Applications of On-Shell Physics

Jacob L. Bourjaily

Nordic Winter School on
Cosmology and Particle Physics

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Organization and Outline

Spiritus Movens: One-Loop Generalized Unitarity

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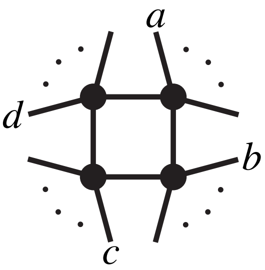

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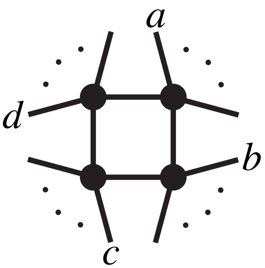
$I_{a,b,c,d} \equiv$  $f_{a,b,c,d}$

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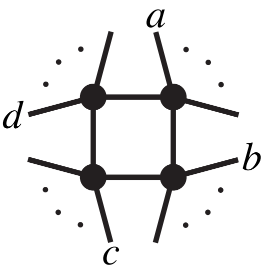

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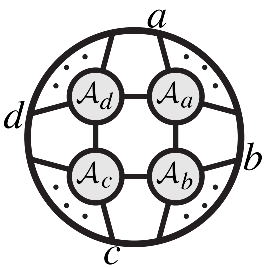

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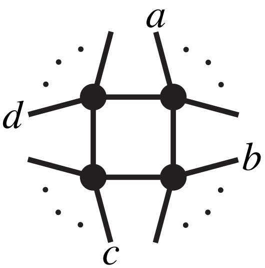
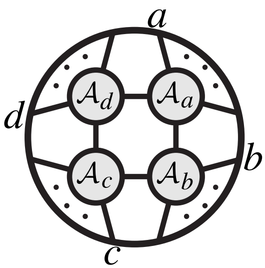
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \equiv$ 

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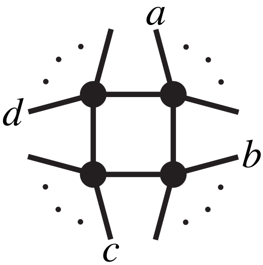

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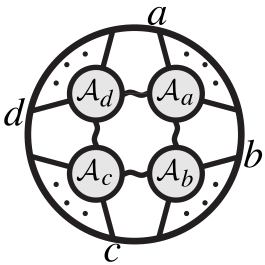

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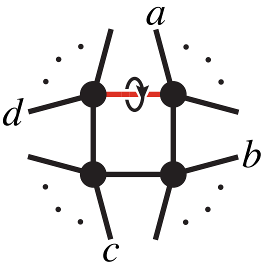

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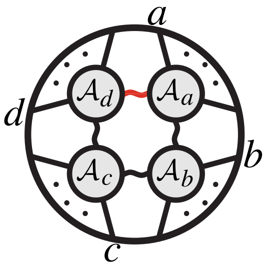

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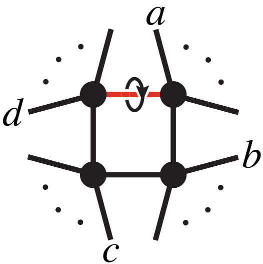

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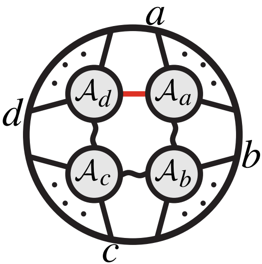

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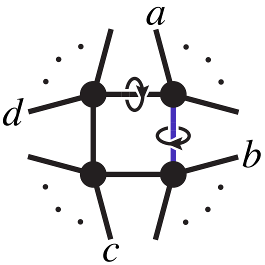
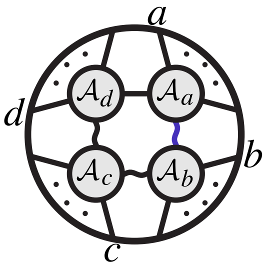
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \equiv$ 

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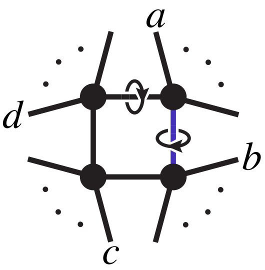
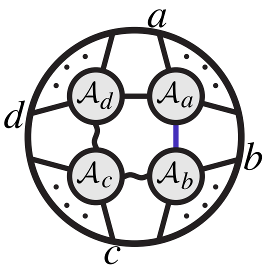
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \equiv$ 

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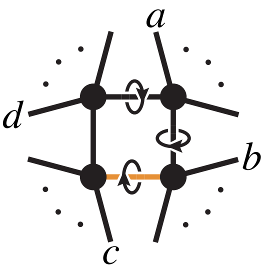

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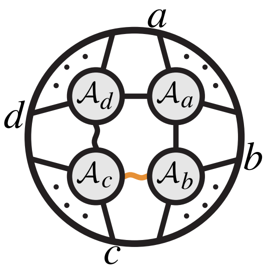

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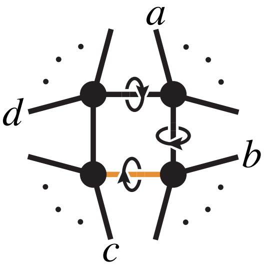

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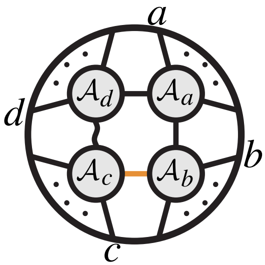

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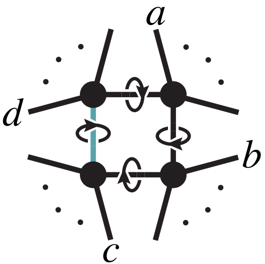
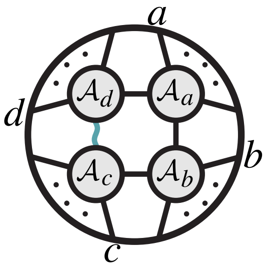
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \equiv$ 

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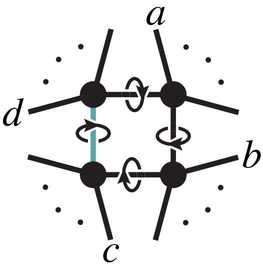
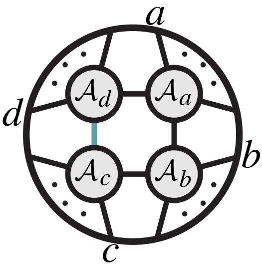
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \equiv$ 

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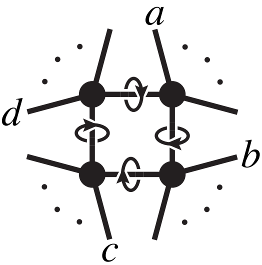
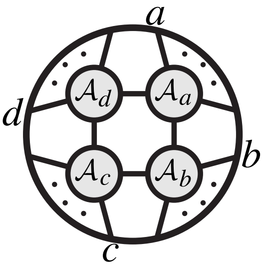
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \equiv$ 

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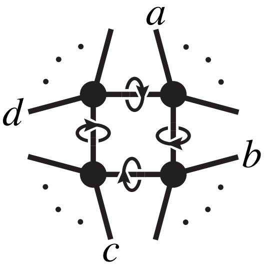

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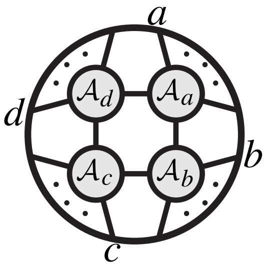

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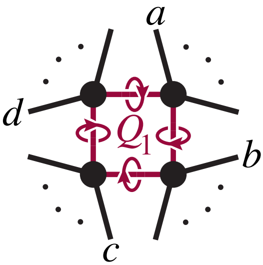

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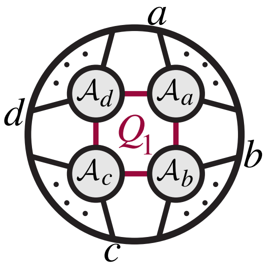

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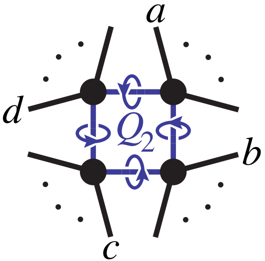
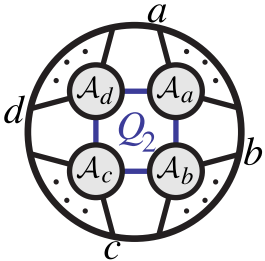
$I_{a,b,c,d} \equiv$ 

$f^2_{a,b,c,d} \equiv$ 

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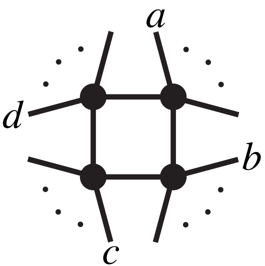

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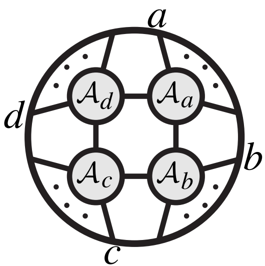

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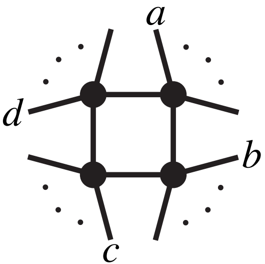

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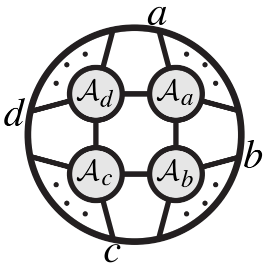

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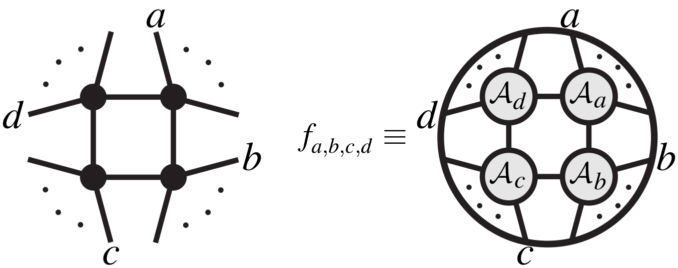
$I_{a,b,c,d} \equiv$  $f_{a,b,c,d} \equiv$

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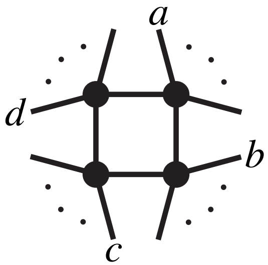

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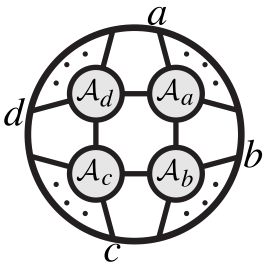

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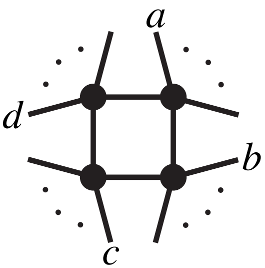
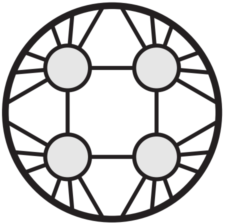
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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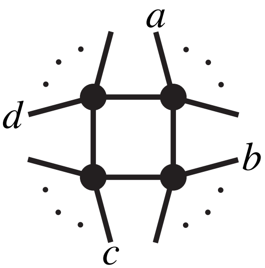
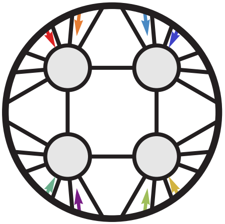
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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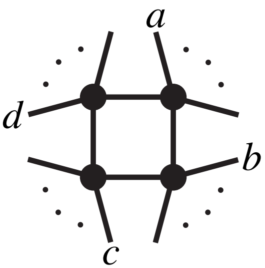
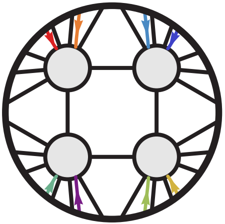
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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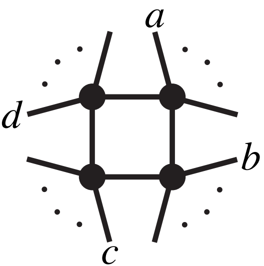
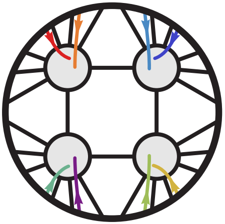
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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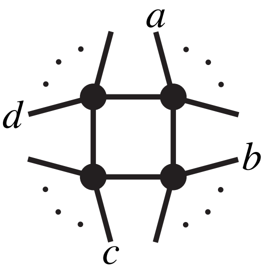
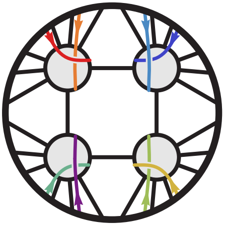
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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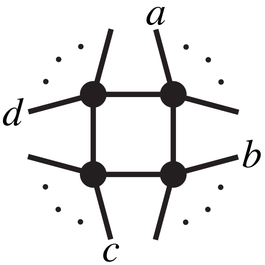
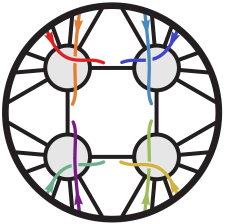
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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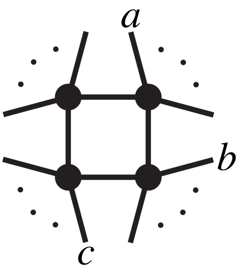
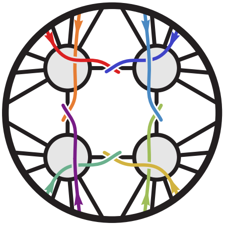
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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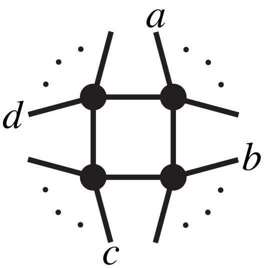
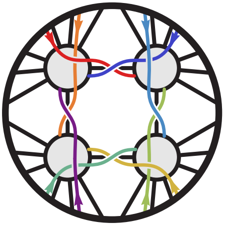
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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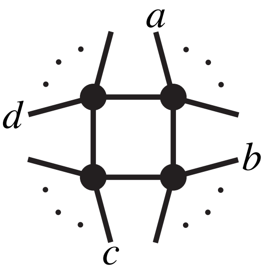
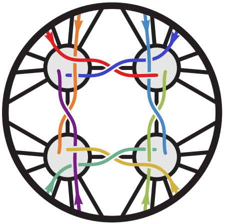
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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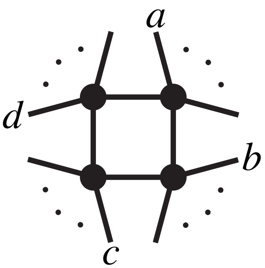
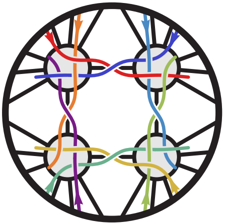
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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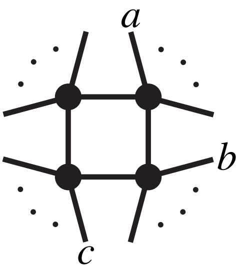
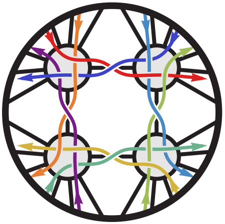
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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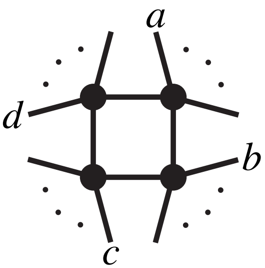
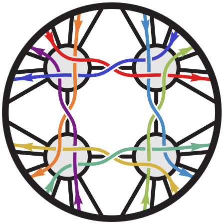
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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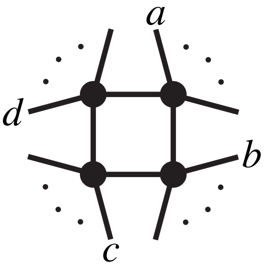
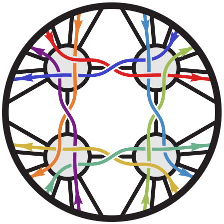
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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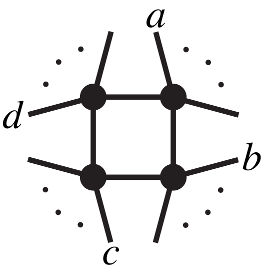
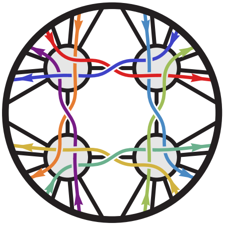
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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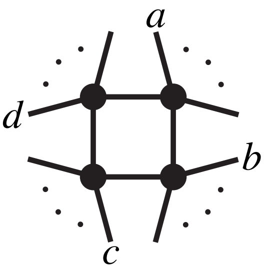
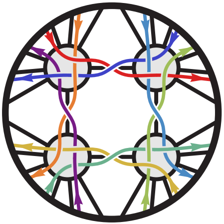
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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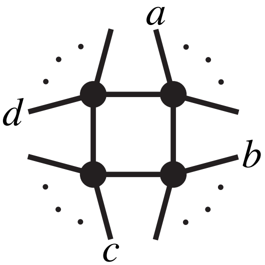
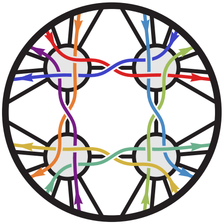
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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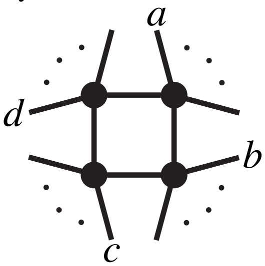

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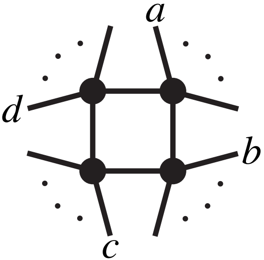
$I_{a,b,c,d} \equiv$ 

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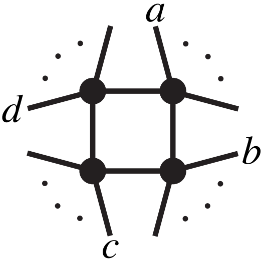

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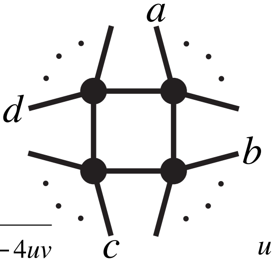

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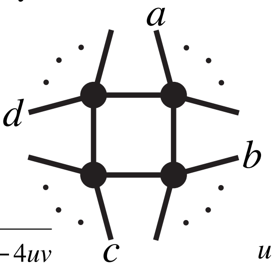

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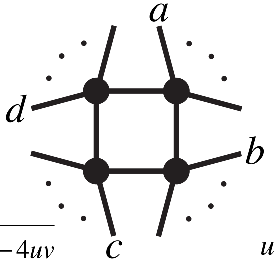

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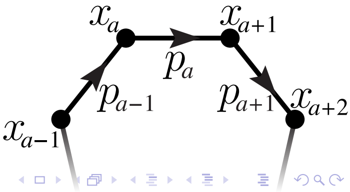

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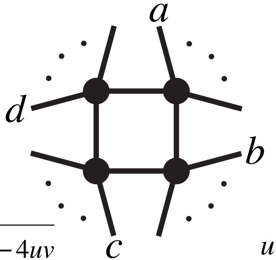

$$\equiv -\int_{\ell\in\mathbb{R}^{3,1}} d^4\ell \, \frac{(a,c)(b,d)\Delta}{(\ell,a)(\ell,b)(\ell,c)(\ell,d)} \equiv \mathcal{I}_{a,b,c,d},$$

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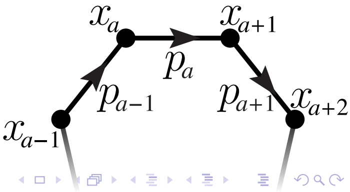

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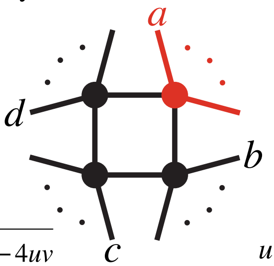

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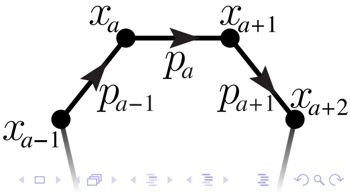

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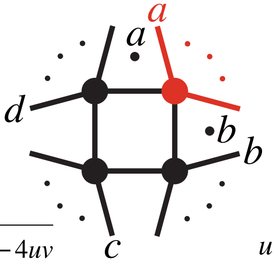

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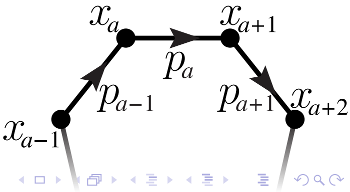

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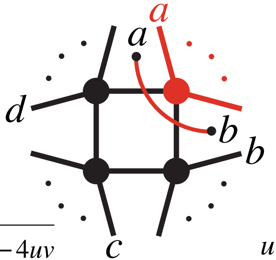

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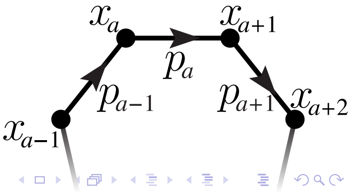

