On-Shell Diagrams, Recursion Relations, & Combinatorics

Jacob L. Bourjaily

Nordic Winter School on
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Organization and Outline

Broadening the Class of Physically Meaningful Functions

We are interested in the class of functions involving **only** observable quantities

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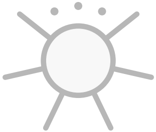

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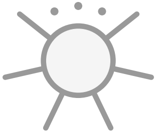

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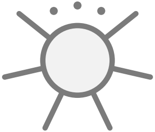

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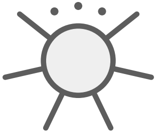

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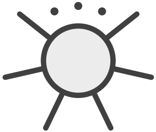

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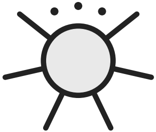

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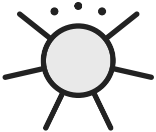

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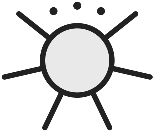

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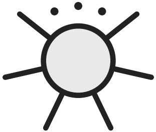

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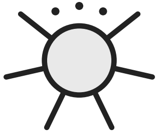

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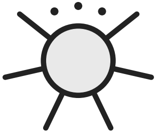

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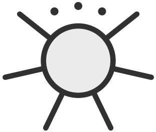

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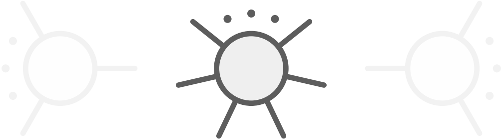

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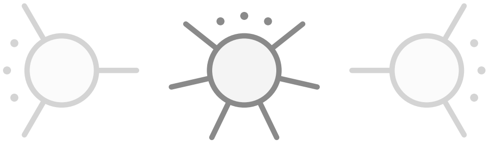

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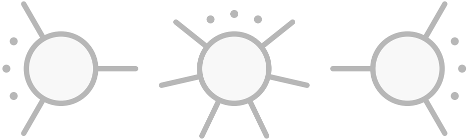

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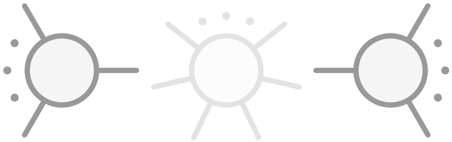

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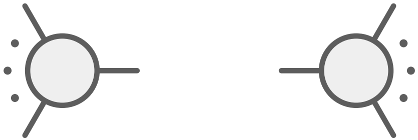

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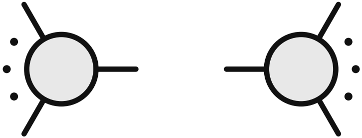

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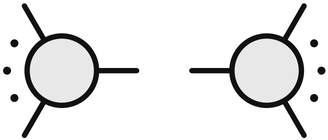

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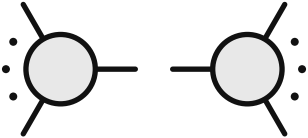

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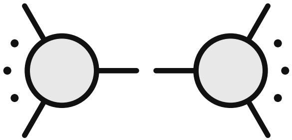

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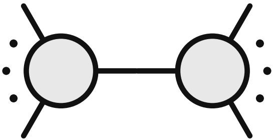

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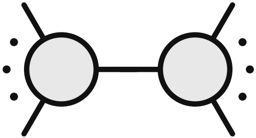

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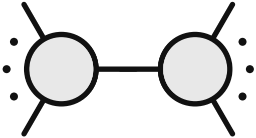

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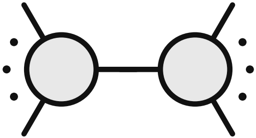

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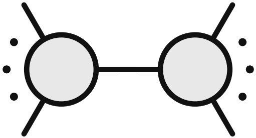

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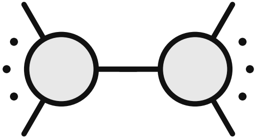

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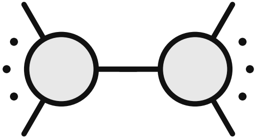

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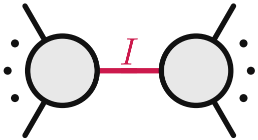

Internal Particles:

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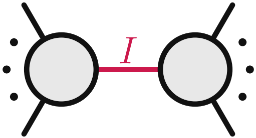

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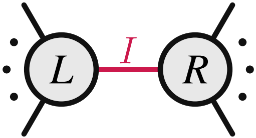

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$$\mathcal{A}_L(\ldots,I)\times\mathcal{A}_R(I,\ldots)$$

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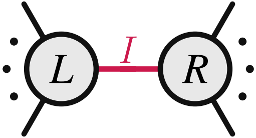

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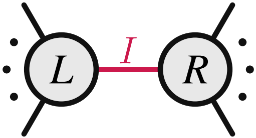

Internal Particles: locality dictates that we multiply each amplitude, and unitarity dictates that we marginalize over unobserved states—integrating over the Lorentz-invariant phase space ("LIPS") for each particle I,

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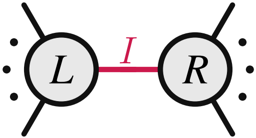

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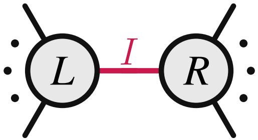

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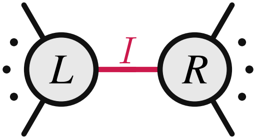

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$$\sum_{\text{states } I} \int d^3\text{LIPS}_I \;\; \mathcal{A}_L(\ldots, I) \times \mathcal{A}_R(I, \ldots)$$

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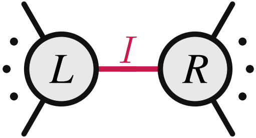

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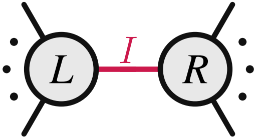

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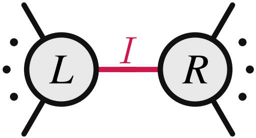

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$$\sum_{\text{states } I}\int\frac{d^2\lambda_I d^2\widetilde{\lambda}_I}{\mathrm{vol}(GL_1)}\,\mathcal{A}_L(\ldots,I)\times\mathcal{A}_R(I,\ldots)$$

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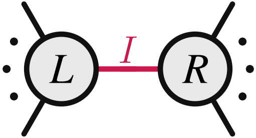

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$$\int d^4\widetilde{\eta}_I \int \frac{d^2\lambda_I d^2\widetilde{\lambda}_I}{\mathrm{vol}(GL_1)} \mathcal{A}_L(\ldots, I) \times \mathcal{A}_R(I, \ldots)$$

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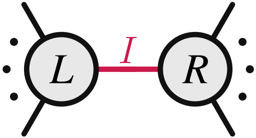

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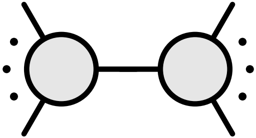

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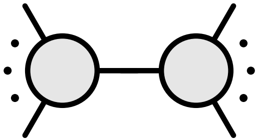

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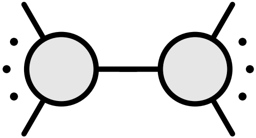

On-Shell Functions: networks of amplitudes, $\mathcal{A}_v$, connected by any number of internal particles, $i \in I$, forming a graph Γ called an "on-shell diagram".

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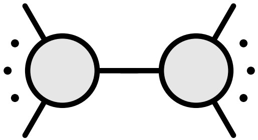

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$$f_\Gamma \equiv \prod_{i \in I} \Bigg(\sum_{\substack{h_i, c_i,\\ m_i, \cdots}} \int d^3\mathrm{LIPS}_i \Bigg) \prod_v \mathcal{A}_v$$

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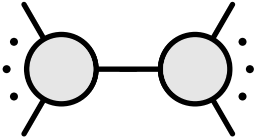

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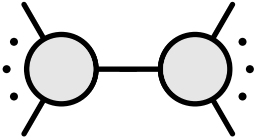

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Counting Constraints:

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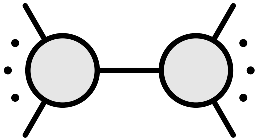

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$$f_\Gamma \equiv \prod_{i\in I}\Bigg(\sum_{\substack{h_i,c_i,\\ m_i,\cdots}}\int d^3\mathrm{LIPS}_i\Bigg)\prod_v \mathcal{A}_v$$

Counting Constraints:

n_δ

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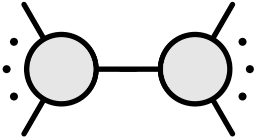

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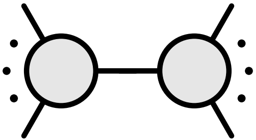

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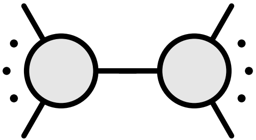

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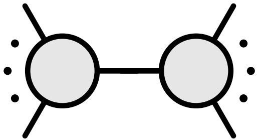

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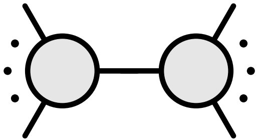

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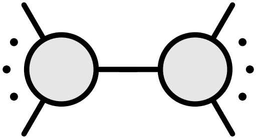

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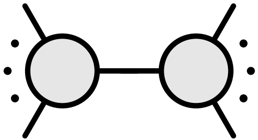

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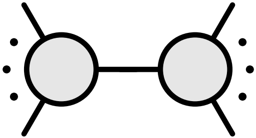

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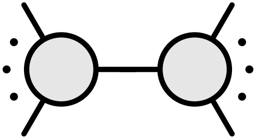

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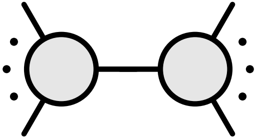

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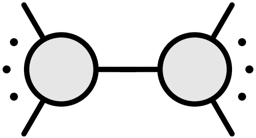

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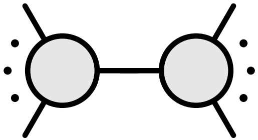

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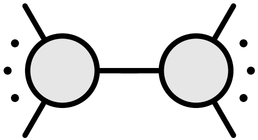

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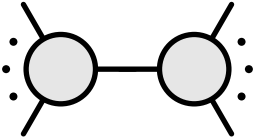

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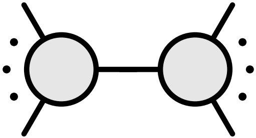

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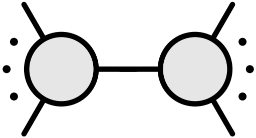

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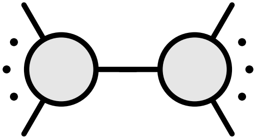

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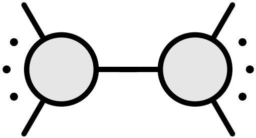

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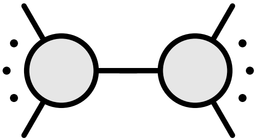

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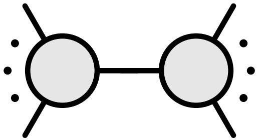

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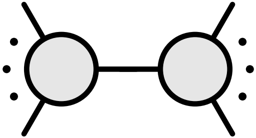

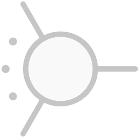

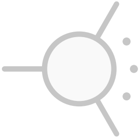

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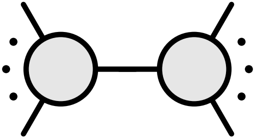

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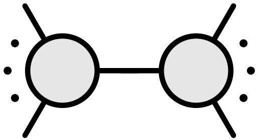

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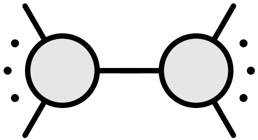

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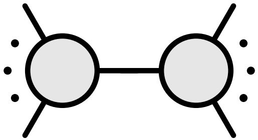

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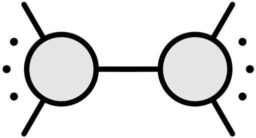

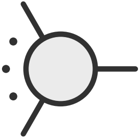

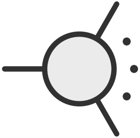

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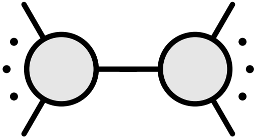

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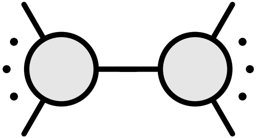

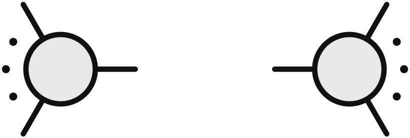

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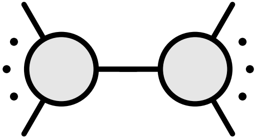

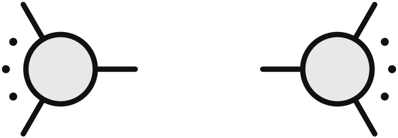

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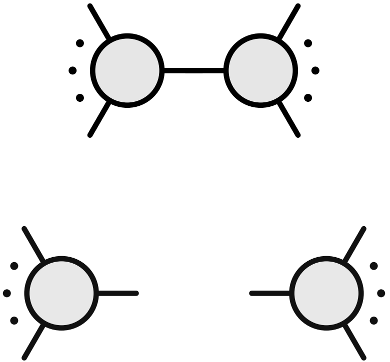

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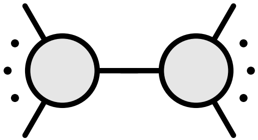

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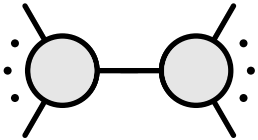

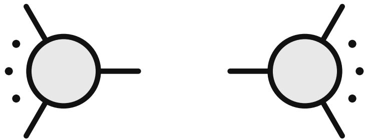

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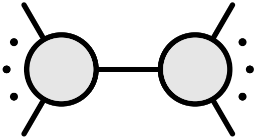

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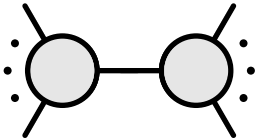

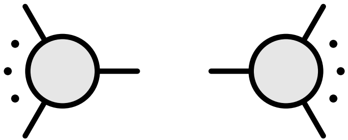

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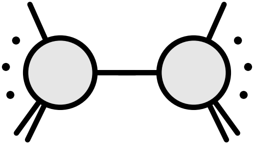

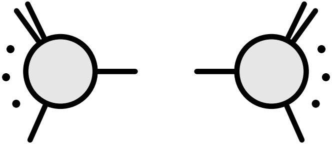

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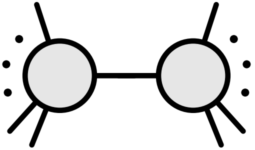

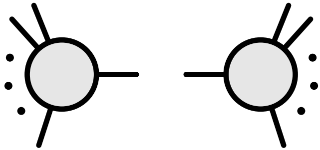

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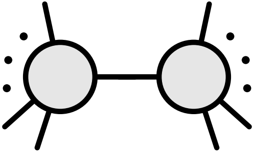

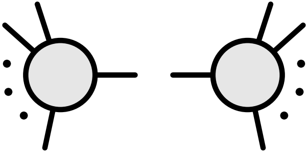

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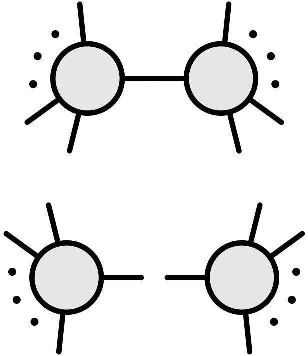

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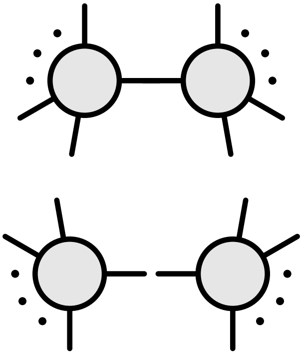

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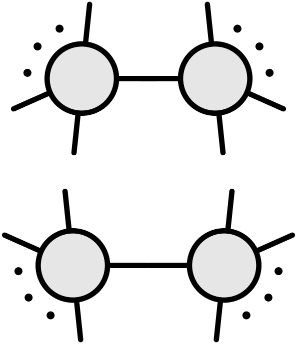

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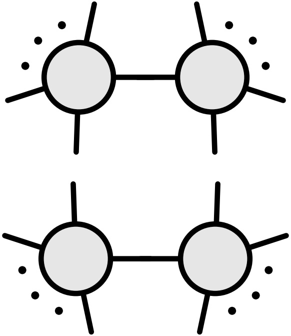

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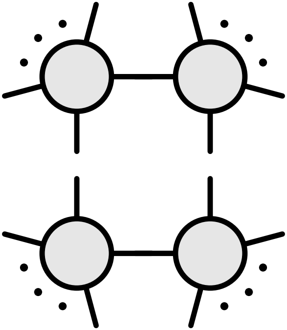

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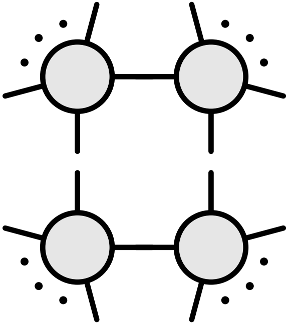

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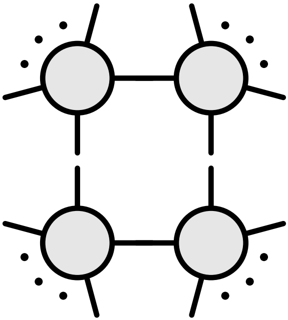

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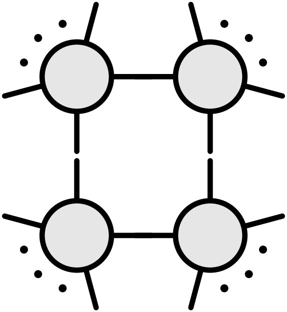

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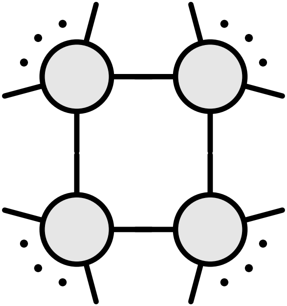

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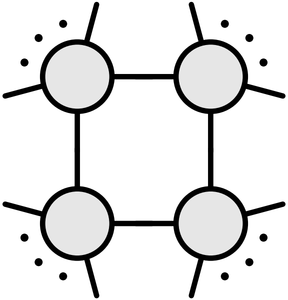

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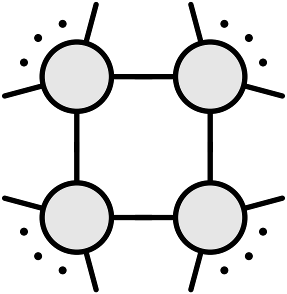

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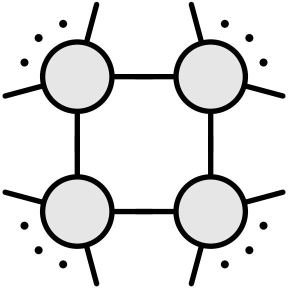

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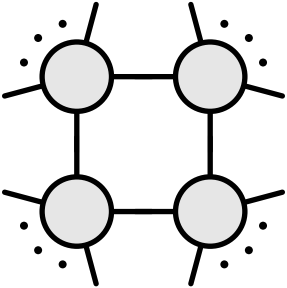

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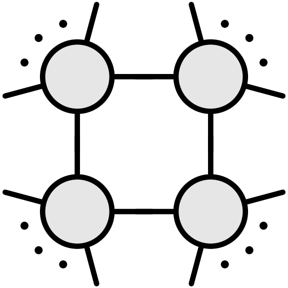

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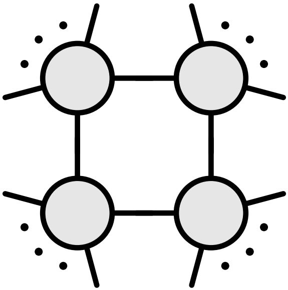

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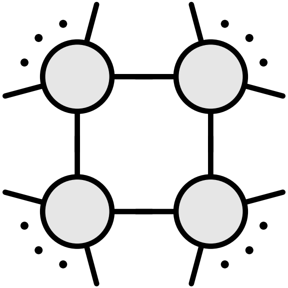

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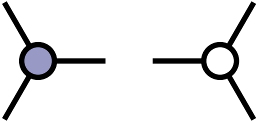

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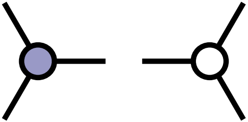

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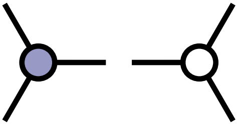

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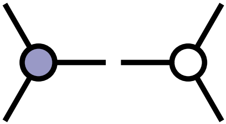

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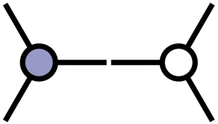

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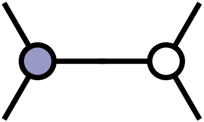

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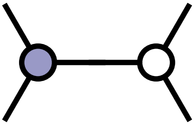

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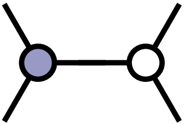

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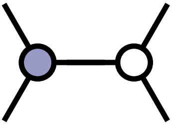

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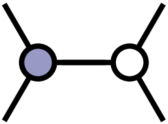

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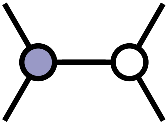

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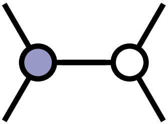

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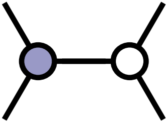

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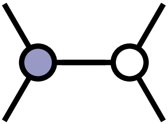

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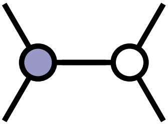

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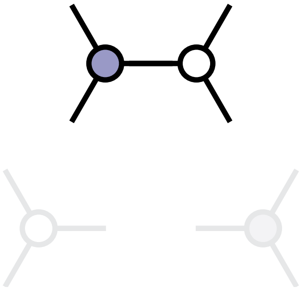

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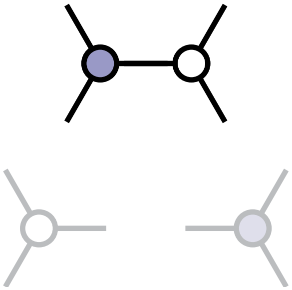

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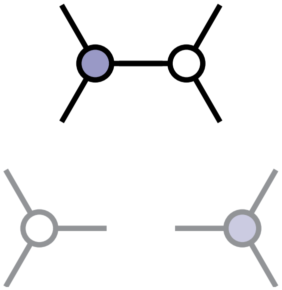

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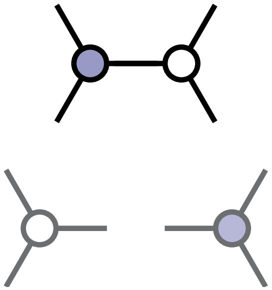

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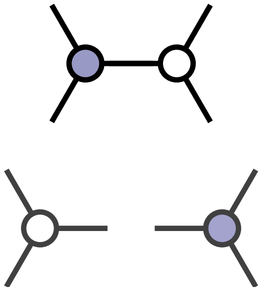

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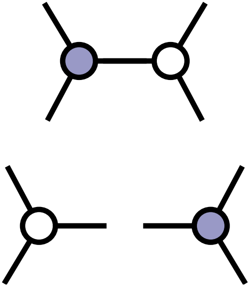

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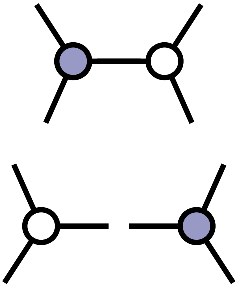

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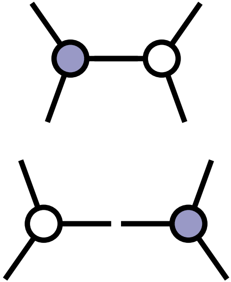

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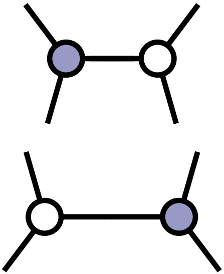

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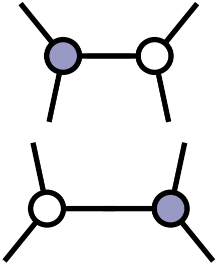

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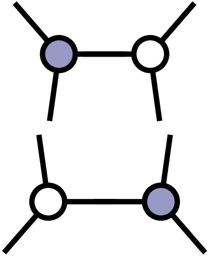

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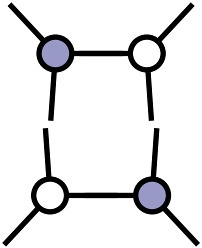

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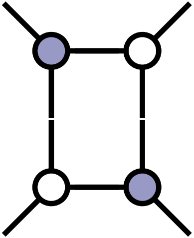

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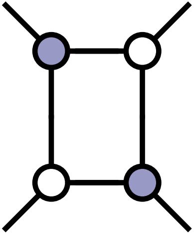

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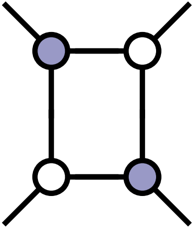

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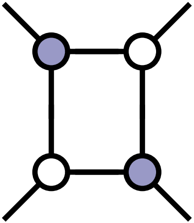

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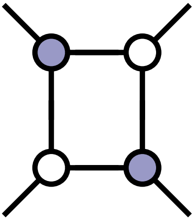

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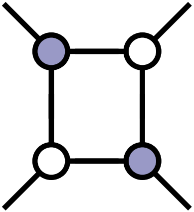

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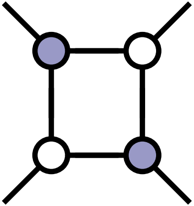

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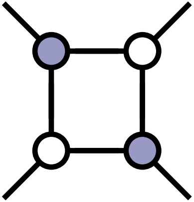

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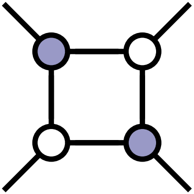

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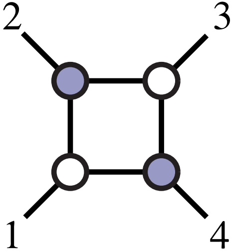

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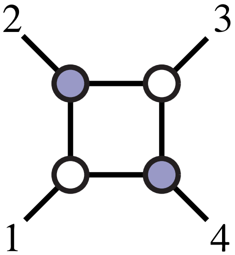

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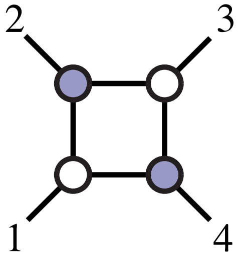

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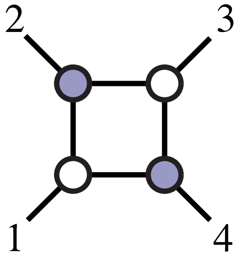

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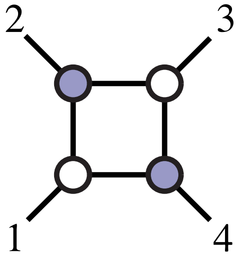

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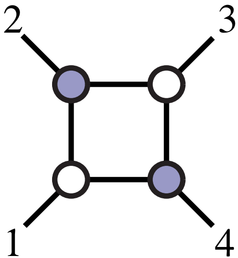

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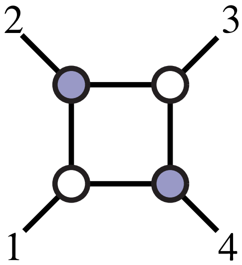

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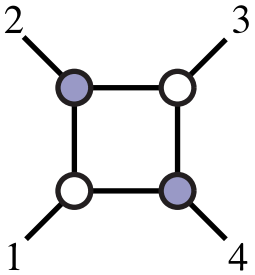

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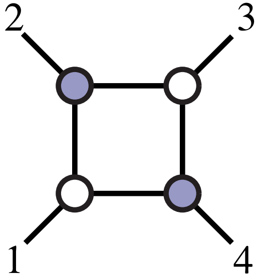

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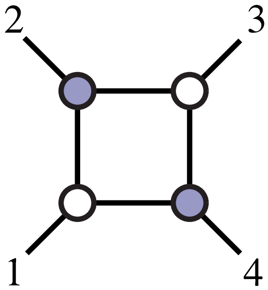

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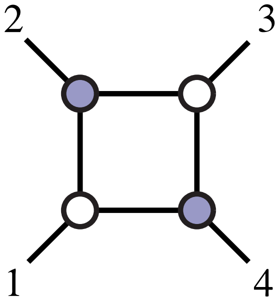

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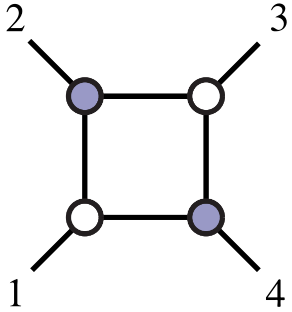

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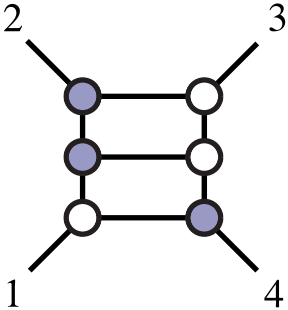

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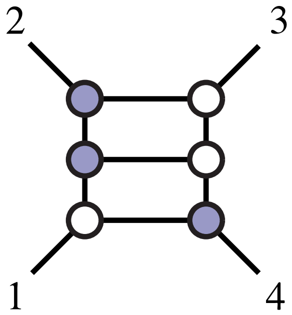

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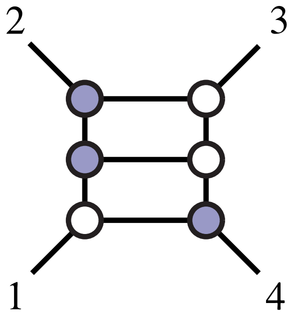

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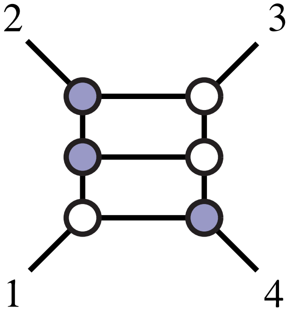

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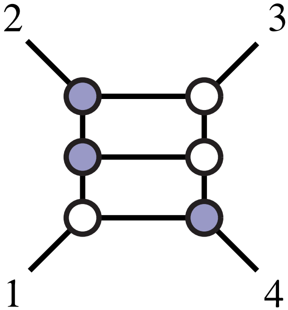

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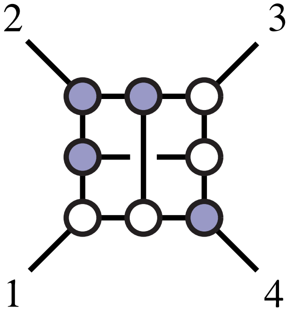

