu\, ,\ \mathbf{r}_\mu$	$\mathcal{O}(p^1)$
$\chi\, ,\ \mathbf{F}^{\mu\nu}_{L,R}$	$\mathcal{O}(p^2)$

$$\mathbf{F}_L^{\mu\nu} \equiv \partial^\mu \mathbf{l}^\nu - \partial^\nu \mathbf{l}^\mu - i\, [\mathbf{l}^\mu, \mathbf{l}^\nu]$$

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General connected diagram with N_d vertices of $\mathcal{O}(p^d)$ and L loops:

$$D \;=\; 2\,L + 2 + \sum_d N_d\,(d-2)$$

Weinberg

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- $D = 2:$ $\quad L=0\ ,\ \ d=2$
- $D = 4:$ $\quad L=0\ ,\ \ d=4\ ,\ \ N_4=1$
  $\quad L=1\ ,\ \ d=2$

$\mathcal{O}(\mathbf{p}^4)$ χPT

i) $\mathcal{L}_4$ **at tree level** (Gasser–Leutwyler)

$$\begin{aligned}
\mathcal{L}_4 \;=\;& L_1 \langle D_\mu U^\dagger D^\mu U\rangle^2 + L_2 \langle D_\mu U^\dagger D_\nu U\rangle \langle D^\mu U^\dagger D^\nu U\rangle \\
&+ L_3 \langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U\rangle + L_4 \langle D_\mu U^\dagger D^\mu U\rangle \langle U^\dagger \chi + \chi \,\dagger\, U\rangle \\
&+ L_5 \langle D_\mu U^\dagger D^\mu U \left(U^\dagger \chi + \chi^\dagger U\right)\rangle + L_6 \langle U^\dagger \chi + \chi^\dagger U\rangle^2 \\
&+ L_7 \langle U^\dagger \chi - \chi^\dagger U\rangle^2 + L_8 \langle \chi^\dagger U \chi^\dagger U + U^\dagger \chi U^\dagger \chi\rangle \\
&- i\, L_9 \langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U\rangle + L_{10} \langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu}\rangle
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\end{aligned}
$$

ii) $\mathcal{L}_2$ **at one loop (unitarity):** $T_4 \sim p^4 \left\{ a \,\log(p^2/\mu^2) + b(\mu) \right\}$

- Chiral Logarithms unambiguously predicted

$\mathcal{O}(\mathbf{p}^4)$ χPT

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$\mathcal{O}(\mathbf{p}^4)$ χPT

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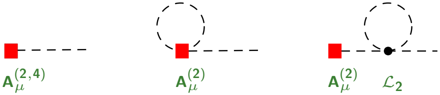
iii) **Wess–Zumino–Witten term (chiral anomaly):** $\pi^0, \eta \to \gamma\gamma$

Meson Decay Constants:



$$\mu_P \equiv \frac{M_P^2}{32\pi^2 f^2}\,\log\left(\frac{M_P^2}{\mu^2}\right)$$

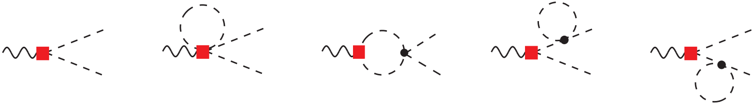
$$f_\pi = f\,\left\{1 - 2\mu_\pi - \mu_K + \frac{4M_\pi^2}{f^2}\,L_5^r(\mu) + \frac{8M_K^2+4M_\pi^2}{f^2}\,L_4^r(\mu)\right\}$$

$$f_K = f\,\left\{1 - \frac{3}{4}\mu_\pi - \frac{3}{2}\mu_K - \frac{3}{4}\mu_{\eta_8} + \frac{4M_K^2}{f^2}\,L_5^r(\mu) + \frac{8M_K^2+4M_\pi^2}{f^2}\,L_4^r(\mu)\right\}$$

$$f_{\eta_8} = f\,\left\{1 - 3\mu_K + \frac{4M_{\eta_8}^2}{f^2}\,L_5^r(\mu) + \frac{8M_K^2+4M_\pi^2}{f^2}\,L_4^r(\mu)\right\}$$

$$\frac{f_K}{f_\pi} = 1.22 \pm 0.01 \;\longrightarrow\; L_5^r(M_\rho) = (1.4 \pm 0.5)\cdot 10^{-3} \;\longrightarrow\; \frac{f_{\eta_8}}{f_\pi} = 1.3 \pm 0.05$$