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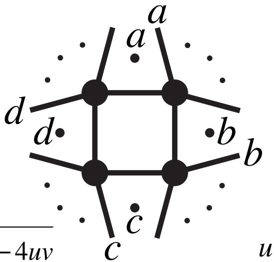

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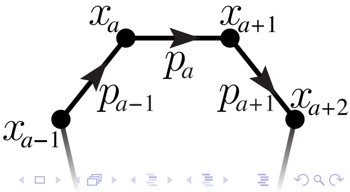

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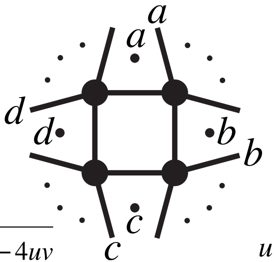

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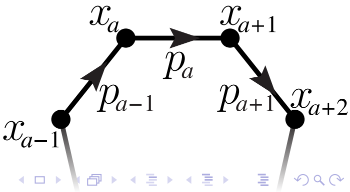

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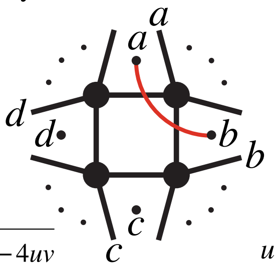

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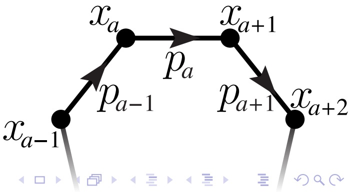

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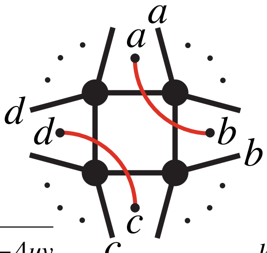

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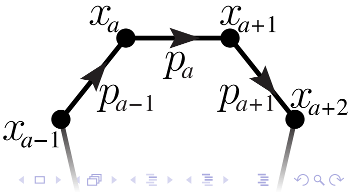

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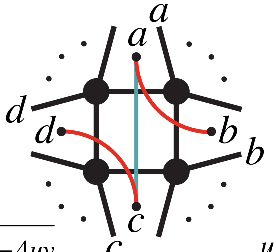

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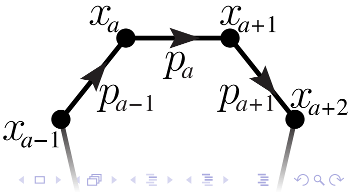

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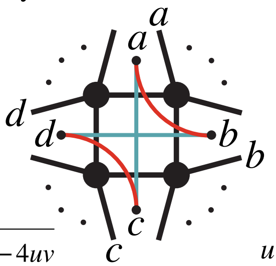

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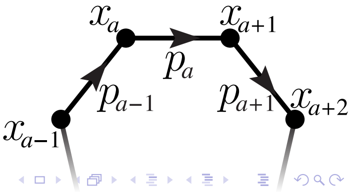

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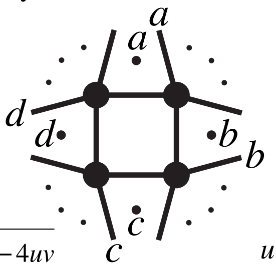

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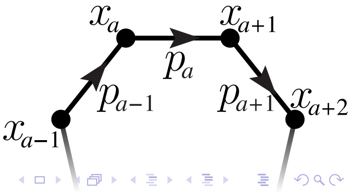

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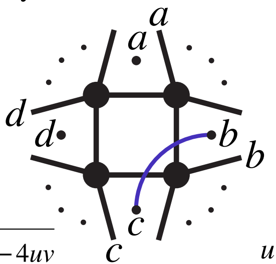

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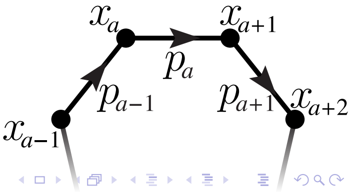

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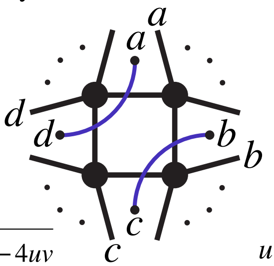

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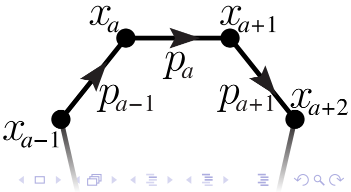

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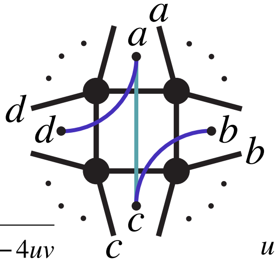

$$\equiv -\int_{\ell\in\mathbb{R}^{3,1}} d^4\ell\, \frac{(a,c)(b,d)\Delta}{(\ell,a)(\ell,b)(\ell,c)(\ell,d)}, \qquad \mathcal{I}_{a,b,c,d} \equiv \text{(the same)}$$

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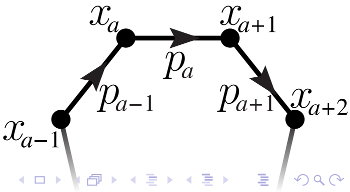

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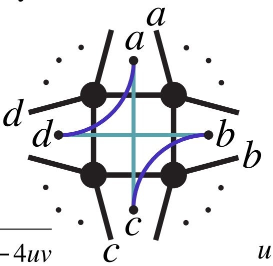

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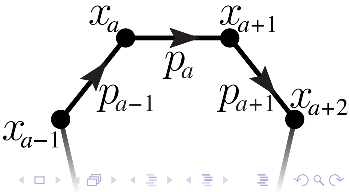

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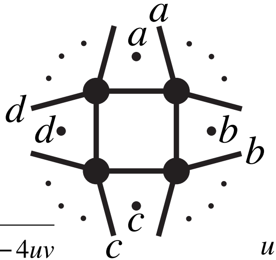

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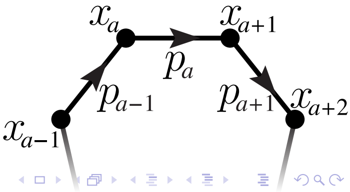

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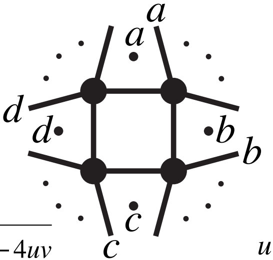

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with $p_a \equiv x_{a+1} - x_a$, and where $1/(\ell,a)$ denotes the standard propagator:

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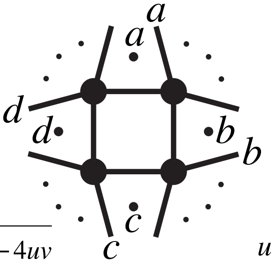
The four-mass box integral is a manifestly finite, symmetric function of two dual-conformally invariant cross ratios, denoted u and v.

$$I_{a,b,c,d} \equiv$$



$$\equiv -\int\limits_{\ell\in\mathbb{R}^{3,1}} d^4\ell\, \frac{(a,c)(b,d)\Delta}{(\ell,a)(\ell,b)(\ell,c)(\ell,d)}, \qquad \mathcal{I}_{a,b,c,d} \equiv \ldots$$

$$\Delta \equiv \sqrt{(1-u-v)^2-4uv} \qquad u \equiv \frac{(a,b)(c,d)}{(a,c)(b,d)}, \quad v \equiv \frac{(b,c)(a,d)}{(a,c)(b,d)}$$

with $p_a \equiv x_{a+1} - x_a$, and where $1/(\ell,a)$ denotes the standard propagator:

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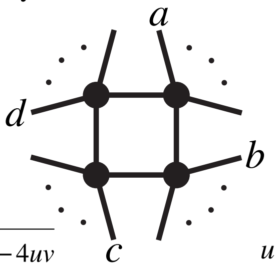

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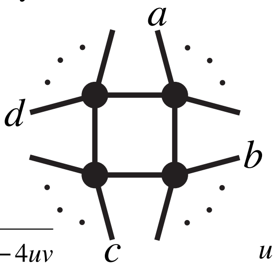

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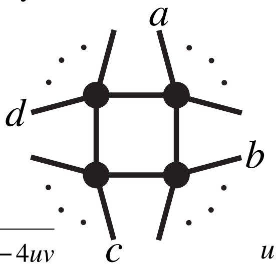

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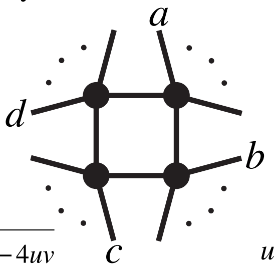

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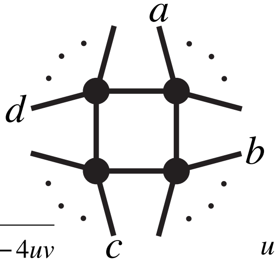

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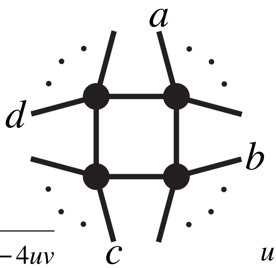

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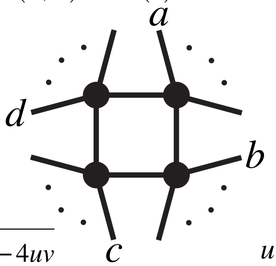

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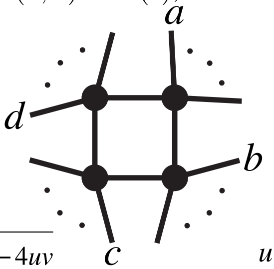

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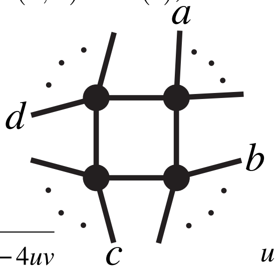

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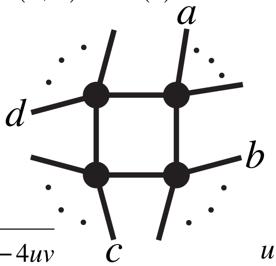

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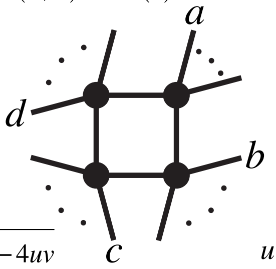

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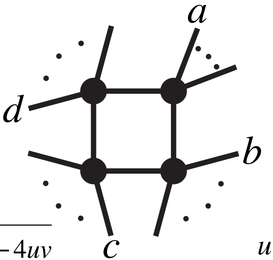

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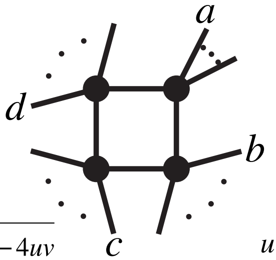

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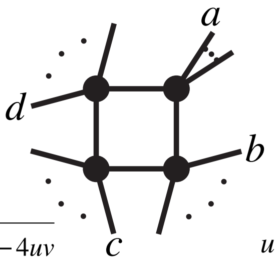

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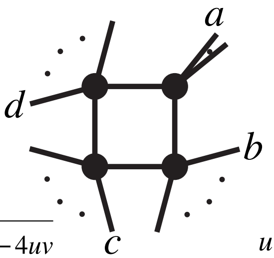
$I_{a,b,c,d}\equiv$ 

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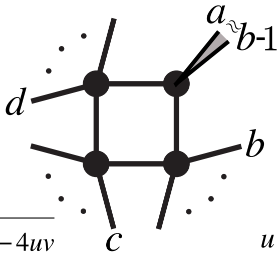

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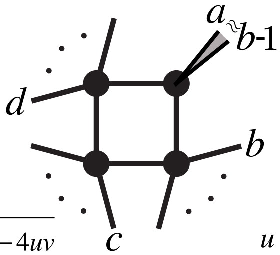

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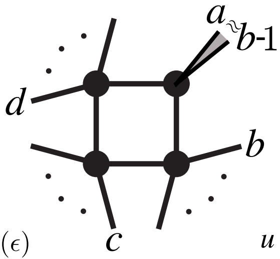
$I_{a,b,c,d}\equiv$ 

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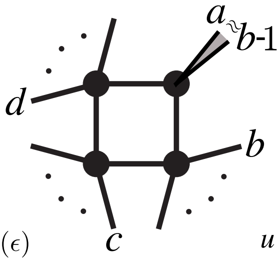
$I_{a,b,c,d}\equiv$ 

$$\equiv -\int\limits_{\ell\in\mathbb{R}^{3,1}} d^4\ell\;\frac{(a,c)(b,d)\Delta}{(\ell,a)(\ell,b)(\ell,c)(\ell,d)},$$

$$\Delta\to(1-v)+\mathcal{O}(\epsilon)\qquad u\to\mathcal{O}(\epsilon),\qquad v\equiv\frac{(b,c)(a,d)}{(a,c)(b,d)}$$

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The Famous "Four-Mass Box" Integral *and its Divergences*

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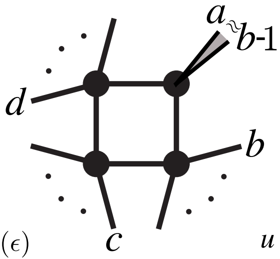

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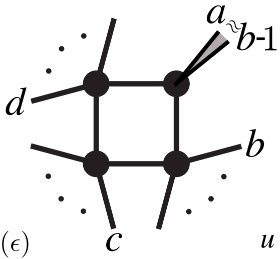

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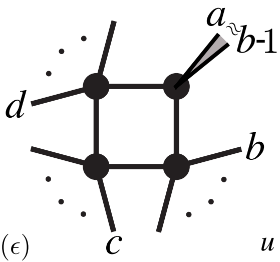

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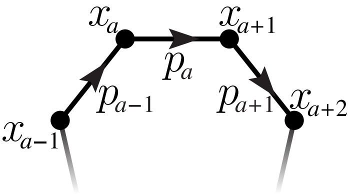

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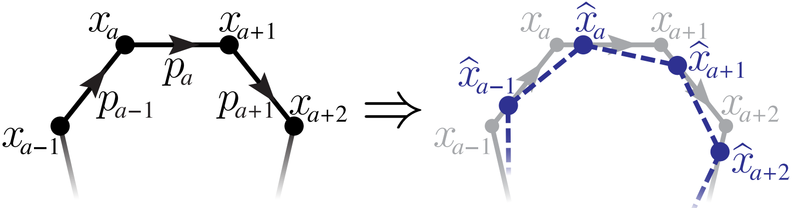

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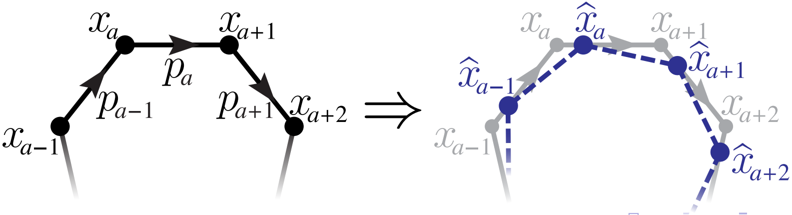

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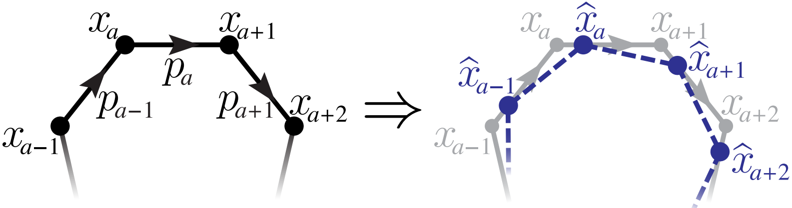

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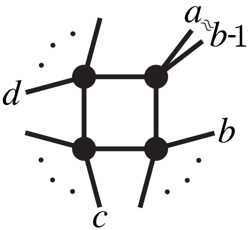

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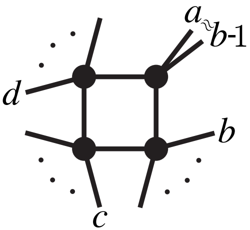

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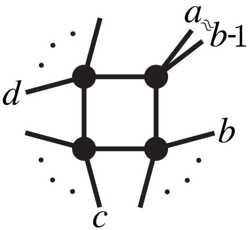

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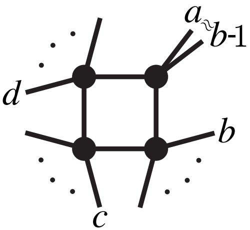

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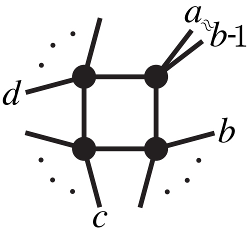

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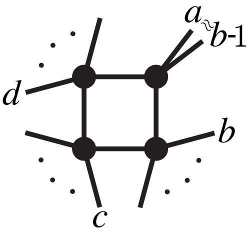

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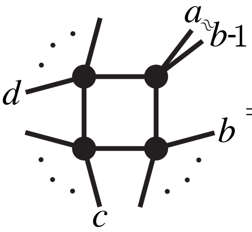

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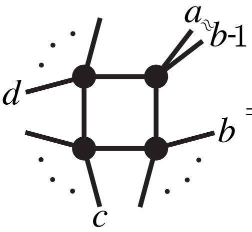

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$$\int d^4\ell\,\mathcal{A}_n^{(k),1} = \sum_{a,b,c,d} I_{\,a,b,c,d}\left(f^1_{a,b,c,d} + f^2_{a,b,c,d}\right)$$

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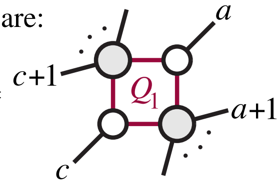

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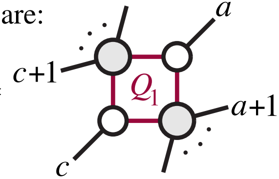

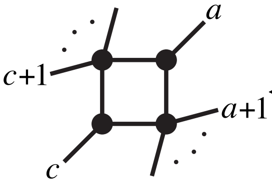

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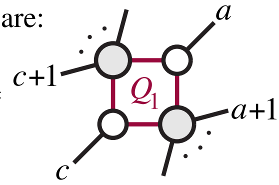

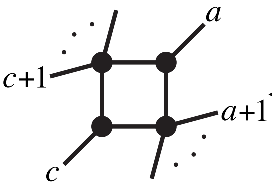

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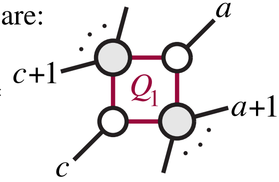

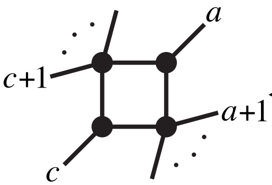

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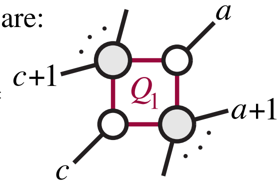
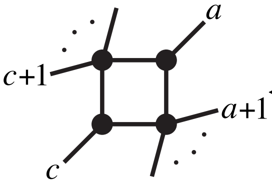




$$\Leftrightarrow \int d^4\ell\, \underbrace{\frac{(a,c)(a,a+1)-(a,c+1)(c,a+1)}{(\ell,a)(\ell,a+1)(\ell,c)(\ell,c+1)}}_{\mathcal{I}_{a,a+1,c,c+1}}$$

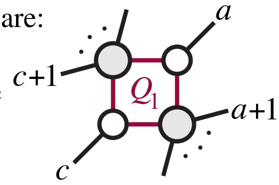
A ‘Box’-Expansion for One-Loop *Integrands*

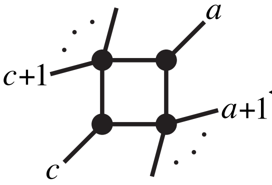
The *Scalar* Box Expansion for the One-Loop Amplitude

$$\int d^4\ell\, \mathcal{A}_n^{(2),1} = \sum_{a,c} I_{a,a+1,c,c+1}\, f^1_{a,a+1,c,c+1}$$

Consider for example the ‘MHV’ amplitude $(k=2)$, for which $f^2_{a,b,c,d} = 0$, and the only non-vanishing $f^1_{a,b,c,d}$ are:

$$f^1_{a,a+1,c,c+1} =$$





$$\Leftrightarrow \int d^4\ell\, \underbrace{\frac{(a,c)(a,a+1)-(a,c+1)(c,a+1)}{(\ell,a)(\ell,a+1)(\ell,c)(\ell,c+1)}}_{\mathcal{I}_{a,a+1,c,c+1}}$$

A ‘Box’-Expansion for One-Loop *Integrands*

The *Scalar* Box Expansion for the One-Loop Amplitude

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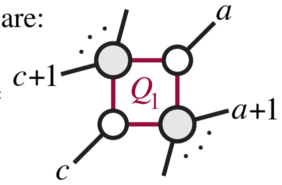

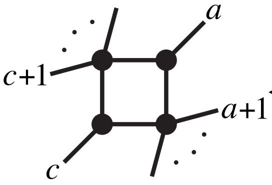

$$\Leftrightarrow \underbrace{\int d^4\ell\, \frac{(a,c)(a,a{+}1)-(a,c{+}1)(c,a{+}1)}{(\ell,a)(\ell,a{+}1)(\ell,c)(\ell,c{+}1)}}_{\mathcal{I}_{a,a+1,c,c+1}}$$

A 'Box'-Expansion for One-Loop *Integrands*

The *Scalar* Box Expansion for the One-Loop Amplitude

$$\int d^4\ell\, \mathcal{A}_n^{(2),1} = \sum_{a,c} I_{a,a+1,c,c+1}\, f^1_{a,a+1,c,c+1}$$