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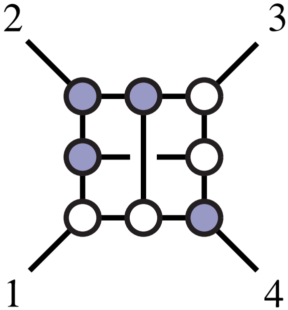

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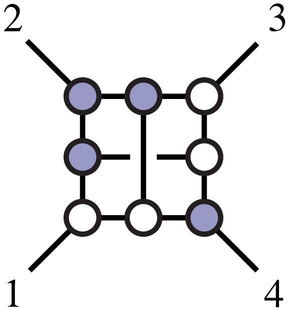

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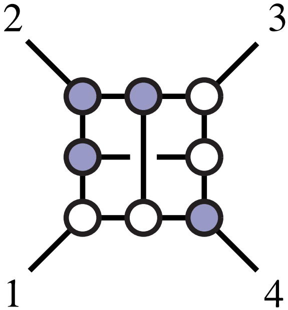

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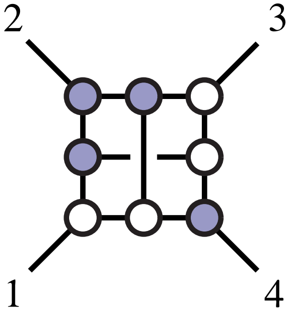

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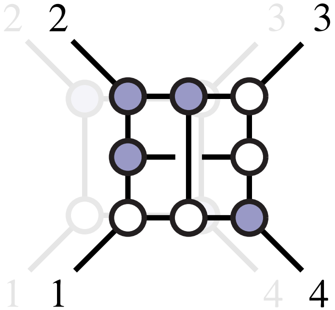

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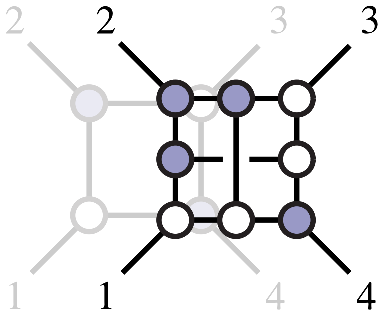

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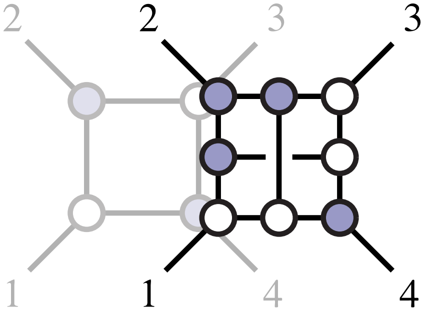

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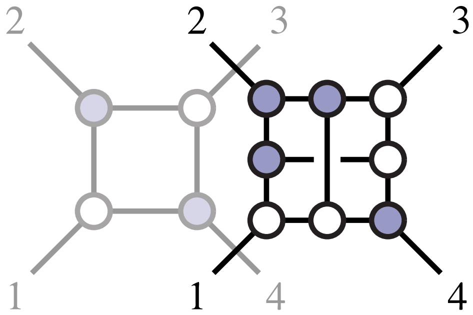

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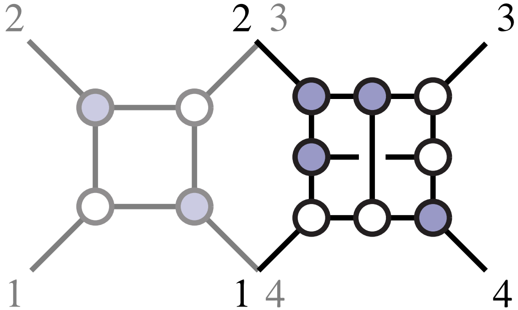

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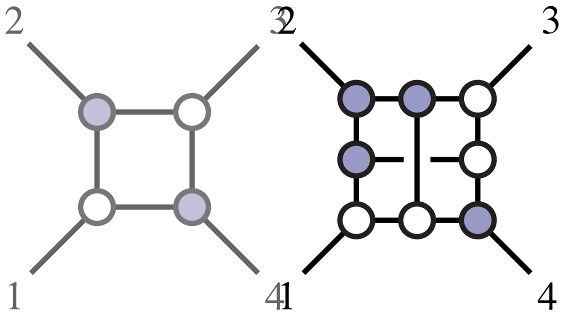

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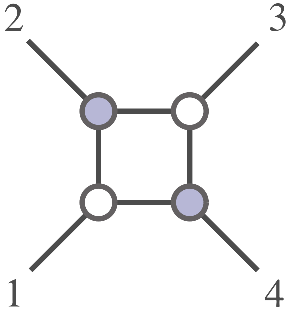

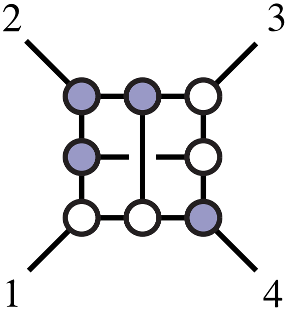

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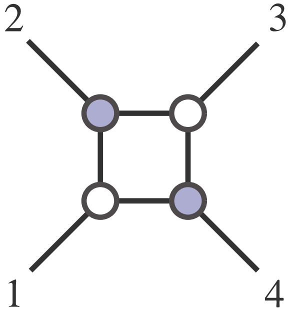

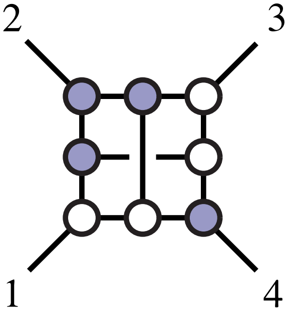

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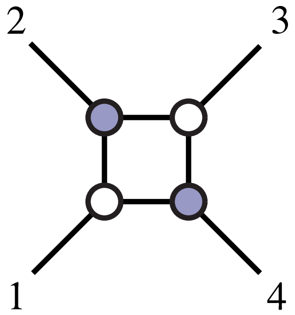

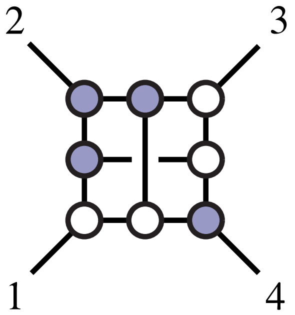

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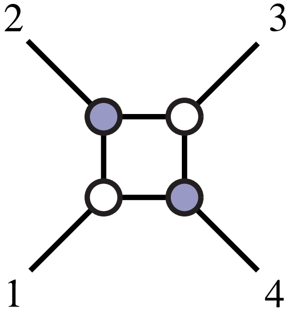

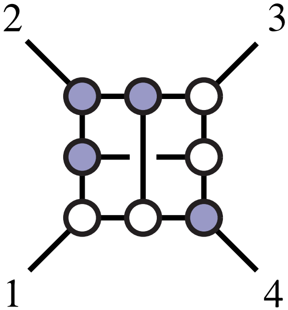

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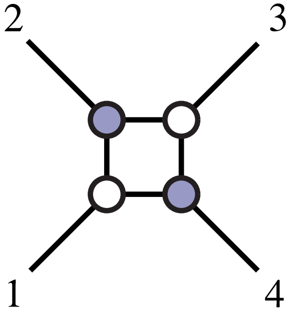

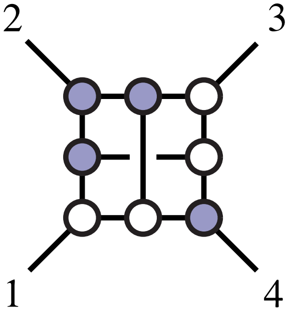

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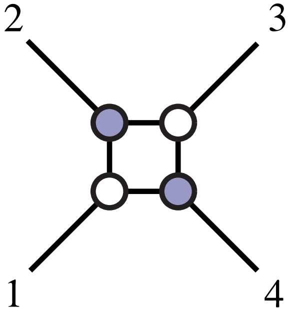

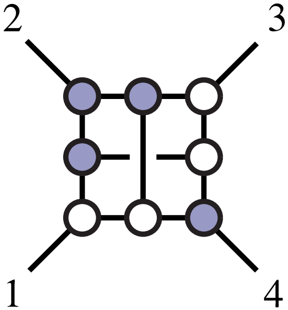

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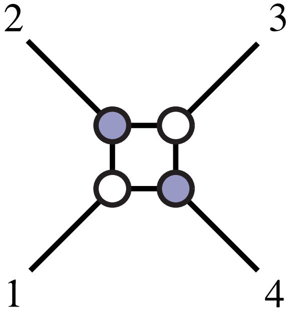

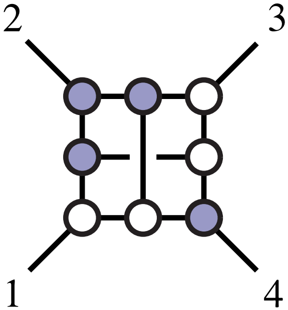

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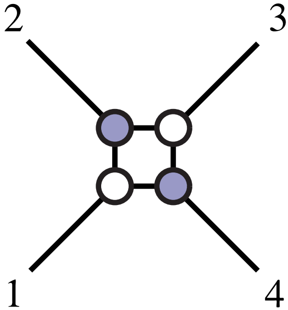

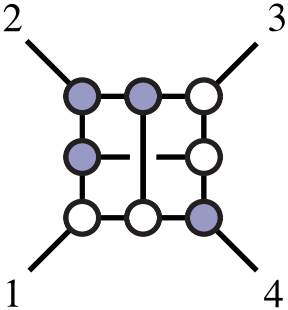

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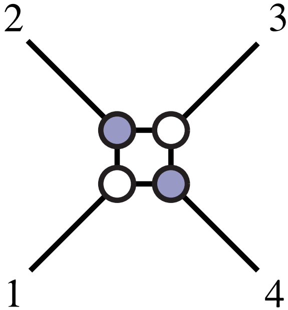

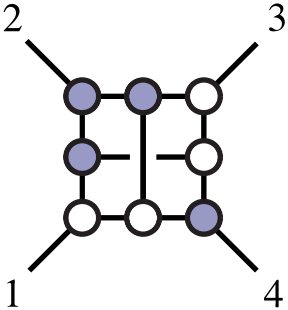

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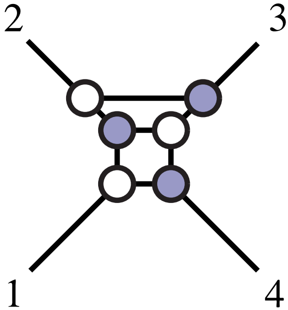

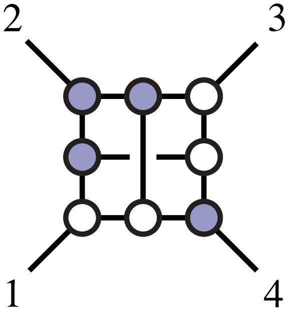

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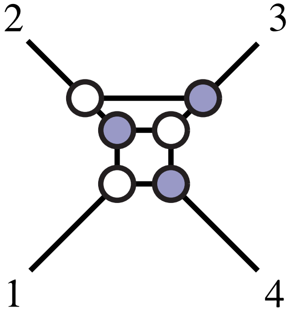

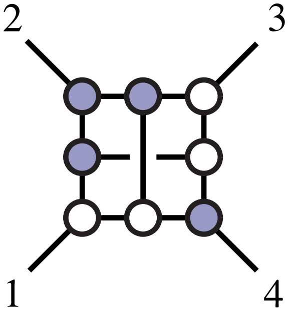

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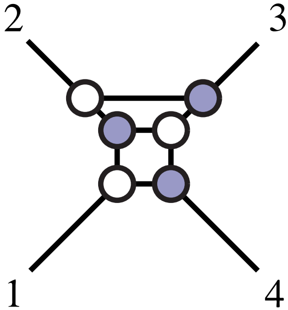

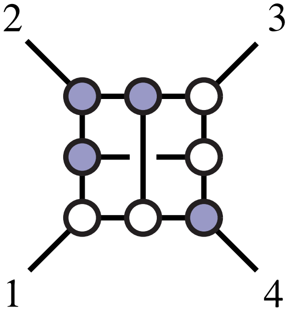

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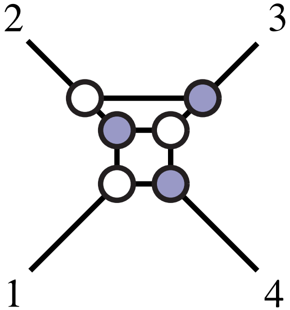

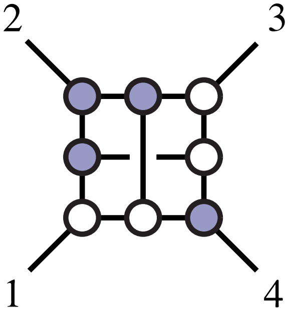

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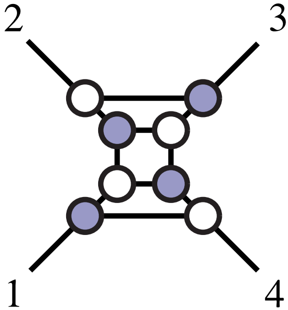

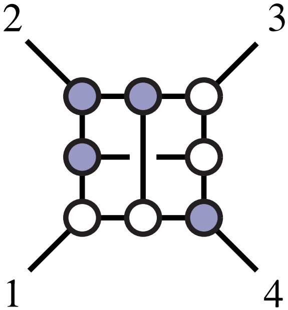

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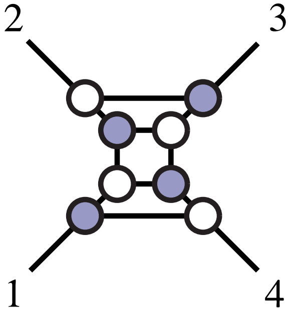

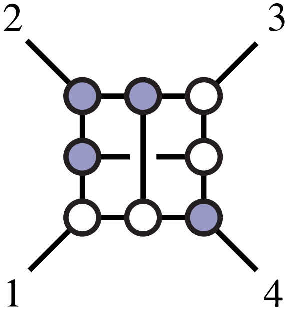

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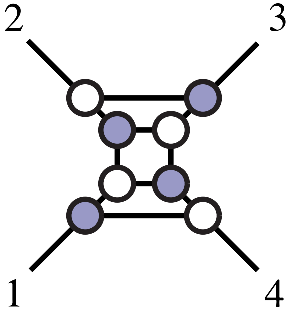

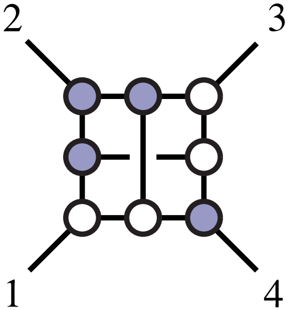

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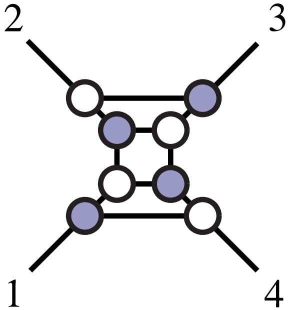

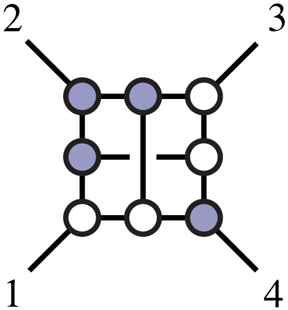

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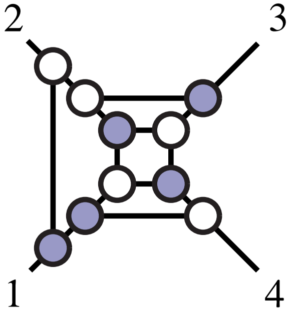

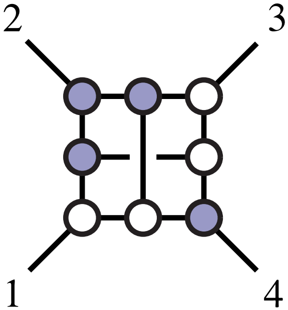

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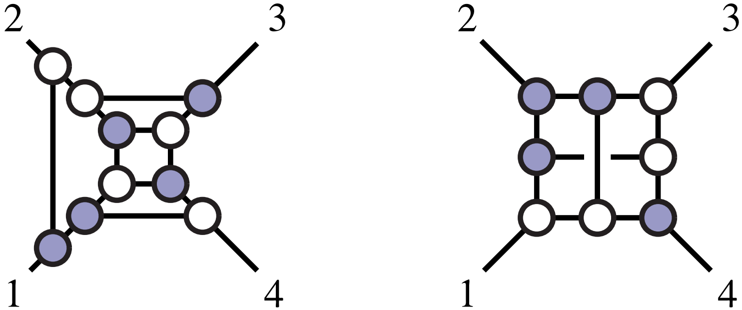

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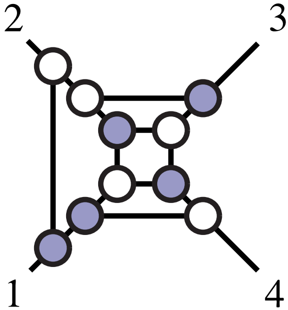

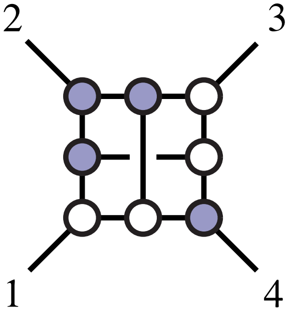

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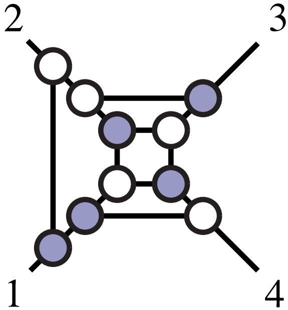

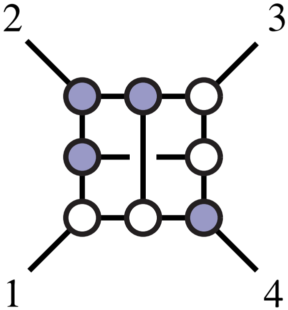

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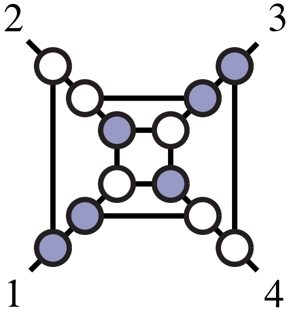

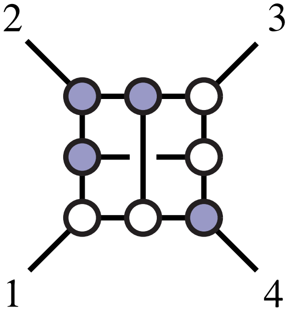

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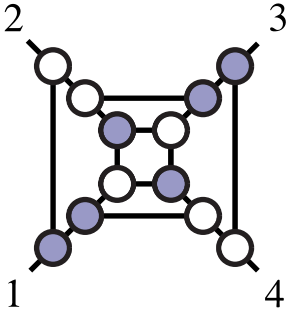

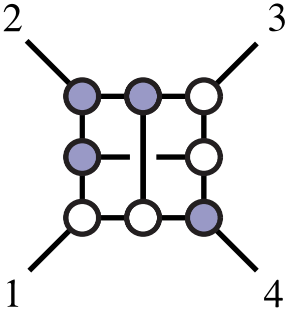

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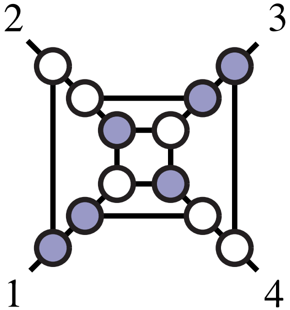

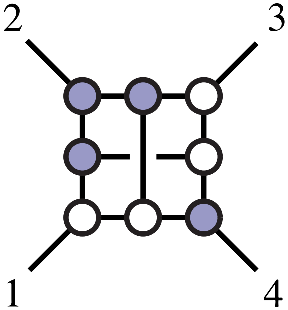

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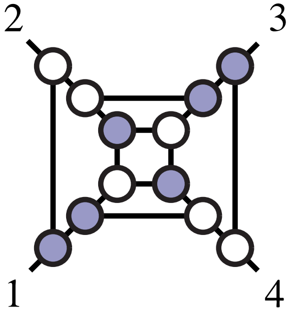

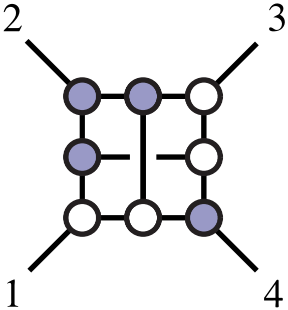

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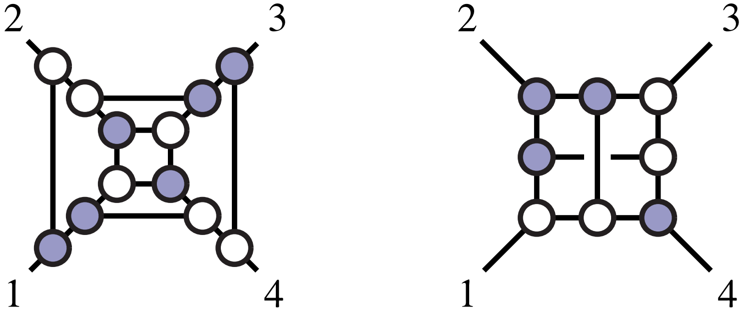

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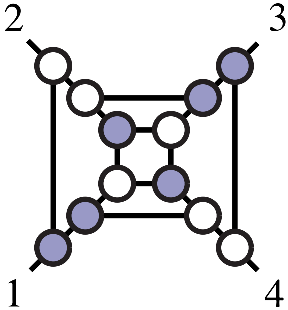

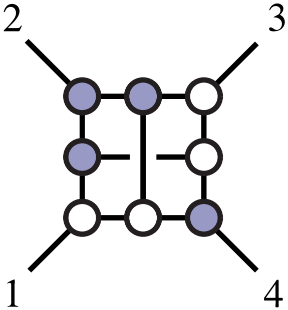

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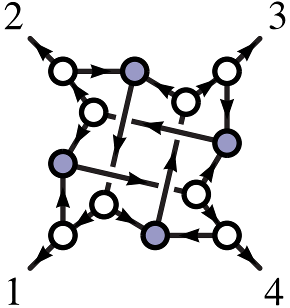

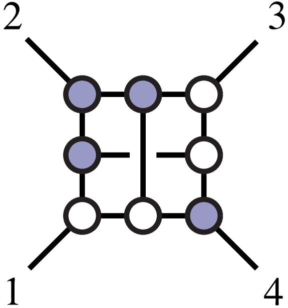

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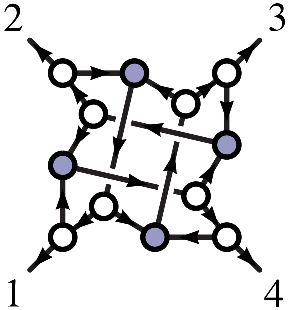

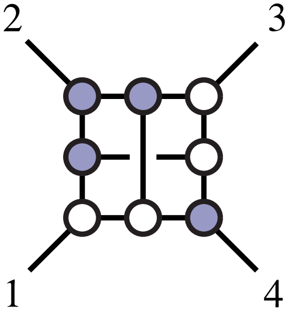

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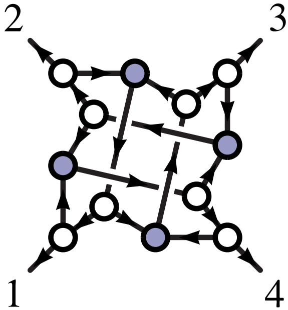

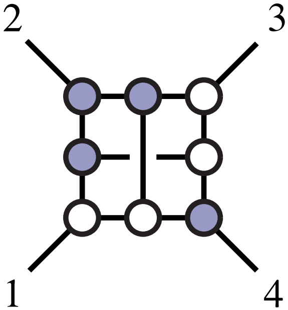

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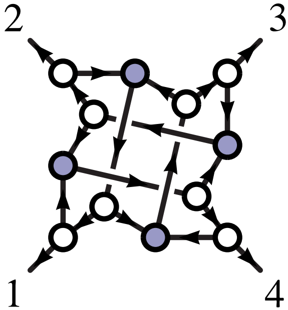

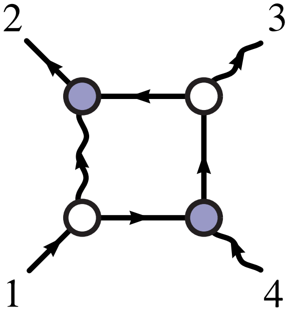

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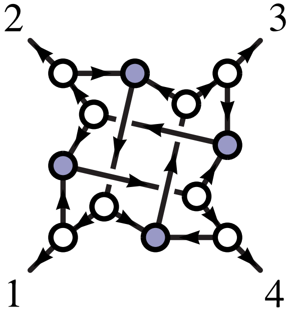

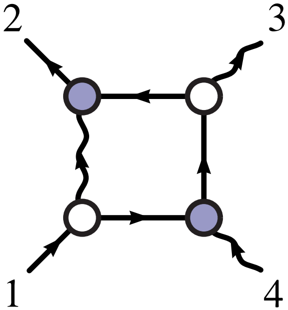

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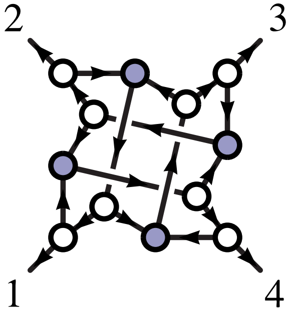

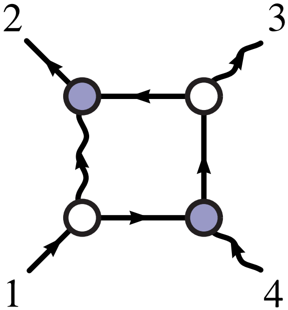

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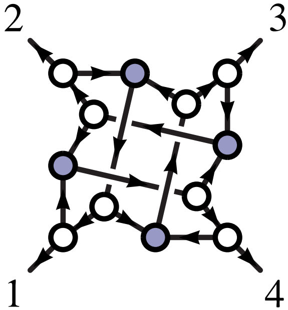

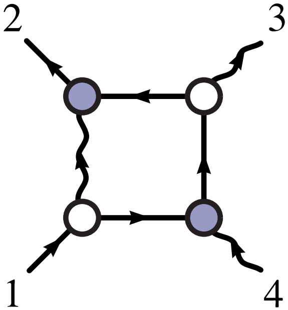

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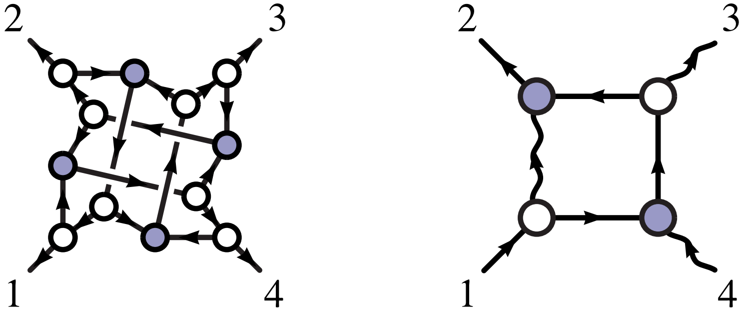

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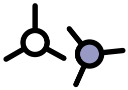

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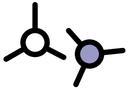

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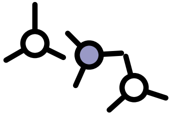

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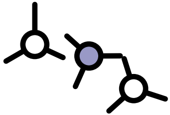

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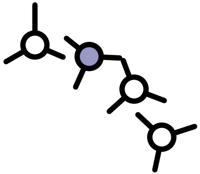

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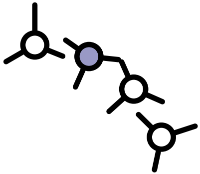

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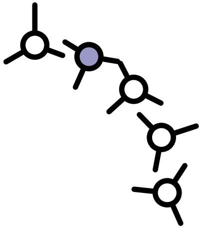

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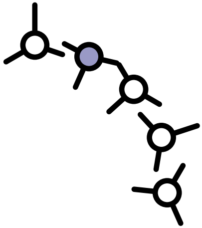

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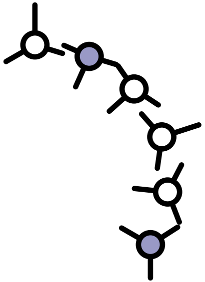

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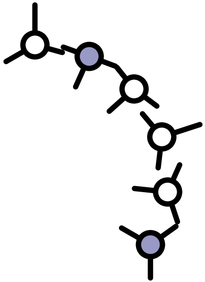

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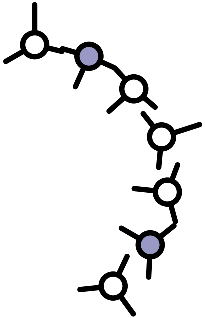

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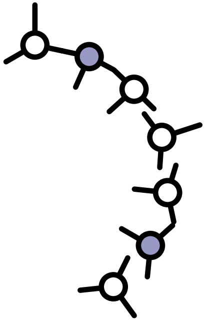

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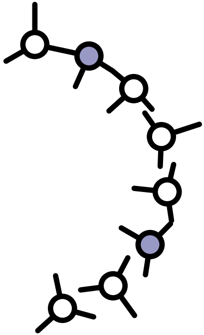

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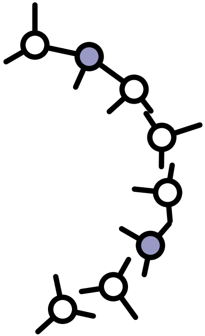

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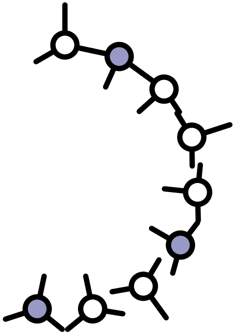

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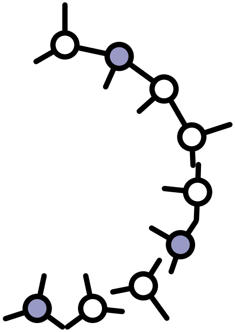

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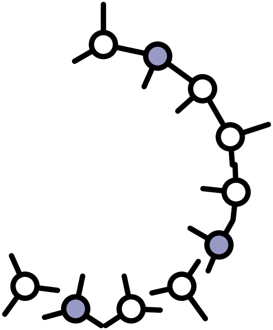