Vector Form Factor: $\langle \pi^+\pi^- | J^\mu_{\rm em} | 0\rangle = (p_+ - p_-)^\mu \, F^V_\pi(s)$

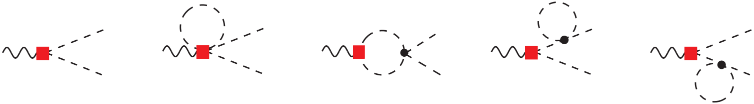


$$F^V_\pi(s) = 1 + \frac{2L_9^r(\mu)}{f^2}\, s - \frac{s}{96\pi^2 f^2}\left[A\left(\frac{m_\pi^2}{s}, \frac{m_\pi^2}{\mu^2}\right) + \frac{1}{2}\, A\left(\frac{m_K^2}{s}, \frac{m_K^2}{\mu^2}\right)\right]$$
$$= 1 + \frac{1}{6}\,\langle r^2\rangle^V_\pi\, s + \cdots$$

$$A\left(\frac{m_P^2}{s}, \frac{m_P^2}{\mu^2}\right) = \log\left(\frac{m_P^2}{\mu^2}\right) + \frac{8m_P^2}{s} - \frac{5}{3} + \sigma_P^3\, \log\left(\frac{\sigma_P+1}{\sigma_P-1}\right) \quad , \quad \sigma_P \equiv \sqrt{1 - \frac{4m_P^2}{s}}$$

Vector Form Factor:

$$\langle \pi^+\pi^- | J^\mu_{\rm em} | 0\rangle \; = \; (p_+ - p_-)^\mu \; F^V_\pi(s)$$



$$F^V_\pi(s) \; = \; 1 \; + \; \frac{2L_9^r(\mu)}{f^2}\, s \; - \; \frac{s}{96\pi^2 f^2}\left[A\left(\frac{m_\pi^2}{s}, \frac{m_\pi^2}{\mu^2}\right) + \frac{1}{2}\, A\left(\frac{m_K^2}{s}, \frac{m_K^2}{\mu^2}\right)\right]$$

$$= \; 1 \; + \; \frac{1}{6}\, \langle r^2\rangle^V_\pi \, s \; + \; \cdots$$

$$A\left(\frac{m_P^2}{s}, \frac{m_P^2}{\mu^2}\right) \; = \; \log\left(\frac{m_P^2}{\mu^2}\right) + \frac{8 m_P^2}{s} - \frac{5}{3} + \sigma_P^3 \, \log\left(\frac{\sigma_P + 1}{\sigma_P - 1}\right) \qquad , \qquad \sigma_P \equiv \sqrt{1 - \frac{4 m_P^2}{s}}$$

$$\langle r^2\rangle^V_\pi \; = \; \frac{12\, L_9^r(\mu)}{f^2} \; - \; \frac{1}{32\pi^2 f^2}\left\{ 2\, \log\left(\frac{M_\pi^2}{\mu^2}\right) + \log\left(\frac{M_K^2}{\mu^2}\right) + 3 \right\}$$

$$\langle r^2\rangle^V_\pi \; = \; (0.439 \pm 0.008)\ \mathrm{fm}^2 \quad \longrightarrow \quad L_9^r(M_\rho) = (6.9 \pm 0.7)\cdot 10^{-3}$$