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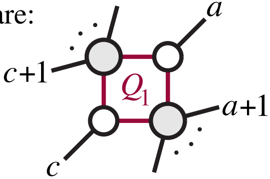

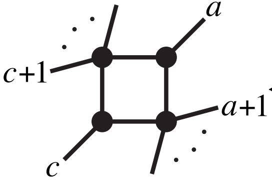

$$\Leftrightarrow \underbrace{\int d^4\ell\, \frac{(a,c)(a,a+1)-(a,c+1)(c,a+1)}{(\ell,a)(\ell,a+1)(\ell,c)(\ell,c+1)}}_{\mathcal{I}_{a,a+1,c,c+1}}$$

A ‘Box’-Expansion for One-Loop *Integrands*

The *Scalar* Box Expansion for the One-Loop Amplitude

$$\int d^4\ell\, \mathcal{A}_n^{(2),1} = \sum_{a,c} I_{a,a+1,c,c+1}\, f^1_{a,a+1,c,c+1}$$

Consider for example the ‘MHV’ amplitude $(k=2)$, for which $f^2_{a,b,c,d} = 0$, and the only non-vanishing $f^1_{a,b,c,d}$ are:

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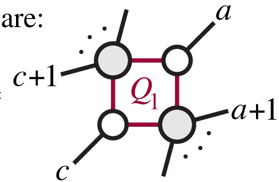

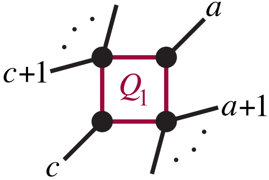

$$\Leftrightarrow \int d^4\ell\, \underbrace{\frac{(a,c)(a,a+1)-(a,c+1)(c,a+1)}{(\ell,a)(\ell,a+1)(\ell,c)(\ell,c+1)}\, \frac{(\ell,Q_2)(X,Q_1)}{(\ell,X)(Q_2,Q_1)}}_{\mathcal{I}^1_{a,a+1,c,c+1}}$$

A 'Box'-Expansion for One-Loop *Integrands*

The *Scalar* Box Expansion for the One-Loop Amplitude

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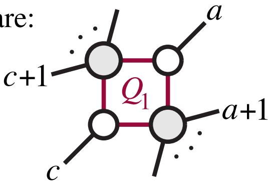

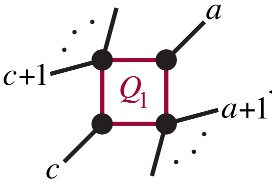

$$\Leftrightarrow \underbrace{\int d^4\ell\, \frac{(a,c)(a,a{+}1)-(a,c{+}1)(c,a{+}1)}{(\ell,a)(\ell,a{+}1)(\ell,c)(\ell,c{+}1)}\,\frac{(\ell,Q_2)(X,Q_1)}{(\ell,X)(Q_2,Q_1)}}_{\mathcal{I}^1_{a,a+1,c,c+1}}$$

A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(2),1} \overset{?}{=} \sum_{a,c} \mathcal{I}^1_{a,a+1,c,c+1}\, f^1_{a,a+1,c,c+1}$$

Consider for example the 'MHV' amplitude $(k=2)$, for which $f^2_{a,b,c,d} = 0$, and the only non-vanishing $f^1_{a,b,c,d}$ are:

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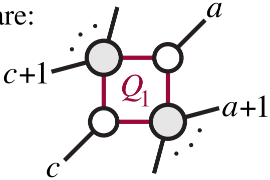

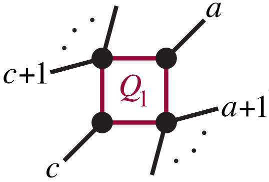

$$\Leftrightarrow \underbrace{\int d^4\ell\, \frac{(a,c)(a,a+1)-(a,c+1)(c,a+1)}{(\ell,a)(\ell,a+1)(\ell,c)(\ell,c+1)}\, \frac{(\ell,Q_2)(X,Q_1)}{(\ell,X)(Q_2,Q_1)}}_{\mathcal{I}^1_{a,a+1,c,c+1}}$$

A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d}\right)$$

Consider for example the 'MHV' amplitude $(k=2)$, for which $f^2_{a,b,c,d} = 0$, and the only non-vanishing $f^1_{a,b,c,d}$ are:

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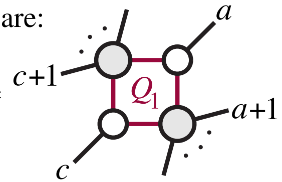

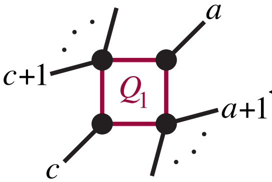

$$\Leftrightarrow \int d^4\ell\, \underbrace{\frac{(a,c)(a,a{+}1) - (a,c{+}1)(c,a{+}1)}{(\ell,a)(\ell,a{+}1)(\ell,c)(\ell,c{+}1)} \frac{(\ell,Q_2)(X,Q_1)}{(\ell,X)(Q_2,Q_1)}}_{\mathcal{I}^1_{a,a+1,c,c+1}}$$

A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d} \right)$$

A ‘Box’-Expansion for One-Loop *Integrands*

A *Chiral* ‘Box’-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d} \right)$$

This ansatz matches the correct integrand on **all** co-dimension four residues

A ‘Box’-Expansion for One-Loop *Integrands*

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This ansatz matches the correct integrand on **all** co-dimension four residues *involving four distinct propagators*.

A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d} \right)$$

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A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \stackrel{¿?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d}\right)$$

This ansatz matches the correct integrand on **all** co-dimension four residues *involving four distinct propagators*. **However**, each chiral box is **IR-finite**!

There are **also** co-dimension four residues involving only three propagators:

A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d} \right)$$

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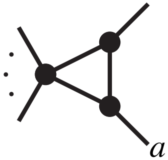

A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d}\left(\mathcal{I}^1_{a,b,c,d}f^1_{a,b,c,d}+\mathcal{I}^2_{a,b,c,d}f^2_{a,b,c,d}\right)$$

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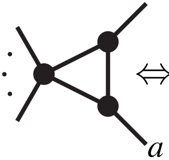

$$\Leftrightarrow \int d^4\ell\,\frac{(a-1,a+1)(a,X)}{(\ell,a-1)(\ell,a)(\ell,a+1)(\ell,X)}$$

A ‘Box’-Expansion for One-Loop *Integrands*

A *Chiral* ‘Box’-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d} \right)$$

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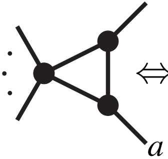
$\mathcal{I}^a_{\text{div}} \equiv$ 

$$\Leftrightarrow \int d^4\ell \, \frac{(a-1,a+1)(a,X)}{(\ell,a-1)(\ell,a)(\ell,a+1)(\ell,X)}$$

A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d} \right)$$

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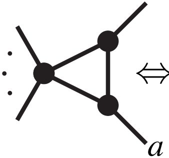
$$\mathcal{I}^a_{\text{div}} \equiv$$  $$\Leftrightarrow \int d^4\ell \, \frac{(a-1, a+1)(a, X)}{(\ell, a-1)(\ell, a)(\ell, a+1)(\ell, X)}$$

and the residue about the point $\ell \to x_a$ must be the tree amplitude: $\mathcal{A}_n^{(k),0}$

A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d} \right)$$

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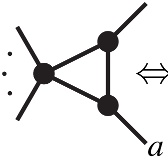
$\mathcal{I}^a_{\text{div}} \equiv$ 

$$\Leftrightarrow \int d^4\ell \, \frac{(a-1, a+1)(a, X)}{(\ell, a-1)(\ell, a)(\ell, a+1)(\ell, X)}$$

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A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d}\right) + \mathcal{A}_n^{(k),0} \sum_a \mathcal{I}^a_{\text{div}}$$

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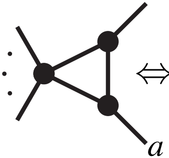
$\mathcal{I}^a_{\text{div}} \equiv$ 

$\Leftrightarrow \int d^4\ell \, \frac{(a-1, a+1)(a, X)}{(\ell, a-1)(\ell, a)(\ell, a+1)(\ell, X)}$

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A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{?}{=} \sum_{a,b,c,d}\left(\mathcal{I}^1_{a,b,c,d}f^1_{a,b,c,d}+\mathcal{I}^2_{a,b,c,d}f^2_{a,b,c,d}\right)+\mathcal{A}_n^{(k),0}\sum_a \mathcal{I}^a_{\text{div}}$$

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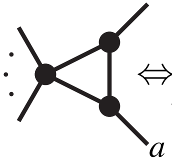

$$\Leftrightarrow \int d^4\ell \, \frac{(a{-}1,a{+}1)(a,X)}{(\ell,a{-}1)(\ell,a)(\ell,a{+}1)(\ell,X)}$$

and the residue about the point $\ell \to x_a$ must be the tree amplitude: $\mathcal{A}_n^{(k),0}$

A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} = \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d} \right) + \mathcal{A}_n^{(k),0} \sum_a \mathcal{I}^a_{\text{div}}$$

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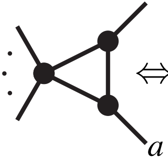
$\mathcal{I}^a_{\text{div}} \equiv$ 

$$\Leftrightarrow \int d^4\ell \, \frac{(a\!-\!1, a\!+\!1)(a, X)}{(\ell, a\!-\!1)(\ell, a)(\ell, a\!+\!1)(\ell, X)}$$

and the residue about the point $\ell \to x_a$ must be the tree amplitude: $\mathcal{A}_n^{(k),0}$

A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} = \sum_{a,b,c,d}\left(\mathcal{I}^1_{a,b,c,d}f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d}f^2_{a,b,c,d}\right) + \mathcal{A}_n^{(k),0}\sum_a \mathcal{I}^a_{\text{div}}$$

A ‘Box’-Expansion for One-Loop *Integrands*

A *Chiral* ‘Box’-Expansion for the One-Loop Amplitude Integrand

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A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

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A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} = \underbrace{\sum_{a,b,c,d}\left(\mathcal{I}^1_{a,b,c,d}f^1_{a,b,c,d}+\mathcal{I}^2_{a,b,c,d}f^2_{a,b,c,d}\right)}_{\mathcal{A}_{n,\text{fin}}^{(k),1}} + \mathcal{A}_n^{(k),0}\underbrace{\sum_a \mathcal{I}^a_{\text{div}}}_{\mathcal{A}_{n,\text{div}}^{(k),1}}$$

A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} = \underbrace{\sum_{a,b,c,d}\left(\mathcal{I}^1_{a,b,c,d}f^1_{a,b,c,d}+\mathcal{I}^2_{a,b,c,d}f^2_{a,b,c,d}\right)}_{\mathcal{A}^{(k),1}_{n,\text{fin}}} + \mathcal{A}_n^{(k),0}\underbrace{\sum_{a}\mathcal{I}^a_{\text{div}}}_{\mathcal{A}^{(k),1}_{n,\text{div}}}$$

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Because the divergences are universal, the **ratio function** is manifestly finite!

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Manifesting the Exponentiation of Divergences to All Orders

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Constructing Local Integrands for Two-Loop Amplitudes

The Two-Loop *Chiral* Expansion

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$$|||$$

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$$\mathcal{A}_{n,\text{div}}^{(k),2} = \mathcal{A}_n^{(k),0}\underbrace{\left(\mathcal{A}_{n,\text{div}}^{(2),1}\bigotimes\mathcal{A}_{n,\text{div}}^{(2),1}\right)}_{} + \left(\mathcal{A}_{n,\text{div}}^{(2),1}\bigotimes\mathcal{A}_{n,\text{fin}}^{(k),1}\right)$$

$$|||$$

$$\frac{(b-1,b+1)(b,a)(a-1,a+1)}{(\ell_1,b-1)(\ell_1,b)(\ell_1,b+1)(\ell_1,\ell_2)(\ell_2,a-1)(\ell_2,a)(\ell_2,a+1)}$$

Constructing Local Integrands for Two-Loop Amplitudes

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$$\equiv$$

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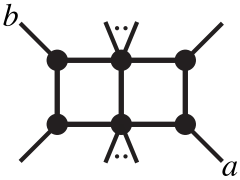

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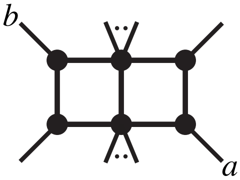

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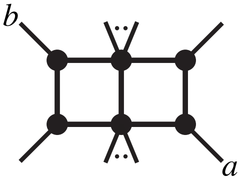

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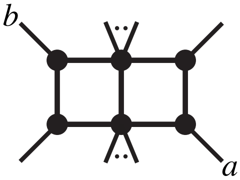

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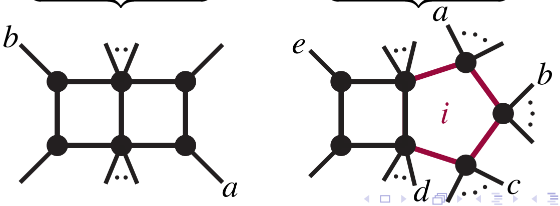

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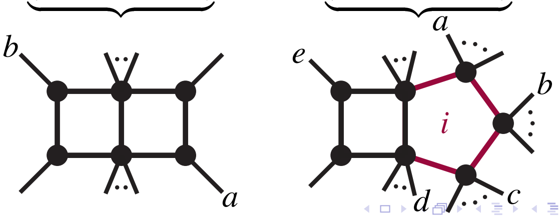

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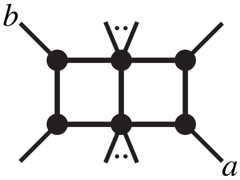

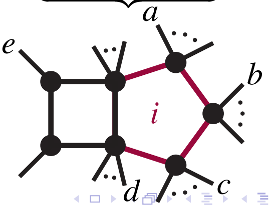

Finite Integrand Contributions to Two-Loop Amplitudes

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1. "Kissing" Boxes:

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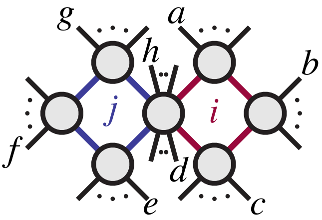

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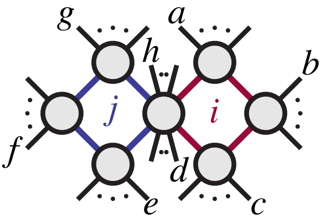

$$\times\left(\mathcal{I}^{j}_{e,f,g,h}\otimes\mathcal{I}^{i}_{a,b,c,d}\right)$$

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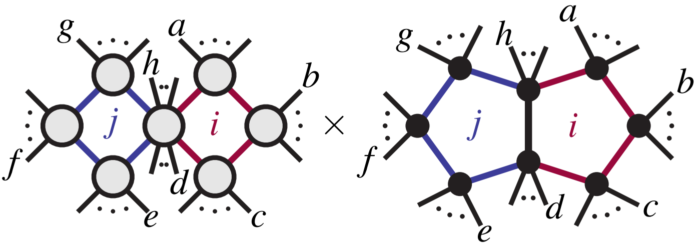

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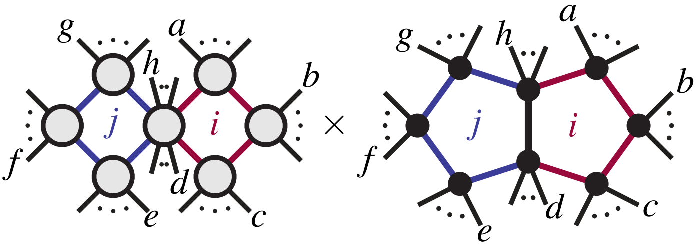

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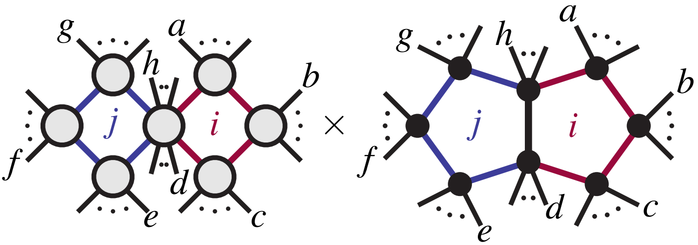

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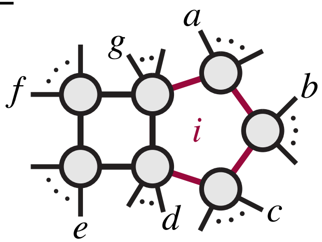

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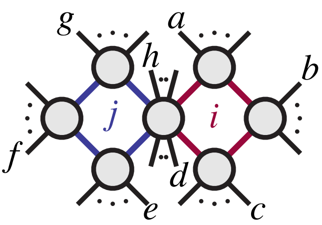
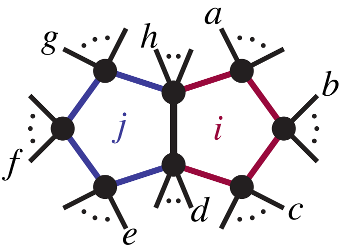


$\times$



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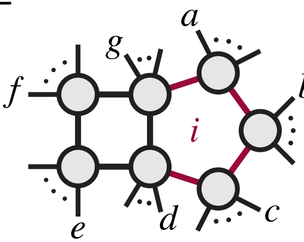

$\times \left(\mathcal{I}_{e,f,g,X} \otimes \mathcal{I}^{i}_{a,b,c,d}\right)$

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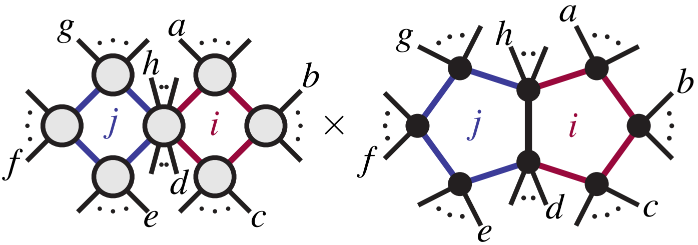

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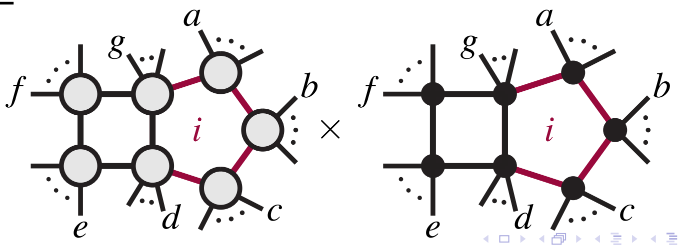

Finite Integrand Contributions to Two-Loop Amplitudes

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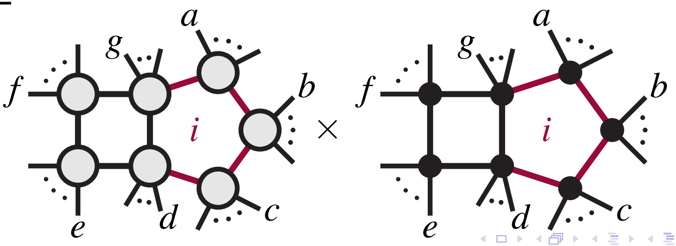

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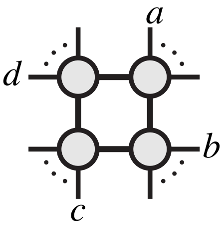

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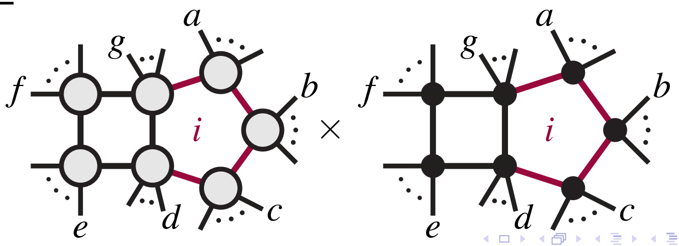

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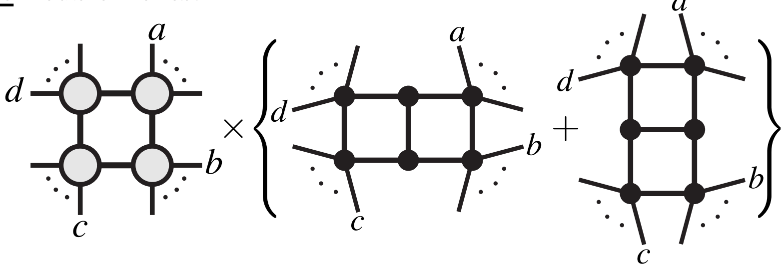

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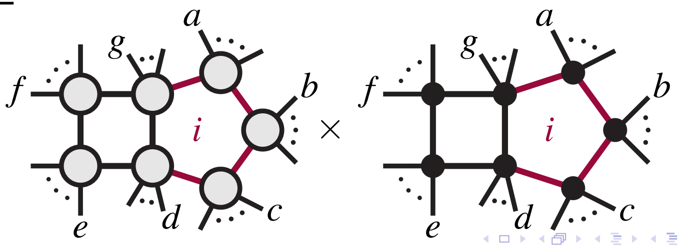

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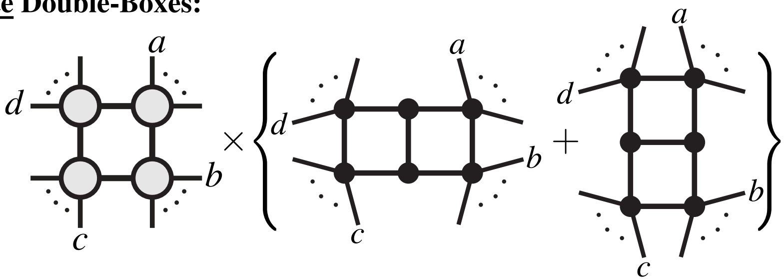