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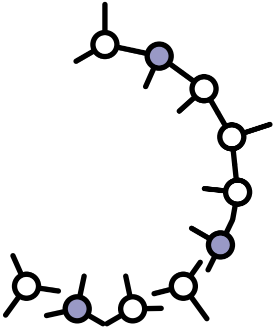

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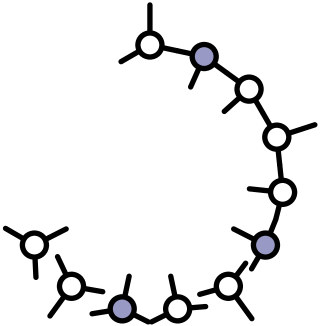

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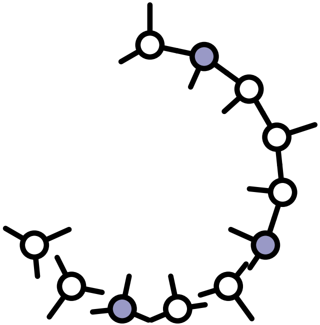

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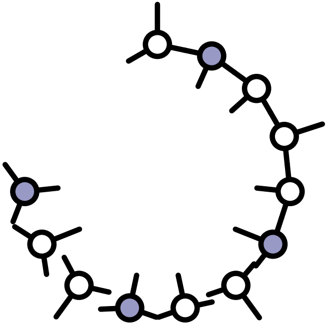

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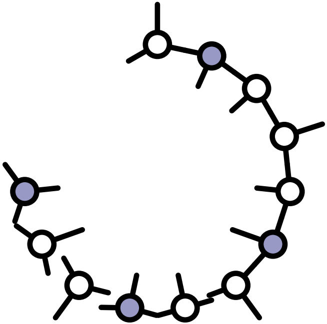

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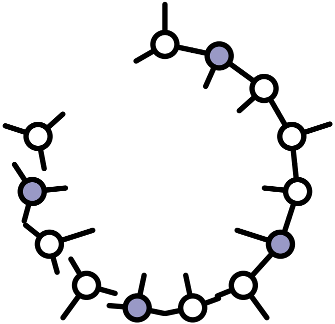

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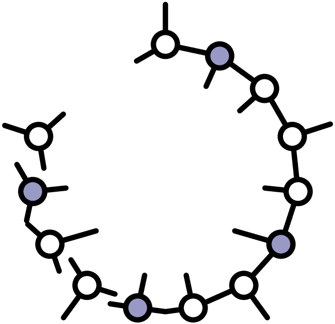

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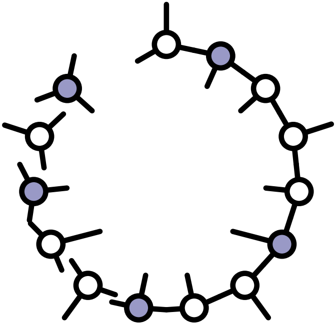

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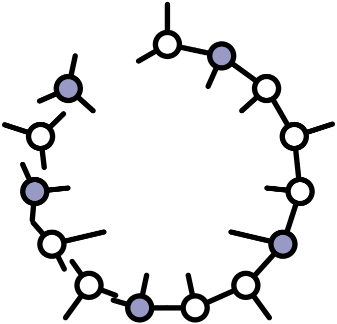

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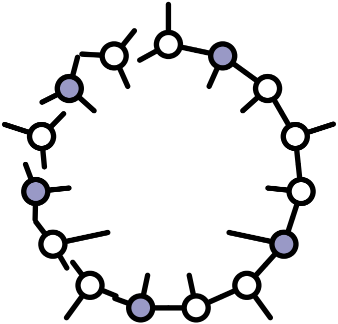

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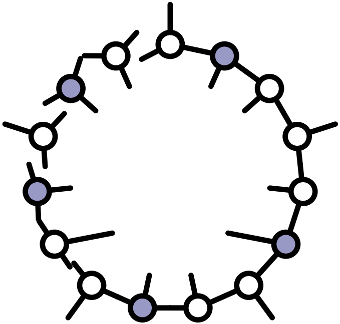

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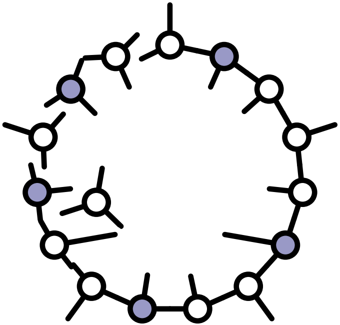

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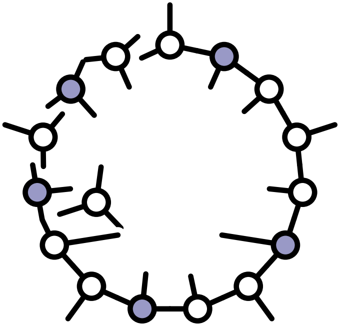

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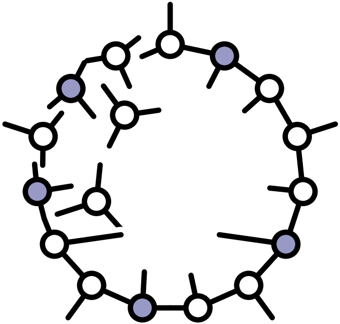

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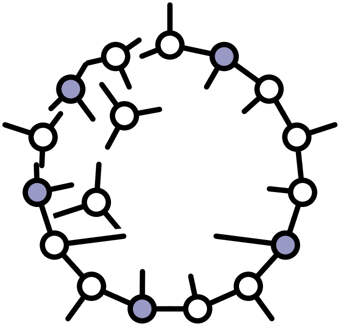

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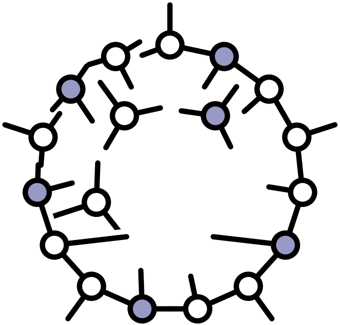

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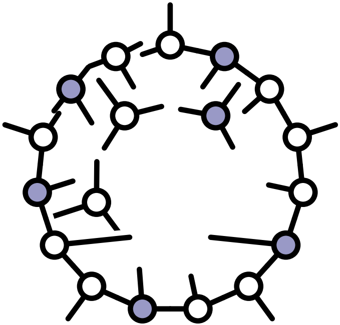

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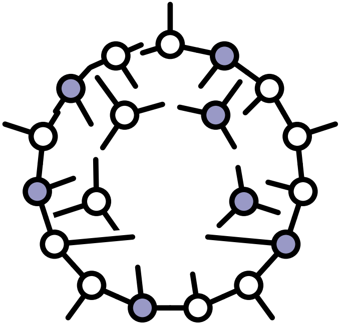

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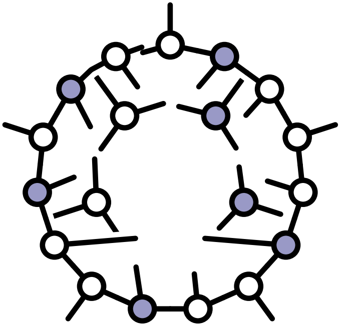

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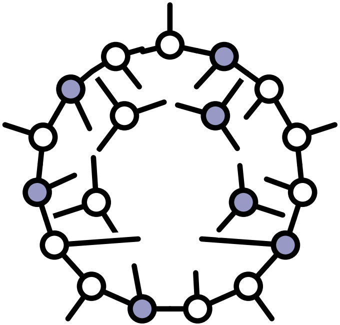

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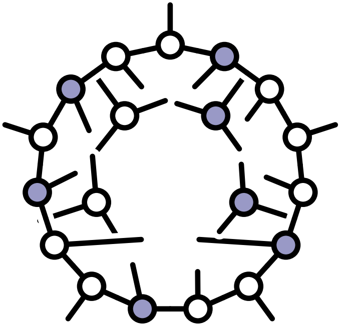

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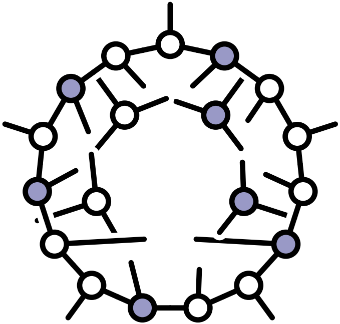

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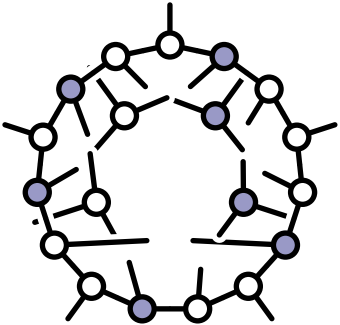

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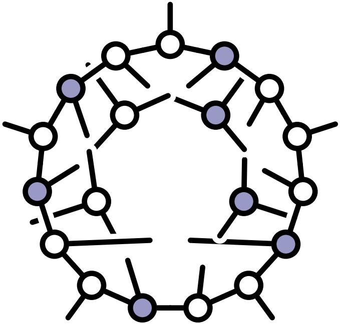

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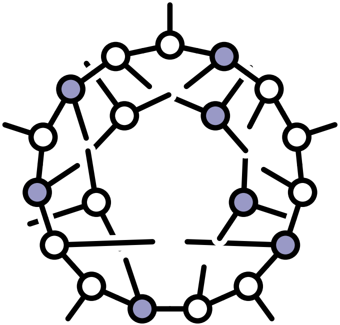

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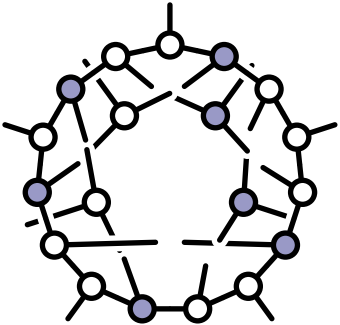

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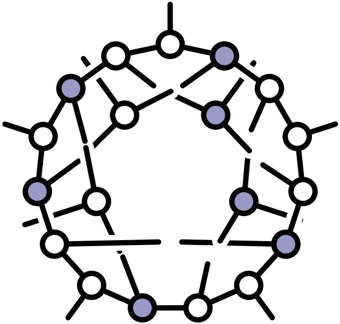

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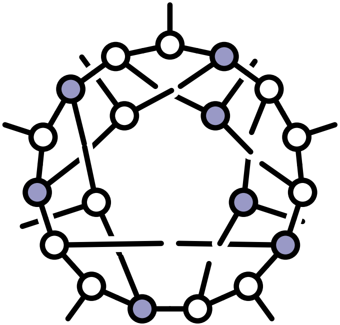

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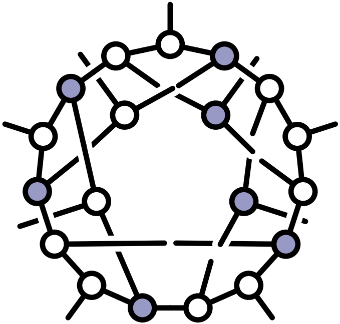

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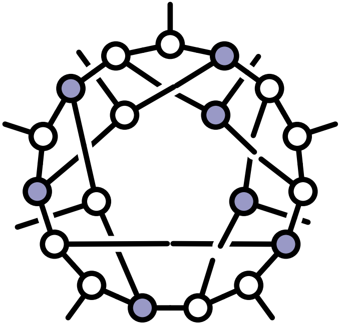

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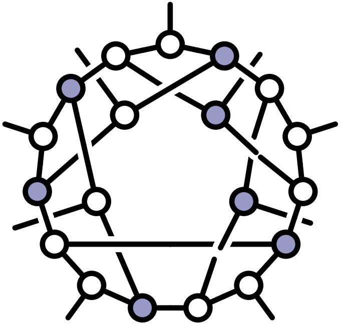

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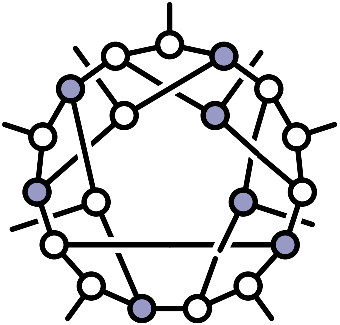

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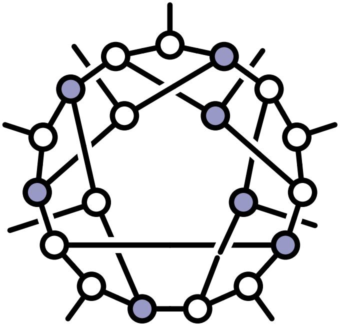

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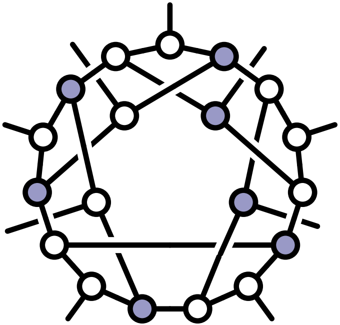

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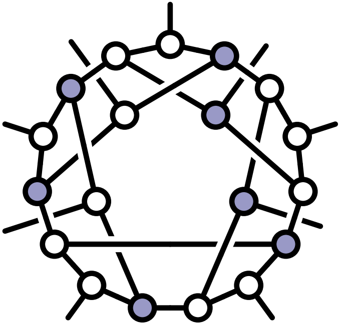

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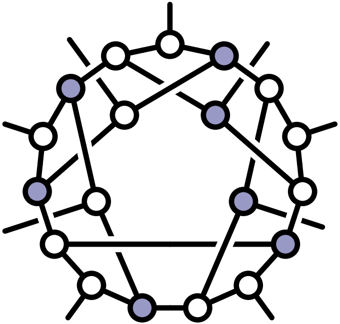

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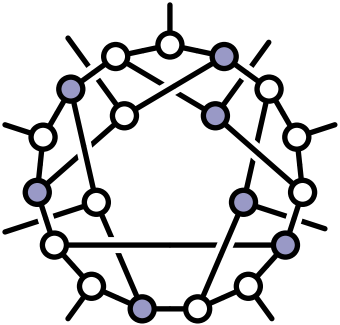

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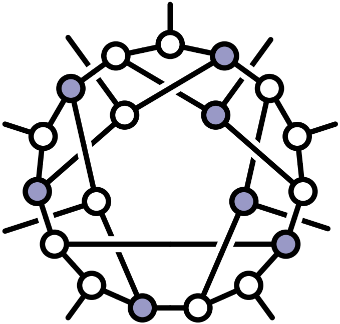

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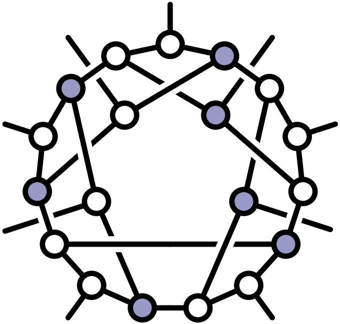

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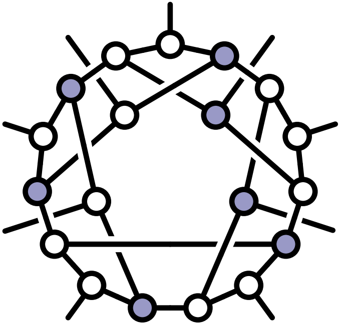

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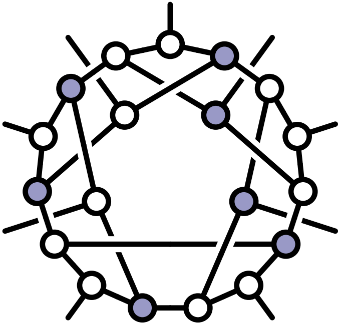

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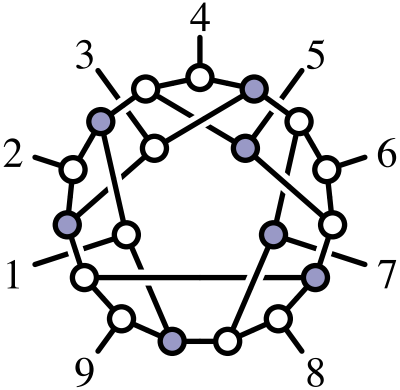

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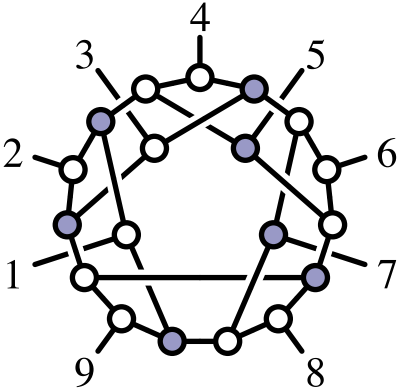

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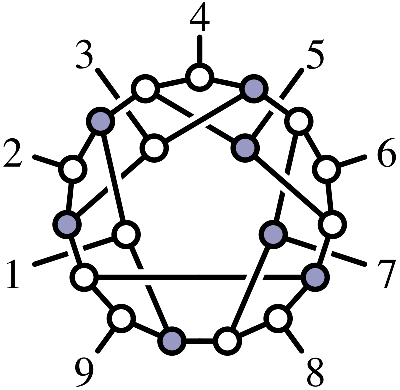

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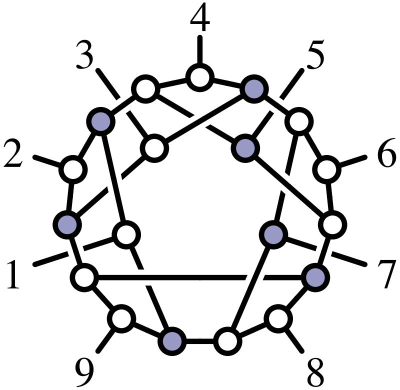

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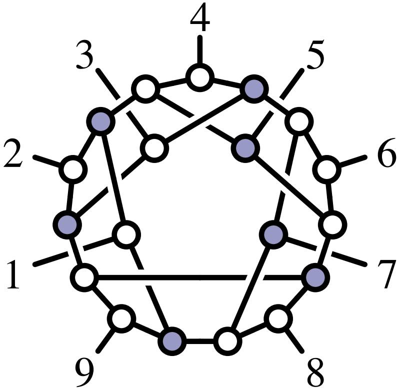

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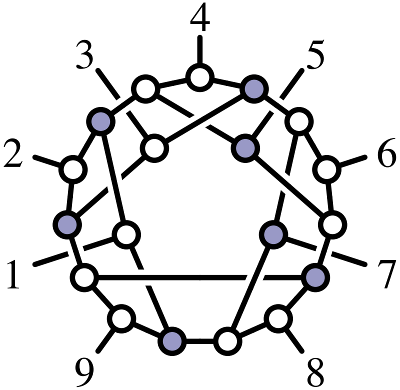

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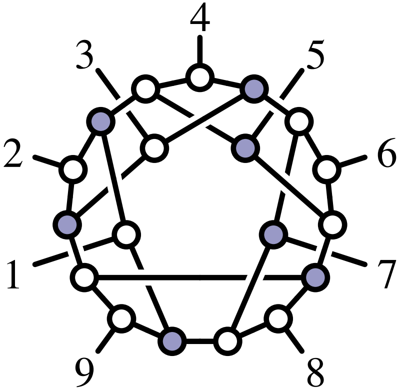

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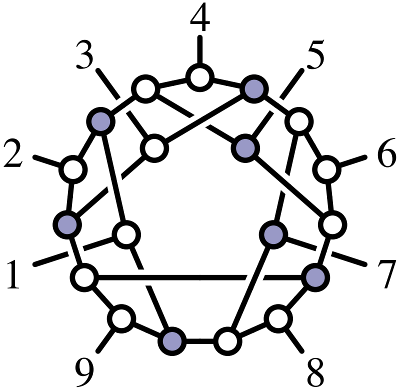

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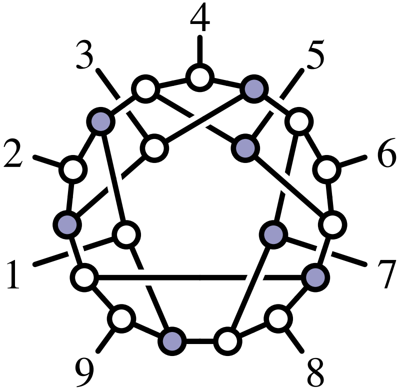

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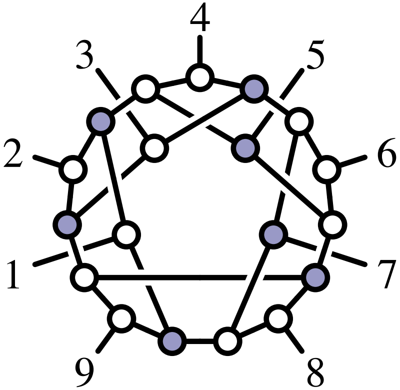

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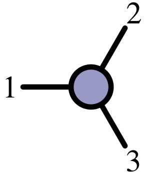

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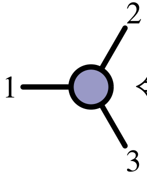

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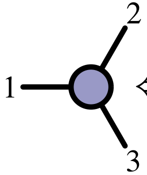

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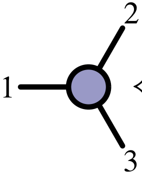

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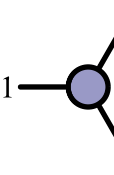

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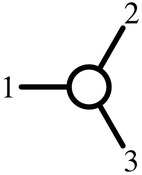

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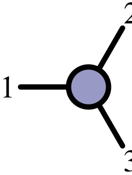

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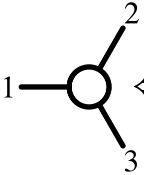

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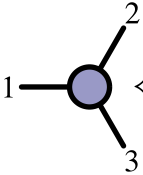

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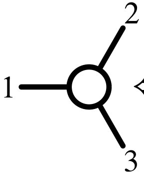

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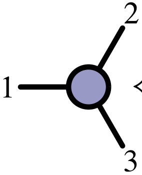

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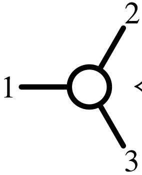

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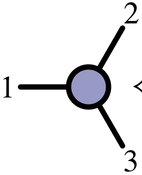

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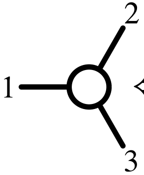

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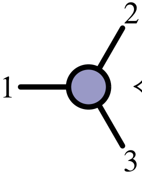

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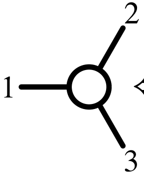

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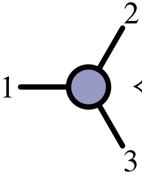

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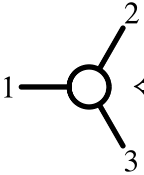

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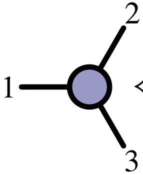

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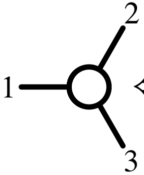

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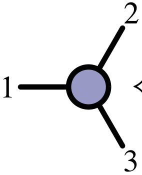

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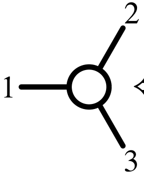

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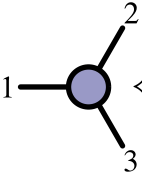

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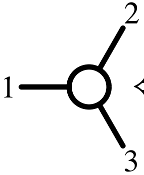

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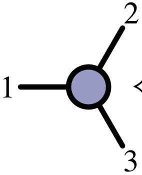
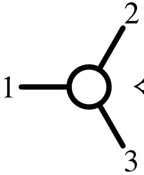


$$\Leftrightarrow B\equiv\begin{pmatrix} b_1^1 & 1 & 0 \\ b_1^2 & 0 & 1\end{pmatrix}$$



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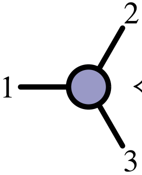

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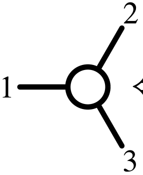

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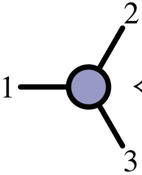

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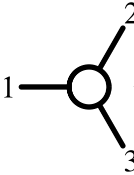

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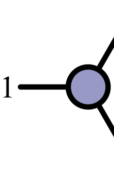

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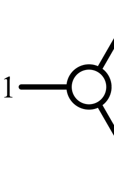

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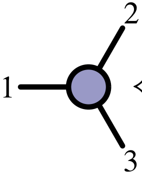

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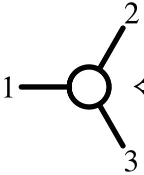

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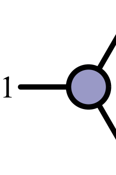

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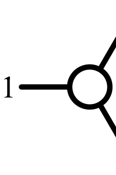

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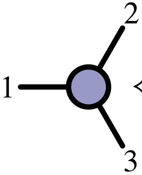

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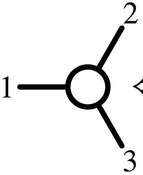

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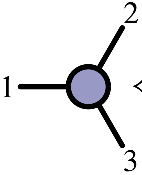

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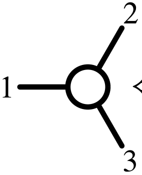

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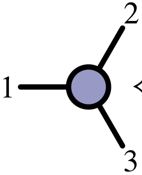

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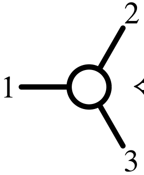

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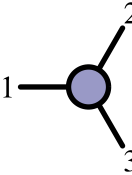

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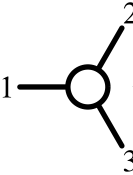

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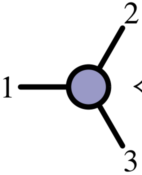

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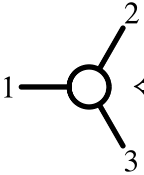

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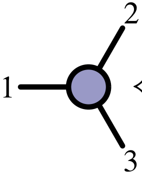

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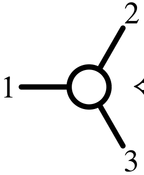

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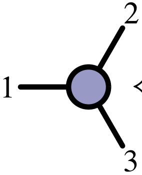

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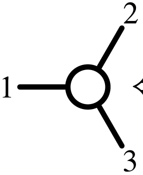

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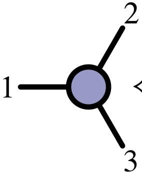

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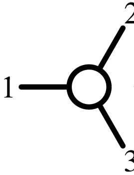

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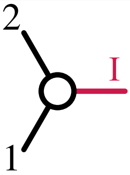

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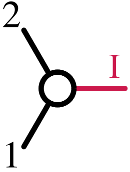
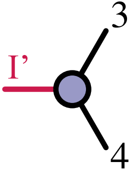


$$\begin{array}{ccc} 1 & 2 & \mathrm{I} \\ \end{array}$$
$$\begin{pmatrix} 1 & w_2 & w_{\mathrm{I}} \end{pmatrix}$$



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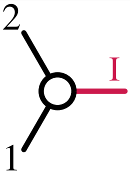

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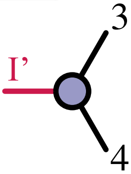

$$\begin{array}{ccc} \mathrm{I}' & 3 & 4 \end{array}$$
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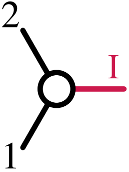

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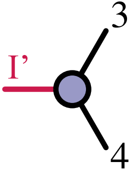

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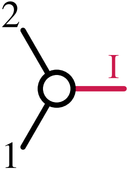

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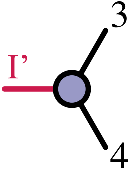

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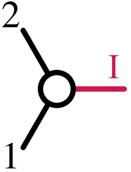

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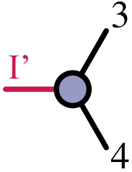

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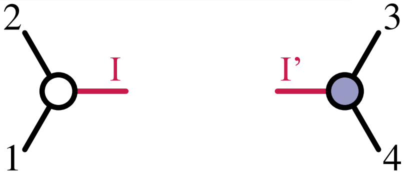

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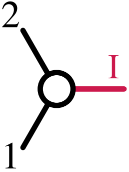

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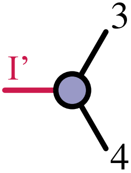

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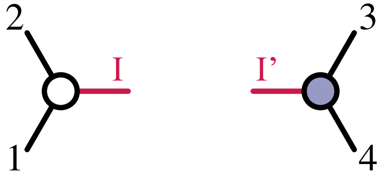

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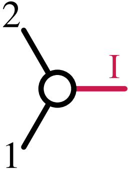

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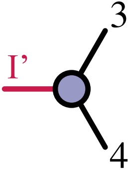

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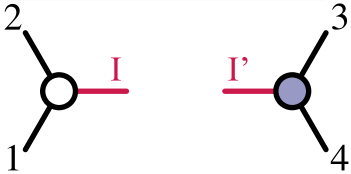

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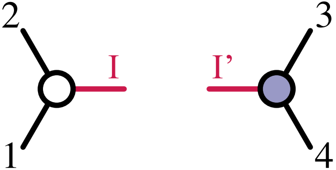

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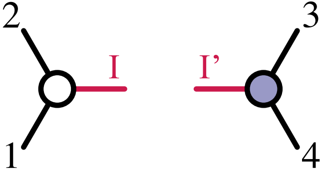

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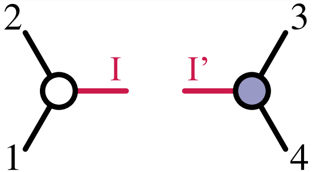

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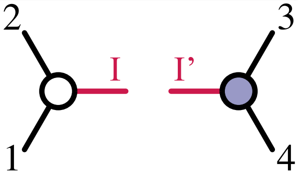

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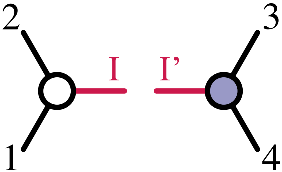

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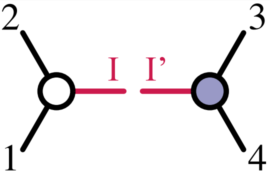

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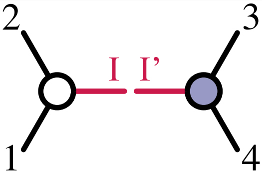

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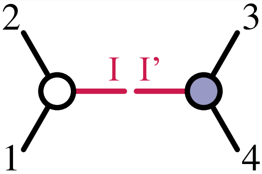

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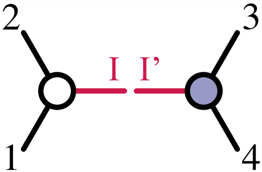

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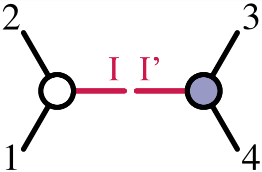