$O(p^4)$ χPT COUPLINGS

i	$L_i^r(M_\rho) \times 10^3$	Source	Γ_i
1	0.4 ± 0.3	K_{e4}, $\pi\pi \to \pi\pi$	$3/32$
2	1.4 ± 0.3	K_{e4}, $\pi\pi \to \pi\pi$	$3/16$
3	-3.5 ± 1.1	K_{e4}, $\pi\pi \to \pi\pi$	0
4	-0.3 ± 0.5	Zweig rule	$1/8$
5	1.4 ± 0.5	F_K/F_π	$3/8$
6	-0.2 ± 0.3	Zweig rule	$11/144$
7	-0.4 ± 0.2	GMO, $L_{5,8}$	0
8	0.9 ± 0.3	$M_{K^0} - M_{K^+}$, L_5, $(m_s - \hat{m})/(m_d - m_u)$	$5/48$
9	6.9 ± 0.7	$\langle r^2 \rangle_V^\pi$	$1/4$
10	-5.5 ± 0.7	$\pi \to e\nu\gamma$	$-1/4$

- $$L_i = L_i^r(\mu) + \Gamma_i \, \frac{\mu^{D-4}}{32\pi^2} \left\{ \frac{2}{D-4} + \gamma_E - \log(4\pi) - 1 \right\}$$

$O(p^4)$ χPT COUPLINGS

i	$L_i^r(M_\rho) \times 10^3$	Source	Γ_i
1	0.4 ± 0.3	K_{e4}, $\pi\pi \to \pi\pi$	$3/32$
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- $\Lambda_\chi \sim 1\,\mathrm{GeV}$ ➡ $L_i \sim \frac{f_\pi^2/4}{\Lambda_\chi^2} \sim 2 \times 10^{-3}$

$O(p^4)$ χPT COUPLINGS

i	$L_i^r(M_\rho)\times 10^3$	Source	Γ_i
1	0.4 ± 0.3	K_{e4}, $\pi\pi\to\pi\pi$	$3/32$
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- $L_i = L_i^r(\mu) + \Gamma_i \,\frac{\mu^{D-4}}{32\pi^2}\left\{\frac{2}{D-4}+\gamma_E-\log(4\pi)-1\right\}$
- $\Lambda_\chi\sim 1\,\mathrm{GeV}$ ➡ $L_i\sim\frac{f_\pi^2/4}{\Lambda_\chi^2}\sim 2\times 10^{-3}$
- χPT Loops $\sim$ $1/(4\pi f_\pi)^2$

$\mathcal{O}(p^6)$ χPT

i) $\mathcal{L}_6 = \sum_i C_i \, O_i^{p^6}$ **at tree level** Bijnens-Colangelo-Ecker, Fearing-Scherer

$90 + 4$ $[53 + 4]$ terms in SU(3) [SU(2)] χPT (even-intrinsic parity only)

ii) $\mathcal{L}_4$ **at one loop,** $\mathcal{L}_2$ **at two loops** Bijnens-Colangelo-Ecker

Double chiral logarithms

Many Calculations: M_ϕ, f_ϕ, $\gamma\gamma \to \pi\pi$, $\pi\pi \to \pi\pi$, $\pi K \to \pi K$, K_{l4},
$\pi \to e\,\bar{\nu}_e\gamma$, $F_V(s)$, $F_S(s)$, $\Pi_{V,A}(s)$, $\cdots$

Amoros-Bijnens-Dhonte-Talavera, Ananthanarayan-Colangelo-Gasser-Leutwyler, Bellucci-Gasser-Sainio, Bürgui, Bijnens et al, Descotes-Genon et al, Golowich-Kambor, Post-Schilcher...

Theoretical Challenge: **QCD calculation of the χPT couplings**

$K^+ \to \pi^0 \ell^+ \nu_\ell$, $K^0 \to \pi^- \ell^+ \nu_\ell$:

$C_{K^+\pi^0} = \frac{1}{\sqrt{2}}$, $C_{K^0\pi^-} = 1$

$$\langle \pi | \bar{s}\gamma^\mu u | K \rangle = C_{K\pi} \left[(P_K + P_\pi)^\mu \, f_+^{K\pi}(t) + (P_K - P_\pi)^\mu \, f_-^{K\pi}(t) \right]$$

- **Lowest order [$\mathcal{O}(p^2)$]:** $f_+^{K\pi}(t) = 1$, $f_-^{K\pi}(t) = 0$

$K^+ \to \pi^0 \ell^+ \nu_\ell$, $K^0 \to \pi^- \ell^+ \nu_\ell$:

$C_{K^+\pi^0} = \frac{1}{\sqrt{2}}$, $C_{K^0\pi^-} = 1$

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- **Lowest order** $[\mathcal{O}(p^2)]$**:** $\quad f_+^{K\pi}(t) = 1 \quad , \quad f_-^{K\pi}(t) = 0$

- **Ademollo-Gatto Theorem:** $\quad f_+^{K^0\pi^-}(0) \, = \, 1 \, + \, \mathcal{O}[(m_s - m_u)^2]$

$K^+ \to \pi^0 \ell^+ \nu_\ell$, $K^0 \to \pi^- \ell^+ \nu_\ell$:

$C_{K^+\pi^0} = \frac{1}{\sqrt{2}}$, $C_{K^0\pi^-} = 1$

$$\langle\pi|\bar{s}\gamma^\mu u|K\rangle \; = \; C_{K\pi} \, \left[(P_K + P_\pi)^\mu \; f_+^{K\pi}(t) \, + \, (P_K - P_\pi)^\mu \; f_-^{K\pi}(t)\right]$$

- **Lowest order** $[\mathcal{O}(p^2)]$**:** $\quad f_+^{K\pi}(t) = 1 \quad , \quad f_-^{K\pi}(t) = 0$
- **Ademollo-Gatto Theorem:** $\quad f_+^{K^0\pi^-}(0) \, = \, 1 \, + \, \mathcal{O}[(m_s - m_u)^2]$
- π^0**–**η **mixing:** $\quad f_+^{K^+\pi^0}(0) \, = \, 1 + \frac{3}{4}\, \frac{m_d - m_u}{m_s - \hat{m}} \, = \, 1.017$

$K^+ \to \pi^0 \ell^+ \nu_\ell$, $K^0 \to \pi^- \ell^+ \nu_\ell$:

$C_{K^+\pi^0} = \frac{1}{\sqrt{2}}$, $C_{K^0\pi^-} = 1$

$$\langle \pi | \bar{s}\gamma^\mu u | K \rangle = C_{K\pi} \left[(P_K + P_\pi)^\mu \, f_+^{K\pi}(t) + (P_K - P_\pi)^\mu \, f_-^{K\pi}(t) \right]$$

- **Lowest order** $[\mathcal{O}(p^2)]$**:** $f_+^{K\pi}(t) = 1$, $f_-^{K\pi}(t) = 0$
- **Ademollo-Gatto Theorem:** $f_+^{K^0\pi^-}(0) = 1 + \mathcal{O}[(m_s - m_u)^2]$
- π^0**–**η **mixing:** $f_+^{K^+\pi^0}(0) = 1 + \frac{3}{4} \frac{m_d - m_u}{m_s - \hat{m}} = 1.017$
- $\mathcal{O}(\mathbf{p^4})$**:** $f_+^{K^0\pi^-}(0) = 0.977$, $\frac{f_+^{K^+\pi^0}(0)}{f_+^{K^0\pi^-}(0)} = 1.022$

Gasser-Leutwyler '85

$K^+ \to \pi^0 \ell^+ \nu_\ell$, $K^0 \to \pi^- \ell^+ \nu_\ell$:

$C_{K^+\pi^0} = \frac{1}{\sqrt{2}}$, $C_{K^0\pi^-} = 1$

$$\langle \pi | \bar{s}\gamma^\mu u | K \rangle \; = \; C_{K\pi} \, \left[(P_K + P_\pi)^\mu \; f_+^{K\pi}(t) \, + \, (P_K - P_\pi)^\mu \; f_-^{K\pi}(t) \right]$$

- **Lowest order** $[\mathcal{O}(p^2)]$**:** $f_+^{K\pi}(t) = 1$, $f_-^{K\pi}(t) = 0$
- **Ademollo-Gatto Theorem:** $f_+^{K^0\pi^-}(0) \, = \, 1 \, + \, \mathcal{O}[(m_s - m_u)^2]$
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Gasser-Leutwyler '85