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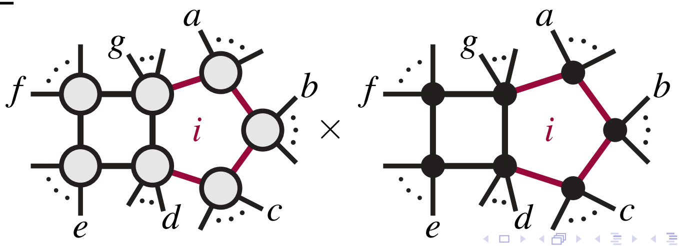

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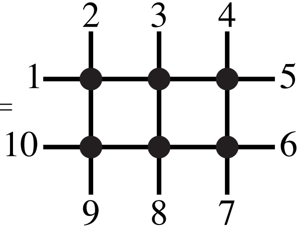
$=$ 

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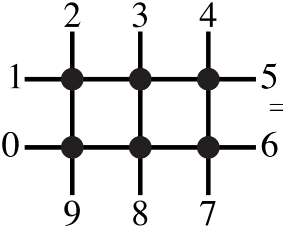

$$= \int \frac{d^4\ell_1 d^4\ell_2}{(\ell_1,9)(\ell_1,1)(\ell_1,3)(\ell_1,\ell_2)(\ell_2,4)(\ell_2,6)(\ell_2,8)}$$

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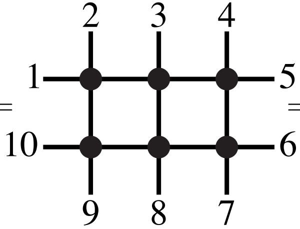

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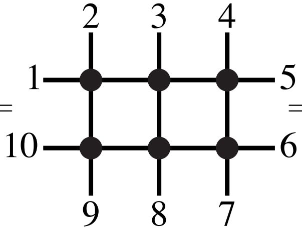
$=$ 

$$= \int \frac{d^4\ell_1 d^4\ell_2}{(\ell_1,9)(\ell_1,1)(\ell_1,3)(\ell_1,\ell_2)(\ell_2,4)(\ell_2,6)(\ell_2,8)}$$

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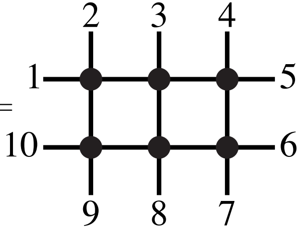
$=$  $\times ?$

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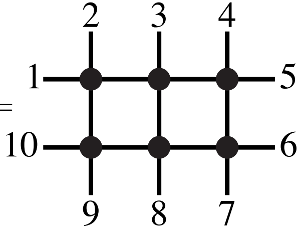
$=$  $\times ?$

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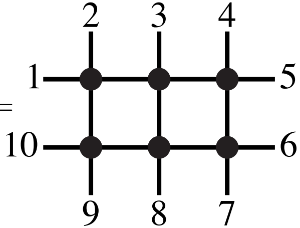
$=$ 

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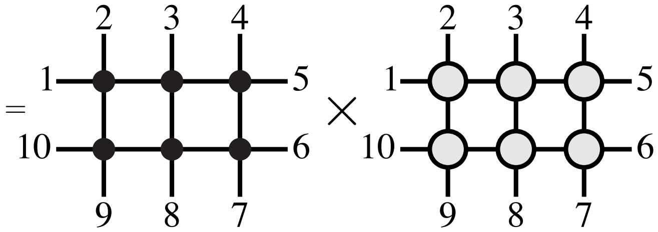

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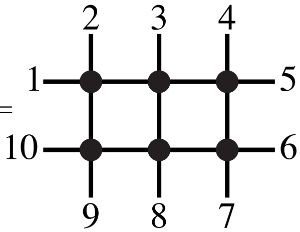

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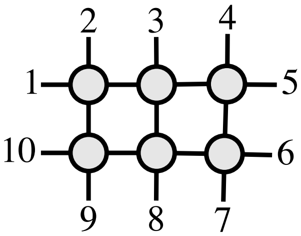

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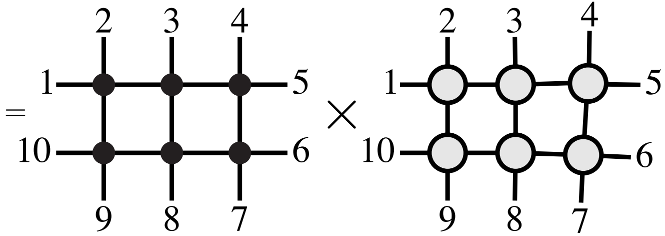

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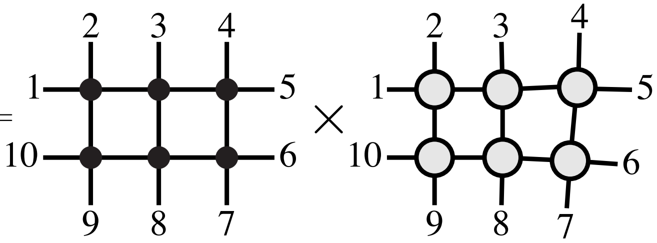
$=$ 

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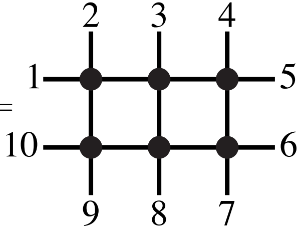
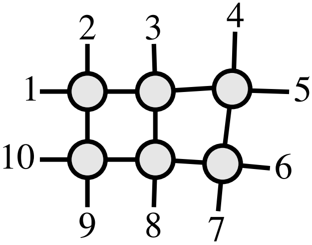
$=$ 

$\times$ 

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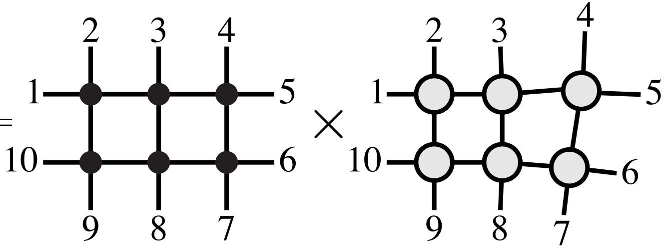
$=$ 

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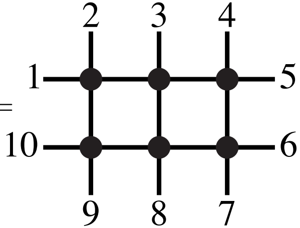
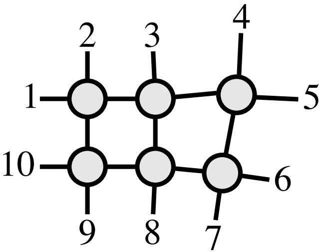
$=$ 

$\times$ 

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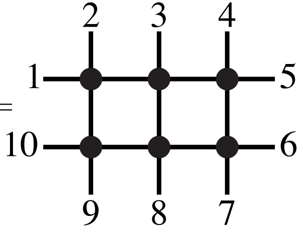
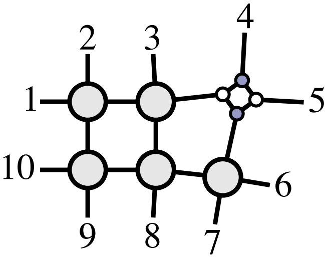
$=$ 

$\times$ 

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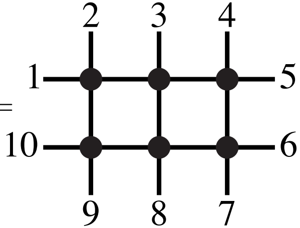
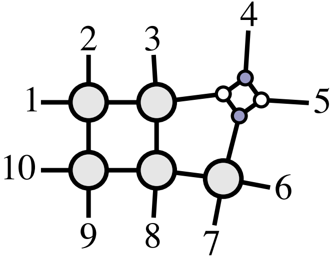
$=$  $\times$ 

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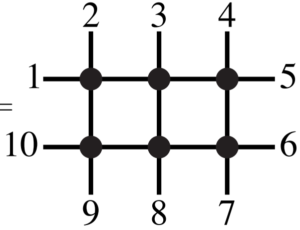
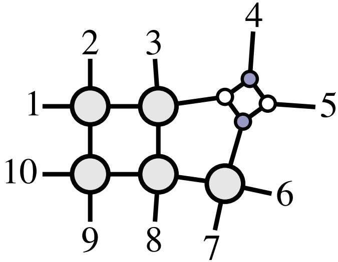
$=$ 

$\times$ 

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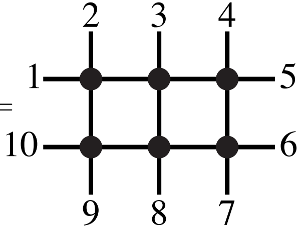
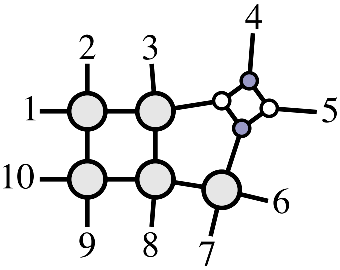
$=$  $\times$ 

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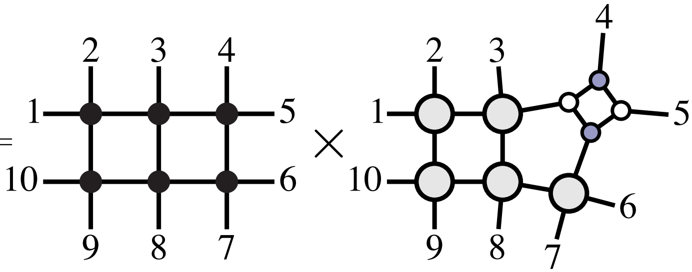
$=$ 

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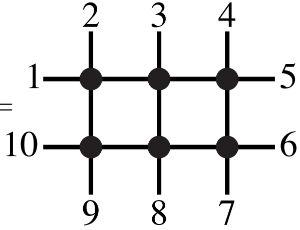
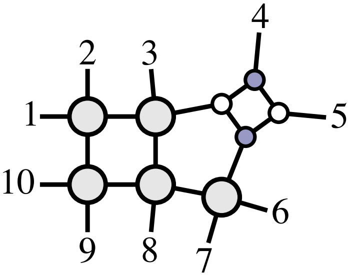
$=$  $\times$ 

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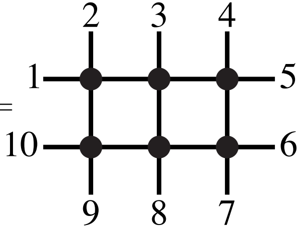
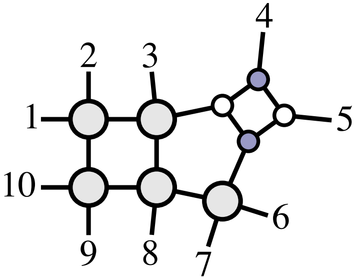
$=$ 

$\times$ 

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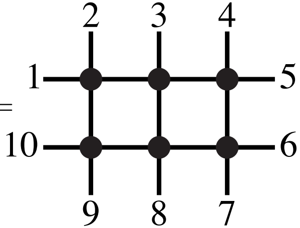
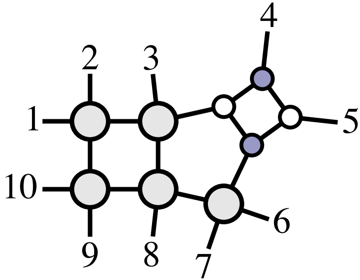
$=$  $\times$ 

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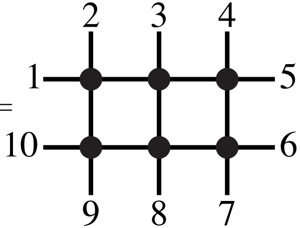
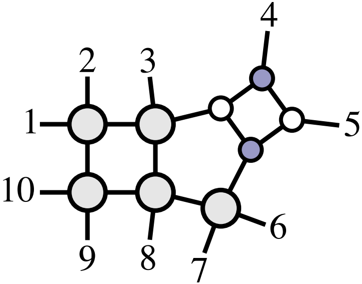
$=$ 

$\times$ 

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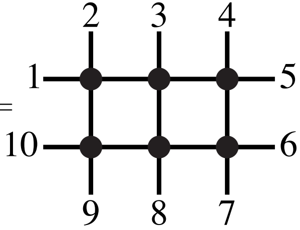
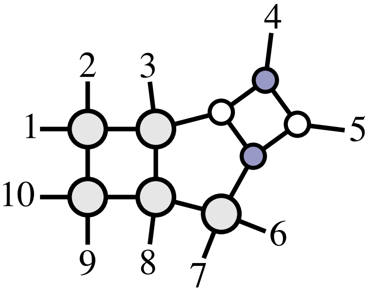
$=$  $\times$ 

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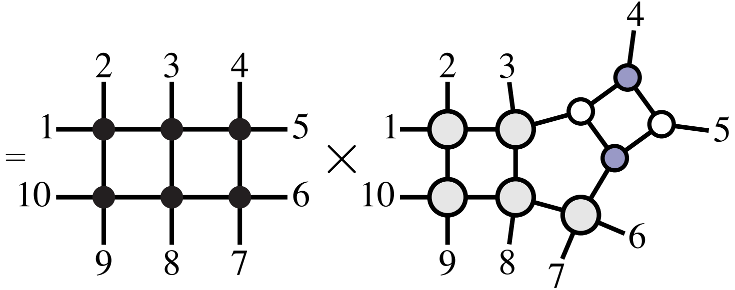

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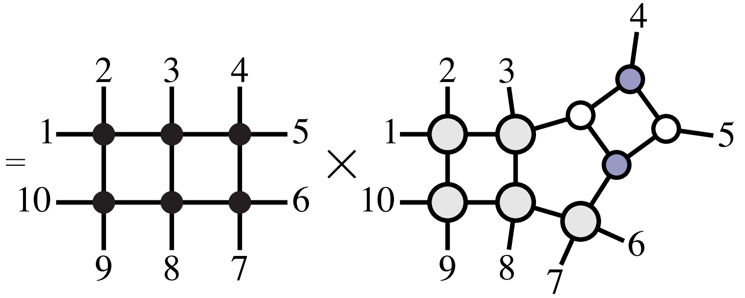

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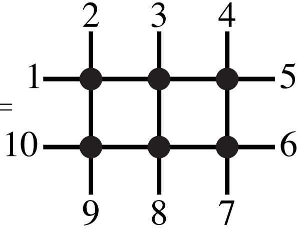
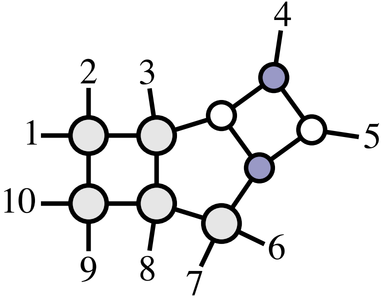
$=$ 

$\times$ 

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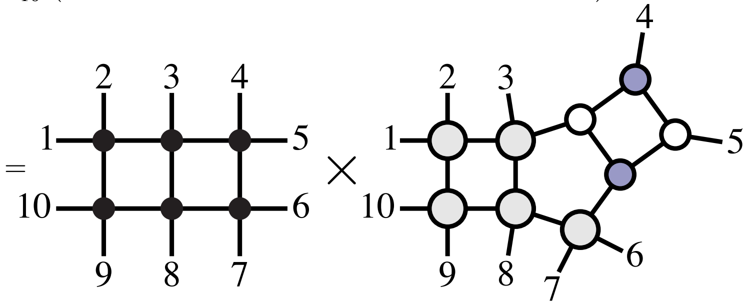

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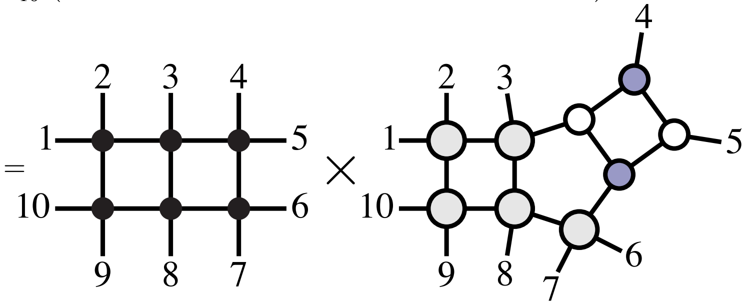

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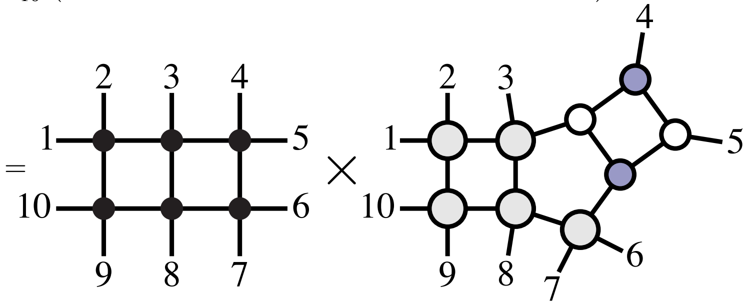

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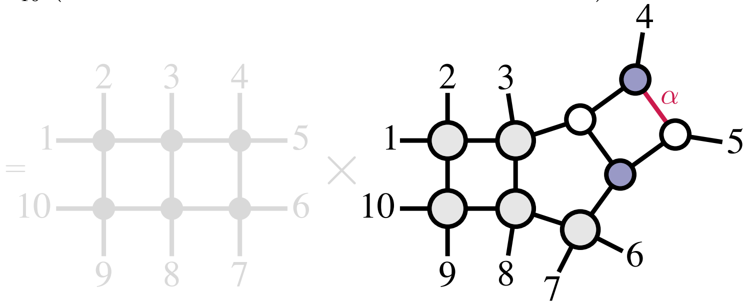

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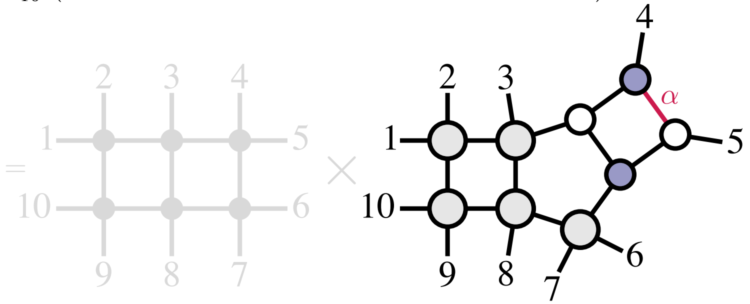

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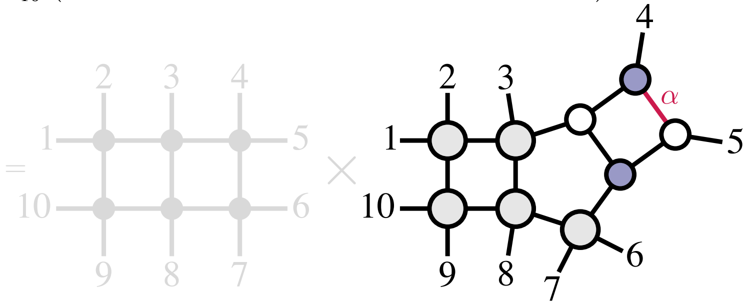

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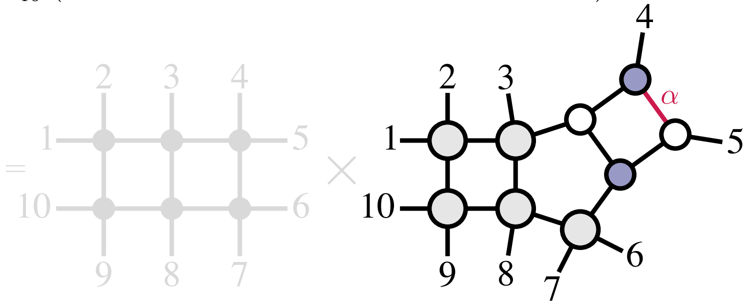

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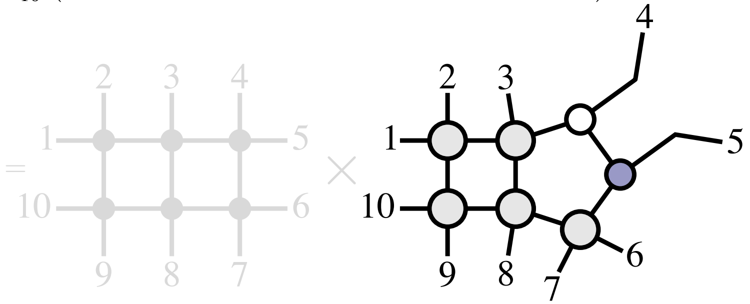

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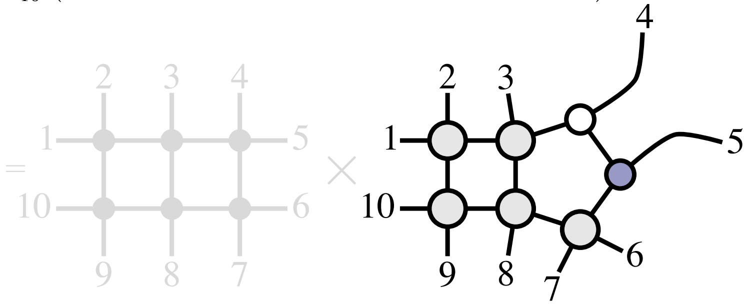