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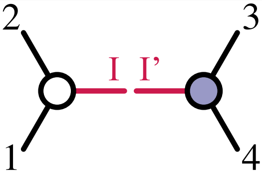

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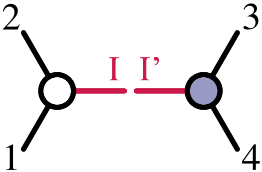

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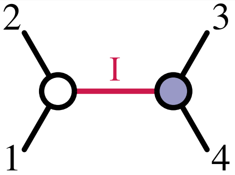

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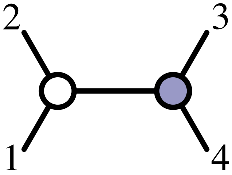

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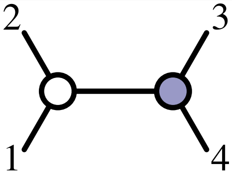

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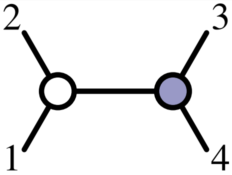

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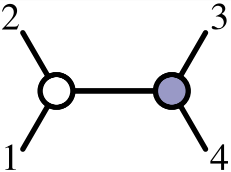

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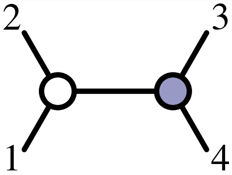

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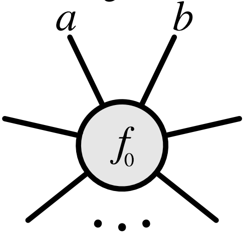

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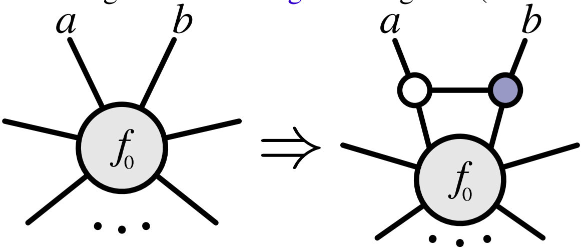

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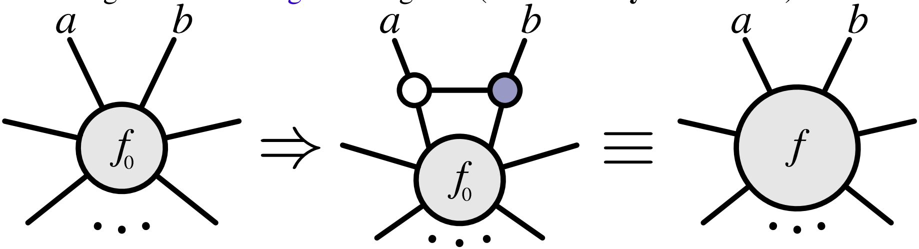

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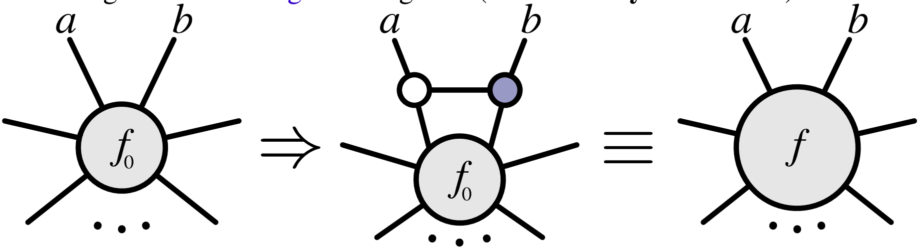

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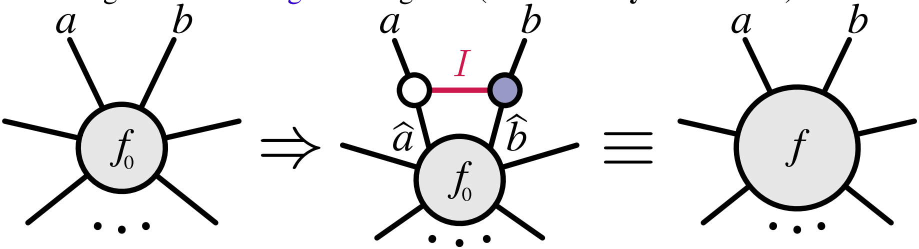

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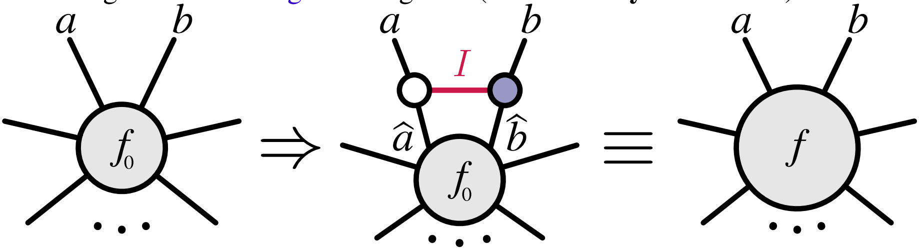

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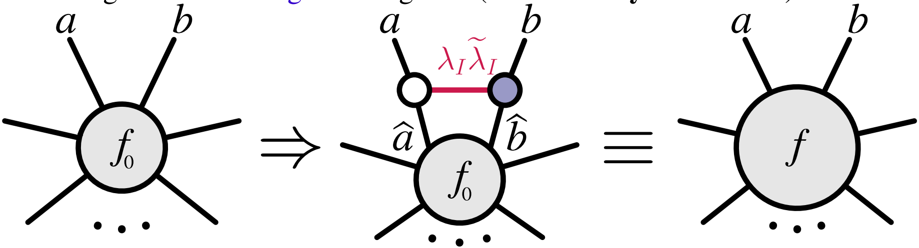

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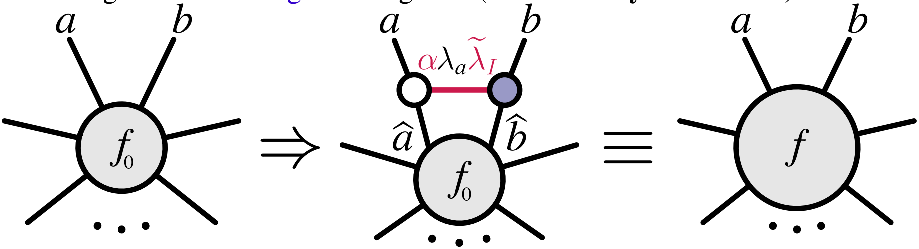

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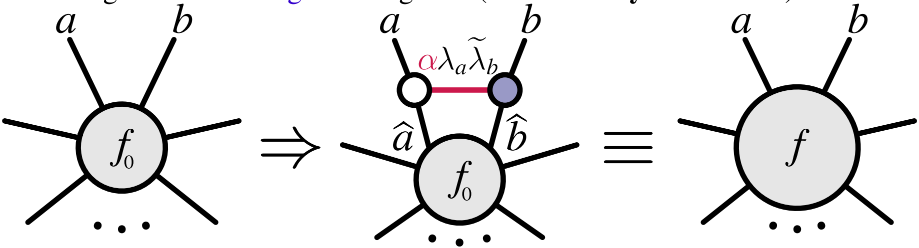

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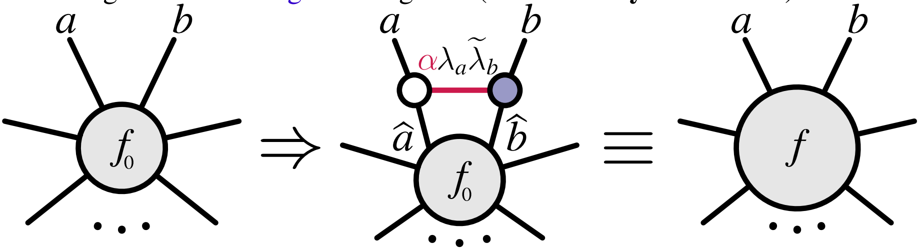

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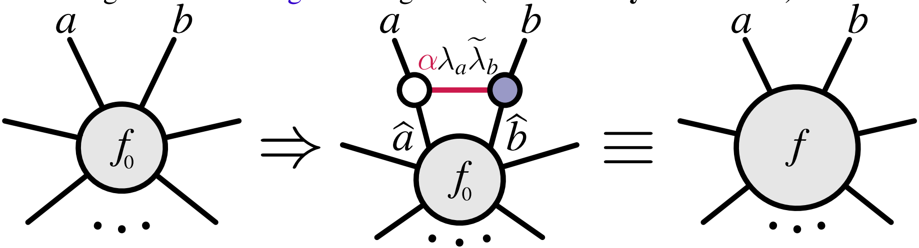

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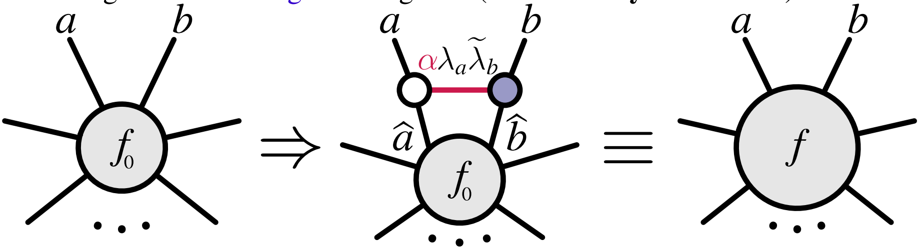

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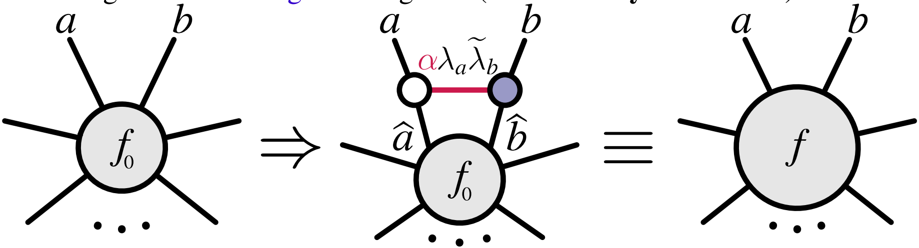

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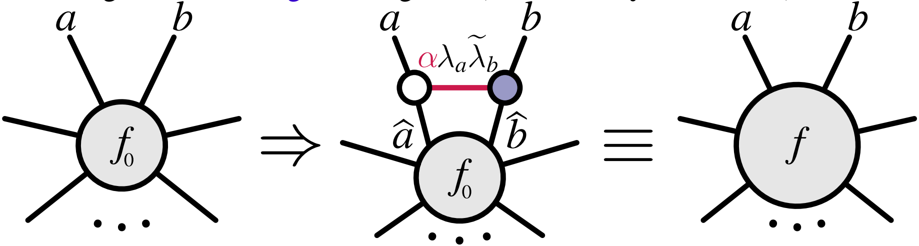

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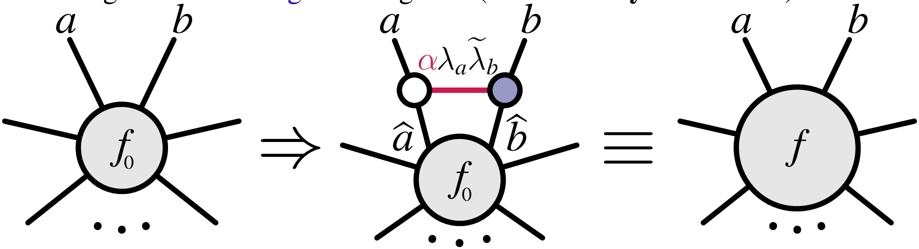

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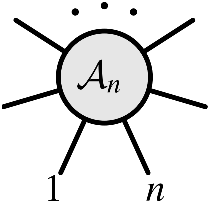

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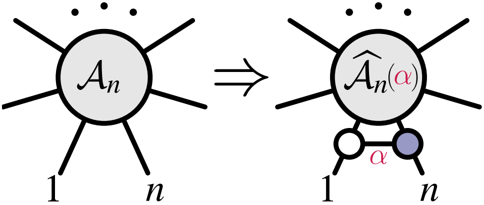

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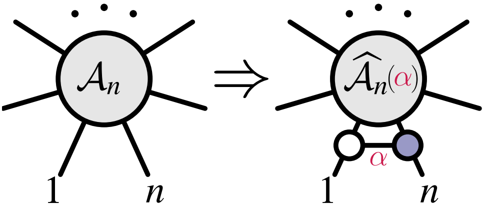

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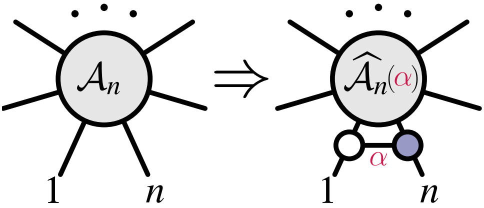

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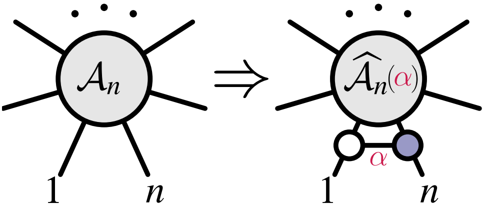

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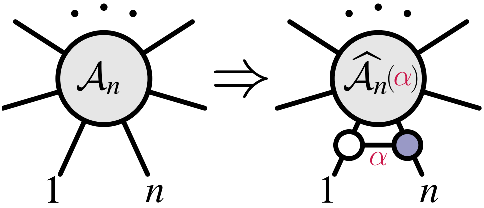

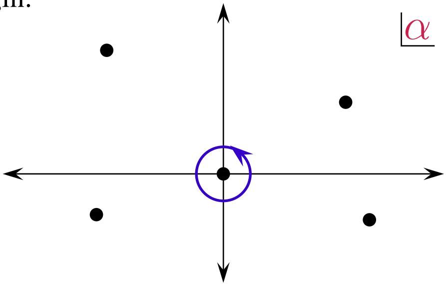

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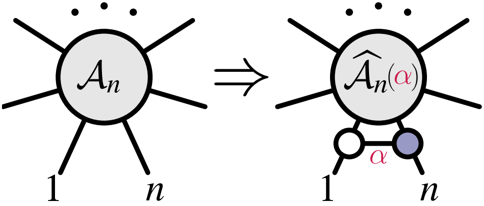

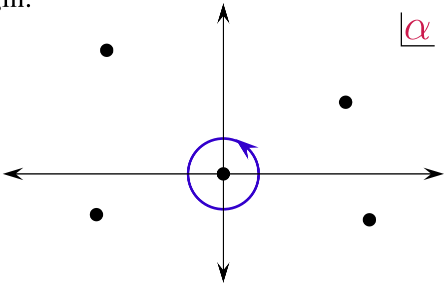

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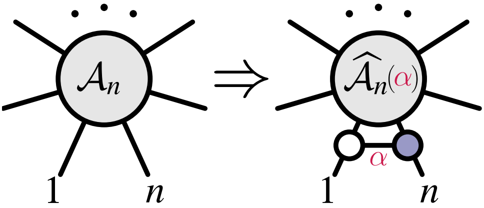

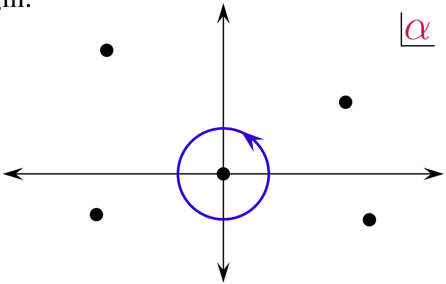

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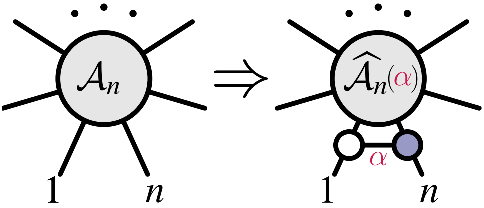

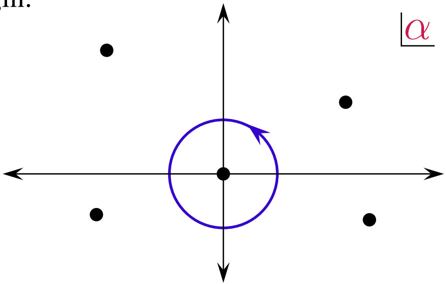

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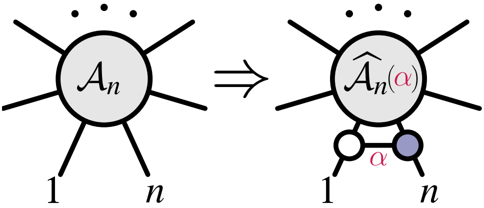

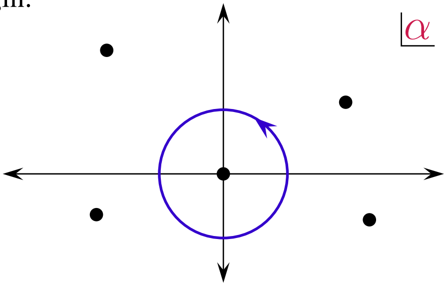

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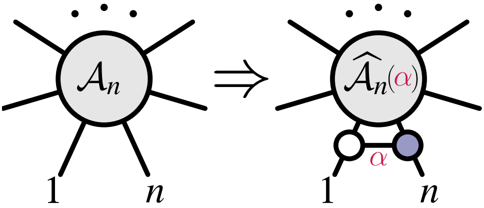

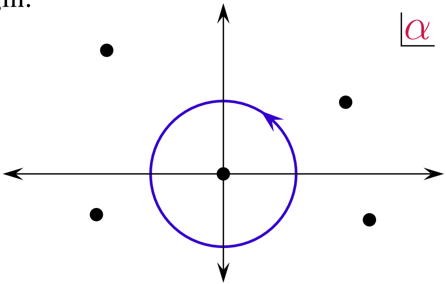

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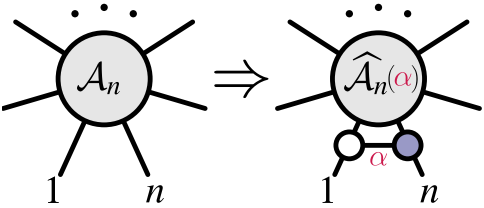

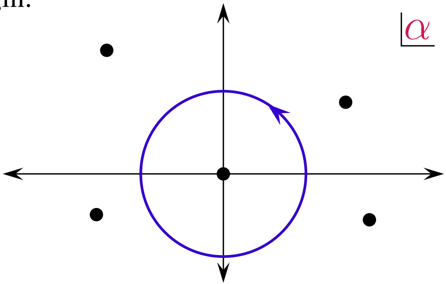

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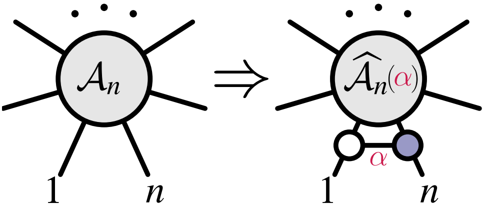

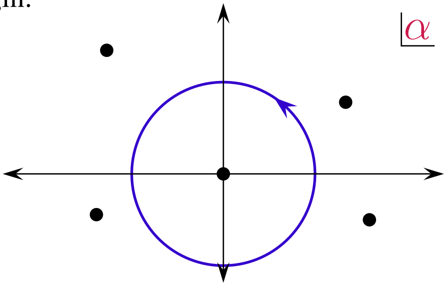

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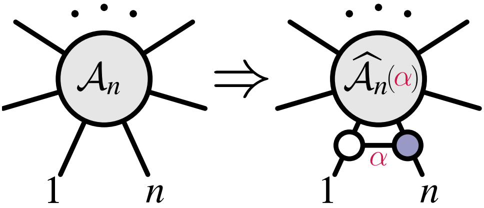

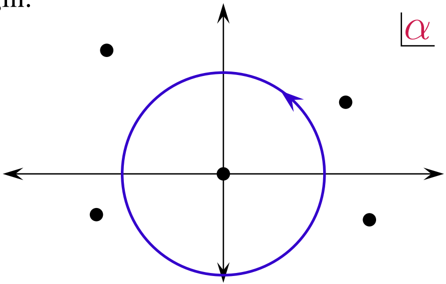

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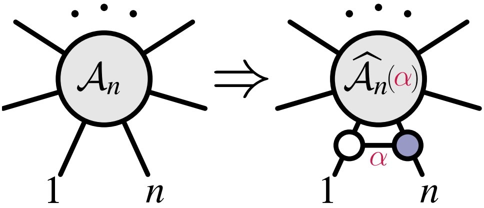

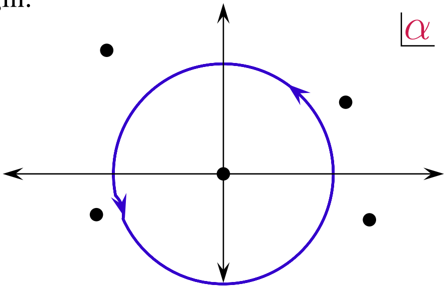

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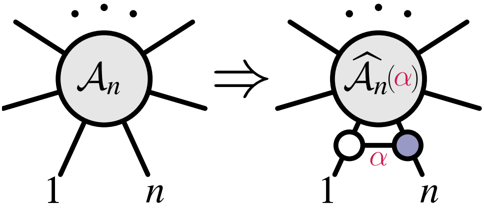

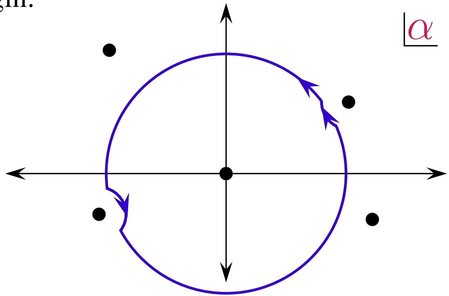

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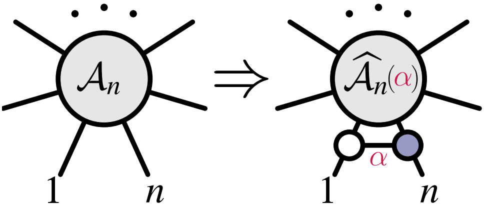

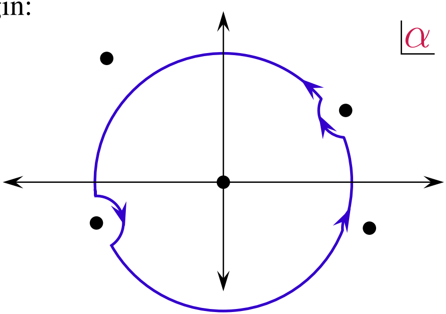

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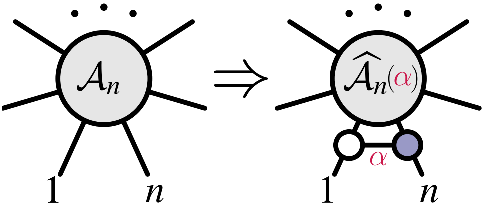

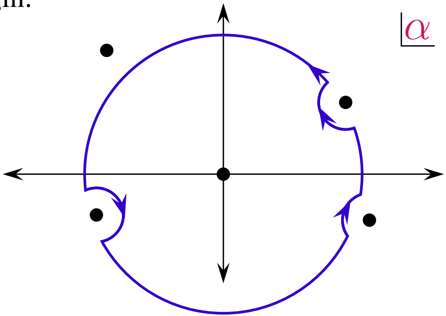

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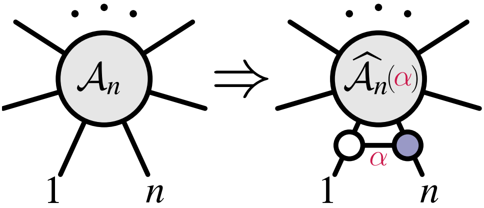

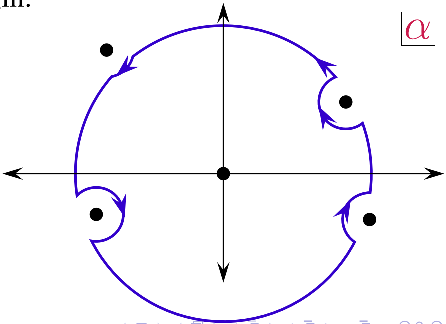

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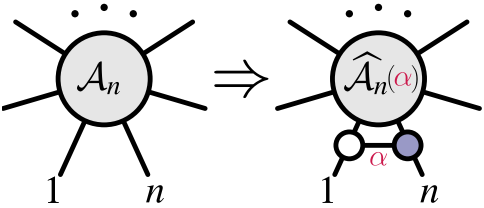

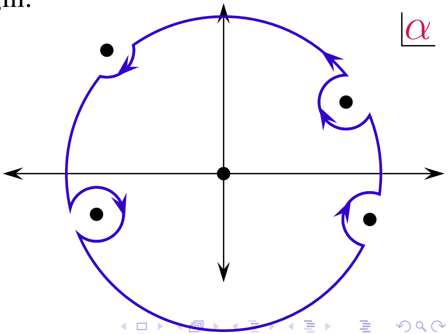

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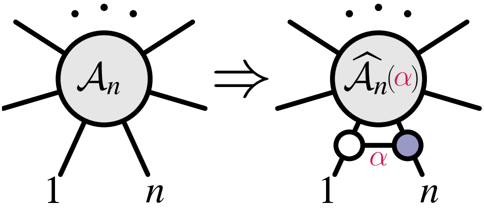

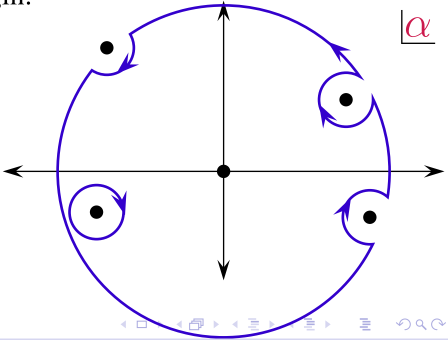

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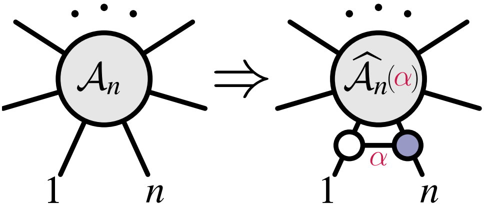

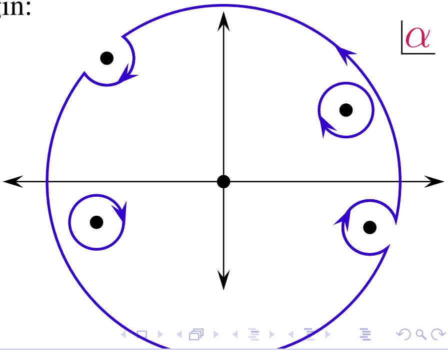

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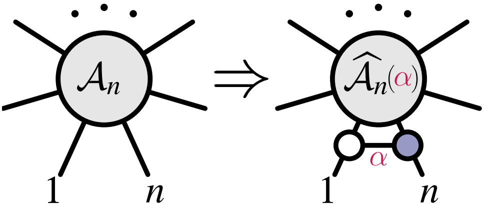

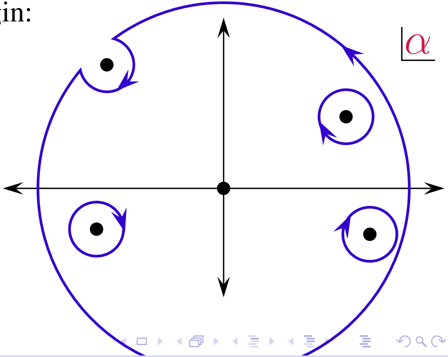

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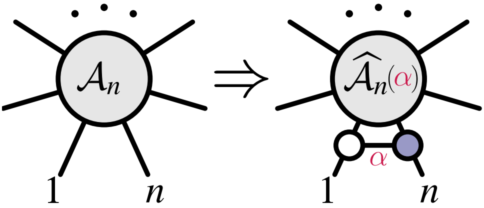

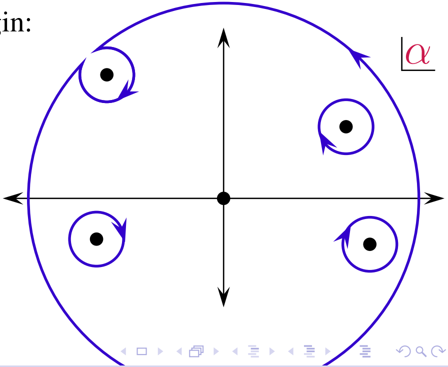

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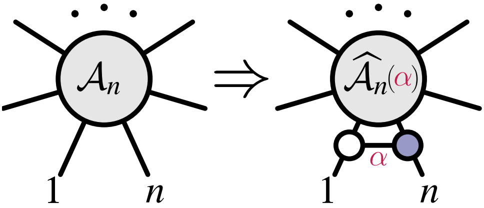

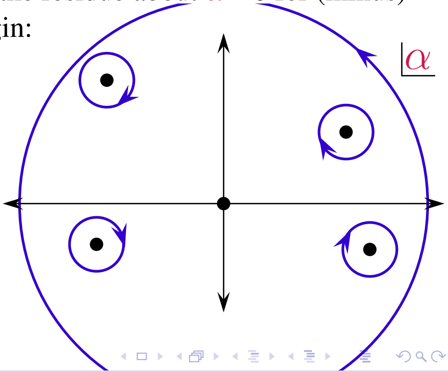

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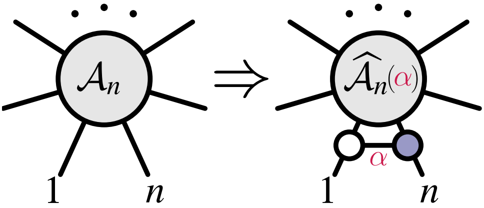

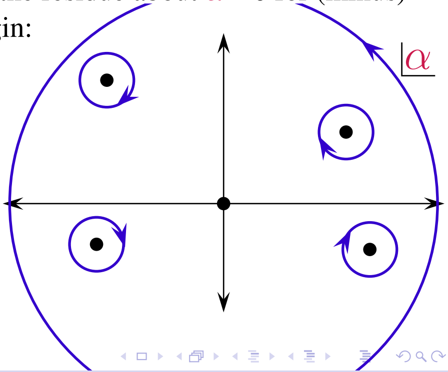

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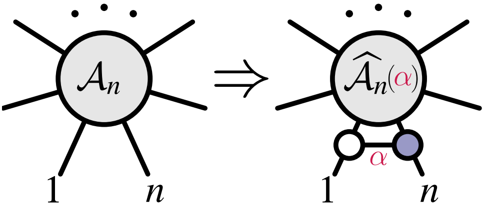