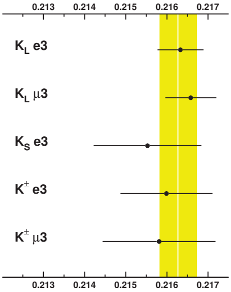
Needed to determine $\mathbf{V_{us}}$

$K \to \pi \, \ell \, \nu_\ell$

$$|\mathbf{V_{us}\, f_+(0)}| = \mathbf{0.2163 \pm 0.0005}$$

Flavianet Kaon WG, arXiv:1005.2323 [hep-ph]

$$\langle \pi^- | \bar{s}\gamma_\mu u | K^0 \rangle = (p_\pi + p_K)_\mu \; f_+(t) + (p_K - p_\pi)_\mu \; f_-(t)$$



$K \to \pi \, \ell \, \nu_\ell$

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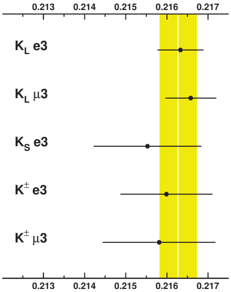
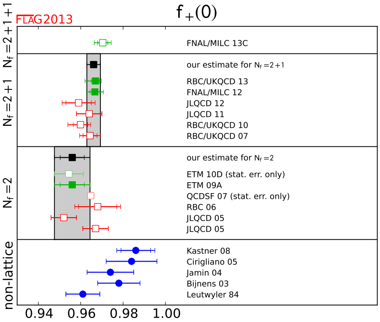


$$\langle \pi^- | \bar{s} \gamma_\mu u | K^0 \rangle \, = \, (p_\pi + p_K)_\mu \; f_+(t) + (p_K - p_\pi)_\mu \; f_-(t)$$



$K \to \pi \ell \nu_\ell$

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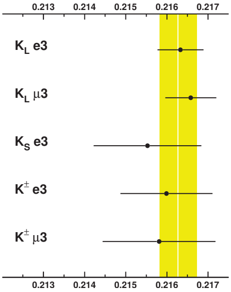

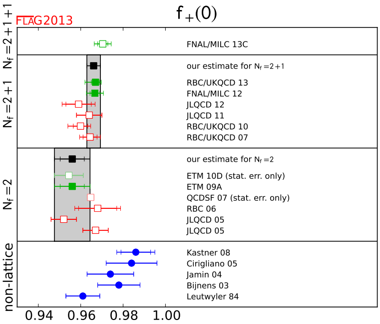

$$f_+(0) = \begin{cases} 0.9704\,(32) & (N_f = 2+1+1) \\ 0.9661\,(32) & (N_f = 2+1) \end{cases}$$

$$\longrightarrow \quad |V_{us}| = \begin{cases} 0.2229\,(9) \\ 0.2239\,(9) \end{cases}$$

$$f_+(0) = 1 + f_2 + f_4 + \cdots$$

Large $\mathcal{O}(p^6)$ χPT correction

Backup Slides

Goldstone Theorem

$$\mathcal{Q} = \int d^3x\, j^0(x) \quad ; \quad \partial_\mu j_a^\mu = 0 \quad ; \quad \exists\, \mathcal{O} : \; v(t) \equiv \langle 0 | \left[\mathcal{Q}(t), \mathcal{O} \right] | 0 \rangle \neq 0$$

$$\Downarrow$$

$$\exists\, |n\rangle : \; \langle 0|\mathcal{O}|n\rangle \, \langle n|j^0|0\rangle \neq 0 \quad ; \quad E_n\, \delta^{(3)}(\vec{p}_n) = 0 \quad ; \quad M_n = 0$$

Goldstone Theorem

$$\mathcal{Q} = \int d^3x\, j^0(x) \quad ; \quad \partial_\mu j_a^\mu = 0 \quad ; \quad \exists\, \mathcal{O} :\; v(t) \equiv \langle 0|\, [\mathcal{Q}(t), \mathcal{O}]\, |0\rangle \neq 0$$

$$\Downarrow$$

$$\exists\, |n\rangle :\; \langle 0|\mathcal{O}|n\rangle\, \langle n|j^0|0\rangle \neq 0 \quad ; \quad E_n\, \delta^{(3)}\,(\vec{p}_n) = 0 \quad ; \quad M_n = 0$$