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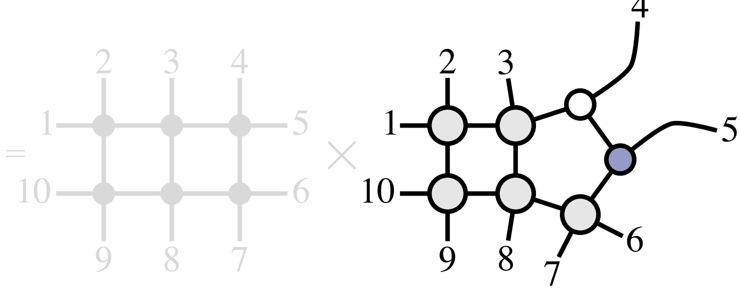

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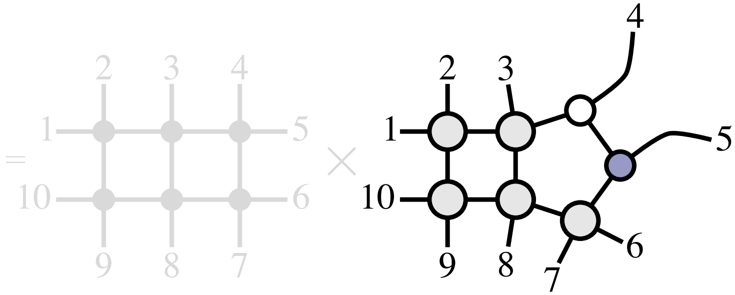

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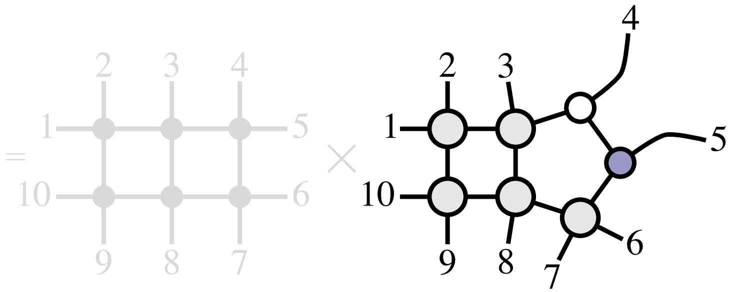

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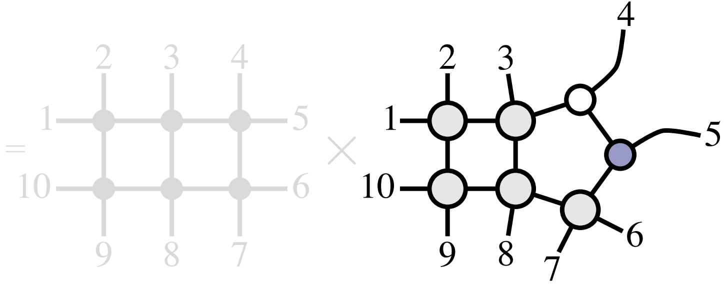

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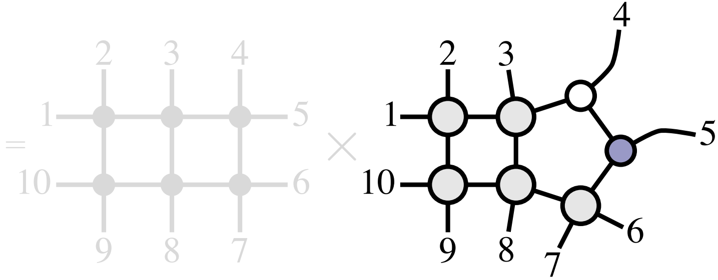

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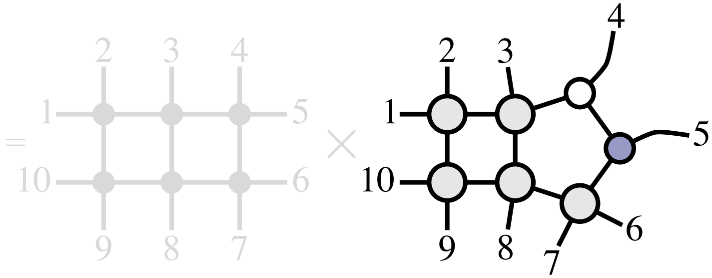

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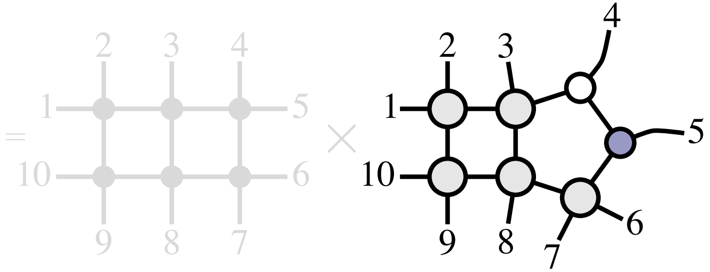

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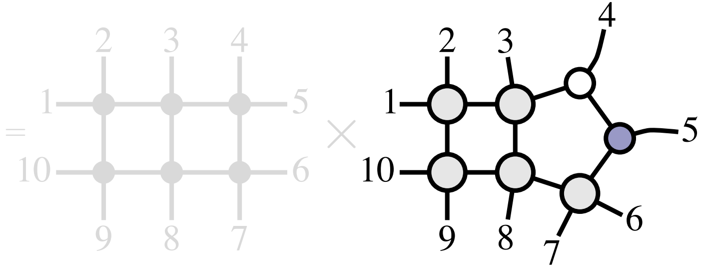

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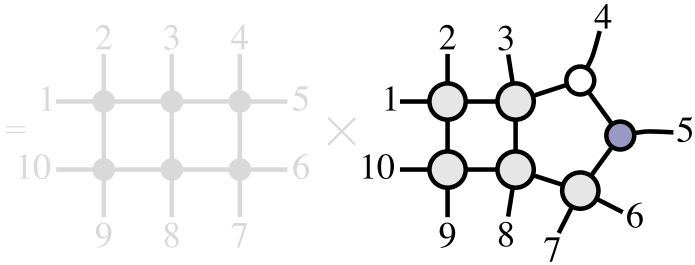

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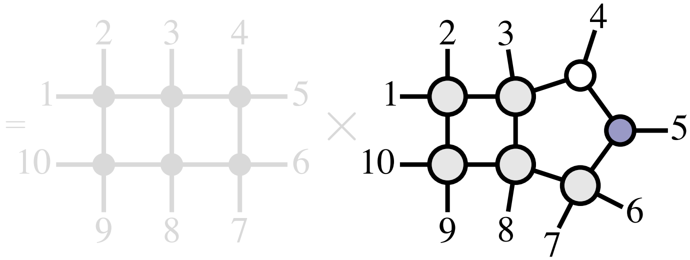

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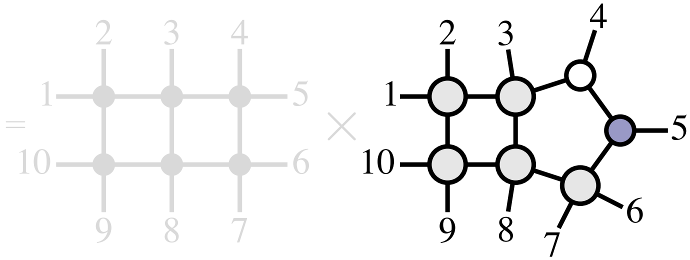

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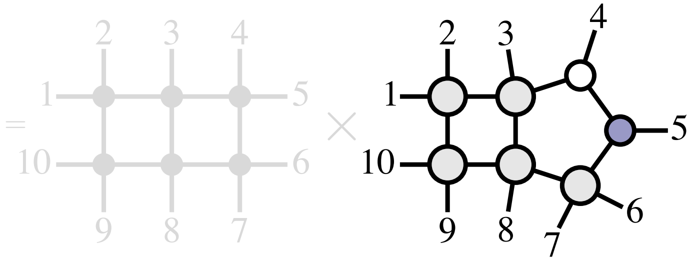

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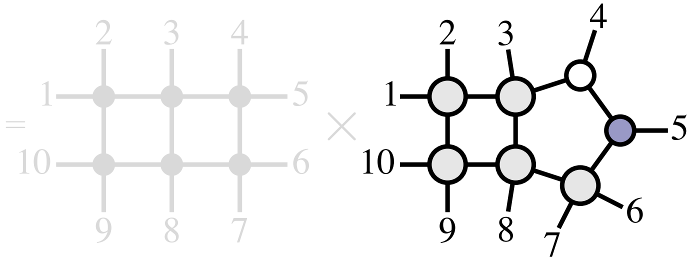

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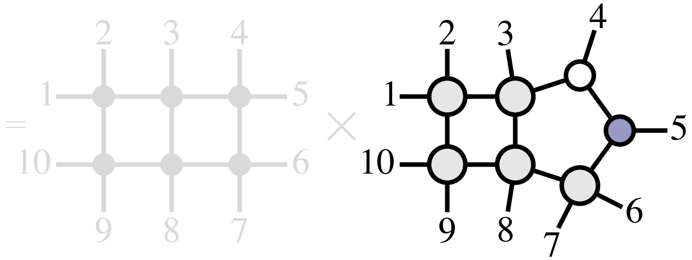

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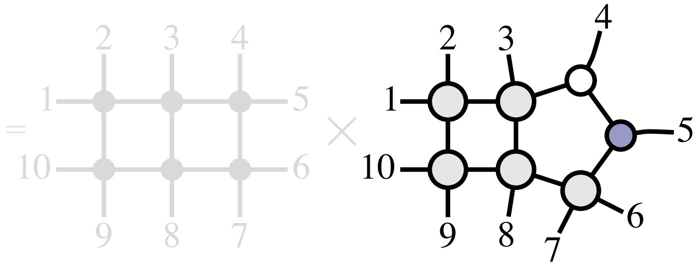

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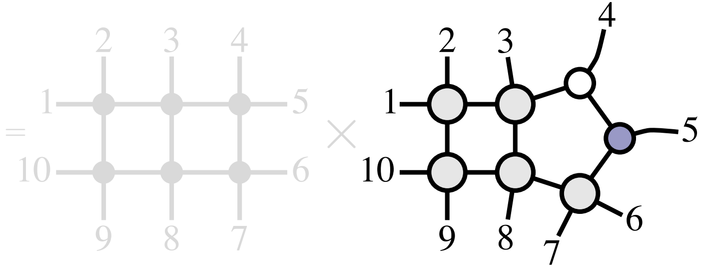

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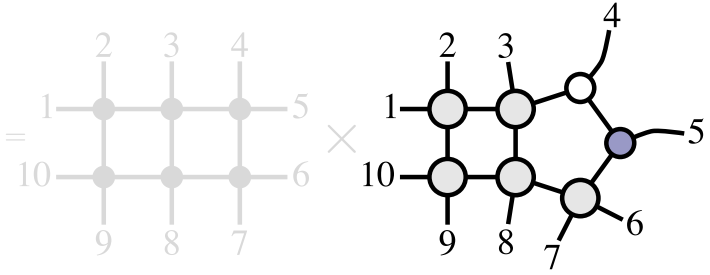

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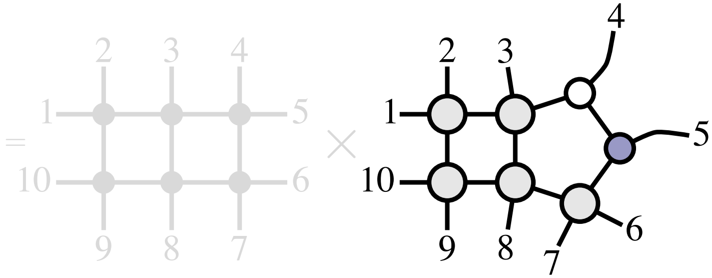

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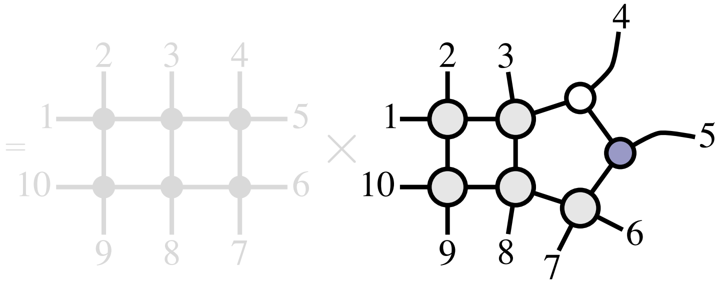

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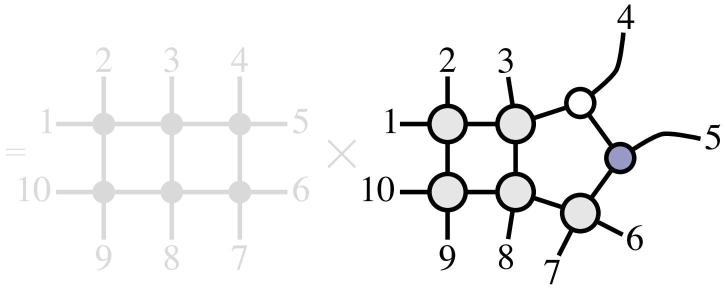

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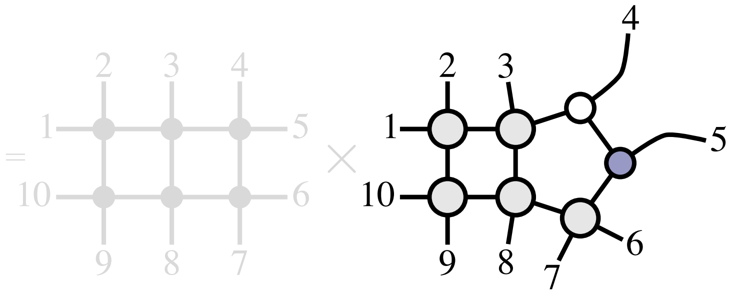

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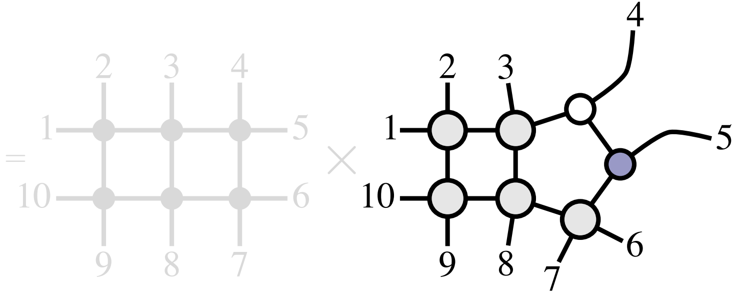

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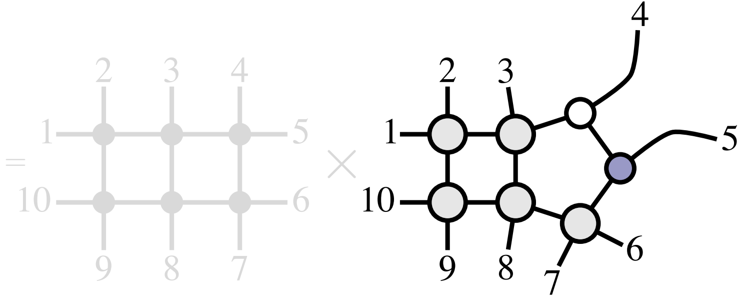

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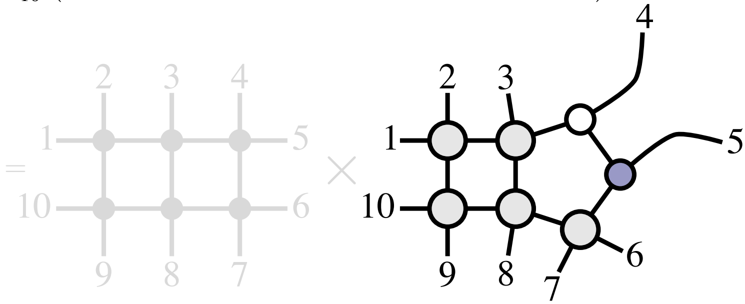

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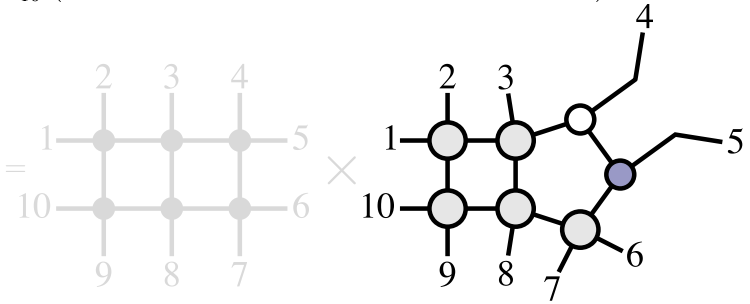

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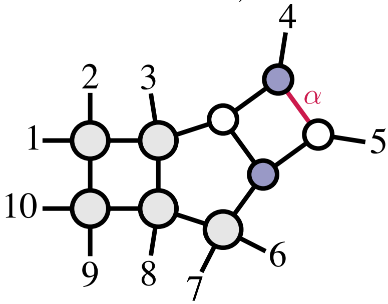

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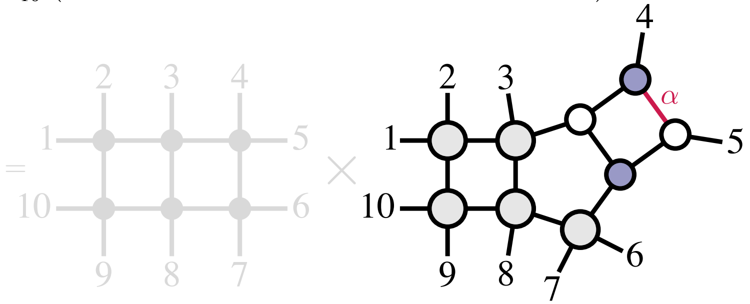

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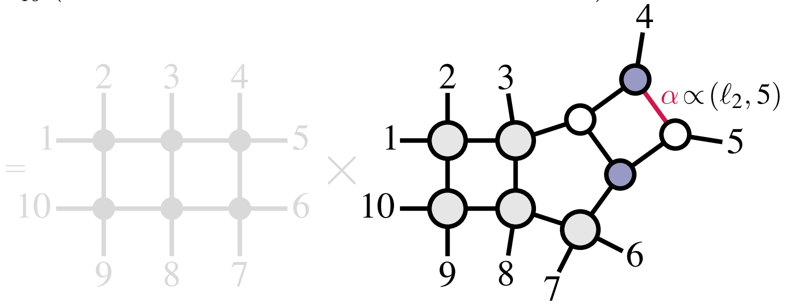

Novel Contributions Required: the *Shifted* Double-Boxes

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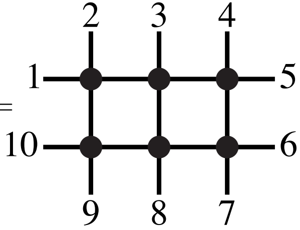
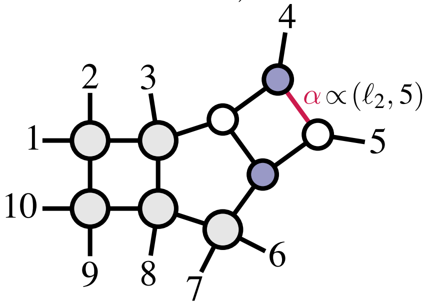
$=$ 

$\times$ 

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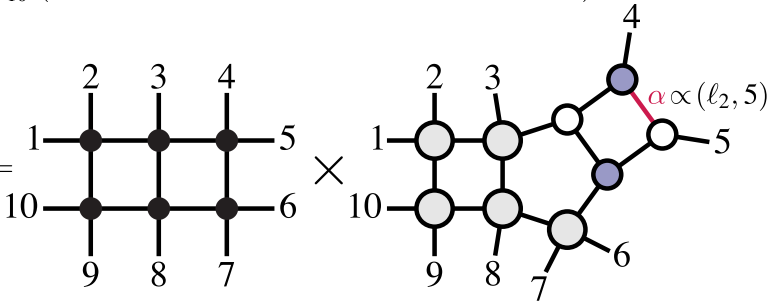
$=$ 

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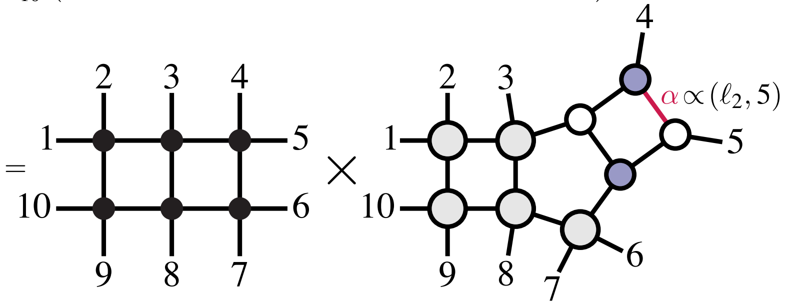

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The Two-Loop Hexagon Wilson Loop in $\mathcal{N}=4$ SYM

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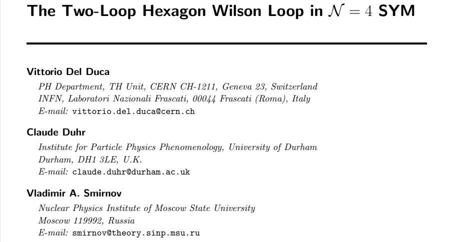