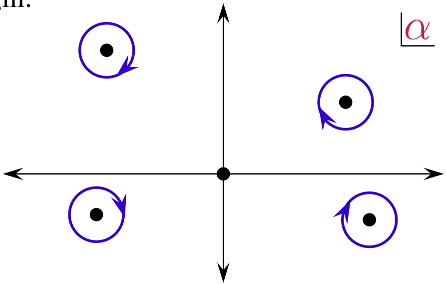

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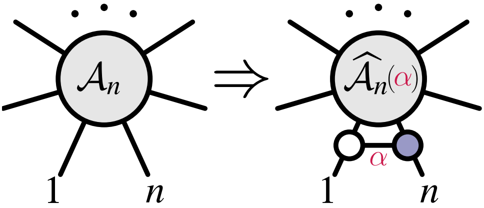

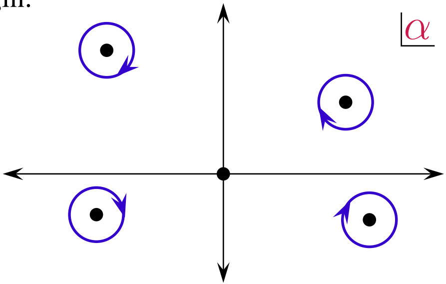

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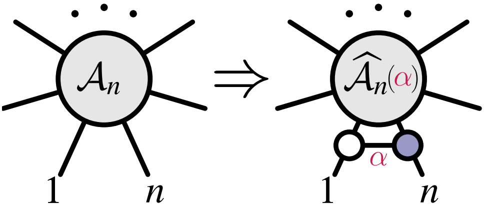

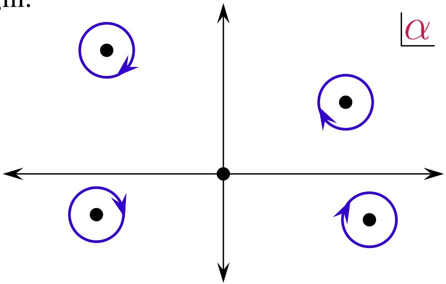

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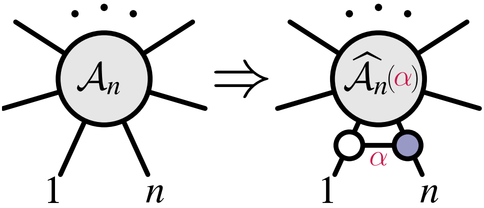

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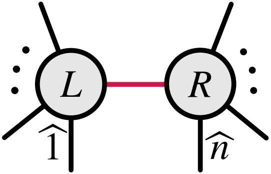

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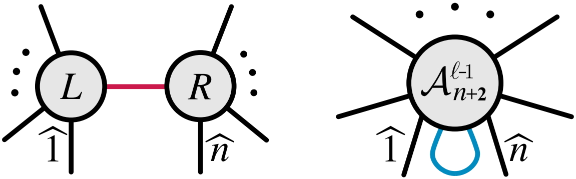

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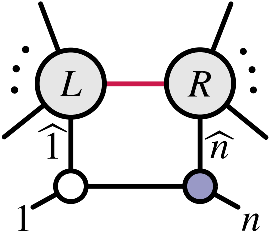

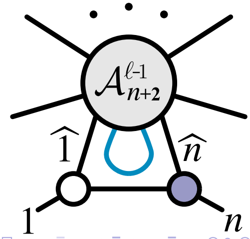

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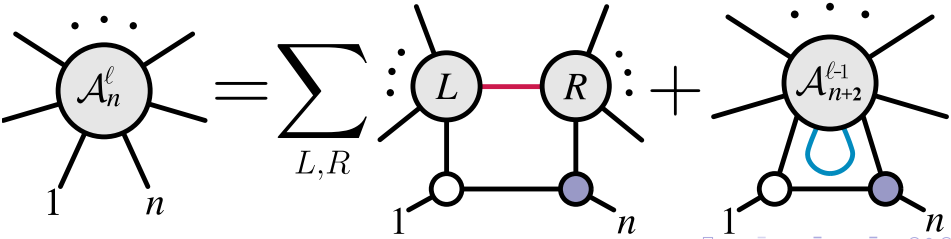

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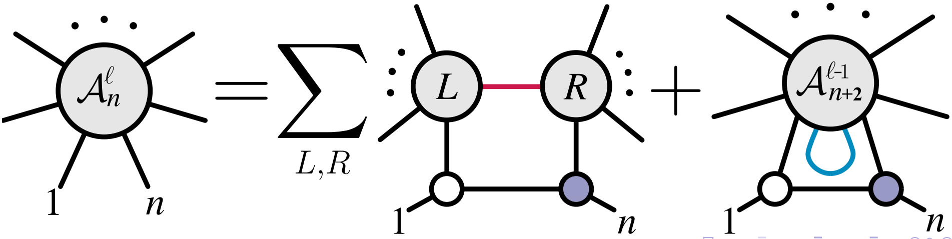

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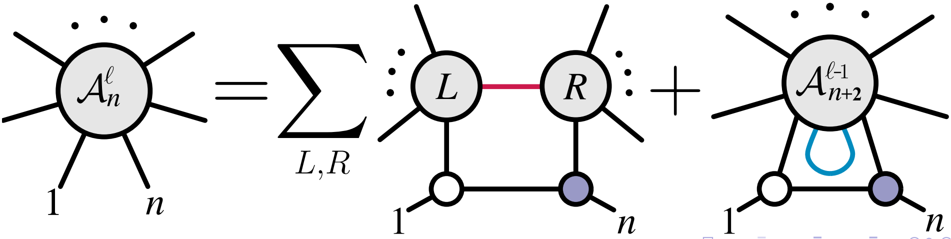

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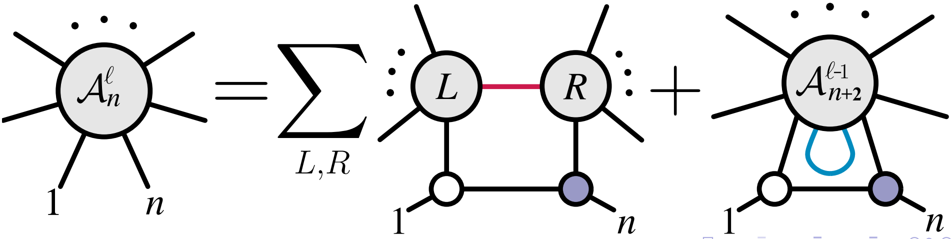

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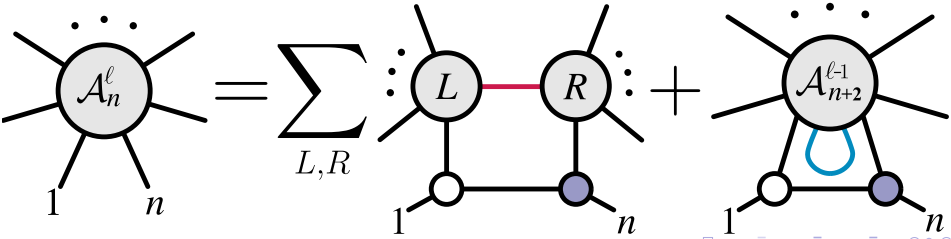

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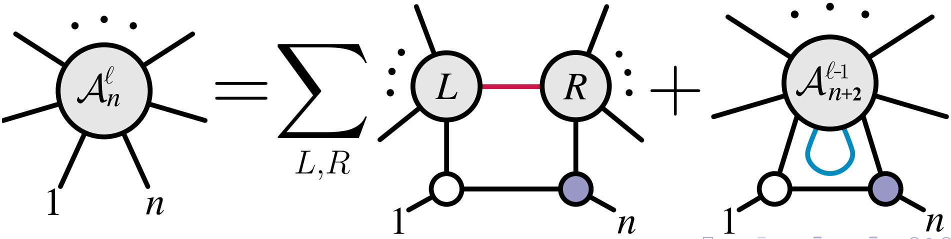

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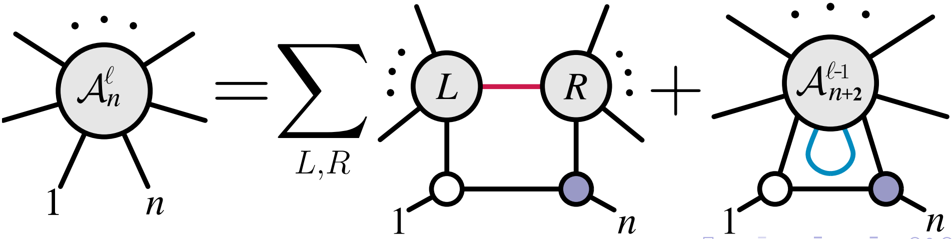

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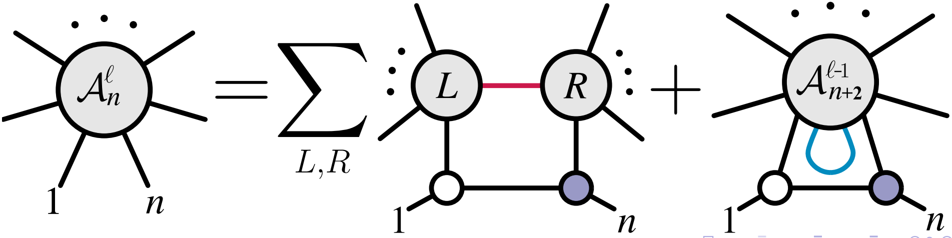

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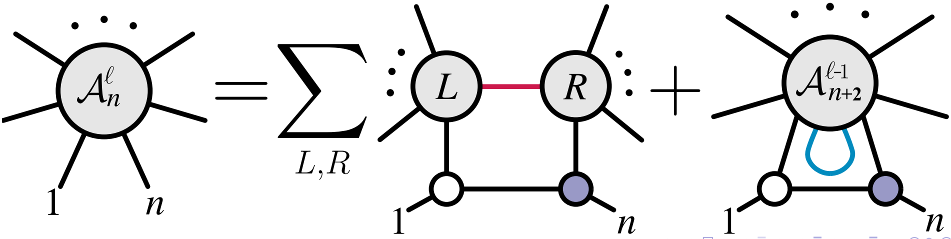

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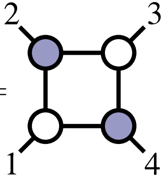
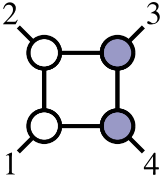
$\mathcal{A}_4^{(2)} =$  $+$ 

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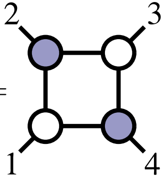
$\mathcal{A}_4^{(2)} =$ 

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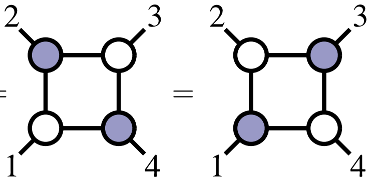
$\mathcal{A}_4^{(2)} =$ 

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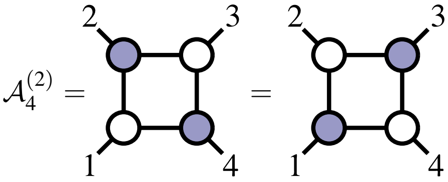

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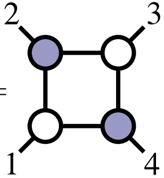
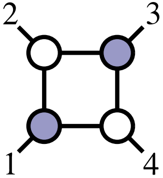
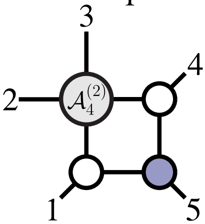
$\mathcal{A}_4^{(2)} =$  $=$ 

$\mathcal{A}_5^{(2)} =$ 

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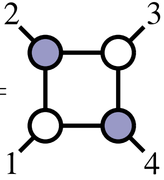
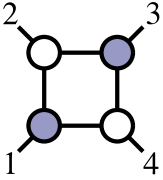
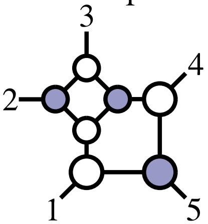
$\mathcal{A}_4^{(2)} =$ 

$=$ 

$\mathcal{A}_5^{(2)} =$ 

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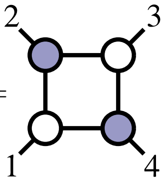

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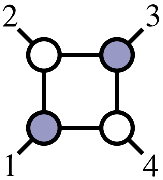

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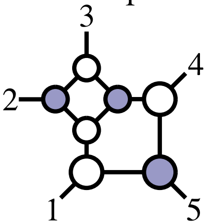

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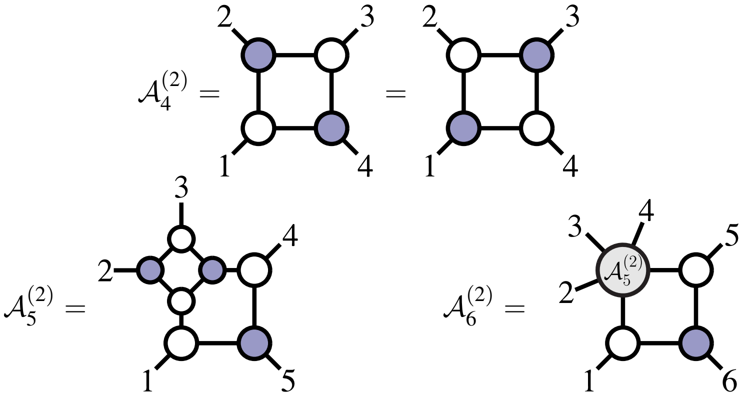

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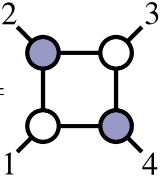
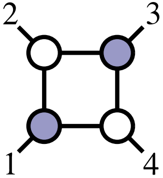
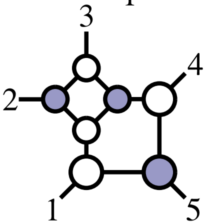
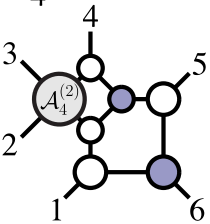
$\mathcal{A}_4^{(2)} =$ 

$=$ 

$\mathcal{A}_5^{(2)} =$ 

$\mathcal{A}_6^{(2)} =$ 

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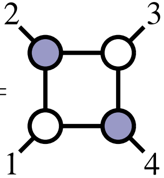
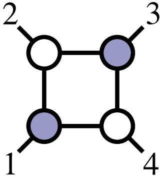
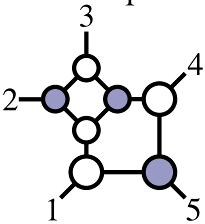
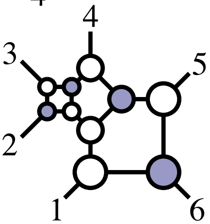
$\mathcal{A}_4^{(2)} =$  $=$ 

$\mathcal{A}_5^{(2)} =$ 

$\mathcal{A}_6^{(2)} =$ 

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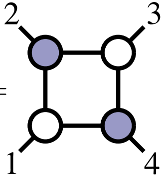
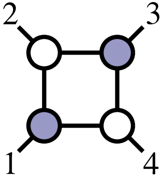
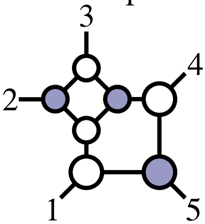
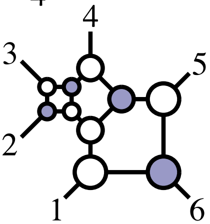
$\mathcal{A}_4^{(2)} =$ 

$=$ 

$\mathcal{A}_5^{(2)} =$ 

$\mathcal{A}_6^{(2)} =$ 

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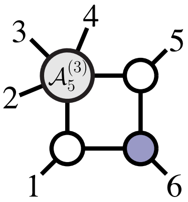
$\mathcal{A}_6^{(3)} =$ 

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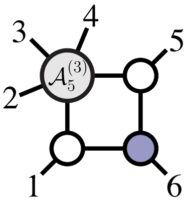
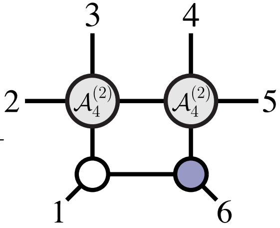
$\mathcal{A}_6^{(3)} =$  $+$ 

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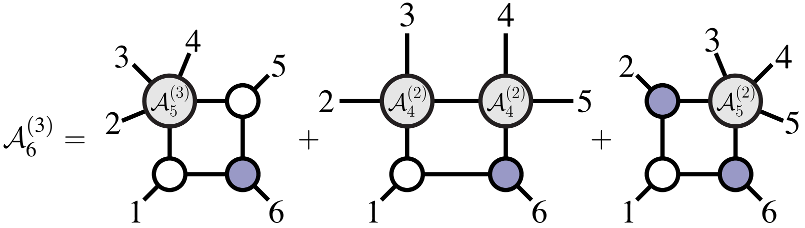

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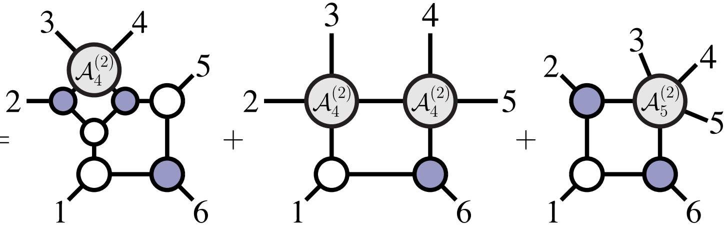
$\mathcal{A}_6^{(3)} =$ 

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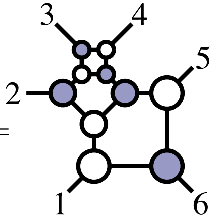
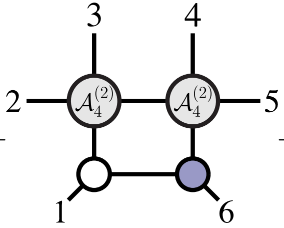
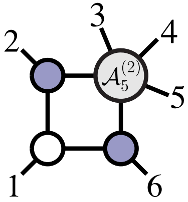
$\mathcal{A}_6^{(3)} =$  $+$  $+$ 

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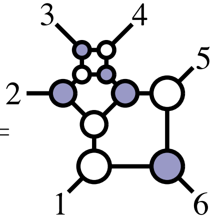
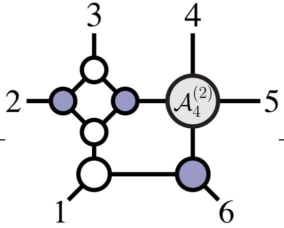
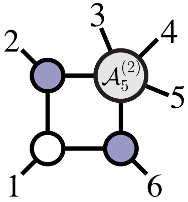
$\mathcal{A}_6^{(3)} =$  $+$  $+$ 

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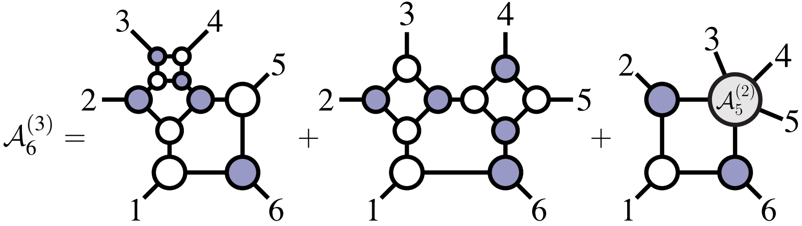

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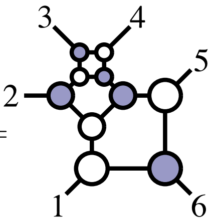
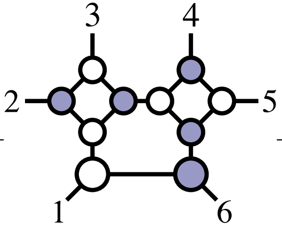
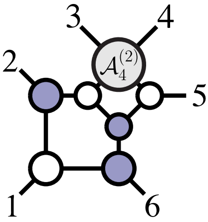
$\mathcal{A}_6^{(3)} =$  $+$  $+$ 

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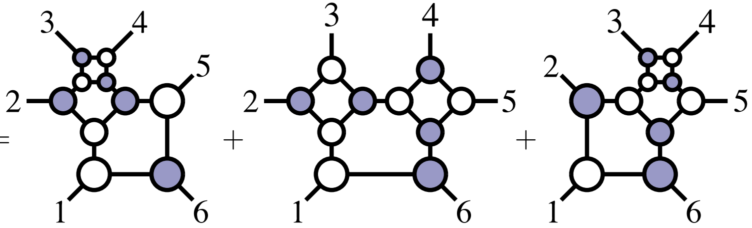
$\mathcal{A}_6^{(3)} =$ 

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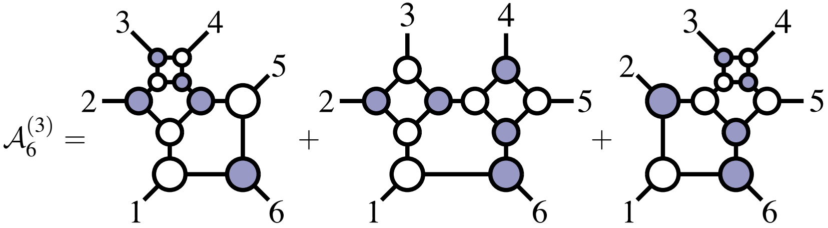

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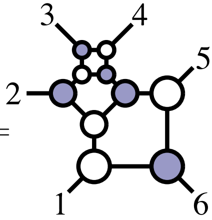
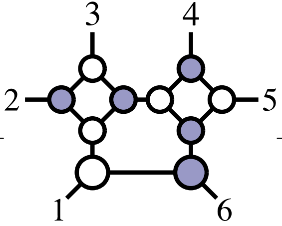
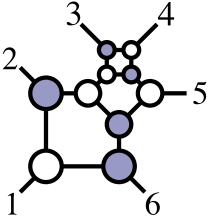
$\mathcal{A}_6^{(3)} =$  $+$  $+$ 

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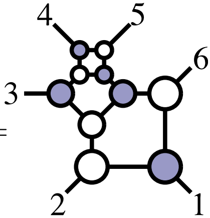
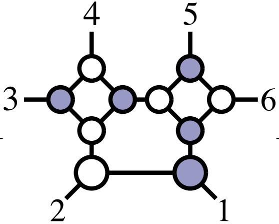
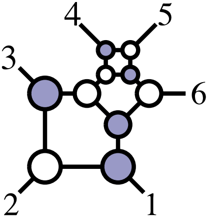
$\mathcal{A}_6^{(3)} =$  $+$  $+$ 

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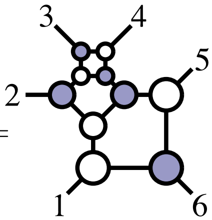
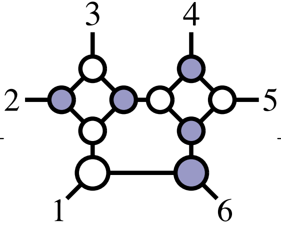
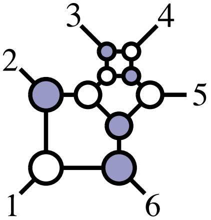
$\mathcal{A}_6^{(3)} =$ 

$+$ 

$+$ 

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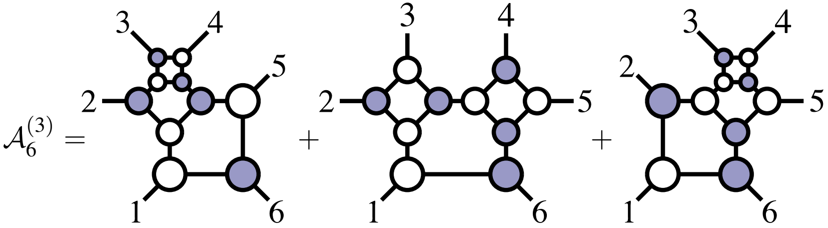

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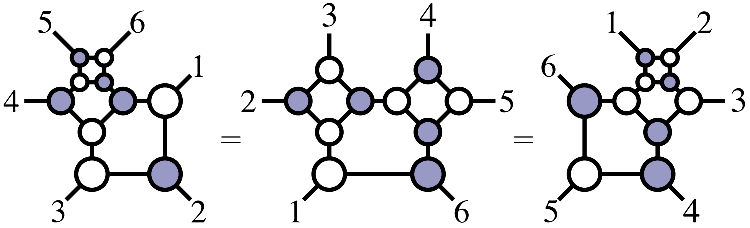

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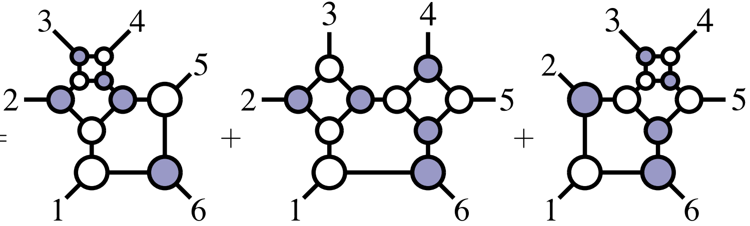

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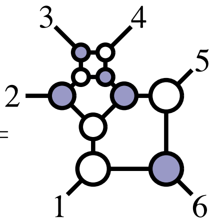
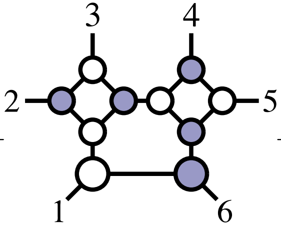
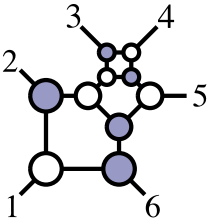
$\mathcal{A}_6^{(3)} =$  $+$  $+$ 

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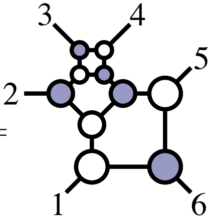
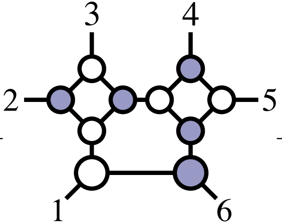
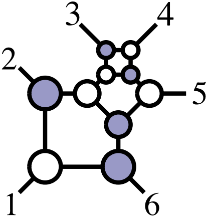
$\mathcal{A}_6^{(3)} =$ 

$+$ 

$+$ 

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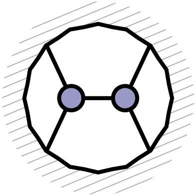

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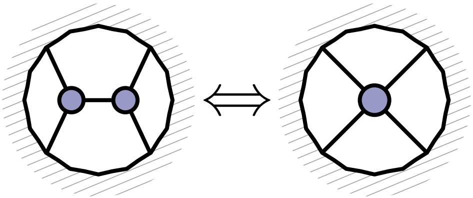

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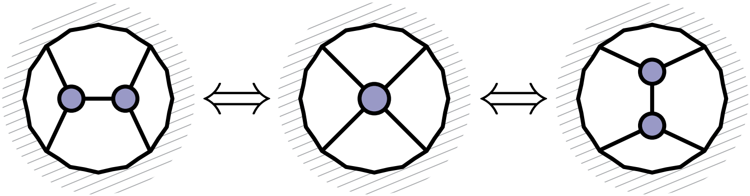

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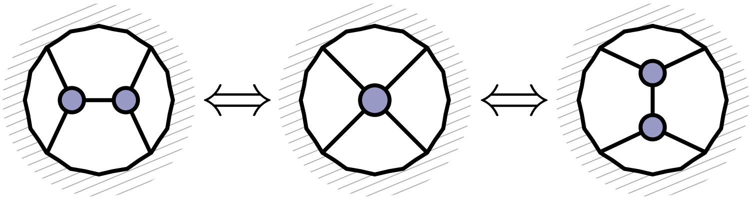

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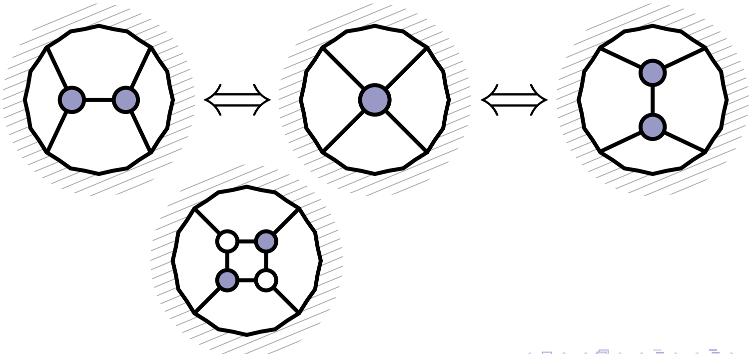

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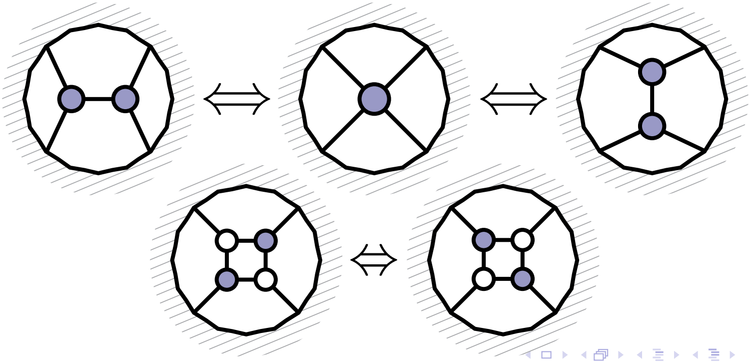

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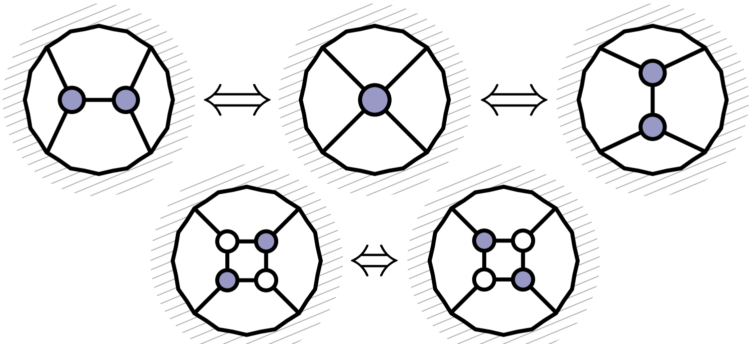

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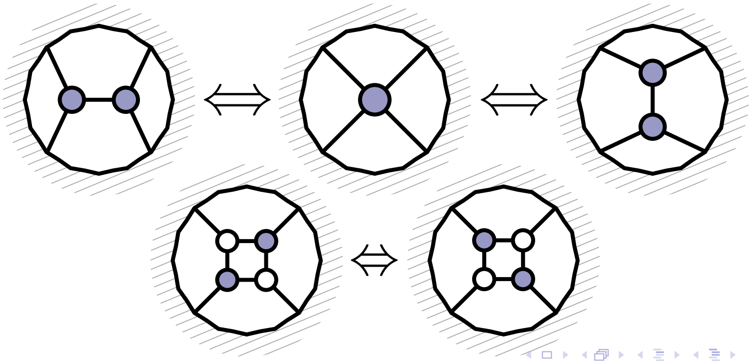