Proof:

$$j^0(x) \,=\, \mathrm{e}^{iP\cdot x}\, j^0(0)\, \mathrm{e}^{-iP\cdot x} \qquad ; \qquad \sum_n |n\rangle\langle n| \,=\, 1$$

Goldstone Theorem

$$\mathcal{Q} = \int d^3x\, j^0(x) \quad ; \quad \partial_\mu j_a^\mu = 0 \quad ; \quad \exists\, \mathcal{O} : \; v(t) \equiv \langle 0 | \left[\mathcal{Q}(t), \mathcal{O} \right] | 0 \rangle \neq 0$$

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Proof: $j^0(x) \,=\, \mathrm{e}^{iP\cdot x}\, j^0(0)\, \mathrm{e}^{-iP\cdot x} \quad ; \quad \sum_n |n\rangle\langle n| \,=\, 1$

$$v(t) \,=\, \sum_n \int d^3x \, \left\{ \langle 0 | j^0(x) | n \rangle \langle n | \mathcal{O} | 0 \rangle \,-\, \langle 0 | \mathcal{O} | n \rangle \langle n | j^0(x) | 0 \rangle \right\}$$

Goldstone Theorem

$$\mathcal{Q} = \int d^3x \, j^0(x) \quad ; \quad \partial_\mu j^\mu_a = 0 \quad ; \quad \exists\, \mathcal{O} : \; v(t) \equiv \langle 0 | \, [\mathcal{Q}(t), \mathcal{O}] \, | 0 \rangle \neq 0$$

$$\Downarrow$$

$$\exists\, |n\rangle : \; \langle 0 | \mathcal{O} | n \rangle \, \langle n | j^0 | 0 \rangle \neq 0 \quad ; \quad E_n \, \delta^{(3)}(\vec{p}_n) = 0 \quad ; \quad M_n = 0$$

Proof:

$$j^0(x) = \mathrm{e}^{iP\cdot x} \, j^0(0) \, \mathrm{e}^{-iP\cdot x} \qquad ; \qquad \sum_n |n\rangle\langle n| = 1$$

$$v(t) = \sum_n \int d^3x \, \left\{ \langle 0 | j^0(x) | n \rangle \langle n | \mathcal{O} | 0 \rangle - \langle 0 | \mathcal{O} | n \rangle \langle n | j^0(x) | 0 \rangle \right\}$$

$$= \sum_n \int d^3x \, \left\{ \mathrm{e}^{-ip_n\cdot x} \, \langle 0 | j^0(0) | n \rangle \langle n | \mathcal{O} | 0 \rangle - \mathrm{e}^{ip_n\cdot x} \, \langle 0 | \mathcal{O} | n \rangle \langle n | j^0(0) | 0 \rangle \right\}$$

Goldstone Theorem

$$\mathcal{Q} = \int d^3x\, j^0(x) \quad ; \quad \partial_\mu j_a^\mu = 0 \quad ; \quad \exists\, \mathcal{O} : \; v(t) \equiv \langle 0 | \left[\mathcal{Q}(t), \mathcal{O} \right] | 0 \rangle \neq 0$$

$$\Downarrow$$

$$\exists\, |n\rangle : \; \langle 0 | \mathcal{O} | n \rangle \, \langle n | j^0 | 0 \rangle \neq 0 \quad ; \quad E_n\, \delta^{(3)}(\vec{p}_n) = 0 \quad ; \quad M_n = 0$$

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$$\begin{aligned} v(t) &= \sum_n \int d^3x \, \left\{ \langle 0|j^0(x)|n\rangle\langle n|\mathcal{O}|0\rangle \,-\, \langle 0|\mathcal{O}|n\rangle\langle n|j^0(x)|0\rangle \right\} \\ &= \sum_n \int d^3x \, \left\{ \mathrm{e}^{-ip_n\cdot x}\, \langle 0|j^0(0)|n\rangle\langle n|\mathcal{O}|0\rangle \,-\, \mathrm{e}^{ip_n\cdot x}\, \langle 0|\mathcal{O}|n\rangle\langle n|j^0(0)|0\rangle \right\} \\ &= (2\pi)^3 \sum_n \delta^{(3)}(\vec{p}_n) \, \left\{ \mathrm{e}^{-iE_n t}\, \langle 0|j^0(0)|n\rangle\langle n|\mathcal{O}|0\rangle \,-\, \mathrm{e}^{iE_n t}\, \langle 0|\mathcal{O}|n\rangle\langle n|j^0(0)|0\rangle \right\} \neq 0 \end{aligned}$$