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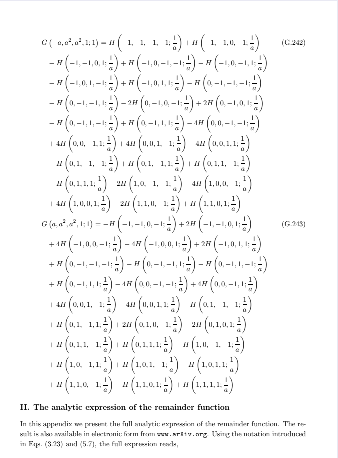

$$\begin{aligned}G\left(-a,a^2,a^2,1;1\right) =\ & H\left(-1,-1,-1,-1;\tfrac{1}{a}\right)+H\left(-1,-1,0,-1;\tfrac{1}{a}\right)\\ &-H\left(-1,-1,0,1;\tfrac{1}{a}\right)+H\left(-1,0,-1,-1;\tfrac{1}{a}\right)-H\left(-1,0,-1,1;\tfrac{1}{a}\right)\\ &-H\left(-1,0,1,-1;\tfrac{1}{a}\right)+H\left(-1,0,1,1;\tfrac{1}{a}\right)-H\left(0,-1,-1,-1;\tfrac{1}{a}\right)\\ &-H\left(0,-1,-1,1;\tfrac{1}{a}\right)-2H\left(0,-1,0,-1;\tfrac{1}{a}\right)+2H\left(0,-1,0,1;\tfrac{1}{a}\right)\\ &-H\left(0,-1,1,-1;\tfrac{1}{a}\right)+H\left(0,-1,1,1;\tfrac{1}{a}\right)-4H\left(0,0,-1,-1;\tfrac{1}{a}\right)\\ &+4H\left(0,0,-1,1;\tfrac{1}{a}\right)+4H\left(0,0,1,-1;\tfrac{1}{a}\right)-4H\left(0,0,1,1;\tfrac{1}{a}\right)\\ &-H\left(0,1,-1,-1;\tfrac{1}{a}\right)+H\left(0,1,-1,1;\tfrac{1}{a}\right)+H\left(0,1,1,-1;\tfrac{1}{a}\right)\\ &-H\left(0,1,1,1;\tfrac{1}{a}\right)-2H\left(1,0,-1,-1;\tfrac{1}{a}\right)-4H\left(1,0,0,-1;\tfrac{1}{a}\right)\\ &+4H\left(1,0,0,1;\tfrac{1}{a}\right)-2H\left(1,1,0,-1;\tfrac{1}{a}\right)+H\left(1,1,0,1;\tfrac{1}{a}\right)\end{aligned} \tag{G.242}$$

$$\begin{aligned}G\left(a,a^2,a^2,1;1\right) =\ & -H\left(-1,-1,0,-1;\tfrac{1}{a}\right)+2H\left(-1,-1,0,1;\tfrac{1}{a}\right)\\ &+4H\left(-1,0,0,-1;\tfrac{1}{a}\right)-4H\left(-1,0,0,1;\tfrac{1}{a}\right)+2H\left(-1,0,1,1;\tfrac{1}{a}\right)\\ &+H\left(0,-1,-1,-1;\tfrac{1}{a}\right)-H\left(0,-1,-1,1;\tfrac{1}{a}\right)-H\left(0,-1,1,-1;\tfrac{1}{a}\right)\\ &+H\left(0,-1,1,1;\tfrac{1}{a}\right)-4H\left(0,0,-1,-1;\tfrac{1}{a}\right)+4H\left(0,0,-1,1;\tfrac{1}{a}\right)\\ &+4H\left(0,0,1,-1;\tfrac{1}{a}\right)-4H\left(0,0,1,1;\tfrac{1}{a}\right)-H\left(0,1,-1,-1;\tfrac{1}{a}\right)\\ &+H\left(0,1,-1,1;\tfrac{1}{a}\right)+2H\left(0,1,0,-1;\tfrac{1}{a}\right)-2H\left(0,1,0,1;\tfrac{1}{a}\right)\\ &+H\left(0,1,1,-1;\tfrac{1}{a}\right)+H\left(0,1,1,1;\tfrac{1}{a}\right)-H\left(1,0,-1,-1;\tfrac{1}{a}\right)\\ &+H\left(1,0,-1,1;\tfrac{1}{a}\right)+H\left(1,0,1,-1;\tfrac{1}{a}\right)-H\left(1,0,1,1;\tfrac{1}{a}\right)\\ &+H\left(1,1,0,-1;\tfrac{1}{a}\right)-H\left(1,1,0,1;\tfrac{1}{a}\right)+H\left(1,1,1,1;\tfrac{1}{a}\right)\end{aligned} \tag{G.243}$$

H. The analytic expression of the remainder function

In this appendix we present the full analytic expression of the remainder function. The result is also available in electronic form from www.arXiv.org. Using the notation introduced in Eqs. (3.23) and (5.7), the full expression reads,

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$$R^{(2)}_{6,WL}(u_1,u_2,u_3) = \tag{H.1}$$
$$\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)+$$
$$\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)+$$
$$\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)+$$
$$\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)+$$
$$\frac{3}{2}G\left(0,0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)-$$
$$\frac{1}{2}G\left(0,\frac{1}{u_1},0,\frac{1}{u_2};1\right)+G\left(0,\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},0,\frac{1}{u_3};1\right)+$$
$$G\left(0,\frac{1}{u_1},0,\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-$$
$$\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},0,\frac{1}{u_1};1\right)+$$
$$G\left(0,\frac{1}{u_2},0,\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},0,\frac{1}{u_3};1\right)+G\left(0,\frac{1}{u_2},0,\frac{1}{u_2+u_3};1\right)-$$
$$\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)-$$
$$\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)+\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},0,\frac{1}{1-u_1};1\right)+$$
$$\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},0;1\right)-\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},1;1\right)+$$
$$\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)-$$
$$\frac{1}{2}G\left(0,\frac{1}{u_3},0,\frac{1}{u_1};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},0,\frac{1}{u_2};1\right)+G\left(0,\frac{1}{u_3},0,\frac{1}{u_1+u_3};1\right)+$$
$$G\left(0,\frac{1}{u_3},0,\frac{1}{u_2+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)-$$
$$\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)+$$
$$\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},0,\frac{1}{1-u_3};1\right)+\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},0;1\right)-$$
$$\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},1;1\right)+\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)-$$
$$\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)+\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},0,\frac{1}{1-u_2};1\right)+$$
$$\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},0;1\right)-\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},1;1\right)+$$
$$\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)-\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},1,\frac{1}{u_3},0;1\right)+\frac{1}{2}G\left(\frac{1}{1-u_1},\frac{1}{1-u_1},1,\frac{1}{1-u_1};1\right)+$$

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$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},0,1;1\right)-\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},0,\frac{1}{1-u_1};1\right)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},1,0;1\right)-\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},0;1\right)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},1;1\right)-\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1},1;1\right)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)-G\left(\frac{1}{u_1},0,0,\frac{1}{u_2};1\right)+$$
$$\frac{1}{2}G\left(\frac{1}{u_1},0,0,\frac{1}{u_1+u_2};1\right)-G\left(\frac{1}{u_1},0,0,\frac{1}{u_3};1\right)+\frac{1}{2}G\left(\frac{1}{u_1},0,0,\frac{1}{u_1+u_3};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{1-u_2},1,\frac{1}{u_1},0;1\right)+$$
$$\frac{1}{2}G\left(\frac{1}{1-u_2},\frac{1}{1-u_2},1,\frac{1}{1-u_2};1\right)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},0,1;1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},0,\frac{1}{1-u_2};1\right)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},1,0;1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},0;1\right)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},1;1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{u_3-1}{u_2+u_3-1},1;1\right)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)-G\left(\frac{1}{u_2},0,0,\frac{1}{u_1};1\right)+$$
$$\frac{1}{2}G\left(\frac{1}{u_2},0,0,\frac{1}{u_1+u_2};1\right)-G\left(\frac{1}{u_2},0,0,\frac{1}{u_3};1\right)+\frac{1}{2}G\left(\frac{1}{u_2},0,0,\frac{1}{u_2+u_3};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{1-u_3},1,\frac{1}{u_2},0;1\right)+$$
$$\frac{1}{2}G\left(\frac{1}{1-u_3},\frac{1}{1-u_3},1,\frac{1}{1-u_3};1\right)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},0,1;1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},0,\frac{1}{1-u_3};1\right)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},1,0;1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},0;1\right)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},1;1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1},1;1\right)-\frac{79\pi^4}{360}+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)-G\left(\frac{1}{u_3},0,0,\frac{1}{u_1};1\right)-$$

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$$
\begin{aligned}
&G\left(\frac{1}{u_3},0,0,\frac{1}{u_2};1\right)+\frac{1}{2}G\left(\frac{1}{u_3},0,0,\frac{1}{u_1+u_3};1\right)+\frac{1}{2}G\left(\frac{1}{u_3},0,0,\frac{1}{u_2+u_3};1\right)-\\
&\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-\\
&\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)-\frac{1}{24}\pi^2\mathcal{G}\left(\frac{1}{1-u_1},u_{123};1\right)+\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)+\\
&\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)-\frac{1}{24}\pi^2\mathcal{G}\left(\frac{1}{1-u_2},u_{231};1\right)+\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_2},v_{213};1\right)+\\
&\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_2},v_{231};1\right)-\frac{1}{24}\pi^2\mathcal{G}\left(\frac{1}{1-u_3},u_{312};1\right)+\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_3},v_{312};1\right)+\\
&\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_3},v_{321};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_1},v_{123};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_1},v_{132};1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_2},v_{213};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_2},v_{231};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_3},v_{312};1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_3},v_{321};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,v_{123},\frac{1}{1-u_1};1\right)+\mathcal{G}\left(0,0,v_{132},0;1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,0,v_{132},\frac{1}{1-u_1};1\right)+\mathcal{G}\left(0,0,v_{213},0;1\right)-\frac{1}{4}\mathcal{G}\left(0,0,v_{213},\frac{1}{1-u_2};1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,0,v_{231},\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,v_{312},\frac{1}{1-u_3};1\right)+\mathcal{G}\left(0,0,v_{321},0;1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,0,v_{321},\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},0,v_{123};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},0,v_{132};1\right)-\\
&\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123};1\right)-\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132};1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{123},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{132},1;1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},0,v_{213};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},0,v_{231};1\right)-\\
&\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213};1\right)-\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231};1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{213},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{231},1;1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},0,v_{312};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},0,v_{321};1\right)-\\
&\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312};1\right)-\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321};1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{312},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{321},1;1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{123},0,\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1},0;1\right)+\\
&\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{u_2-1}{u_1+u_2-1},1;1\right)+\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{u_3},0;1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{231},0,\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{u_1},0;1\right)-
\end{aligned}
$$

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$$\begin{aligned}
&\tfrac{1}{4}\mathcal{G}\left(0,u_{231},\tfrac{1}{1-u_2},0;1\right)+\tfrac{1}{4}\mathcal{G}\left(0,u_{231},\tfrac{1}{1-u_2},1;1\right)-\tfrac{1}{4}\mathcal{G}\left(0,u_{231},\tfrac{1}{1-u_2},\tfrac{1}{1-u_2};1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(0,u_{231},\tfrac{u_3-1}{u_2+u_3-1},1;1\right)+\tfrac{1}{4}\mathcal{G}\left(0,u_{231},\tfrac{u_3-1}{u_2+u_3-1},\tfrac{1}{1-u_2};1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(0,u_{312},0,\tfrac{1}{1-u_3};1\right)-\tfrac{1}{4}\mathcal{G}\left(0,u_{312},\tfrac{1}{u_2},0;1\right)-\tfrac{1}{4}\mathcal{G}\left(0,u_{312},\tfrac{1}{1-u_3},0;1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(0,u_{312},\tfrac{1}{1-u_3},1;1\right)-\tfrac{1}{4}\mathcal{G}\left(0,u_{312},\tfrac{1}{1-u_3},\tfrac{1}{1-u_3};1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(0,u_{312},\tfrac{u_1-1}{u_1+u_3-1},1;1\right)+\tfrac{1}{4}\mathcal{G}\left(0,u_{312},\tfrac{u_1-1}{u_1+u_3-1},\tfrac{1}{1-u_3};1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(0,v_{123},0,\tfrac{1}{1-u_1};1\right)-\tfrac{1}{2}\mathcal{G}\left(0,v_{123},1,\tfrac{1}{1-u_1};1\right)+\tfrac{1}{4}\mathcal{G}\left(0,v_{123},\tfrac{1}{1-u_1},0;1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(0,v_{123},\tfrac{1}{1-u_1},1;1\right)+\tfrac{1}{4}\mathcal{G}\left(0,v_{123},\tfrac{1}{1-u_1},\tfrac{1}{1-u_1};1\right)-\tfrac{1}{4}\mathcal{G}\left(0,v_{132},0,\tfrac{1}{1-u_1};1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(0,v_{132},\tfrac{1}{1-u_1},0;1\right)-\tfrac{1}{4}\mathcal{G}\left(0,v_{132},\tfrac{1}{1-u_1},\tfrac{1}{1-u_1};1\right)-\tfrac{1}{4}\mathcal{G}\left(0,v_{213},0,\tfrac{1}{1-u_2};1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(0,v_{213},\tfrac{1}{1-u_2},0;1\right)-\tfrac{1}{4}\mathcal{G}\left(0,v_{213},\tfrac{1}{1-u_2},\tfrac{1}{1-u_2};1\right)+\tfrac{1}{4}\mathcal{G}\left(0,v_{231},0,\tfrac{1}{1-u_2};1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(0,v_{231},1,\tfrac{1}{1-u_2};1\right)+\tfrac{1}{4}\mathcal{G}\left(0,v_{231},\tfrac{1}{1-u_2},0;1\right)-\tfrac{1}{2}\mathcal{G}\left(0,v_{231},\tfrac{1}{1-u_2},1;1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(0,v_{231},\tfrac{1}{1-u_2},\tfrac{1}{1-u_2};1\right)+\tfrac{1}{4}\mathcal{G}\left(0,v_{312},0,\tfrac{1}{1-u_3};1\right)-\tfrac{1}{2}\mathcal{G}\left(0,v_{312},1,\tfrac{1}{1-u_3};1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(0,v_{312},\tfrac{1}{1-u_3},0;1\right)-\tfrac{1}{2}\mathcal{G}\left(0,v_{312},\tfrac{1}{1-u_3},1;1\right)+\tfrac{1}{4}\mathcal{G}\left(0,v_{312},\tfrac{1}{1-u_3},\tfrac{1}{1-u_3};1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(0,v_{321},0,\tfrac{1}{1-u_3};1\right)-\tfrac{1}{4}\mathcal{G}\left(0,v_{321},\tfrac{1}{1-u_3},0;1\right)-\tfrac{1}{4}\mathcal{G}\left(0,v_{321},\tfrac{1}{1-u_3},\tfrac{1}{1-u_3};1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},0,0,v_{123};1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},0,0,v_{132};1\right)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},0,\tfrac{1}{1-u_1},v_{123};1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},0,\tfrac{1}{1-u_1},v_{132};1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},0,v_{123},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},0,v_{123},\tfrac{1}{1-u_1};1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},0,v_{132},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},0,v_{132},\tfrac{1}{1-u_1};1\right)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},\tfrac{1}{1-u_1},0,v_{123};1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},\tfrac{1}{1-u_1},0,v_{132};1\right)-\tfrac{3}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},\tfrac{1}{1-u_1},\tfrac{1}{1-u_1},v_{123};1\right)-\\
&\tfrac{3}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},\tfrac{1}{1-u_1},\tfrac{1}{1-u_1},v_{132};1\right)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},\tfrac{1}{1-u_1},v_{123},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},\tfrac{1}{1-u_1},v_{123},\tfrac{1}{1-u_1};1\right)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},\tfrac{1}{1-u_1},v_{132},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},\tfrac{1}{1-u_1},v_{132},\tfrac{1}{1-u_1};1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},u_{123},0,1;1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},u_{123},0,\tfrac{1}{1-u_1};1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},u_{123},1,0;1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},u_{123},\tfrac{1}{1-u_1},0;1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},u_{123},\tfrac{1}{1-u_1},1;1\right)+
\end{aligned}$$

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$$
\begin{aligned}
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},u_{123},\tfrac{1}{1-u_1},\tfrac{1}{1-u_1};1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},u_{123},\tfrac{u_2-1}{u_1+u_2-1},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},u_{123},\tfrac{u_2-1}{u_1+u_2-1},\tfrac{1}{1-u_1};1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},u_{123},\tfrac{1}{u_3},0;1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{123},0,0;1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{123},0,1;1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{123},0,\tfrac{1}{1-u_1};1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{123},1,0;1\right)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{123},1,\tfrac{1}{1-u_1};1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{123},\tfrac{1}{1-u_1},0;1\right)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{123},\tfrac{1}{1-u_1},1;1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{123},\tfrac{1}{1-u_1},\tfrac{1}{1-u_1};1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{132},0,0;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{132},0,1;1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{132},0,\tfrac{1}{1-u_1};1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{132},1,0;1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{132},1,\tfrac{1}{1-u_1};1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{132},\tfrac{1}{1-u_1},0;1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{132},\tfrac{1}{1-u_1},1;1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_1},v_{132},\tfrac{1}{1-u_1},\tfrac{1}{1-u_1};1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},0,0,v_{213};1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},0,0,v_{231};1\right)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_2},0,\tfrac{1}{1-u_2},v_{213};1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_2},0,\tfrac{1}{1-u_2},v_{231};1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},0,v_{213},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},0,v_{213},\tfrac{1}{1-u_2};1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},0,v_{231},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},0,v_{231},\tfrac{1}{1-u_2};1\right)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_2},\tfrac{1}{1-u_2},0,v_{213};1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_2},\tfrac{1}{1-u_2},0,v_{231};1\right)-\tfrac{3}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},\tfrac{1}{1-u_2},\tfrac{1}{1-u_2},v_{213};1\right)-\\
&\tfrac{3}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},\tfrac{1}{1-u_2},\tfrac{1}{1-u_2},v_{231};1\right)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_2},\tfrac{1}{1-u_2},v_{213},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},\tfrac{1}{1-u_2},v_{213},\tfrac{1}{1-u_2};1\right)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_2},\tfrac{1}{1-u_2},v_{231},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},\tfrac{1}{1-u_2},v_{231},\tfrac{1}{1-u_2};1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},u_{231},0,1;1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},u_{231},0,\tfrac{1}{1-u_2};1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},u_{231},1,0;1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},u_{231},\tfrac{1}{u_1},0;1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},u_{231},\tfrac{1}{1-u_2},0;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},u_{231},\tfrac{1}{1-u_2},1;1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},u_{231},\tfrac{1}{1-u_2},\tfrac{1}{1-u_2};1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},u_{231},\tfrac{u_3-1}{u_2+u_3-1},1;1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},u_{231},\tfrac{u_3-1}{u_2+u_3-1},\tfrac{1}{1-u_2};1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{213},0,0;1\right)-\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{213},0,1;1\right)+\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{213},0,\tfrac{1}{1-u_2};1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{213},1,0;1\right)-\tfrac{1}{2}\mathcal{G}\left(\tfrac{1}{1-u_2},v_{213},1,\tfrac{1}{1-u_2};1\right)+
\end{aligned}
$$

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$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2},0;1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2},1;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},0,0;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},0,1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},0,\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},1,0;1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},1,\frac{1}{1-u_2};1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2},0;1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2},1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,0,v_{312};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,0,v_{321};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},0,\frac{1}{1-u_3},v_{312};1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},0,\frac{1}{1-u_3},v_{321};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{312},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{312},\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{321},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{321},\frac{1}{1-u_3};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},0,v_{312};1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},0,v_{321};1\right)-\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312};1\right)-$$
$$\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},0,1;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},0,\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},1,0;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{u_2},0;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{1-u_3},0;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{1-u_3},1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{u_1-1}{u_1+u_3-1},1;1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},0,0;1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},0,1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},0,\frac{1}{1-u_3};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},1,0;1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},1,\frac{1}{1-u_3};1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3},0;1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3},1;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},0,0;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},0,1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},0,\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},1,0;1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},1,\frac{1}{1-u_3};1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3},0;1\right)-$$

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$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3},1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{123},0,1,\frac{1}{1-u_1};1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},0,\frac{1}{1-u_1},1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},1,0,\frac{1}{1-u_1};1\right)-$$
$$\frac{5}{4}\mathcal{G}\left(v_{123},1,1,\frac{1}{1-u_1};1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1},0;1\right)-\frac{5}{4}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1},1;1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},0,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1,0;1\right)-$$
$$\frac{5}{4}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1,\frac{1}{1-u_1};1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},\frac{1}{1-u_1},1;1\right)-\frac{1}{4}\mathcal{G}\left(v_{132},1,1,\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(v_{132},1,\frac{1}{1-u_1},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(v_{132},\frac{1}{1-u_1},1,1;1\right)-\frac{1}{4}\mathcal{G}\left(v_{213},1,1,\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(v_{213},1,\frac{1}{1-u_2},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(v_{213},\frac{1}{1-u_2},1,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},0,1,\frac{1}{1-u_2};1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},0,\frac{1}{1-u_2},1;1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{231},1,0,\frac{1}{1-u_2};1\right)-\frac{5}{4}\mathcal{G}\left(v_{231},1,1,\frac{1}{1-u_2};1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2},0;1\right)-$$
$$\frac{5}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2},1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},0,1;1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1,0;1\right)-\frac{5}{4}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1,\frac{1}{1-u_2};1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},\frac{1}{1-u_2},1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},0,1,\frac{1}{1-u_3};1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},0,\frac{1}{1-u_3},1;1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{312},1,0,\frac{1}{1-u_3};1\right)-\frac{5}{4}\mathcal{G}\left(v_{312},1,1,\frac{1}{1-u_3};1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3},0;1\right)-$$
$$\frac{5}{4}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3},1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},0,1;1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1,0;1\right)-\frac{5}{4}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1,\frac{1}{1-u_3};1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},\frac{1}{1-u_3},1;1\right)-\frac{1}{4}\mathcal{G}\left(v_{321},1,1,\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(v_{321},1,\frac{1}{1-u_3},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(v_{321},\frac{1}{1-u_3},1,1;1\right)-\frac{3}{4}G\left(0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H(0;u_1)-$$
$$\frac{3}{4}G\left(0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H(0;u_1)-\frac{1}{4}G\left(0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H(0;u_1)-$$
$$\frac{1}{4}G\left(0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H(0;u_1)-\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)H(0;u_1)+$$
$$\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)H(0;u_1)-\frac{3}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right)H(0;u_1)-$$
$$\frac{3}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_3};1\right)H(0;u_1)+\frac{1}{2}G\left(\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H(0;u_1)+$$
$$\frac{1}{2}G\left(\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H(0;u_1)+\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H(0;u_1)+$$
$$\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H(0;u_1)-\frac{1}{4}G\left(\frac{1}{1-u_2},1,\frac{1}{u_1};1\right)H(0;u_1)+$$