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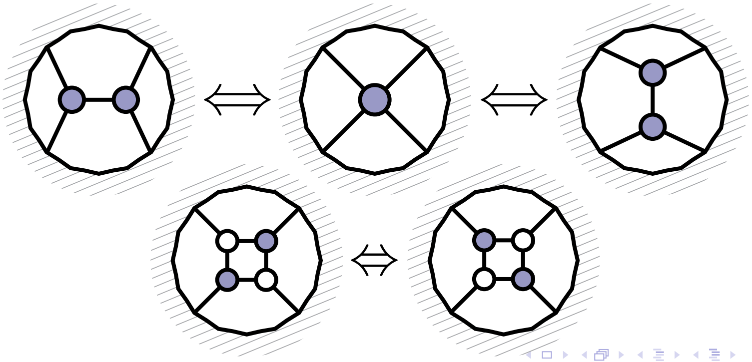

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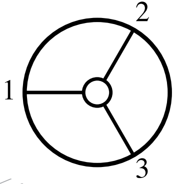

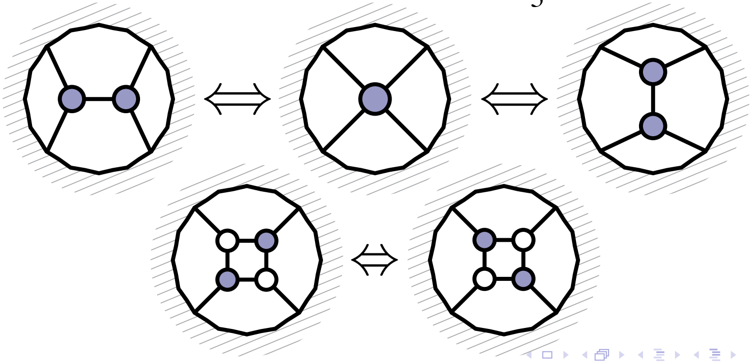

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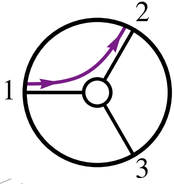

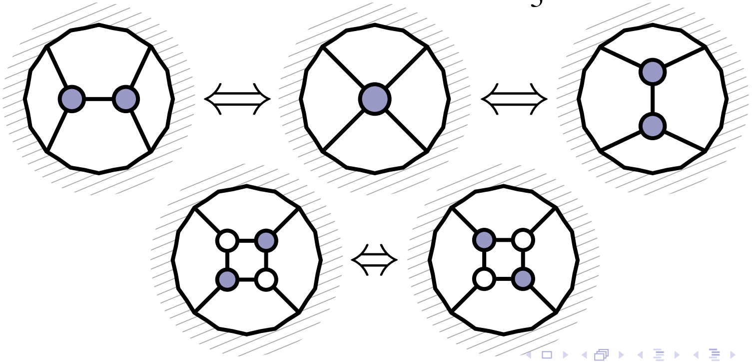

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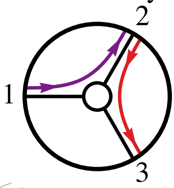

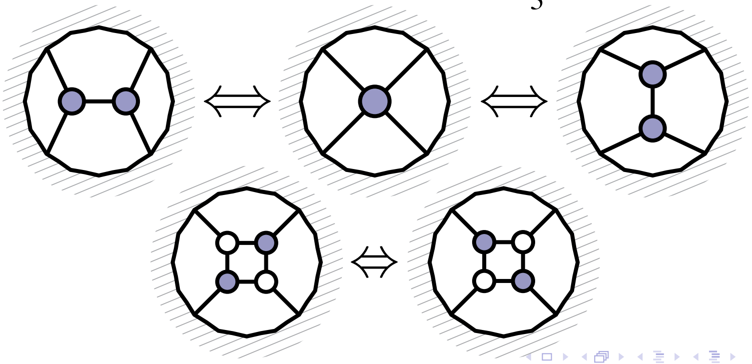

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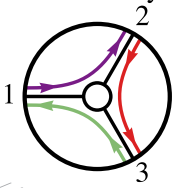

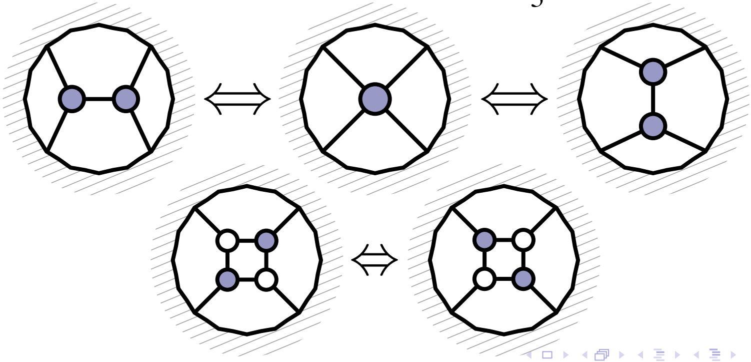

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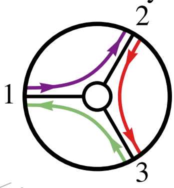

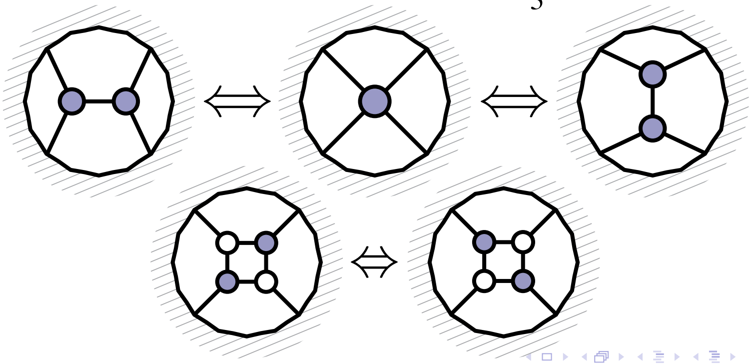

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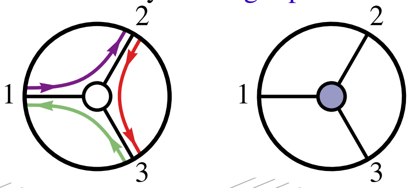

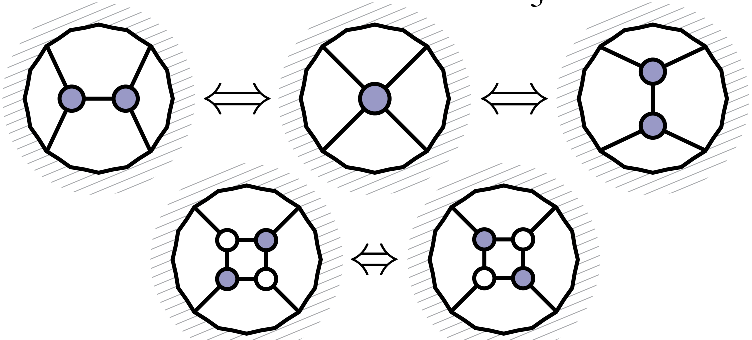

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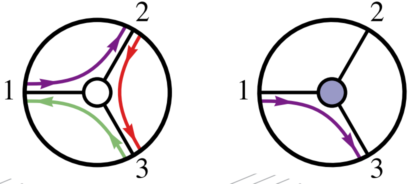

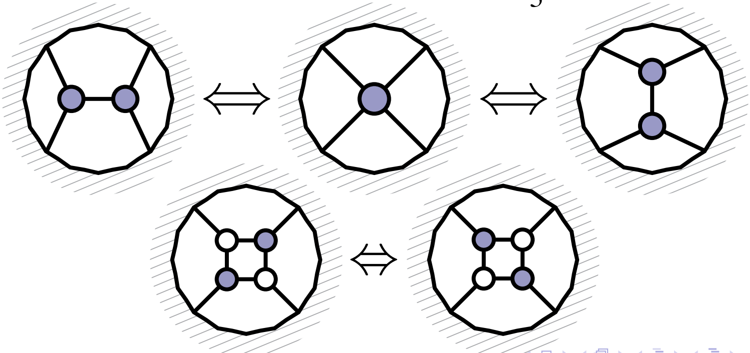

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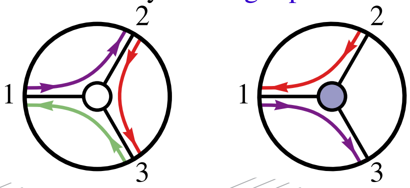

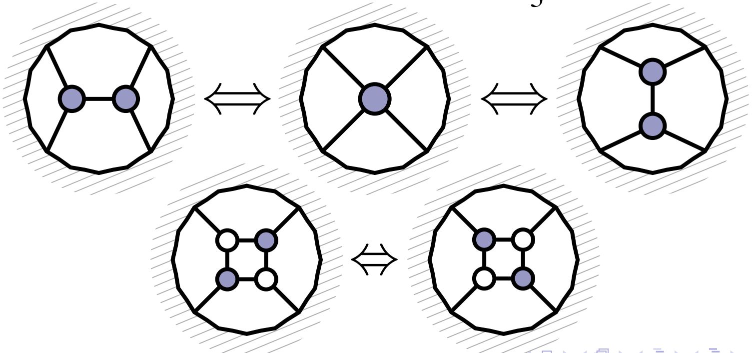

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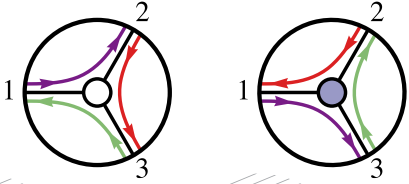

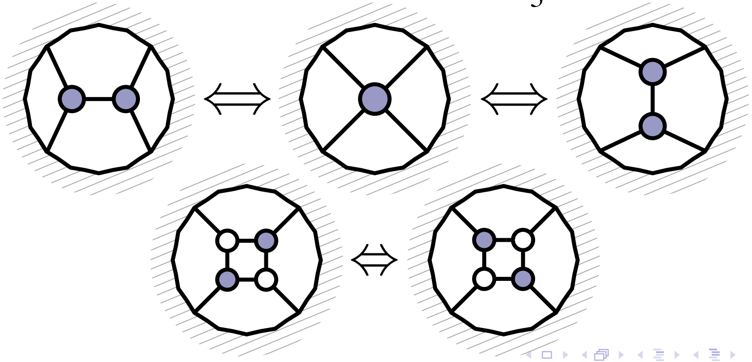

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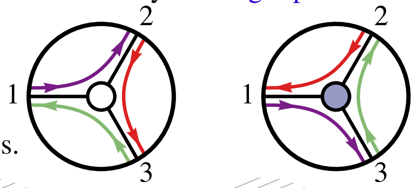

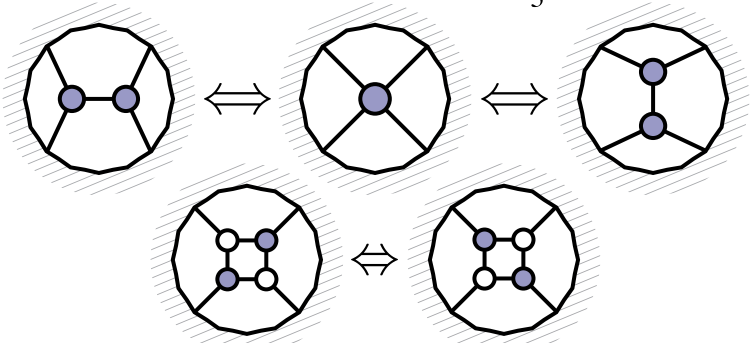

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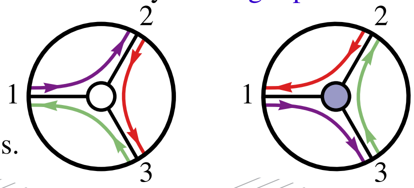

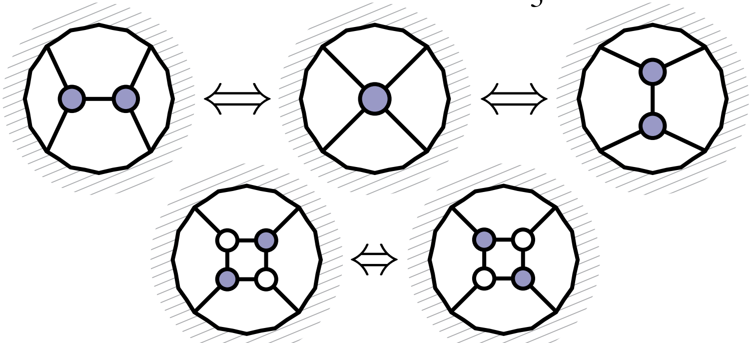

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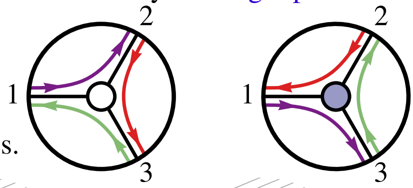

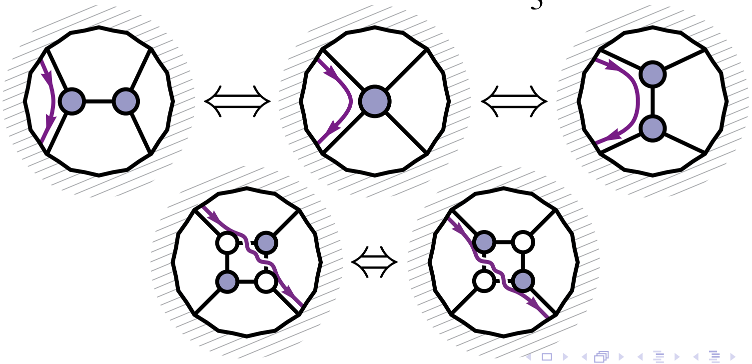

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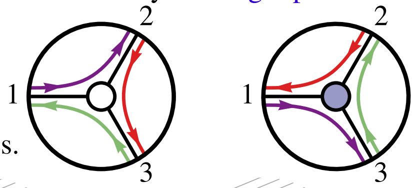

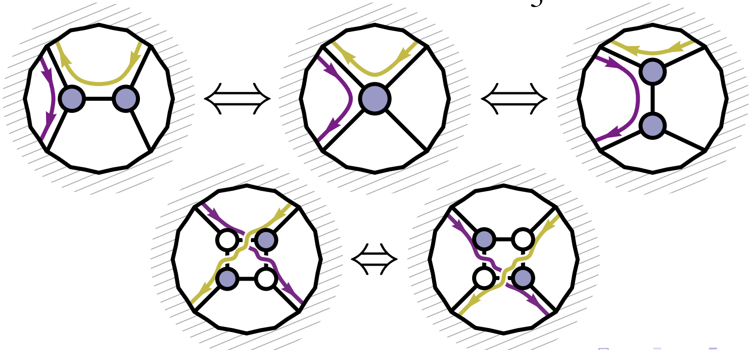

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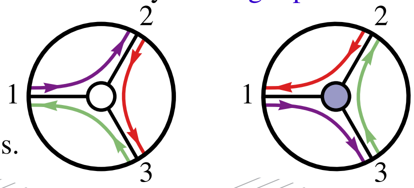

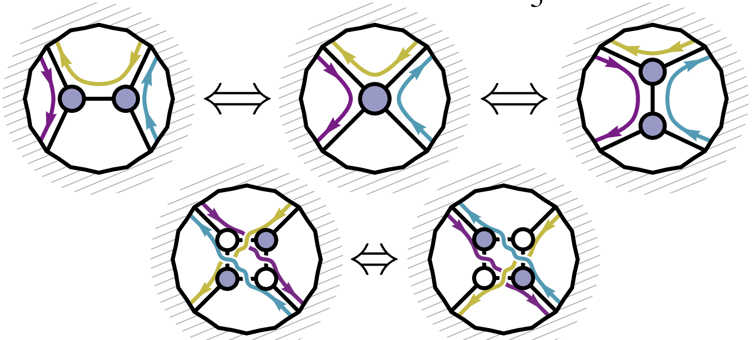

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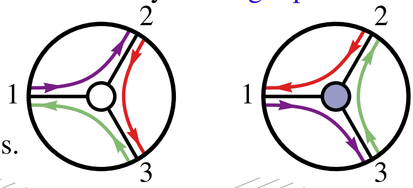

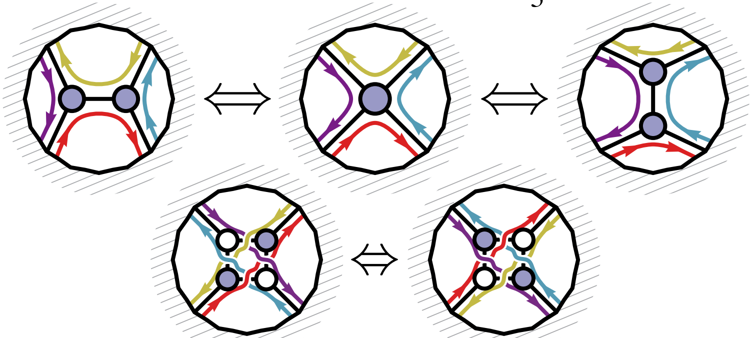

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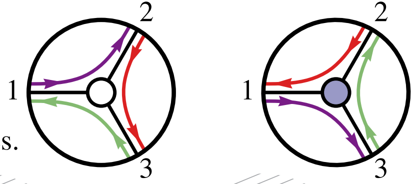

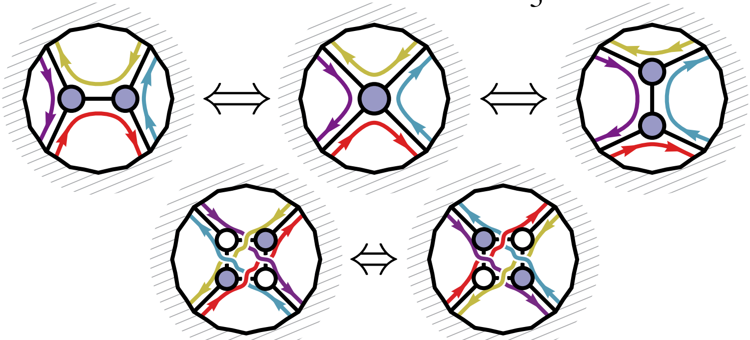

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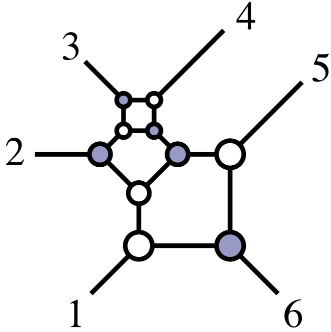

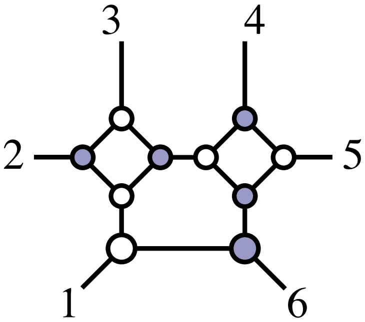

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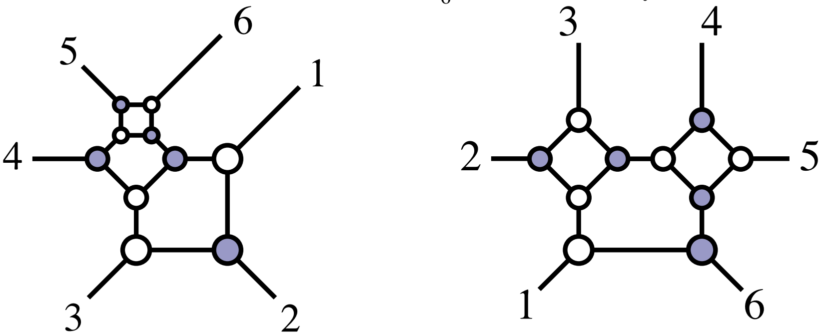

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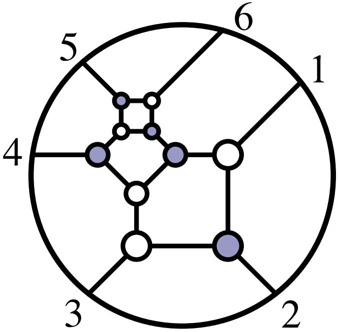

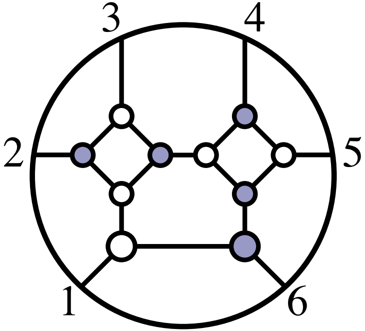

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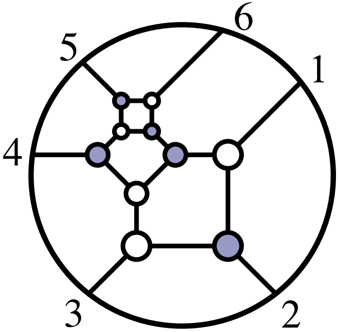

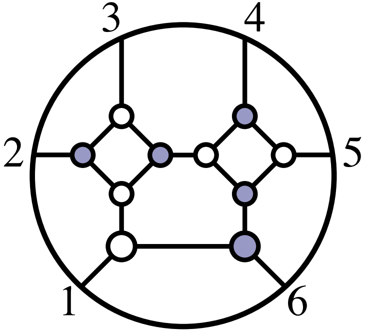

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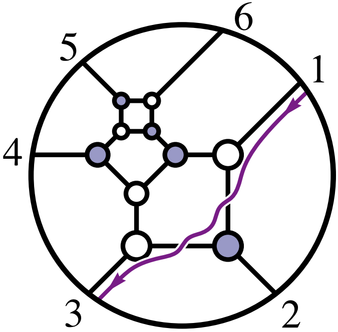

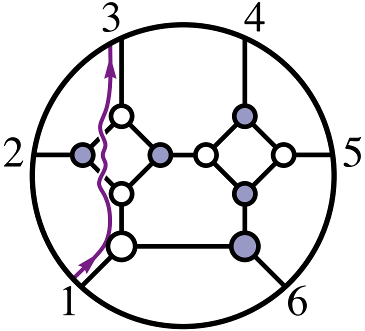

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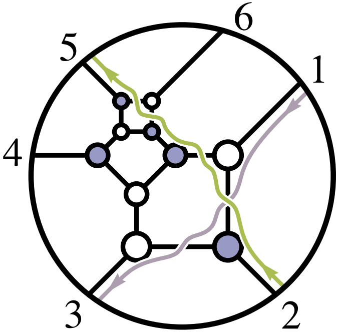

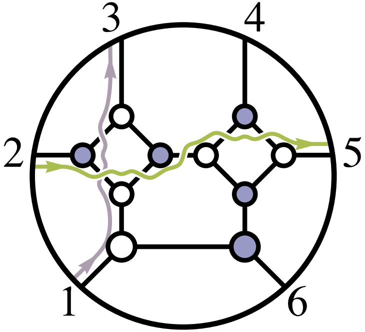

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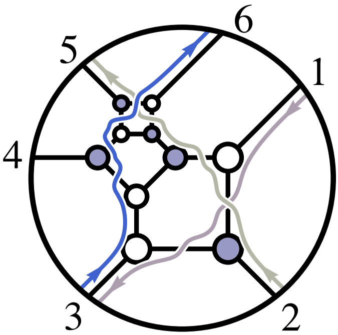

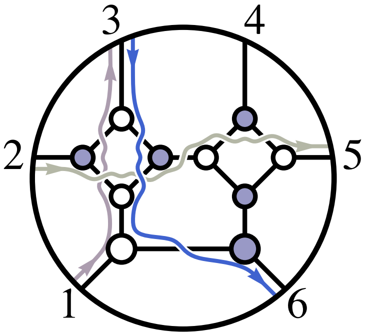

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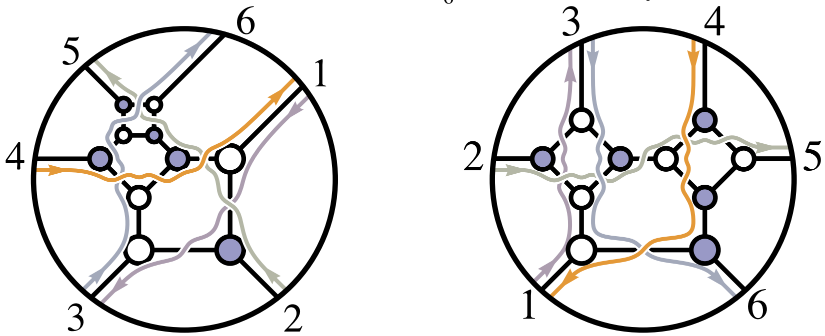

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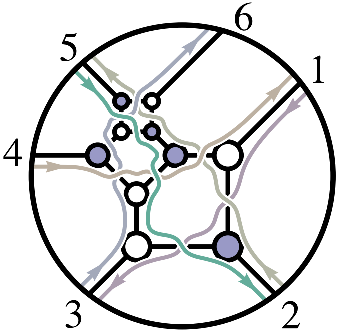

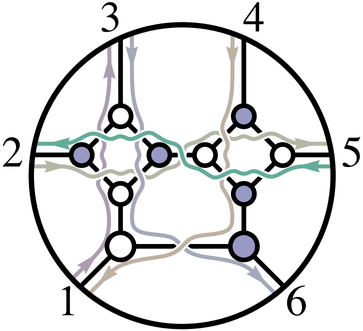

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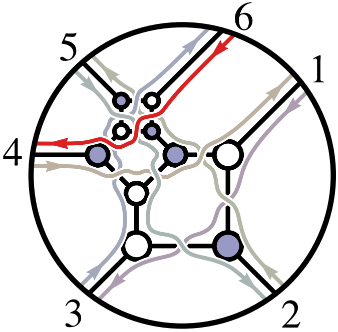

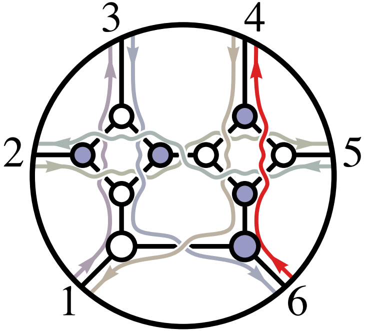

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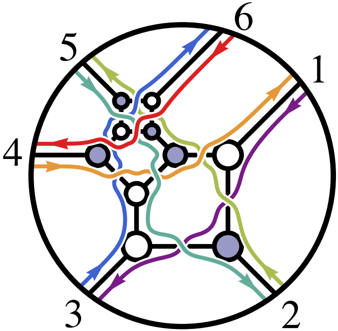

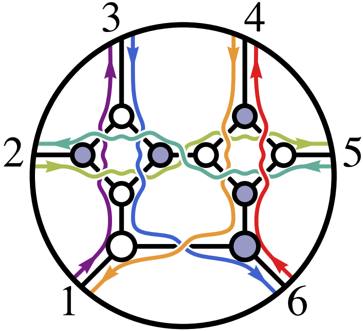

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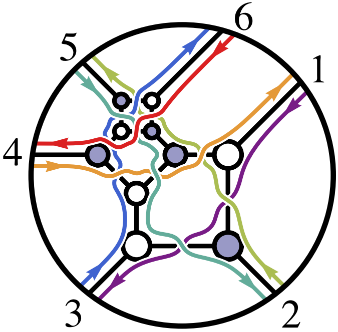

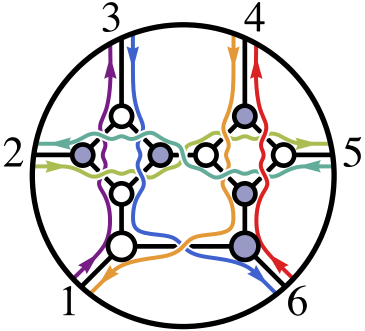

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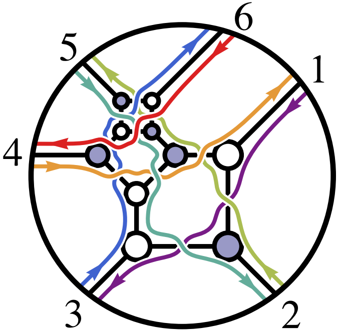

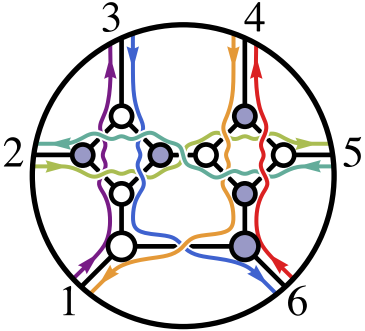

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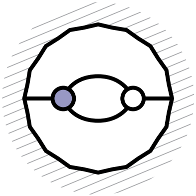

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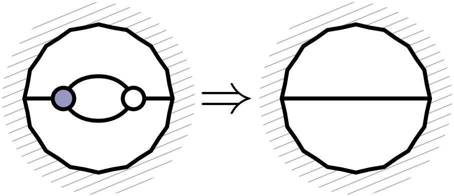
Notice that the merge and square moves leave the number of 'faces' of an on-shell diagram invariant. Diagrams with different numbers of faces can be related by 'reduction'—also known as 'bubble deletion':



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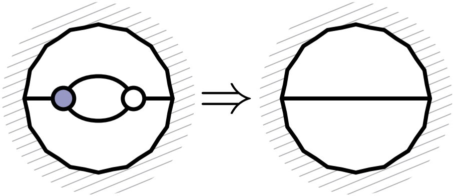

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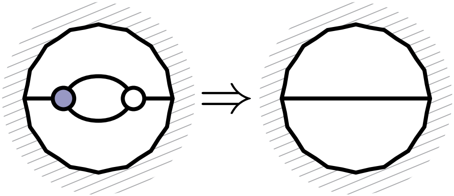

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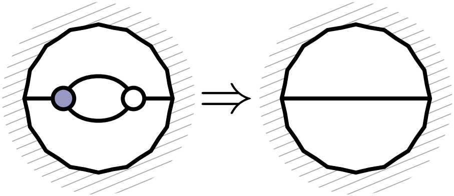

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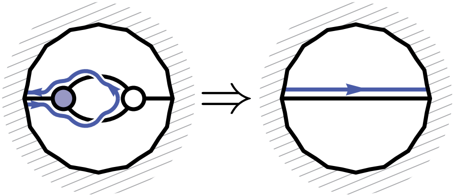

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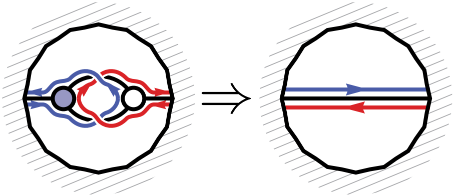

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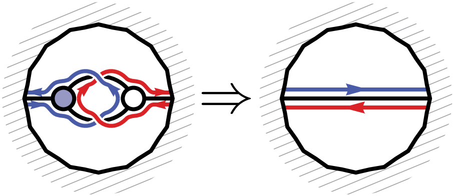
Such factors of $d\alpha/\alpha$ arising from bubble deletion encode loop integrands!