Goldstone Theorem

$$\mathcal{Q} = \int d^3x\, j^0(x) \quad ; \quad \partial_\mu j_a^\mu = 0 \quad ; \quad \exists\, \mathcal{O} : \; v(t) \equiv \langle 0 | \left[\mathcal{Q}(t), \mathcal{O} \right] | 0 \rangle \neq 0$$

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Proof:

$$j^0(x) \,=\, \mathrm{e}^{iP\cdot x}\, j^0(0)\, \mathrm{e}^{-iP\cdot x} \qquad ; \qquad \sum_n |n\rangle\langle n| \,=\, 1$$

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$$\frac{d}{dt}\, v(t) \,=\, 0$$

Goldstone Theorem

$$\mathcal{Q} = \int d^3x\, j^0(x) \quad ; \quad \partial_\mu j_a^\mu = 0 \quad ; \quad \exists\, \mathcal{O} : \; v(t) \equiv \langle 0 | \, [\mathcal{Q}(t), \mathcal{O}] \, | 0 \rangle \neq 0$$

$$\Downarrow$$

$$\exists\, |n\rangle : \; \langle 0 | \mathcal{O} | n \rangle \, \langle n | j^0 | 0 \rangle \neq 0 \quad ; \quad E_n \, \delta^{(3)}(\vec{p}_n) = 0 \quad ; \quad M_n = 0$$

Proof: $\quad j^0(x) \, = \, \mathrm{e}^{iP\cdot x} \, j^0(0) \, \mathrm{e}^{-iP\cdot x} \quad ; \quad \sum_n |n\rangle\langle n| \, = \, 1$

$$v(t) \, = \sum_n \int d^3x \, \left\{ \langle 0 | j^0(x) | n \rangle \langle n | \mathcal{O} | 0 \rangle \, - \, \langle 0 | \mathcal{O} | n \rangle \langle n | j^0(x) | 0 \rangle \right\}$$

$$= \sum_n \int d^3x \, \left\{ \mathrm{e}^{-ip_n\cdot x} \, \langle 0 | j^0(0) | n \rangle \langle n | \mathcal{O} | 0 \rangle \, - \, \mathrm{e}^{ip_n\cdot x} \, \langle 0 | \mathcal{O} | n \rangle \langle n | j^0(0) | 0 \rangle \right\}$$

$$= (2\pi)^3 \sum_n \delta^{(3)}(\vec{p}_n) \, \left\{ \mathrm{e}^{-iE_n t} \, \langle 0 | j^0(0) | n \rangle \langle n | \mathcal{O} | 0 \rangle \, - \, \mathrm{e}^{iE_n t} \, \langle 0 | \mathcal{O} | n \rangle \langle n | j^0(0) | 0 \rangle \right\} \neq 0$$

$$\frac{d}{dt} v(t) \, = \, 0 \, = \, -i \, (2\pi)^3 \sum_n \delta^{(3)}(\vec{p}_n) \, E_n \, \Big\{ \, \mathrm{e}^{-iE_n t} \, \langle 0 | j^0(0) | n \rangle \langle n | \mathcal{O} | 0 \rangle + \, \mathrm{e}^{iE_n t} \, \langle 0 | \mathcal{O} | n \rangle \langle n | j^0(0) | 0 \rangle \Big\} \qquad \square$$

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- $\langle 0 | J_A^{a\mu} | \pi^b(p) \rangle = i \, \delta^{ab} \, \sqrt{2} \, f_\pi \, p^\mu$

Chiral Anomaly: $\delta Z[v,a,s,p] = -\frac{N_C}{16\pi^2}\int d^4x\,\langle\delta\beta(x)\,\Omega(x)\rangle$