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$$
\begin{aligned}
&\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},1;1\right)H(0;u_1)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)H(0;u_1)+\frac{1}{2}G\left(\frac{1}{u_2},0,\frac{1}{u_1};1\right)H(0;u_1)-\\
&\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_1+u_2};1\right)H(0;u_1)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H(0;u_1)+\\
&\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},0;1\right)H(0;u_1)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)H(0;u_1)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1};1\right)H(0;u_1)+\frac{1}{2}G\left(\frac{1}{u_3},0,\frac{1}{u_1};1\right)H(0;u_1)-\\
&\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_1+u_3};1\right)H(0;u_1)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H(0;u_1)+\\
&\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{123};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{132};1\right)H(0;u_1)+\\
&\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{213};1\right)H(0;u_1)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{231};1\right)H(0;u_1)+\\
&\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{312};1\right)H(0;u_1)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{321};1\right)H(0;u_1)-\\
&\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{u_1};1\right)H(0;u_1)-\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{1-u_2};1\right)H(0;u_1)+\\
&\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{1-u_3};1\right)H(0;u_1)-\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{u_1-1}{u_1+u_3-1};1\right)H(0;u_1)+\\
&\frac{1}{4}\mathcal{G}\left(0,v_{123},\frac{1}{1-u_1};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(0,v_{132},\frac{1}{1-u_1};1\right)H(0;u_1)-\\
&\frac{1}{2}\mathcal{G}\left(0,v_{231},\frac{1}{1-u_2};1\right)H(0;u_1)+\frac{1}{2}\mathcal{G}\left(0,v_{312},\frac{1}{1-u_3};1\right)H(0;u_1)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{123};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{132};1\right)H(0;u_1)+\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123};1\right)H(0;u_1)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132};1\right)H(0;u_1)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},1;1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1};1\right)H(0;u_1)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},1;1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1};1\right)H(0;u_1)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{213};1\right)H(0;u_1)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{231};1\right)H(0;u_1)+\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213};1\right)H(0;u_1)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231};1\right)H(0;u_1)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},1;1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{u_1};1\right)H(0;u_1)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{1-u_2};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},0;1\right)H(0;u_1)+\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2};1\right)H(0;u_1)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},0;1\right)H(0;u_1)-\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{312};1\right)H(0;u_1)-
\end{aligned}
$$

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The Two-Loop Hexagon Wilson Loop in $\mathcal{N}=4$ SYM

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$$
\begin{aligned}
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{321};1\right)H(0;u_1)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312};1\right)H(0;u_1)-\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321};1\right)H(0;u_1)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},0;1\right)H(0;u_1)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{1-u_3};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{u_1-1}{u_1+u_3-1};1\right)H(0;u_1)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},0;1\right)H(0;u_1)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3};1\right)H(0;u_1)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},0;1\right)H(0;u_1)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3};1\right)H(0;u_1)+\\
&\frac{1}{4}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1;1\right)H(0;u_1)+\\
&\frac{1}{4}\mathcal{G}\left(v_{132},1,\frac{1}{1-u_1};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(v_{132},\frac{1}{1-u_1},1;1\right)H(0;u_1)+\\
&\frac{1}{4}\mathcal{G}\left(v_{213},1,\frac{1}{1-u_2};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(v_{213},\frac{1}{1-u_2},1;1\right)H(0;u_1)-\\
&\frac{3}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2};1\right)H(0;u_1)-\frac{3}{4}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1;1\right)H(0;u_1)+\\
&\frac{3}{4}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3};1\right)H(0;u_1)+\frac{3}{4}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1;1\right)H(0;u_1)-\\
&\frac{1}{4}\mathcal{G}\left(v_{321},1,\frac{1}{1-u_3};1\right)H(0;u_1)-\frac{1}{4}\mathcal{G}\left(v_{321},\frac{1}{1-u_3},1;1\right)H(0;u_1)-\\
&\frac{1}{4}G\left(0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H(0;u_2)-\frac{3}{4}G\left(0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H(0;u_2)-\\
&\frac{3}{4}G\left(0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(0;u_2)-\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)H(0;u_2)-\\
&\frac{1}{4}G\left(0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(0;u_2)+\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)H(0;u_2)+\\
&\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},0;1\right)H(0;u_2)+\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)H(0;u_2)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1};1\right)H(0;u_2)+\frac{1}{2}G\left(\frac{1}{u_1},0,\frac{1}{u_2};1\right)H(0;u_2)-\\
&\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right)H(0;u_2)+\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H(0;u_2)-\\
&\frac{3}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_1+u_2};1\right)H(0;u_2)-\frac{3}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_2+u_3};1\right)H(0;u_2)+\\
&\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H(0;u_2)+\frac{1}{2}G\left(\frac{1}{u_2},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H(0;u_2)+\\
&\frac{1}{2}G\left(\frac{1}{u_2},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(0;u_2)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(0;u_2)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_3},1,\frac{1}{u_2};1\right)H(0;u_2)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},1;1\right)H(0;u_2)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)H(0;u_2)+\frac{1}{2}G\left(\frac{1}{u_3},0,\frac{1}{u_2};1\right)H(0;u_2)-\\
&\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_2+u_3};1\right)H(0;u_2)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(0;u_2)+
\end{aligned}
$$

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$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{123};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{132};1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{213};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{231};1\right)H(0;u_2)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{312};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{321};1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{u_2-1}{u_1+u_2-1};1\right)H(0;u_2)-$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{u_2};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{1-u_3};1\right)H(0;u_2)+$$
$$\frac{1}{2}\mathcal{G}\left(0,v_{123},\frac{1}{1-u_1};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(0,v_{213},\frac{1}{1-u_2};1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(0,v_{231},\frac{1}{1-u_2};1\right)H(0;u_2)-\frac{1}{2}\mathcal{G}\left(0,v_{312},\frac{1}{1-u_3};1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{123};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{132};1\right)H(0;u_2)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123};1\right)H(0;u_2)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132};1\right)H(0;u_2)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},0;1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{1-u_1};1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{u_2-1}{u_1+u_2-1};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},0;1\right)H(0;u_2)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},0;1\right)H(0;u_2)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{213};1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{231};1\right)H(0;u_2)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213};1\right)H(0;u_2)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},1;1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},1;1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{312};1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{321};1\right)H(0;u_2)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312};1\right)H(0;u_2)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},1;1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{u_2};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{1-u_3};1\right)H(0;u_2)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},0;1\right)H(0;u_2)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3};1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},0;1\right)H(0;u_2)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3};1\right)H(0;u_2)+$$
$$\frac{3}{4}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1};1\right)H(0;u_2)+\frac{3}{4}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1;1\right)H(0;u_2)-$$

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$$
\begin{aligned}
&\frac{1}{4}\mathcal{G}\left(v_{132},1,\frac{1}{1-u_1};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(v_{132},\frac{1}{1-u_1},1;1\right)H(0;u_2)+\\
&\frac{1}{4}\mathcal{G}\left(v_{213},1,\frac{1}{1-u_2};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(v_{213},\frac{1}{1-u_2},1;1\right)H(0;u_2)+\\
&\frac{1}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1;1\right)H(0;u_2)-\\
&\frac{3}{4}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3};1\right)H(0;u_2)-\frac{3}{4}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1;1\right)H(0;u_2)+\\
&\frac{1}{4}\mathcal{G}\left(v_{321},1,\frac{1}{1-u_3};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(v_{321},\frac{1}{1-u_3},1;1\right)H(0;u_2)+\\
&\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H(0;u_1)H(0;u_2)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H(0;u_1)H(0;u_2)+\\
&\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1};1\right)H(0;u_1)H(0;u_2)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312};1\right)H(0;u_1)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312};1\right)H(0;u_1)H(0;u_2)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321};1\right)H(0;u_1)H(0;u_2)+\frac{5}{24}\pi^2H(0;u_1)H(0;u_2)-\\
&\frac{1}{4}G\left(0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H(0;u_3)-\frac{1}{4}G\left(0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(0;u_3)+\\
&\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)H(0;u_3)-\frac{3}{4}G\left(0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H(0;u_3)-\\
&\frac{3}{4}G\left(0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(0;u_3)-\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)H(0;u_3)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_1},1,\frac{1}{u_3};1\right)H(0;u_3)+\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},1;1\right)H(0;u_3)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)H(0;u_3)+\frac{1}{2}G\left(\frac{1}{u_1},0,\frac{1}{u_3};1\right)H(0;u_3)-\\
&\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_3};1\right)H(0;u_3)+\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H(0;u_3)+\\
&\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},0;1\right)H(0;u_3)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)H(0;u_3)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{u_3-1}{u_2+u_3-1};1\right)H(0;u_3)+\frac{1}{2}G\left(\frac{1}{u_2},0,\frac{1}{u_3};1\right)H(0;u_3)-\\
&\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_2+u_3};1\right)H(0;u_3)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(0;u_3)-\\
&\frac{3}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_1+u_3};1\right)H(0;u_3)-\frac{3}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_2+u_3};1\right)H(0;u_3)+\\
&\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H(0;u_3)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(0;u_3)+\\
&\frac{1}{2}G\left(\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H(0;u_3)+\frac{1}{2}G\left(\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(0;u_3)-\\
&\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{123};1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{132};1\right)H(0;u_3)-\\
&\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{213};1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{231};1\right)H(0;u_3)+
\end{aligned}
$$

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The Two-Loop Hexagon Wilson Loop in $\mathcal{N}=4$ SYM

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$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{312};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{321};1\right)H\left(0;u_3\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1};1\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{u_3};1\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{1-u_2};1\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{u_3-1}{u_2+u_3-1};1\right)H\left(0;u_3\right)-$$
$$\frac{1}{2}\mathcal{G}\left(0,v_{123},\frac{1}{1-u_1};1\right)H\left(0;u_3\right)+\frac{1}{2}\mathcal{G}\left(0,v_{231},\frac{1}{1-u_2};1\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,v_{312},\frac{1}{1-u_3};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(0,v_{321},\frac{1}{1-u_3};1\right)H\left(0;u_3\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{123};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{132};1\right)H\left(0;u_3\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123};1\right)H\left(0;u_3\right)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132};1\right)H\left(0;u_3\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},1;1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{1-u_1};1\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{u_3};1\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},0;1\right)H\left(0;u_3\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},0;1\right)H\left(0;u_3\right)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1};1\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{213};1\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{231};1\right)H\left(0;u_3\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213};1\right)H\left(0;u_3\right)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231};1\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},0;1\right)H\left(0;u_3\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{1-u_2};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{u_3-1}{u_2+u_3-1};1\right)H\left(0;u_3\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},0;1\right)H\left(0;u_3\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2};1\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},0;1\right)H\left(0;u_3\right)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2};1\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{312};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{321};1\right)H\left(0;u_3\right)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312};1\right)H\left(0;u_3\right)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321};1\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},1;1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3};1\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},1;1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3};1\right)H\left(0;u_3\right)-$$
$$\frac{3}{4}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1};1\right)H\left(0;u_3\right)-\frac{3}{4}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1;1\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}\mathcal{G}\left(v_{132},1,\frac{1}{1-u_1};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(v_{132},\frac{1}{1-u_1},1;1\right)H\left(0;u_3\right)-$$
$$\frac{1}{4}\mathcal{G}\left(v_{213},1,\frac{1}{1-u_2};1\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(v_{213},\frac{1}{1-u_2},1;1\right)H\left(0;u_3\right)+$$

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$$
\begin{aligned}
&\frac{3}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2};1\right)H(0;u_3)+\frac{3}{4}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1;1\right)H(0;u_3)+\\
&\frac{1}{4}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3};1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1;1\right)H(0;u_3)+\\
&\frac{1}{4}\mathcal{G}\left(v_{321},1,\frac{1}{1-u_3};1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(v_{321},\frac{1}{1-u_3},1;1\right)H(0;u_3)+\\
&\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H(0;u_1)H(0;u_3)+\\
&\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1};1\right)H(0;u_1)H(0;u_3)+\\
&\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H(0;u_1)H(0;u_3)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231};1\right)H(0;u_1)H(0;u_3)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213};1\right)H(0;u_1)H(0;u_3)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231};1\right)H(0;u_1)H(0;u_3)+\\
&\frac{5}{24}\pi^2H(0;u_1)H(0;u_3)+\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right)H(0;u_2)H(0;u_3)+\\
&\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(0;u_2)H(0;u_3)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(0;u_2)H(0;u_3)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123};1\right)H(0;u_2)H(0;u_3)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)H(0;u_2)H(0;u_3)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)H(0;u_2)H(0;u_3)+\frac{5}{24}\pi^2H(0;u_2)H(0;u_3)+\\
&3H(0;u_2)H(0,0;u_1)H(0;u_3)+3H(0;u_1)H(0,0;u_2)H(0;u_3)+\\
&\frac{1}{4}H(0;u_2)H\left(0,1;\frac{u_1+u_2-1}{u_2-1}\right)H(0;u_3)+\frac{1}{2}H(0;u_1)H(0,1;(u_1+u_3))H(0;u_3)+\\
&\frac{1}{4}H(0;u_1)H\left(0,1;\frac{u_2+u_3-1}{u_3-1}\right)H(0;u_3)+\frac{1}{2}H(0;u_2)H(0,1;(u_2+u_3))H(0;u_3)+\\
&\frac{3}{4}H(0;u_2)H(1,0;u_1)H(0;u_3)+\frac{3}{4}H(0;u_1)H(1,0;u_2)H(0;u_3)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213};1\right)H(0,0;u_1)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231};1\right)H(0,0;u_1)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312};1\right)H(0,0;u_1)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321};1\right)H(0,0;u_1)-\frac{23}{24}\pi^2H(0,0;u_1)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)H(0,0;u_2)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)H(0,0;u_2)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312};1\right)H(0,0;u_2)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321};1\right)H(0,0;u_2)-\\
&\frac{25}{4}H(0,0;u_1)H(0,0;u_2)-\frac{23}{24}\pi^2H(0,0;u_2)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)H(0,0;u_3)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)H(0,0;u_3)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213};1\right)H(0,0;u_3)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231};1\right)H(0,0;u_3)+3H(0;u_1)H(0;u_2)H(0,0;u_3)-\\
&\frac{25}{4}H(0,0;u_1)H(0,0;u_3)-\frac{25}{4}H(0,0;u_2)H(0,0;u_3)-\frac{23}{24}\pi^2H(0,0;u_3)+\frac{1}{12}\pi^2H(0,1;u_1)+\\
&\frac{1}{12}\pi^2H(0,1;u_2)-\frac{1}{24}\pi^2H\left(0,1;\frac{u_1+u_2-1}{u_2-1}\right)+\frac{1}{2}H(0;u_1)H(0;u_2)H(0,1;(u_1+u_2))+
\end{aligned}
$$

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$$
\begin{aligned}
&\frac{1}{12}\pi^2 H(0,1;(u_1+u_2)) + \frac{1}{12}\pi^2 H(0,1;u_3) + \frac{1}{4}H(0;u_1)H(0;u_2)H\left(0,1;\frac{u_1+u_3-1}{u_1-1}\right) - \\
&\frac{1}{24}\pi^2 H\left(0,1;\frac{u_1+u_3-1}{u_1-1}\right) + \frac{1}{12}\pi^2 H(0,1;(u_1+u_3)) - \frac{1}{24}\pi^2 H\left(0,1;\frac{u_2+u_3-1}{u_3-1}\right) + \\
&\frac{1}{12}\pi^2 H(0,1;(u_2+u_3)) - \frac{1}{2}G\left(0,\frac{1}{u_1+u_2};1\right)H(1,0;u_1) - \\
&\frac{1}{2}G\left(0,\frac{1}{u_1+u_3};1\right)H(1,0;u_1) + \frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H(1,0;u_1) + \\
&\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H(1,0;u_1) + \frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H(1,0;u_1) + \\
&\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1};1\right)H(1,0;u_1) + \frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H(1,0;u_1) - \\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312};1\right)H(1,0;u_1) - \frac{3}{4}H(0,0;u_2)H(1,0;u_1) - \frac{3}{4}H(0,0;u_3)H(1,0;u_1) + \\
&\frac{1}{4}H\left(0,1;\frac{u_1+u_3-1}{u_1-1}\right)H(1,0;u_1) - \frac{1}{3}\pi^2 H(1,0;u_1) - \frac{1}{2}G\left(0,\frac{1}{u_1+u_2};1\right)H(1,0;u_2) - \\
&\frac{1}{2}G\left(0,\frac{1}{u_2+u_3};1\right)H(1,0;u_2) + \frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right)H(1,0;u_2) + \\
&\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H(1,0;u_2) + \frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H(1,0;u_2) + \\
&\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(1,0;u_2) + \frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(1,0;u_2) - \\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123};1\right)H(1,0;u_2) - \frac{3}{4}H(0,0;u_1)H(1,0;u_2) - \frac{3}{4}H(0,0;u_3)H(1,0;u_2) + \\
&\frac{1}{4}H\left(0,1;\frac{u_1+u_2-1}{u_2-1}\right)H(1,0;u_2) - \frac{1}{4}H(1,0;u_1)H(1,0;u_2) - \frac{1}{3}\pi^2 H(1,0;u_2) - \\
&\frac{1}{2}G\left(0,\frac{1}{u_1+u_3};1\right)H(1,0;u_3) - \frac{1}{2}G\left(0,\frac{1}{u_2+u_3};1\right)H(1,0;u_3) + \\
&\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H(1,0;u_3) + \frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1};1\right)H(1,0;u_3) + \\
&\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(1,0;u_3) - \frac{1}{3}\pi^2 H(1,0;u_3) + \frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H(1,0;u_3) + \\
&\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(1,0;u_3) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231};1\right)H(1,0;u_3) + \\
&\frac{3}{4}H(0;u_1)H(0;u_2)H(1,0;u_3) - \frac{3}{4}H(0,0;u_1)H(1,0;u_3) - \frac{3}{4}H(0,0;u_2)H(1,0;u_3) + \\
&\frac{1}{4}H\left(0,1;\frac{u_2+u_3-1}{u_3-1}\right)H(1,0;u_3) - \frac{1}{4}H(1,0;u_1)H(1,0;u_3) - \frac{1}{4}H(1,0;u_2)H(1,0;u_3) + \\
&\frac{1}{24}\pi^2 H(1,1;u_1) + \frac{1}{24}\pi^2 H(1,1;u_2) + \frac{1}{24}\pi^2 H(1,1;u_3) + \frac{1}{2}H(0;u_2)H(0,0,0;u_1) + \\
&\frac{1}{2}H(0;u_3)H(0,0,0;u_2) + \frac{1}{2}H(0;u_1)H(0,0,0;u_3) - \frac{1}{2}H(0;u_2)H\left(0,0,1;\frac{u_1+u_2-1}{u_2-1}\right) - \\
&\frac{1}{2}H(0;u_3)H\left(0,0,1;\frac{u_1+u_2-1}{u_2-1}\right) - H(0;u_1)H(0,0,1;(u_1+u_2)) - \\
&H(0;u_2)H(0,0,1;(u_1+u_2)) - \frac{1}{2}H(0;u_1)H\left(0,0,1;\frac{u_1+u_3-1}{u_1-1}\right) -
\end{aligned}
$$