Canonical Coordinates for Computing On-Shell Functions

Recall that attaching 'BCFW bridges' can lead to very rich on-shell diagrams.

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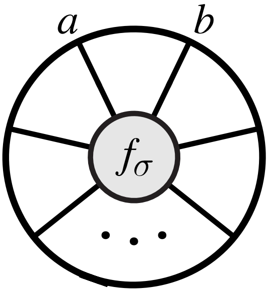

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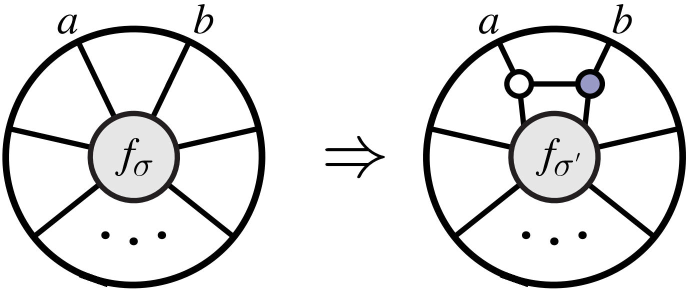

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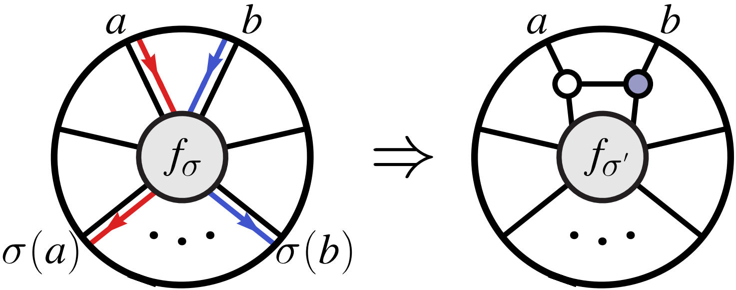

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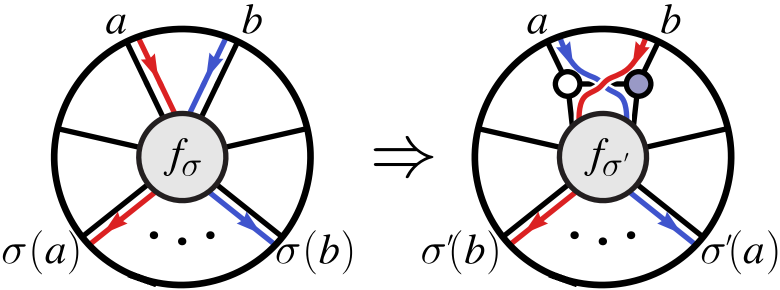

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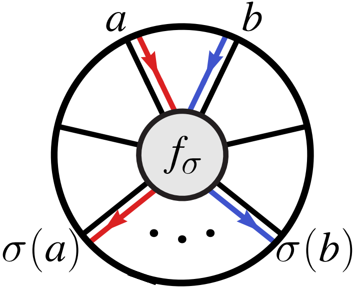

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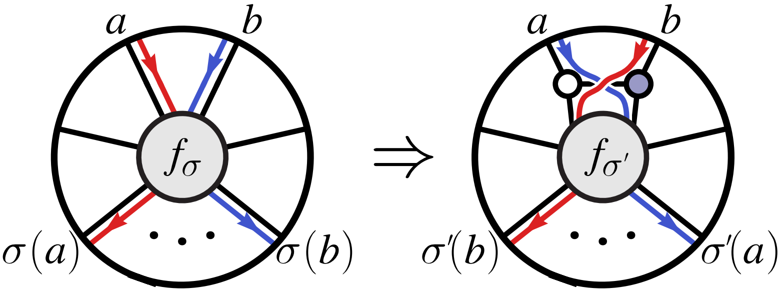

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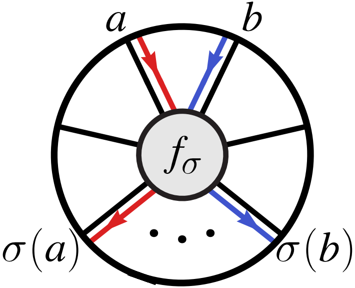

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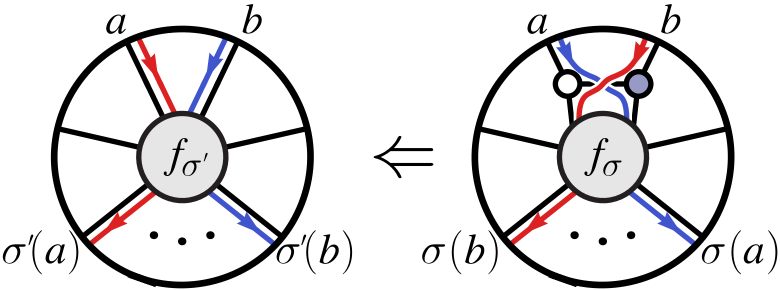

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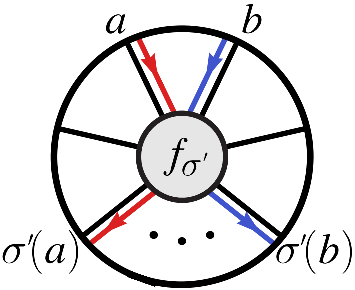
Recall that attaching 'BCFW bridges' can lead to very rich on-shell diagrams.

Read the other way, we can 'peel-off' bridges and thereby decompose a permutation into transpositions according to $\sigma = (a\,b) \circ \sigma'$



Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions

'Bridge' Decomposition

$$\sigma:\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow\\ 3 & 5 & 6 & 7 & 8 & 10\end{pmatrix}$$

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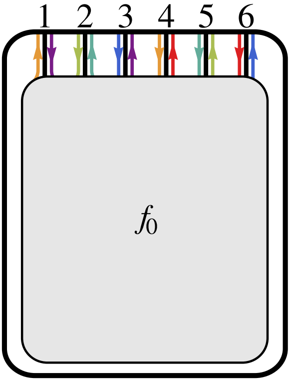

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$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \\ f_0 & \{3 & 5 & 6 & 7 & 8 & 10\} & \end{array}$$

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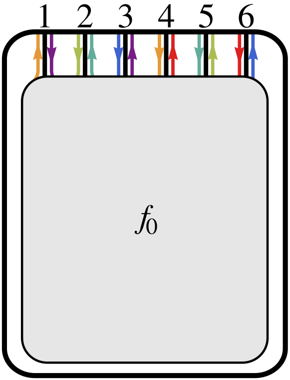

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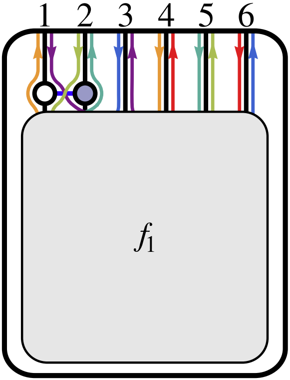

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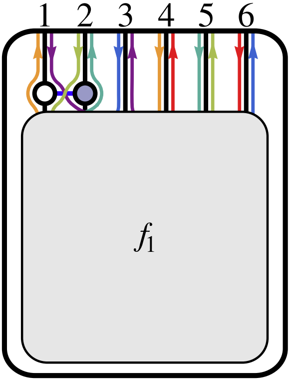

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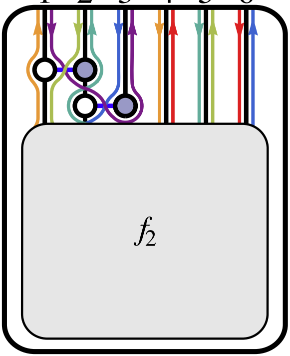

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	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	$(1\,2)$
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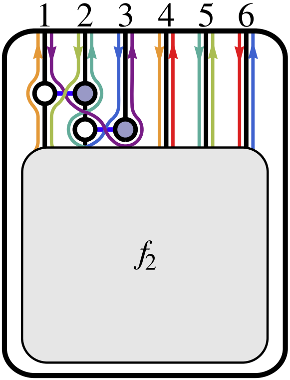

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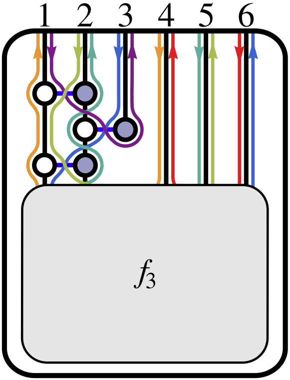

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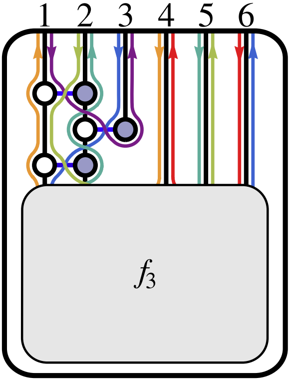

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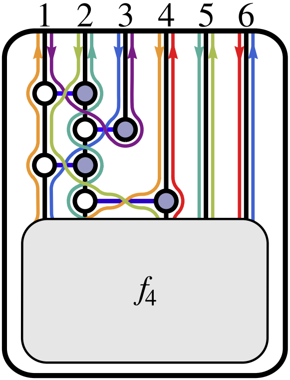

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
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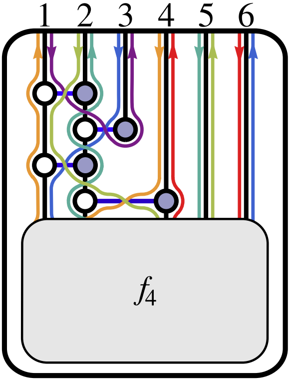

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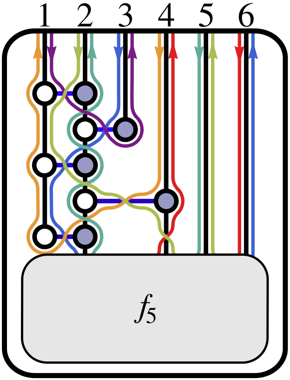

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	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
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f_1	{5	3	6	7	8	10}	$(2\,3)$
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f_3	{6	5	3	7	8	10}	$(2\,4)$
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f_5	{7	6	3	5	8	10}	

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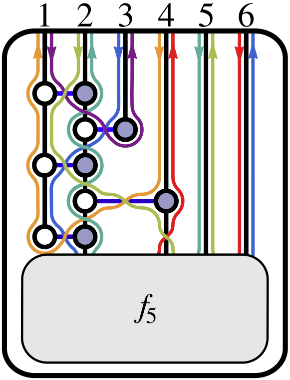

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							$(2\,3)$
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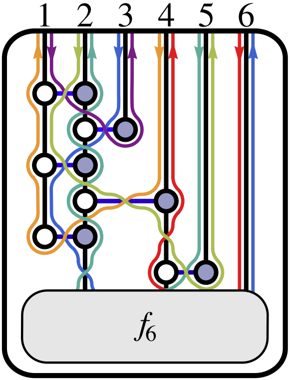

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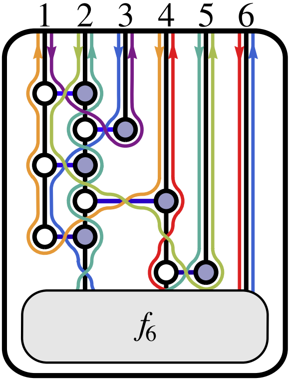

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)

Canonical Coordinates for Computing On-Shell Functions

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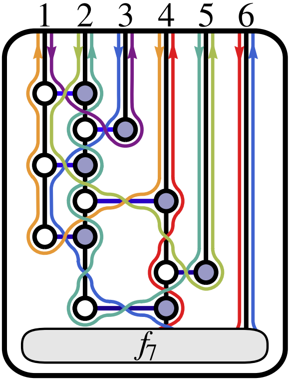

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	
							(1 2)
f_1	{5	3	6	7	8	10}	
							(2 3)
f_2	{5	6	3	7	8	10}	
							(1 2)
f_3	{6	5	3	7	8	10}	
							(2 4)
f_4	{6	7	3	5	8	10}	
							(1 2)
f_5	{7	6	3	5	8	10}	
							(4 5)
f_6	{7	6	3	8	5	10}	
							(2 4)
f_7	{7	8	3	6	5	10}	

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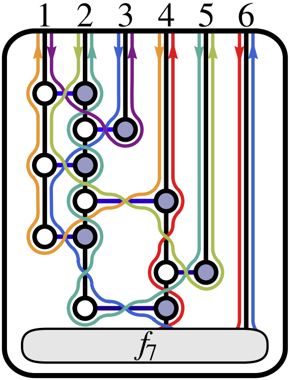

'Bridge' Decomposition

	1	2	3	4	5	6	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)

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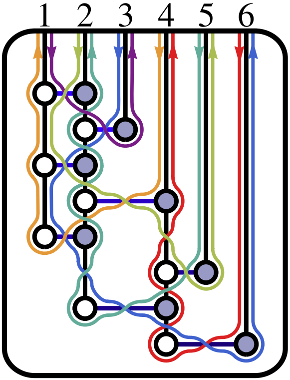

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6}	

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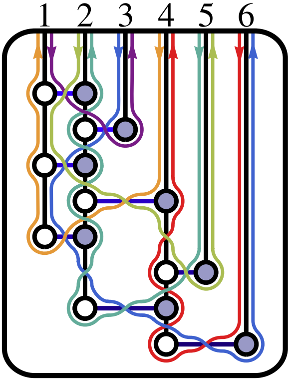

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
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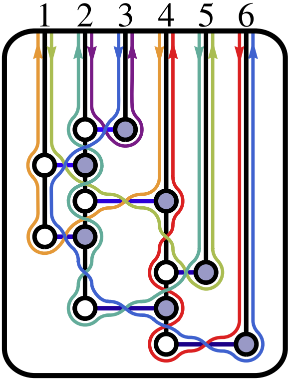

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6 }	

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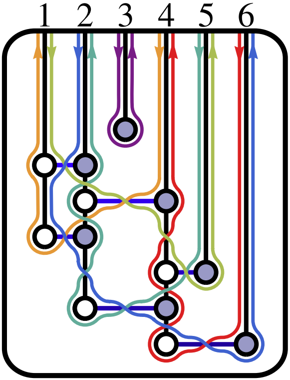

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	$(4\,6)$
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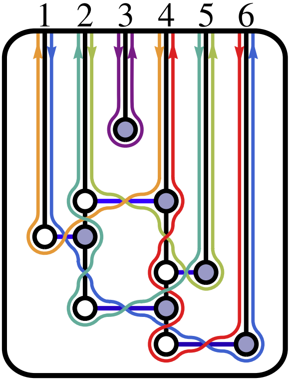

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	$(4\,6)$
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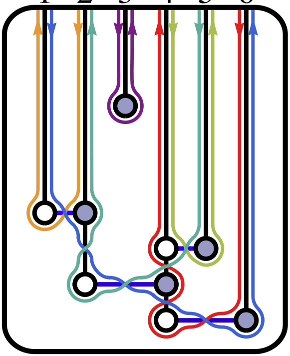

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_4	{6	7	3	5	8	10}	
							(1 2)
f_5	{7	6	3	5	8	10}	
							(4 5)
f_6	{7	6	3	8	5	10}	
							(2 4)
f_7	{7	8	3	6	5	10}	
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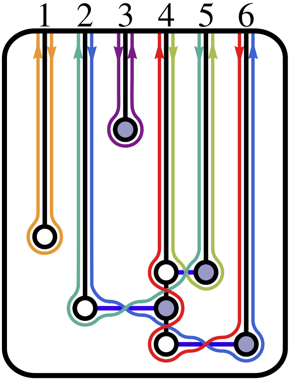

'Bridge' Decomposition

	1	2	3	4	5	6		τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$		
f_5	{7	6	3	5	8	10	}	
								(4 5)
f_6	{7	6	3	8	5	10	}	
								(2 4)
f_7	{7	8	3	6	5	10	}	
								(4 6)
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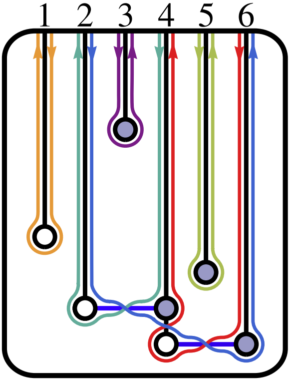

'Bridge' Decomposition

$$\begin{array}{lcccccccc} & & 1 & 2 & 3 & 4 & 5 & 6 & \\ & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \\ f_6 & \{ & 7 & 6 & 3 & 8 & 5 & 10 & \} \\ & & & & & & & & (2\,4) \\ f_7 & \{ & 7 & 8 & 3 & 6 & 5 & 10 & \} \\ & & & & & & & & (4\,6) \\ f_8 & \{ & 7 & 8 & 3 & 10 & 5 & 6 & \} \end{array}$$

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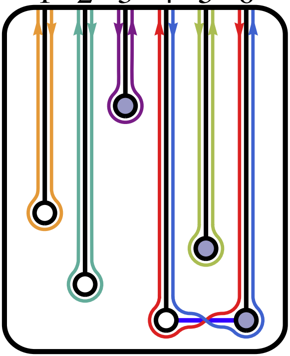

'Bridge' Decomposition

$$\begin{array}{lcccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & & \tau \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\ f_7 & \{7 & 8 & 3 & 6 & 5 & 10\} & & \\ f_8 & \{7 & 8 & 3 & 10 & 5 & 6\} & & (4\,6) \end{array}$$

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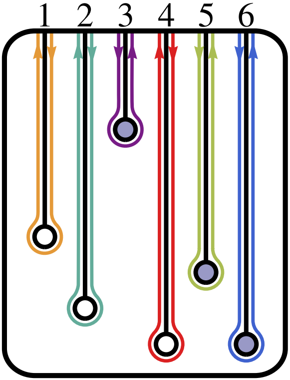

'Bridge' Decomposition

1	2	3	4	5	6	
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	τ

$f_8\ \{7\ \ 8\ \ 3\ \ 10\ \ 5\ \ 6\ \}$

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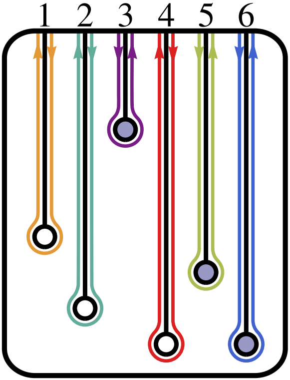

‘Bridge’ Decomposition

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \end{array}$$

$$f_8 \; \{7 \;\; 8 \;\; 3 \;\; 10 \;\; 5 \;\; 6\}$$

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$$f_8 = \prod_{a=\sigma(a)+n}\Big(\delta^4(\widetilde{\eta}_a)\,\delta^2(\widetilde{\lambda}_a)\Big)\prod_{b=\sigma(b)}\Big(\delta^2(\lambda_b)\Big)$$

'Bridge' Decomposition

$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \\ f_8 & \{7 & 8 & 3 & 10 & 5 & 6\,\} & \end{array}$$

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$$f_8 = \prod_{a=\sigma(a)+n}\Big(\delta^4(\widetilde{\eta}_a)\,\delta^2(\widetilde{\lambda}_a)\Big)\prod_{b=\sigma(b)}\Big(\delta^2(\lambda_b)\Big)$$

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

‘Bridge’ Decomposition

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \tau \end{array}$$

$f_8\ \{7\ \ 8\ \ 3\ \ 10\ \ 5\ \ 6\ \}$

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$$f_8 = \delta^{3\times4}(C\cdot\widetilde{\eta})\,\delta^{3\times2}(C\cdot\widetilde{\lambda})\,\delta^{2\times3}(\lambda\cdot C^{\perp})$$

$$C \equiv \begin{array}{c} \\ \\ \end{array}\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

$$C \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

'Bridge' Decomposition

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \tau \end{array}$$

$$f_8\ \{7\ \ 8\ \ 3\ \ 10\ \ 5\ \ 6\ \}$$

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$$C \equiv \begin{array}{c} \\ \\ \end{array}\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$
$$\left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

‘Bridge’ Decomposition

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \end{array}$$

$f_8\ \{7\ \ 8\ \ 3\ \ 10\ \ 5\ \ 6\ \}$

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$$f_7 = \frac{d\alpha_8}{\alpha_8}\delta^{3\times4}(C\cdot\widetilde{\eta})\,\delta^{3\times2}(C\cdot\widetilde{\lambda})\,\delta^{2\times3}(\lambda\cdot C^{\perp})$$

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_8 \end{array}\right) \end{array}$$

$$(4\,6):\quad c_6 \mapsto c_6 + \alpha_8\, c_4$$

‘Bridge’ Decomposition

$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \\ f_7 & \{7 & 8 & 3 & 6 & 5 & 10\} & \\ f_8 & \{7 & 8 & 3 & 10 & 5 & 6\} & (4\,6) \end{array}$$

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:

$$f_0 = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4}\frac{d\alpha_5}{\alpha_5}\frac{d\alpha_6}{\alpha_6}\frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8} f_8$$

$$f_6 = \frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}(C\cdot\widetilde{\eta})\,\delta^{3\times2}(C\cdot\widetilde{\lambda})\,\delta^{2\times3}(\lambda\cdot C^{\perp})$$

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_8 \end{pmatrix} \end{array}$$

$$(2\,4):\quad c_4 \mapsto c_4 + \alpha_7\, c_2$$

'Bridge' Decomposition

$$\begin{array}{lcccccccl} & 1 & 2 & 3 & 4 & 5 & 6 & & \tau \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\ f_6 & \{7 & 6 & 3 & 8 & 5 & 10\} & & (2\,4) \\ f_7 & \{7 & 8 & 3 & 6 & 5 & 10\} & & (4\,6) \\ f_8 & \{7 & 8 & 3 & 10 & 5 & 6\} & & \end{array}$$

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:

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$$f_5 = \frac{d\alpha_6}{\alpha_6}\frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}(C\cdot\widetilde{\eta})\,\delta^{3\times2}(C\cdot\widetilde{\lambda})\,\delta^{2\times3}(\lambda\cdot C^{\perp})$$

$$C \equiv \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{pmatrix} \end{array}$$

$(4\,5)$: $c_5 \mapsto c_5 + \alpha_6\, c_4$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	$(4\,6)$
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:

$$f_0 = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4}\frac{d\alpha_5}{\alpha_5}\frac{d\alpha_6}{\alpha_6}\frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8} f_8$$

$$f_4 = \frac{d\alpha_5}{\alpha_5}\cdots\frac{d\alpha_8}{\alpha_8}\,\delta^{3\times4}(C\cdot\widetilde{\eta})\,\delta^{3\times2}(C\cdot\widetilde{\lambda})\,\delta^{2\times3}(\lambda\cdot C^{\perp})$$

$$C \equiv \begin{array}{c} \begin{matrix} \;1 & \;\;2 & \;\;3 & \;\;4 & \;\;5 & \;\;6 \end{matrix} \\ \begin{pmatrix} 1 & \alpha_5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{pmatrix} \end{array}$$

$$(1\,2):\quad c_2 \mapsto c_2 + \alpha_5\, c_1$$

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	$(4\,6)$
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:

$$f_0 = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4}\frac{d\alpha_5}{\alpha_5}\frac{d\alpha_6}{\alpha_6}\frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8} f_8$$

$$f_3 = \frac{d\alpha_4}{\alpha_4} \cdots \frac{d\alpha_8}{\alpha_8}\delta^{3\times4}(C\cdot\widetilde{\eta})\,\delta^{3\times2}(C\cdot\widetilde{\lambda})\,\delta^{2\times3}(\lambda\cdot C^{\perp})$$

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & \alpha_5 & 0 & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & 0 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

$$(2\,4): \quad c_4 \mapsto c_4 + \alpha_4\, c_2$$

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6 }	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:

$$f_0 = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4}\frac{d\alpha_5}{\alpha_5}\frac{d\alpha_6}{\alpha_6}\frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8}f_8$$

$$f_2 = \frac{d\alpha_3}{\alpha_3}\cdots\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}(C\cdot\widetilde{\eta})\,\delta^{3\times2}(C\cdot\widetilde{\lambda})\,\delta^{2\times3}(\lambda\cdot C^{\perp})$$

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & (\alpha_3+\alpha_5) & 0 & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & 0 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

$$(1\,2):\quad c_2 \mapsto c_2 + \alpha_3\, c_1$$

‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	$(4\,6)$
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:

$$f_0 = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4}\frac{d\alpha_5}{\alpha_5}\frac{d\alpha_6}{\alpha_6}\frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8} f_8$$

$$f_1 = \frac{d\alpha_2}{\alpha_2}\cdots\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}(C\cdot\widetilde{\eta})\,\delta^{3\times2}(C\cdot\widetilde{\lambda})\,\delta^{2\times3}(\lambda\cdot C^{\perp})$$

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & (\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

$$(2\,3):\quad c_3 \mapsto c_3 + \alpha_2\, c_2$$

‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	$(4\,6)$
f_8	{7	8	3	10	5	6 }	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:

$$f_0 = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4}\frac{d\alpha_5}{\alpha_5}\frac{d\alpha_6}{\alpha_6}\frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8}f_8$$

$$f_0 = \frac{d\alpha_1}{\alpha_1}\cdots\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}(C\cdot\widetilde{\eta})\,\delta^{3\times2}(C\cdot\widetilde{\lambda})\,\delta^{2\times3}(\lambda\cdot C^{\perp})$$

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{pmatrix} \end{array}$$

$$(1\,2):\quad c_2 \mapsto c_2 + \alpha_1\, c_1$$

‘Bridge’ Decomposition

	1	2	3	4	5	6		τ
f_0	{3	5	6	7	8	10}		$(1\,2)$
f_1	{5	3	6	7	8	10}		$(2\,3)$
f_2	{5	6	3	7	8	10}		$(1\,2)$
f_3	{6	5	3	7	8	10}		$(2\,4)$
f_4	{6	7	3	5	8	10}		$(1\,2)$
f_5	{7	6	3	5	8	10}		$(4\,5)$
f_6	{7	6	3	8	5	10}		$(2\,4)$
f_7	{7	8	3	6	5	10}		$(4\,6)$
f_8	{7	8	3	10	5	6}		

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:

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$$f_0 = \frac{d\alpha_1}{\alpha_1}\cdots\frac{d\alpha_8}{\alpha_8}\,\delta^{3\times4}\big(C\cdot\widetilde{\eta}\big)\,\delta^{3\times2}\big(C\cdot\widetilde{\lambda}\big)\,\delta^{2\times3}\big(\lambda\cdot C^{\perp}\big)$$

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	$(4\,6)$
f_8	{7	8	3	10	5	6 }	

Canonical Coordinates for Computing On-Shell Functions

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$$f_0 = \frac{d\alpha_1}{\alpha_1} \cdots \frac{d\alpha_8}{\alpha_8}\delta^{3\times4}(C\cdot\widetilde{\eta})\,\delta^{3\times2}(C\cdot\widetilde{\lambda})\,\delta^{2\times3}(\lambda\cdot C^{\perp})$$

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{pmatrix} \end{array}$$

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
f_0	3	5	6	7	8	10	$(1\,2)$
f_1	5	3	6	7	8	10	$(2\,3)$
f_2	5	6	3	7	8	10	$(1\,2)$
f_3	6	5	3	7	8	10	$(2\,4)$
f_4	6	7	3	5	8	10	$(1\,2)$
f_5	7	6	3	5	8	10	$(4\,5)$
f_6	7	6	3	8	5	10	$(2\,4)$
f_7	7	8	3	6	5	10	$(4\,6)$
f_8	7	8	3	10	5	6	

Canonical Coordinates for Computing On-Shell Functions

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$$C \equiv \begin{array}{c} \\ \\ \end{array}\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

$$C \equiv \begin{pmatrix} 1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{pmatrix}$$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	
							(1 2)
f_1	{5	3	6	7	8	10}	
							(2 3)
f_2	{5	6	3	7	8	10}	
							(1 2)
f_3	{6	5	3	7	8	10}	
							(2 4)
f_4	{6	7	3	5	8	10}	
							(1 2)
f_5	{7	6	3	5	8	10}	
							(4 5)
f_6	{7	6	3	8	5	10}	
							(2 4)
f_7	{7	8	3	6	5	10}	
							(4 6)
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

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$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ \alpha_0\alpha_8 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

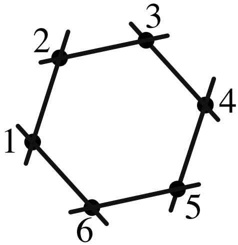
$$(6\,1):\quad c_1 \mapsto c_1 + \alpha_0\, c_6$$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	$(4\,6)$
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:



$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ \alpha_0\alpha_8 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

$$(6\,1): \quad c_1 \mapsto c_1 + \alpha_0\, c_6$$

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6 }	

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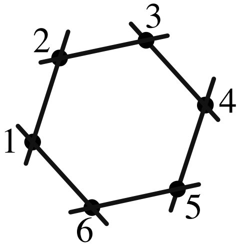

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ \alpha_0\alpha_8 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

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'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	
							$(1\,2)$
f_1	{5	3	6	7	8	10}	
							$(2\,3)$
f_2	{5	6	3	7	8	10}	
							$(1\,2)$
f_3	{6	5	3	7	8	10}	
							$(2\,4)$
f_4	{6	7	3	5	8	10}	
							$(1\,2)$
f_5	{7	6	3	5	8	10}	
							$(4\,5)$
f_6	{7	6	3	8	5	10}	
							$(2\,4)$
f_7	{7	8	3	6	5	10}	
							$(4\,6)$
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

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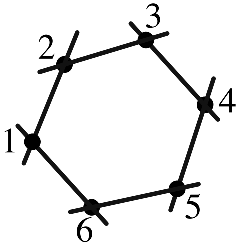

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ \alpha_0\alpha_8 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

$$(6\,1): \quad c_1 \mapsto c_1 + \alpha_0\, c_6$$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
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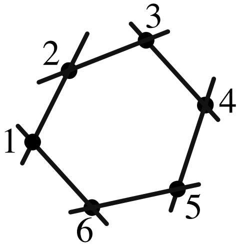

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'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
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f_3	{6	5	3	7	8	10}	$(2\,4)$
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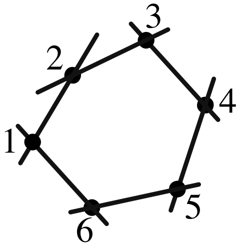

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'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
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f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
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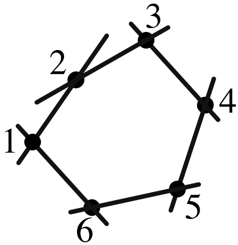

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'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
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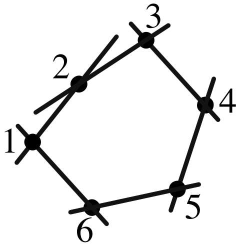

$$
\begin{array}{c} \\ C \equiv \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{pmatrix}
1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\
0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\
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\end{pmatrix}
\end{array}
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	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
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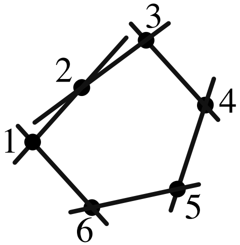

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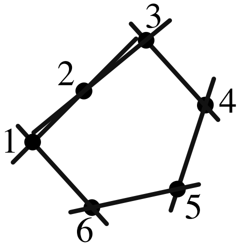

$$C\equiv\begin{array}{c}\begin{matrix}1 & 2 & 3 & 4 & 5 & 6\end{matrix}\\ \begin{pmatrix}1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0\\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0\\ \alpha_0\alpha_8 & 0 & 0 & 1 & \alpha_6 & \alpha_8\end{pmatrix}\end{array}$$

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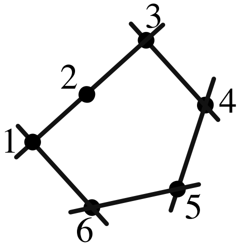

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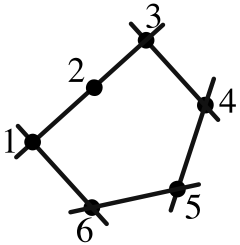

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'Bridge' Decomposition

	$1\downarrow$	$2\downarrow$	$3\downarrow$	$4\downarrow$	$5\downarrow$	$6\downarrow$	τ
f_0	{3	5	6	7	8	10}	(1 2)
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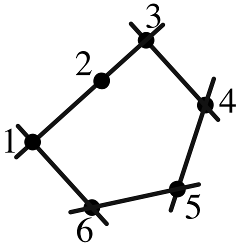

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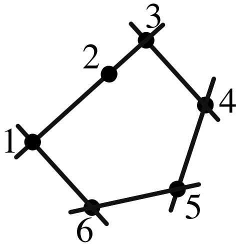

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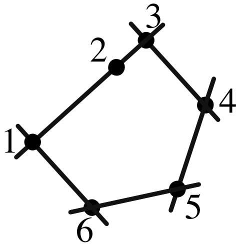

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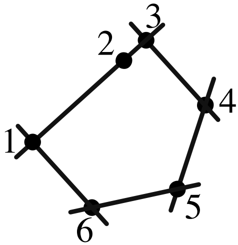

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	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
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f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
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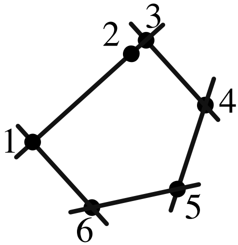

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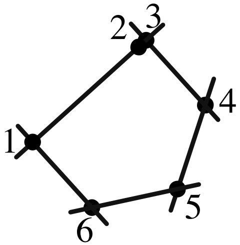
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	$(4\,6)$
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:



$$C \equiv \begin{array}{c} \\ \end{array}\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

$$C \equiv \left(\begin{array}{cccccc} 1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)$$

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
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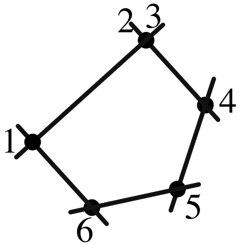

$$C \equiv \begin{array}{c} \\ \end{array}\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline \end{array}$$

$$C \equiv \left(\begin{array}{cccccc} 1 & (\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)$$

with columns labelled $1, 2, 3, 4, 5, 6$.