$g_{L,R} \approx 1 + i\delta\alpha \mp i\delta\beta$

$$\Omega(x) = \varepsilon^{\mu\nu\sigma\rho}\left[v_{\mu\nu}v_{\sigma\rho} + \tfrac{4}{3}\,\nabla_\mu a_\nu \nabla_\sigma a_\rho + \tfrac{2}{3} i\,\{v_{\mu\nu}, a_\sigma a_\rho\} + \tfrac{8}{3} i\, a_\sigma v_{\mu\nu} a_\rho + \tfrac{4}{3}\, a_\mu a_\nu a_\sigma a_\rho\right]$$

$$v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - i\,[v_\mu, v_\nu] \quad , \qquad \nabla_\mu a_\nu = \partial_\mu a_\nu - i\,[v_\mu, a_\nu] \quad , \qquad \varepsilon_{0123} = 1$$

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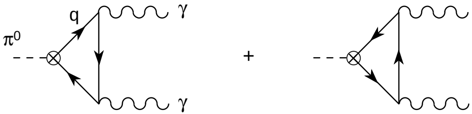
$$S[U,\ell,r]_{WZW} = -\frac{iN_C}{240\pi^2}\int d\sigma^{ijklm}\,\left\langle \Sigma^L_i\Sigma^L_j\Sigma^L_k\Sigma^L_l\Sigma^L_m\right\rangle - \frac{iN_C}{48\pi^2}\int d^4x\;\varepsilon_{\mu\nu\alpha\beta}\left(W(U,\ell,r)^{\mu\nu\alpha\beta} - W(\mathbf{1},\ell,r)^{\mu\nu\alpha\beta}\right)$$

Wess–Zumino–Witten

$$W(U,\ell,r)_{\mu\nu\alpha\beta} = \Big\langle U\ell_\mu\ell_\nu\ell_\alpha U^\dagger r_\beta + \frac{1}{4}U\ell_\mu U^\dagger r_\nu U\ell_\alpha U^\dagger r_\beta + iU\partial_\mu\ell_\nu\ell_\alpha U^\dagger r_\beta$$
$$+\, i\partial_\mu r_\nu U\ell_\alpha U^\dagger r_\beta - i\Sigma^L_\mu\ell_\nu U^\dagger r_\alpha U\ell_\beta + \Sigma^L_\mu U^\dagger\partial_\nu r_\alpha U\ell_\beta - \Sigma^L_\mu\Sigma^L_\nu U^\dagger r_\alpha U\ell_\beta$$
$$+\, \Sigma^L_\mu\ell_\nu\partial_\alpha\ell_\beta + \Sigma^L_\mu\partial_\nu\ell_\alpha\ell_\beta - i\Sigma^L_\mu\ell_\nu\ell_\alpha\ell_\beta + \frac{1}{2}\Sigma^L_\mu\ell_\nu\Sigma^L_\alpha\ell_\beta - i\Sigma^L_\mu\Sigma^L_\nu\Sigma^L_\alpha\ell_\beta\Big\rangle$$
$$-\,(L\leftrightarrow R)$$

$$\Sigma^L_\mu = U^\dagger\partial_\mu U \qquad , \qquad \Sigma^R_\mu = U\partial_\mu U^\dagger$$

$\pi^0 \to \gamma\gamma$:



$A_3^\mu \equiv \bar{u}\gamma^\mu\gamma_5 u - \bar{d}\gamma^\mu\gamma_5 d$

$$\Gamma(\pi^0 \to \gamma\gamma) = \left(\frac{N_c}{3}\right)^2 \frac{\alpha^2 M_\pi^3}{64\,\pi^3 f_\pi^2} = 7.73\ \mathrm{eV}$$

Exp: $(7.7 \pm 0.6)\ \mathrm{eV}$

There are no QCD corrections

The chiral anomaly contributes to: $\pi^0 \to \gamma\gamma$, $\eta \to \gamma\gamma$

$\gamma\, 3\pi$, $\gamma\, \pi^+\pi^-\eta$, $K\bar{K}3\pi$, $\cdots$