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$$
\begin{aligned}
&\frac{1}{2}H(0;u_2)\,H\left(0,0,1;\frac{u_1+u_3-1}{u_1-1}\right)-H(0;u_1)\,H(0,0,1;(u_1+u_3))-\\
&H(0;u_3)\,H(0,0,1;(u_1+u_3))-\frac{1}{2}H(0;u_1)\,H\left(0,0,1;\frac{u_2+u_3-1}{u_3-1}\right)-\\
&\frac{1}{2}H(0;u_3)\,H\left(0,0,1;\frac{u_2+u_3-1}{u_3-1}\right)-H(0;u_2)\,H(0,0,1;(u_2+u_3))-\\
&H(0;u_3)\,H(0,0,1;(u_2+u_3))-\frac{1}{2}H(0;u_2)\,H(0,1,0;u_1)-\frac{1}{2}H(0;u_3)\,H(0,1,0;u_2)-\\
&\frac{1}{2}H(0;u_1)\,H(0,1,0;u_3)+\frac{1}{4}H(0;u_2)\,H\left(0,1,1;\frac{u_1+u_2-1}{u_2-1}\right)-\\
&\frac{1}{4}H(0;u_3)\,H\left(0,1,1;\frac{u_1+u_2-1}{u_2-1}\right)+\frac{1}{4}H(0;u_1)\,H\left(0,1,1;\frac{u_1+u_3-1}{u_1-1}\right)-\\
&\frac{1}{4}H(0;u_2)\,H\left(0,1,1;\frac{u_1+u_3-1}{u_1-1}\right)-\frac{1}{4}H(0;u_1)\,H\left(0,1,1;\frac{u_2+u_3-1}{u_3-1}\right)+\\
&\frac{1}{4}H(0;u_3)\,H\left(0,1,1;\frac{u_2+u_3-1}{u_3-1}\right)+\frac{1}{2}H(0;u_2)\,H(1,0,0;u_1)-\frac{1}{2}H(0;u_3)\,H(1,0,0;u_1)-\\
&\frac{1}{2}H(0;u_1)\,H(1,0,0;u_2)+\frac{1}{2}H(0;u_3)\,H(1,0,0;u_2)+\frac{1}{2}H(0;u_1)\,H(1,0,0;u_3)-\\
&\frac{1}{2}H(0;u_2)\,H(1,0,0;u_3)-\frac{1}{4}H(0;u_3)\,H\left(1,0,1;\frac{u_1+u_2-1}{u_2-1}\right)-\\
&\frac{1}{4}H(0;u_2)\,H\left(1,0,1;\frac{u_1+u_3-1}{u_1-1}\right)-\frac{1}{4}H(0;u_1)\,H\left(1,0,1;\frac{u_2+u_3-1}{u_3-1}\right)-\\
&7H(0,0,0,0;u_1)-7H(0,0,0,0;u_2)-7H(0,0,0,0;u_3)+\frac{3}{2}H\left(0,0,0,1;\frac{u_1+u_2-1}{u_2-1}\right)+\\
&3H(0,0,0,1;(u_1+u_2))+\frac{3}{2}H\left(0,0,0,1;\frac{u_1+u_3-1}{u_1-1}\right)+3H(0,0,0,1;(u_1+u_3))+\\
&\frac{3}{2}H\left(0,0,0,1;\frac{u_2+u_3-1}{u_3-1}\right)+3H(0,0,0,1;(u_2+u_3))+\frac{9}{4}H(0,0,1,0;u_1)+\\
&\frac{9}{4}H(0,0,1,0;u_2)+\frac{9}{4}H(0,0,1,0;u_3)-\frac{1}{2}H(0,1,0,0;u_1)-\frac{1}{2}H(0,1,0,0;u_2)-\\
&\frac{1}{2}H(0,1,0,0;u_3)+\frac{1}{2}H\left(0,1,0,1;\frac{u_1+u_2-1}{u_2-1}\right)+\frac{1}{2}H\left(0,1,0,1;\frac{u_1+u_3-1}{u_1-1}\right)+\\
&\frac{1}{2}H\left(0,1,0,1;\frac{u_2+u_3-1}{u_3-1}\right)+H(0,1,1,0;u_1)+H(0,1,1,0;u_2)+H(0,1,1,0;u_3)-\\
&\frac{1}{4}H\left(0,1,1,1;\frac{u_1+u_2-1}{u_2-1}\right)-\frac{1}{4}H\left(0,1,1,1;\frac{u_1+u_3-1}{u_1-1}\right)-\\
&\frac{1}{4}H\left(0,1,1,1;\frac{u_2+u_3-1}{u_3-1}\right)+H\left(1,0,0,1;\frac{u_1+u_2-1}{u_2-1}\right)+H\left(1,0,0,1;\frac{u_1+u_3-1}{u_1-1}\right)+\\
&H\left(1,0,0,1;\frac{u_2+u_3-1}{u_3-1}\right)+2H(1,0,1,0;u_1)+2H(1,0,1,0;u_2)+2H(1,0,1,0;u_3)+\\
&\frac{1}{4}H\left(1,1,0,1;\frac{u_1+u_2-1}{u_2-1}\right)+\frac{1}{4}H\left(1,1,0,1;\frac{u_1+u_3-1}{u_1-1}\right)+\\
&\frac{1}{4}H\left(1,1,0,1;\frac{u_2+u_3-1}{u_3-1}\right)+\frac{1}{2}H(1,1,1,0;u_1)+\frac{1}{2}H(1,1,1,0;u_2)+\frac{1}{2}H(1,1,1,0;u_3)-\\
&\frac{1}{24}\pi^2H(0;u_3)\,\mathcal{H}\left(1;\frac{1}{u_{123}}\right)-\frac{1}{24}\pi^2H(0;u_1)\,\mathcal{H}\left(1;\frac{1}{u_{231}}\right)-\frac{1}{24}\pi^2H(0;u_2)\,\mathcal{H}\left(1;\frac{1}{u_{312}}\right)+\\
&\frac{1}{8}\pi^2H(0;u_2)\,\mathcal{H}\left(1;\frac{1}{v_{123}}\right)-\frac{1}{8}\pi^2H(0;u_3)\,\mathcal{H}\left(1;\frac{1}{v_{123}}\right)+\frac{1}{24}\pi^2H(0;u_2)\,\mathcal{H}\left(1;\frac{1}{v_{132}}\right)-
\end{aligned}
$$

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$$\frac{1}{24}\pi^2 H(0;u_3)\,\mathcal{H}\left(1;\frac{1}{v_{132}}\right)-\frac{1}{24}\pi^2 H(0;u_1)\,\mathcal{H}\left(1;\frac{1}{v_{213}}\right)+\frac{1}{24}\pi^2 H(0;u_3)\,\mathcal{H}\left(1;\frac{1}{v_{213}}\right)-$$
$$\frac{1}{8}\pi^2 H(0;u_1)\,\mathcal{H}\left(1;\frac{1}{v_{231}}\right)+\frac{1}{8}\pi^2 H(0;u_3)\,\mathcal{H}\left(1;\frac{1}{v_{231}}\right)+\frac{1}{8}\pi^2 H(0;u_1)\,\mathcal{H}\left(1;\frac{1}{v_{312}}\right)-$$
$$\frac{1}{8}\pi^2 H(0;u_2)\,\mathcal{H}\left(1;\frac{1}{v_{312}}\right)+\frac{1}{24}\pi^2 H(0;u_1)\,\mathcal{H}\left(1;\frac{1}{v_{321}}\right)-\frac{1}{24}\pi^2 H(0;u_2)\,\mathcal{H}\left(1;\frac{1}{v_{321}}\right)-$$
$$\frac{1}{4}H(0;u_2)\,H(0;u_3)\,\mathcal{H}\left(0,1;\frac{1}{u_{123}}\right)-\frac{1}{4}H(1,0;u_2)\,\mathcal{H}\left(0,1;\frac{1}{u_{123}}\right)+\frac{1}{24}\pi^2\mathcal{H}\left(0,1;\frac{1}{u_{123}}\right)+$$
$$\frac{1}{24}\pi^2\mathcal{H}\left(0,1;\frac{1}{u_{231}}\right)-\frac{1}{4}H(0;u_1)\,H(0;u_3)\,\mathcal{H}\left(0,1;\frac{1}{u_{231}}\right)-\frac{1}{4}H(1,0;u_3)\,\mathcal{H}\left(0,1;\frac{1}{u_{231}}\right)-$$
$$\frac{1}{4}H(0;u_1)\,H(0;u_2)\,\mathcal{H}\left(0,1;\frac{1}{u_{312}}\right)-\frac{1}{4}H(1,0;u_1)\,\mathcal{H}\left(0,1;\frac{1}{u_{312}}\right)+\frac{1}{24}\pi^2\mathcal{H}\left(0,1;\frac{1}{u_{312}}\right)-$$
$$\frac{1}{4}H(0;u_2)\,H(0;u_3)\,\mathcal{H}\left(0,1;\frac{1}{v_{123}}\right)+\frac{1}{4}H(0,0;u_2)\,\mathcal{H}\left(0,1;\frac{1}{v_{123}}\right)+$$
$$\frac{1}{4}H(0,0;u_3)\,\mathcal{H}\left(0,1;\frac{1}{v_{123}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{123}}\right)-\frac{1}{4}H(0;u_2)\,H(0;u_3)\,\mathcal{H}\left(0,1;\frac{1}{v_{132}}\right)+$$
$$\frac{1}{4}H(0,0;u_2)\,\mathcal{H}\left(0,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H(0,0;u_3)\,\mathcal{H}\left(0,1;\frac{1}{v_{132}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{132}}\right)-$$
$$\frac{1}{4}H(0;u_1)\,H(0;u_3)\,\mathcal{H}\left(0,1;\frac{1}{v_{213}}\right)+\frac{1}{4}H(0,0;u_1)\,\mathcal{H}\left(0,1;\frac{1}{v_{213}}\right)+$$
$$\frac{1}{4}H(0,0;u_3)\,\mathcal{H}\left(0,1;\frac{1}{v_{213}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{213}}\right)-\frac{1}{4}H(0;u_1)\,H(0;u_3)\,\mathcal{H}\left(0,1;\frac{1}{v_{231}}\right)+$$
$$\frac{1}{4}H(0,0;u_1)\,\mathcal{H}\left(0,1;\frac{1}{v_{231}}\right)+\frac{1}{4}H(0,0;u_3)\,\mathcal{H}\left(0,1;\frac{1}{v_{231}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{231}}\right)-$$
$$\frac{1}{4}H(0;u_1)\,H(0;u_2)\,\mathcal{H}\left(0,1;\frac{1}{v_{312}}\right)+\frac{1}{4}H(0,0;u_1)\,\mathcal{H}\left(0,1;\frac{1}{v_{312}}\right)+$$
$$\frac{1}{4}H(0,0;u_2)\,\mathcal{H}\left(0,1;\frac{1}{v_{312}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H(0;u_1)\,H(0;u_2)\,\mathcal{H}\left(0,1;\frac{1}{v_{321}}\right)+$$
$$\frac{1}{4}H(0,0;u_1)\,\mathcal{H}\left(0,1;\frac{1}{v_{321}}\right)+\frac{1}{4}H(0,0;u_2)\,\mathcal{H}\left(0,1;\frac{1}{v_{321}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{321}}\right)-$$
$$\frac{1}{2}H(0;u_2)\,H(0;u_3)\,\mathcal{H}\left(1,1;\frac{1}{v_{123}}\right)+\frac{1}{2}H(0,0;u_2)\,\mathcal{H}\left(1,1;\frac{1}{v_{123}}\right)+$$
$$\frac{1}{2}H(0,0;u_3)\,\mathcal{H}\left(1,1;\frac{1}{v_{123}}\right)+\frac{11}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{123}}\right)-\frac{1}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{132}}\right)-$$
$$\frac{1}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{213}}\right)-\frac{1}{2}H(0;u_1)\,H(0;u_3)\,\mathcal{H}\left(1,1;\frac{1}{v_{231}}\right)+\frac{1}{2}H(0,0;u_1)\,\mathcal{H}\left(1,1;\frac{1}{v_{231}}\right)+$$
$$\frac{1}{2}H(0,0;u_3)\,\mathcal{H}\left(1,1;\frac{1}{v_{231}}\right)+\frac{11}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{231}}\right)-\frac{1}{2}H(0;u_1)\,H(0;u_2)\,\mathcal{H}\left(1,1;\frac{1}{v_{312}}\right)+$$
$$\frac{1}{2}H(0,0;u_1)\,\mathcal{H}\left(1,1;\frac{1}{v_{312}}\right)+\frac{1}{2}H(0,0;u_2)\,\mathcal{H}\left(1,1;\frac{1}{v_{312}}\right)+\frac{11}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{312}}\right)-$$
$$\frac{1}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{321}}\right)+\frac{1}{2}H(0;u_2)\,\mathcal{H}\left(0,0,1;\frac{1}{u_{123}}\right)+\frac{1}{2}H(0;u_3)\,\mathcal{H}\left(0,0,1;\frac{1}{u_{123}}\right)+$$
$$\frac{1}{2}H(0;u_1)\,\mathcal{H}\left(0,0,1;\frac{1}{u_{231}}\right)+\frac{1}{2}H(0;u_3)\,\mathcal{H}\left(0,0,1;\frac{1}{u_{231}}\right)+\frac{1}{2}H(0;u_1)\,\mathcal{H}\left(0,0,1;\frac{1}{u_{312}}\right)+$$
$$\frac{1}{2}H(0;u_2)\,\mathcal{H}\left(0,0,1;\frac{1}{u_{312}}\right)+\frac{1}{4}H(0;u_3)\,\mathcal{H}\left(0,1,1;\frac{1}{u_{123}}\right)+\frac{1}{4}H(0;u_1)\,\mathcal{H}\left(0,1,1;\frac{1}{u_{231}}\right)+$$

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$$
\begin{aligned}
&\tfrac{1}{4}H(0;u_2)\mathcal{H}\left(0,1,1;\tfrac{1}{u_{312}}\right)+\tfrac{1}{4}H(0;u_2)\mathcal{H}\left(0,1,1;\tfrac{1}{v_{123}}\right)-\tfrac{1}{4}H(0;u_3)\mathcal{H}\left(0,1,1;\tfrac{1}{v_{123}}\right)-\\
&\tfrac{1}{4}H(0;u_2)\mathcal{H}\left(0,1,1;\tfrac{1}{v_{132}}\right)+\tfrac{1}{4}H(0;u_3)\mathcal{H}\left(0,1,1;\tfrac{1}{v_{132}}\right)+\tfrac{1}{4}H(0;u_1)\mathcal{H}\left(0,1,1;\tfrac{1}{v_{213}}\right)-\\
&\tfrac{1}{4}H(0;u_3)\mathcal{H}\left(0,1,1;\tfrac{1}{v_{213}}\right)-\tfrac{1}{4}H(0;u_1)\mathcal{H}\left(0,1,1;\tfrac{1}{v_{231}}\right)+\tfrac{1}{4}H(0;u_3)\mathcal{H}\left(0,1,1;\tfrac{1}{v_{231}}\right)+\\
&\tfrac{1}{4}H(0;u_1)\mathcal{H}\left(0,1,1;\tfrac{1}{v_{312}}\right)-\tfrac{1}{4}H(0;u_2)\mathcal{H}\left(0,1,1;\tfrac{1}{v_{312}}\right)-\tfrac{1}{4}H(0;u_1)\mathcal{H}\left(0,1,1;\tfrac{1}{v_{321}}\right)+\\
&\tfrac{1}{4}H(0;u_2)\mathcal{H}\left(0,1,1;\tfrac{1}{v_{321}}\right)+\tfrac{1}{4}H(0;u_3)\mathcal{H}\left(1,0,1;\tfrac{1}{u_{123}}\right)+\tfrac{1}{4}H(0;u_1)\mathcal{H}\left(1,0,1;\tfrac{1}{u_{231}}\right)+\\
&\tfrac{1}{4}H(0;u_2)\mathcal{H}\left(1,0,1;\tfrac{1}{u_{312}}\right)+\tfrac{1}{4}H(0;u_2)\mathcal{H}\left(1,0,1;\tfrac{1}{v_{123}}\right)-\tfrac{1}{4}H(0;u_3)\mathcal{H}\left(1,0,1;\tfrac{1}{v_{123}}\right)-\\
&\tfrac{1}{4}H(0;u_2)\mathcal{H}\left(1,0,1;\tfrac{1}{v_{132}}\right)+\tfrac{1}{4}H(0;u_3)\mathcal{H}\left(1,0,1;\tfrac{1}{v_{132}}\right)+\tfrac{1}{4}H(0;u_1)\mathcal{H}\left(1,0,1;\tfrac{1}{v_{213}}\right)-\\
&\tfrac{1}{4}H(0;u_3)\mathcal{H}\left(1,0,1;\tfrac{1}{v_{213}}\right)-\tfrac{1}{4}H(0;u_1)\mathcal{H}\left(1,0,1;\tfrac{1}{v_{231}}\right)+\tfrac{1}{4}H(0;u_3)\mathcal{H}\left(1,0,1;\tfrac{1}{v_{231}}\right)+\\
&\tfrac{1}{4}H(0;u_1)\mathcal{H}\left(1,0,1;\tfrac{1}{v_{312}}\right)-\tfrac{1}{4}H(0;u_2)\mathcal{H}\left(1,0,1;\tfrac{1}{v_{312}}\right)-\tfrac{1}{4}H(0;u_1)\mathcal{H}\left(1,0,1;\tfrac{1}{v_{321}}\right)+\\
&\tfrac{1}{4}H(0;u_2)\mathcal{H}\left(1,0,1;\tfrac{1}{v_{321}}\right)+H(0;u_2)\mathcal{H}\left(1,1,1;\tfrac{1}{v_{123}}\right)-H(0;u_3)\mathcal{H}\left(1,1,1;\tfrac{1}{v_{123}}\right)-\\
&H(0;u_1)\mathcal{H}\left(1,1,1;\tfrac{1}{v_{231}}\right)+H(0;u_3)\mathcal{H}\left(1,1,1;\tfrac{1}{v_{231}}\right)+H(0;u_1)\mathcal{H}\left(1,1,1;\tfrac{1}{v_{312}}\right)-\\
&H(0;u_2)\mathcal{H}\left(1,1,1;\tfrac{1}{v_{312}}\right)-\tfrac{3}{2}\mathcal{H}\left(0,0,0,1;\tfrac{1}{u_{123}}\right)-\tfrac{3}{2}\mathcal{H}\left(0,0,0,1;\tfrac{1}{u_{231}}\right)-\\
&\tfrac{3}{2}\mathcal{H}\left(0,0,0,1;\tfrac{1}{u_{312}}\right)-3\mathcal{H}\left(0,0,0,1;\tfrac{1}{v_{132}}\right)-3\mathcal{H}\left(0,0,0,1;\tfrac{1}{v_{213}}\right)-3\mathcal{H}\left(0,0,0,1;\tfrac{1}{v_{321}}\right)-\\
&\tfrac{1}{2}\mathcal{H}\left(0,0,1,1;\tfrac{1}{u_{123}}\right)-\tfrac{1}{2}\mathcal{H}\left(0,0,1,1;\tfrac{1}{u_{231}}\right)-\tfrac{1}{2}\mathcal{H}\left(0,0,1,1;\tfrac{1}{u_{312}}\right)-\\
&\tfrac{1}{2}\mathcal{H}\left(0,1,0,1;\tfrac{1}{u_{123}}\right)-\tfrac{1}{2}\mathcal{H}\left(0,1,0,1;\tfrac{1}{u_{231}}\right)-\tfrac{1}{2}\mathcal{H}\left(0,1,0,1;\tfrac{1}{u_{312}}\right)+\\
&\tfrac{1}{4}\mathcal{H}\left(0,1,1,1;\tfrac{1}{v_{123}}\right)+\tfrac{1}{4}\mathcal{H}\left(0,1,1,1;\tfrac{1}{v_{132}}\right)+\zeta_3H(0;u_1)+\zeta_3H(0;u_2)+\zeta_3H(0;u_3)+\\
&\tfrac{5}{2}\zeta_3H(1;u_1)+\tfrac{5}{2}\zeta_3H(1;u_2)+\tfrac{5}{2}\zeta_3H(1;u_3)+\tfrac{1}{2}\zeta_3\mathcal{H}\left(1;\tfrac{1}{u_{123}}\right)+\tfrac{1}{2}\zeta_3\mathcal{H}\left(1;\tfrac{1}{u_{231}}\right)+\\
&\tfrac{1}{2}\zeta_3\mathcal{H}\left(1;\tfrac{1}{u_{312}}\right)-\tfrac{1}{2}\mathcal{H}\left(1,0,0,1;\tfrac{1}{u_{123}}\right)-\tfrac{1}{2}\mathcal{H}\left(1,0,0,1;\tfrac{1}{u_{231}}\right)-\tfrac{1}{2}\mathcal{H}\left(1,0,0,1;\tfrac{1}{u_{312}}\right)+\\
&\tfrac{1}{4}\zeta_3\mathcal{H}\left(1;\tfrac{1}{v_{123}}\right)+\tfrac{1}{4}\zeta_3\mathcal{H}\left(1;\tfrac{1}{v_{132}}\right)+\tfrac{1}{4}\zeta_3\mathcal{H}\left(1;\tfrac{1}{v_{213}}\right)+\tfrac{1}{4}\zeta_3\mathcal{H}\left(1;\tfrac{1}{v_{231}}\right)+\tfrac{1}{4}\zeta_3\mathcal{H}\left(1;\tfrac{1}{v_{312}}\right)+\\
&\tfrac{1}{4}\zeta_3\mathcal{H}\left(1;\tfrac{1}{v_{321}}\right)+\tfrac{1}{4}\mathcal{H}\left(0,1,1,1;\tfrac{1}{v_{213}}\right)+\tfrac{1}{4}\mathcal{H}\left(0,1,1,1;\tfrac{1}{v_{231}}\right)+\tfrac{1}{4}\mathcal{H}\left(0,1,1,1;\tfrac{1}{v_{312}}\right)+\\
&\tfrac{1}{4}\mathcal{H}\left(0,1,1,1;\tfrac{1}{v_{321}}\right)+\tfrac{1}{4}\mathcal{H}\left(1,0,1,1;\tfrac{1}{v_{123}}\right)+\tfrac{1}{4}\mathcal{H}\left(1,0,1,1;\tfrac{1}{v_{132}}\right)+\tfrac{1}{4}\mathcal{H}\left(1,0,1,1;\tfrac{1}{v_{213}}\right)+\\
&\tfrac{1}{4}\mathcal{H}\left(1,0,1,1;\tfrac{1}{v_{231}}\right)+\tfrac{1}{4}\mathcal{H}\left(1,0,1,1;\tfrac{1}{v_{312}}\right)+\tfrac{1}{4}\mathcal{H}\left(1,0,1,1;\tfrac{1}{v_{321}}\right)+\tfrac{1}{4}\mathcal{H}\left(1,1,0,1;\tfrac{1}{v_{123}}\right)+\\
&\tfrac{1}{4}\mathcal{H}\left(1,1,0,1;\tfrac{1}{v_{132}}\right)+\tfrac{1}{4}\mathcal{H}\left(1,1,0,1;\tfrac{1}{v_{213}}\right)+\tfrac{1}{4}\mathcal{H}\left(1,1,0,1;\tfrac{1}{v_{231}}\right)+\tfrac{1}{4}\mathcal{H}\left(1,1,0,1;\tfrac{1}{v_{312}}\right)+
\end{aligned}
$$

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- dimensionally regulating thousands of separately divergent integrals
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The Two-Loop Hexagon Wilson Loop in $\mathcal{N}=4$ SYM

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