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
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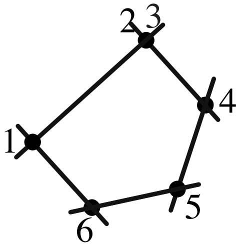

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'Bridge' Decomposition

	$1\downarrow$	$2\downarrow$	$3\downarrow$	$4\downarrow$	$5\downarrow$	$6\downarrow$	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
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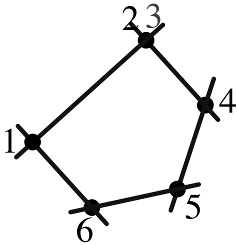

$$C \equiv \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$
$$C \equiv \left(\begin{array}{cccccc} 1 & (\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)$$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
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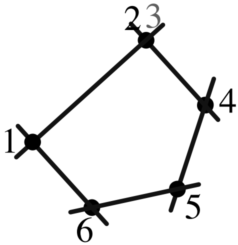

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'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
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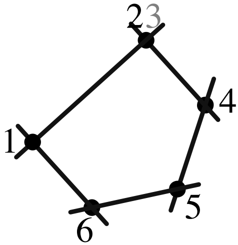

$$C \equiv \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

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‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
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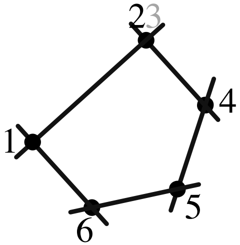

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & (\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
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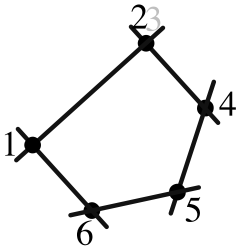

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'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
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f_4	{6	7	3	5	8	10}	(1 2)
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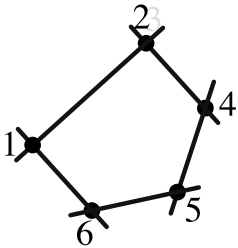

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'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
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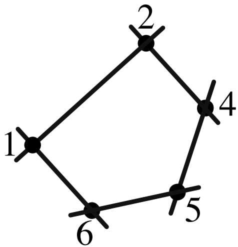

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'Bridge' Decomposition

	1 $\downarrow$	2 $\downarrow$	3 $\downarrow$	4 $\downarrow$	5 $\downarrow$	6 $\downarrow$	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
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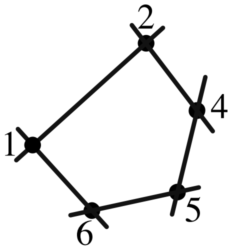

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'Bridge' Decomposition

	$1\downarrow$	$2\downarrow$	$3\downarrow$	$4\downarrow$	$5\downarrow$	$6\downarrow$	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
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f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
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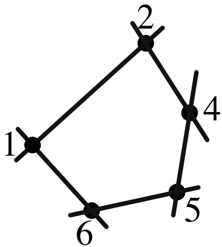

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'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
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f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
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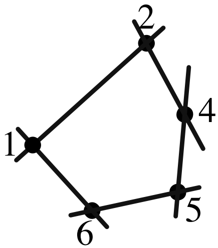

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'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	
							$(1\,2)$
f_1	{5	3	6	7	8	10}	
							$(2\,3)$
f_2	{5	6	3	7	8	10}	
							$(1\,2)$
f_3	{6	5	3	7	8	10}	
							$(2\,4)$
f_4	{6	7	3	5	8	10}	
							$(1\,2)$
f_5	{7	6	3	5	8	10}	
							$(4\,5)$
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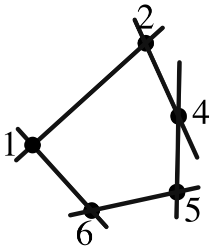

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'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
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							$(1\,2)$
f_1	{5	3	6	7	8	10}	
							$(2\,3)$
f_2	{5	6	3	7	8	10}	
							$(1\,2)$
f_3	{6	5	3	7	8	10}	
							$(2\,4)$
f_4	{6	7	3	5	8	10}	
							$(1\,2)$
f_5	{7	6	3	5	8	10}	
							$(4\,5)$
f_6	{7	6	3	8	5	10}	
							$(2\,4)$
f_7	{7	8	3	6	5	10}	
							$(4\,6)$
f_8	{7	8	3	10	5	6 }	

Canonical Coordinates for Computing On-Shell Functions

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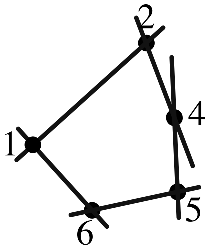

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & (\alpha_3+\alpha_5) & 0 & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & 0 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
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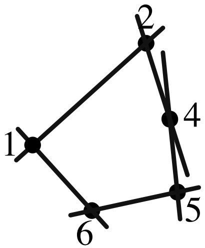

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'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
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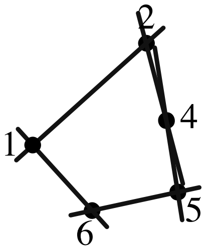

$$C \equiv \begin{array}{c}\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6\end{array}\\ \left(\begin{array}{cccccc} 1 & (\alpha_3+\alpha_5) & 0 & \alpha_4\alpha_5 & 0 & 0\\ 0 & 1 & 0 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0\\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8\end{array}\right)\end{array}$$

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
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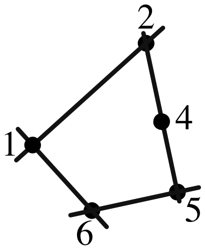

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'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
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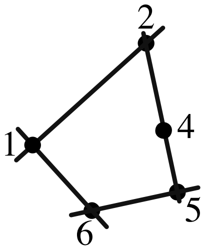

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‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
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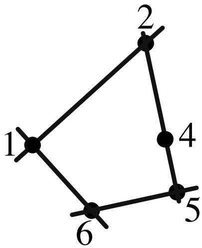

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'Bridge' Decomposition

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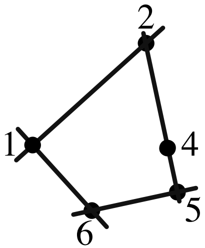

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & \alpha_5 & 0 & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & 0 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

'Bridge' Decomposition

	$1\downarrow$	$2\downarrow$	$3\downarrow$	$4\downarrow$	$5\downarrow$	$6\downarrow$	τ
f_0	{3	5	6	7	8	10}	(1 2)
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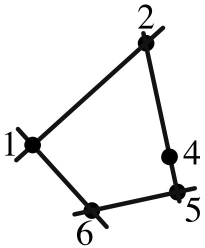

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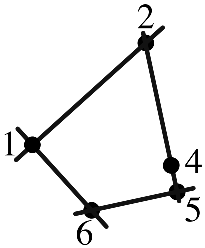

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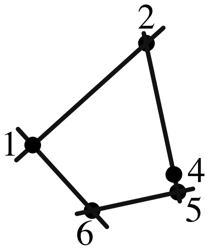

$$C\equiv\begin{array}{c}\begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ \left(\begin{array}{cccccc}1&\alpha_5&0&\alpha_4\alpha_5&0&0\\0&1&0&(\alpha_4+\alpha_7)&\alpha_6\alpha_7&0\\0&0&0&1&\alpha_6&\alpha_8\end{array}\right)\end{array}$$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
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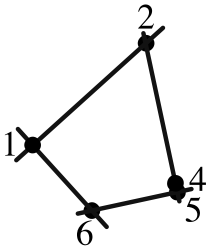

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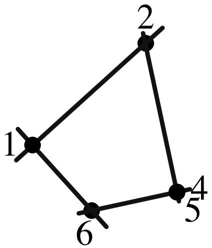

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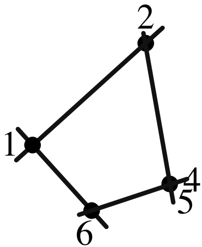

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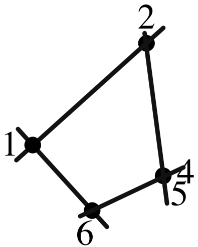

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'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
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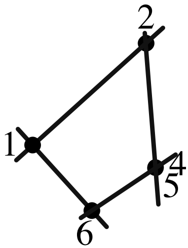

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$$C \equiv \left(\begin{array}{cccccc} 1 & \alpha_5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)$$

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	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
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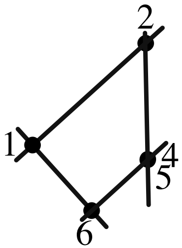
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	$(4\,6)$
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:



$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & \alpha_5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
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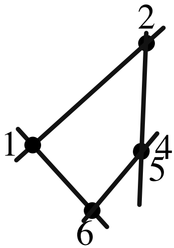

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'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
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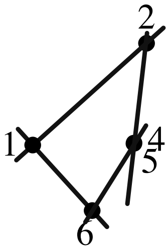

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'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
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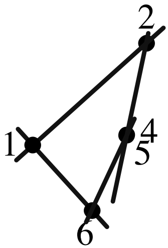

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'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
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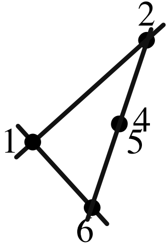

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'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
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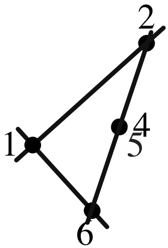

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	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
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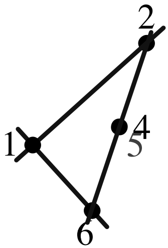

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'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	$(1\,2)$
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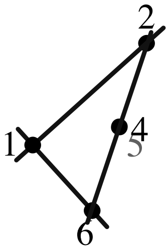

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‘Bridge’ Decomposition

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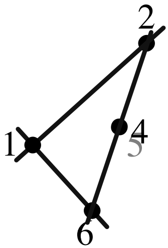

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'Bridge' Decomposition

	$1\downarrow$	$2\downarrow$	$3\downarrow$	$4\downarrow$	$5\downarrow$	$6\downarrow$	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
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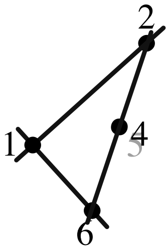

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'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
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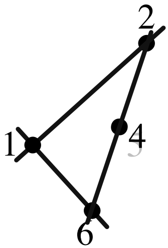

$$C \equiv \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \end{array}$$

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'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
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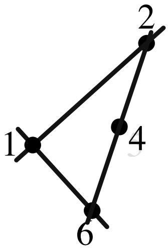

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'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	
							$(1\,2)$
f_1	{5	3	6	7	8	10}	
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							$(2\,4)$
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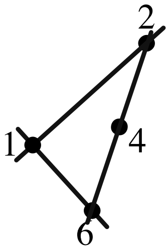

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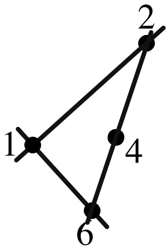

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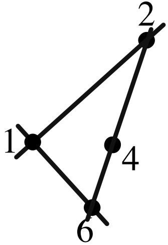

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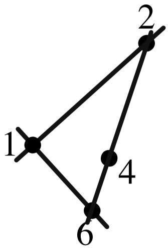

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_8 \end{array}\right) \end{array}$$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
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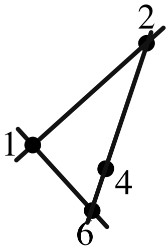

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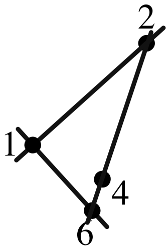

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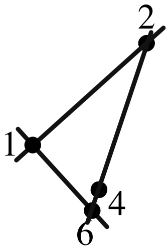

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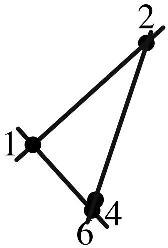

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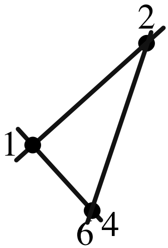

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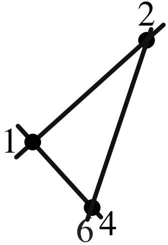

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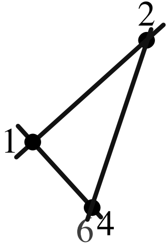

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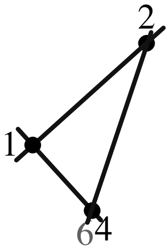

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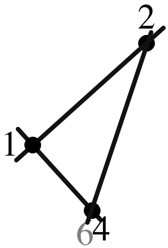

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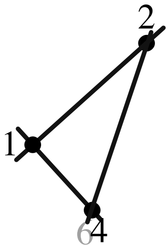

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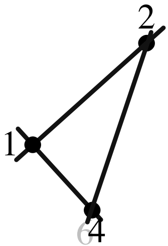

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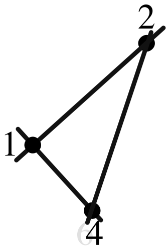

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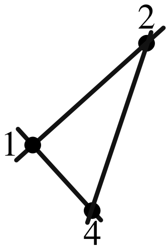

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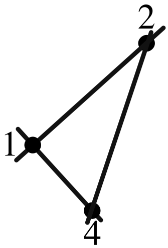

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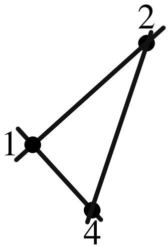

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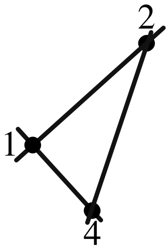
							$(2\,3)$
f_2	{5	6	3	7	8	10}	
							$(1\,2)$
f_3	{6	5	3	7	8	10}	
							$(2\,4)$
f_4	{6	7	3	5	8	10}	
							$(1\,2)$
f_5	{7	6	3	5	8	10}	
							$(4\,5)$
f_6	{7	6	3	8	5	10}	
							$(2\,4)$
f_7	{7	8	3	6	5	10}	
							$(4\,6)$
f_8	{7	8	3	10	5	6 }	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\, b)$ such that $\sigma(a) < \sigma(b)$:



$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right) \end{array}$$

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6}	

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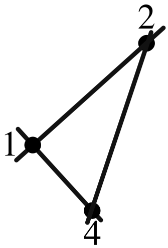

$$C\equiv\begin{array}{c}\begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ \left(\begin{array}{cccccc}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&1&0&0\end{array}\right)\end{array}$$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
f_0	3	5	6	7	8	10	$(1\,2)$
f_1	5	3	6	7	8	10	$(2\,3)$
f_2	5	6	3	7	8	10	$(1\,2)$
f_3	6	5	3	7	8	10	$(2\,4)$
f_4	6	7	3	5	8	10	$(1\,2)$
f_5	7	6	3	5	8	10	$(4\,5)$
f_6	7	6	3	8	5	10	$(2\,4)$
f_7	7	8	3	6	5	10	$(4\,6)$
f_8	7	8	3	10	5	6	

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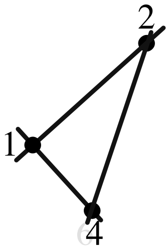

$$C \equiv \begin{array}{c}\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}\\ \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_8 \end{array}\right)\end{array}$$

$$(4\,6): \quad c_6 \mapsto c_6 + \alpha_8\, c_4$$

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	$(4\,6)$
f_8	{7	8	3	10	5	6}	

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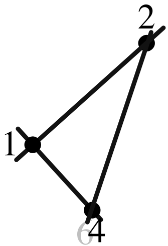

$$C \equiv \begin{array}{c} \\ \\ \end{array}\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

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'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
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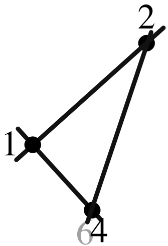

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	1	2	3	4	5	6	τ
f_0	3	5	6	7	8	10	$(1\,2)$
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f_3	6	5	3	7	8	10	$(2\,4)$
f_4	6	7	3	5	8	10	$(1\,2)$
f_5	7	6	3	5	8	10	$(4\,5)$
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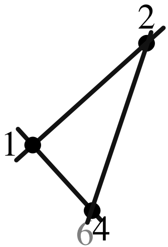

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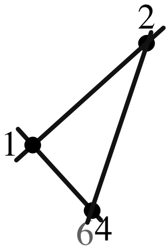

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	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	$(1\,2)$
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f_3	{6	5	3	7	8	10}	$(2\,4)$
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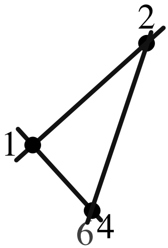

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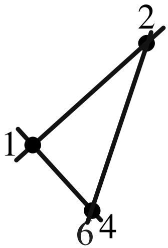

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'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	$(1\,2)$
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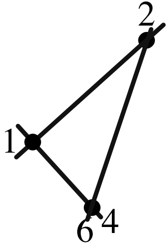

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'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
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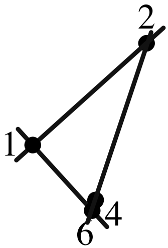

$$\begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ C\equiv & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_8 \end{pmatrix} \end{array}$$

$$(2\,4):\quad c_4 \mapsto c_4+\alpha_7\,c_2$$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	
							$(1\,2)$
f_1	{5	3	6	7	8	10}	
							$(2\,3)$
f_2	{5	6	3	7	8	10}	
							$(1\,2)$
f_3	{6	5	3	7	8	10}	
							$(2\,4)$
f_4	{6	7	3	5	8	10}	
							$(1\,2)$
f_5	{7	6	3	5	8	10}	
							$(4\,5)$
f_6	{7	6	3	8	5	10}	
							$(2\,4)$
f_7	{7	8	3	6	5	10}	
							$(4\,6)$
f_8	{7	8	3	10	5	6}	

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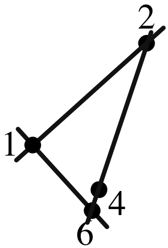

$$C \equiv \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$
$$C \equiv \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_8 \end{array}\right)$$

$$(2\,4): \quad c_4 \mapsto c_4 + \alpha_7\, c_2$$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
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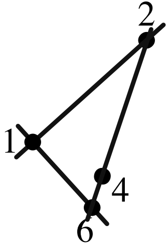

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_8 \end{array}\right) \end{array}$$

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'Bridge' Decomposition

	1	2	3	4	5	6	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
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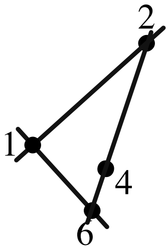

$$C \equiv \begin{array}{c} \\ \end{array}\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

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'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	$(1\,2)$
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f_3	{6	5	3	7	8	10}	$(2\,4)$
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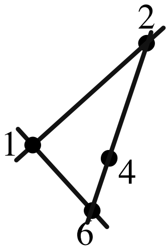

$$C \equiv \begin{array}{c} \\ \\ \end{array}\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

$$C \equiv \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_8 \end{array}\right)$$

$$(2\,4): \quad c_4 \mapsto c_4 + \alpha_7\, c_2$$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	$(4\,6)$
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

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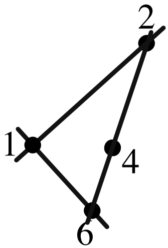

$$C \equiv \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \end{array}$$

$$C \equiv \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_8 \end{array}\right)$$

$$(2\,4): \quad c_4 \mapsto c_4 + \alpha_7\, c_2$$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	$(4\,6)$
f_8	{7	8	3	10	5	6}	

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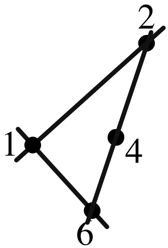

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_8 \end{array}\right) \end{array}$$

$$(2\,4): \quad c_4 \mapsto c_4 + \alpha_7\, c_2$$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	
							$(1\,2)$
f_1	{5	3	6	7	8	10}	
							$(2\,3)$
f_2	{5	6	3	7	8	10}	
							$(1\,2)$
f_3	{6	5	3	7	8	10}	
							$(2\,4)$
f_4	{6	7	3	5	8	10}	
							$(1\,2)$
f_5	{7	6	3	5	8	10}	
							$(4\,5)$
f_6	{7	6	3	8	5	10}	
							$(2\,4)$
f_7	{7	8	3	6	5	10}	
							$(4\,6)$
f_8	{7	8	3	10	5	6 }	

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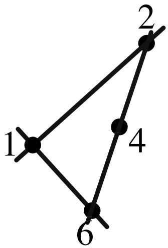

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_8 \end{array}\right) \end{array}$$

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'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
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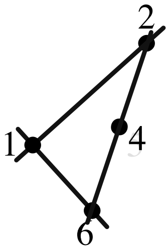

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

$$(4\,5):\quad c_5 \mapsto c_5 + \alpha_6\, c_4$$

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
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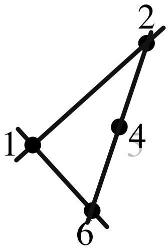

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'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
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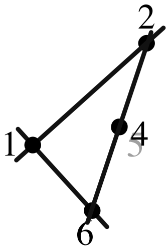

$$
\begin{array}{cc}
 & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
C \equiv & \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)
\end{array}
$$

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'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
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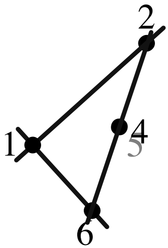

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'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
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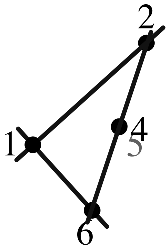

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'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	$(1\,2)$
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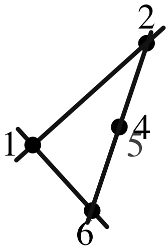

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$$C \equiv \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)$$

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‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
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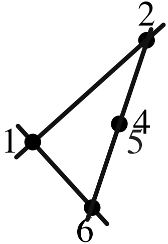

$$C \equiv \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline \end{array}$$
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'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
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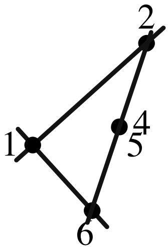

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

$(4\,5)$: $\quad c_5 \mapsto c_5 + \alpha_6\, c_4$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
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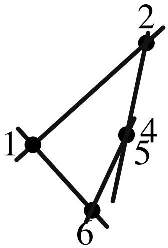

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & \alpha_5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

$$(1\,2):\quad c_2 \mapsto c_2 + \alpha_5\, c_1$$

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
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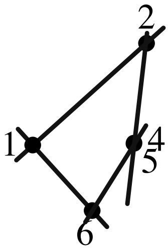

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & \alpha_5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

$$(1\,2): \quad c_2 \mapsto c_2 + \alpha_5\, c_1$$

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	$(4\,6)$
f_8	{7	8	3	10	5	6}	

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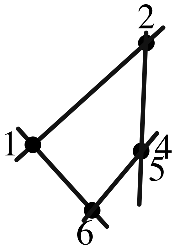

$$C \equiv \begin{array}{c} \\ \\ \end{array}\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$
$$C \equiv \left(\begin{array}{cccccc} 1 & \alpha_5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)$$

$(1\,2): \quad c_2 \mapsto c_2 + \alpha_5\, c_1$

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
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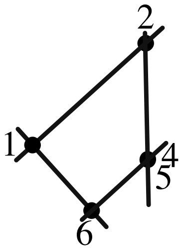

$$C \equiv \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{pmatrix} 1 & \alpha_5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{pmatrix} \end{array}$$

$$(1\,2):\quad c_2 \mapsto c_2 + \alpha_5\, c_1$$

‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
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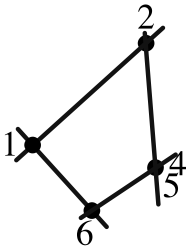

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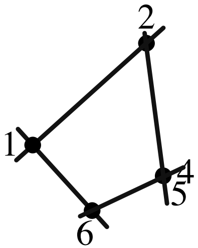

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‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
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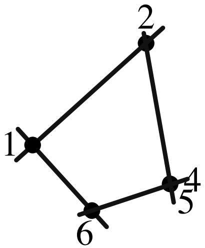

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	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
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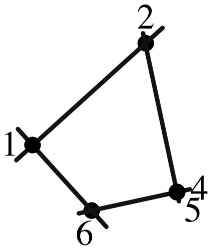

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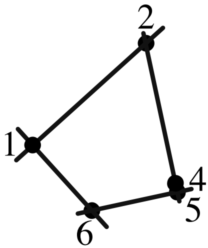

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$$(2\,4): \quad c_4 \mapsto c_4 + \alpha_4\, c_2$$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	
							$(1\,2)$
f_1	{5	3	6	7	8	10}	
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							$(1\,2)$
f_3	{6	5	3	7	8	10}	
							$(2\,4)$
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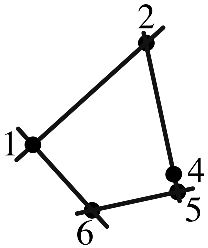

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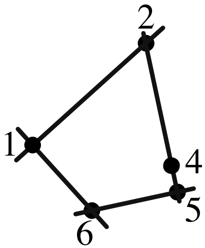

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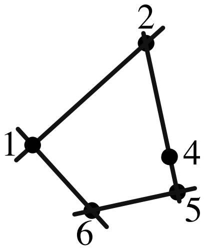

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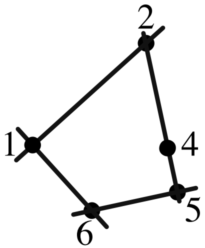

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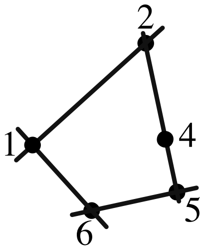

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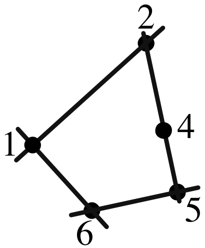

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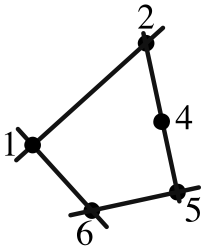

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	1	2	3	4	5	6	τ
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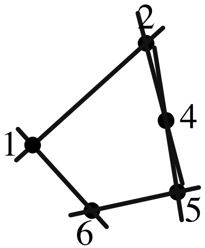

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$$C \equiv \left(\begin{array}{cccccc} 1 & (\alpha_3+\alpha_5) & 0 & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & 0 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)$$

$$(1\,2): \quad c_2 \mapsto c_2 + \alpha_3\, c_1$$

‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
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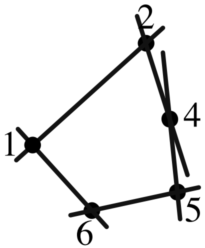

$$C \equiv \begin{array}{c} \begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ \left(\begin{array}{cccccc} 1 & (\alpha_3+\alpha_5) & 0 & \alpha_4\alpha_5 & 0 & 0\\ 0&1&0&(\alpha_4+\alpha_7)&\alpha_6\alpha_7&0\\ 0&0&0&1&\alpha_6&\alpha_8\end{array}\right)\end{array}$$

$$(1\,2):\quad c_2 \mapsto c_2 + \alpha_3\, c_1$$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
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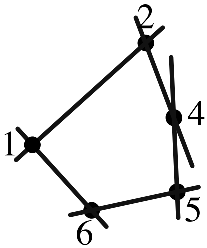

$$C \equiv \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \left(\begin{matrix} 1 & (\alpha_3+\alpha_5) & 0 & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & 0 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{matrix}\right) \end{array}$$

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'Bridge' Decomposition

	1	2	3	4	5	6	τ
f_0	3	5	6	7	8	10	$(1\,2)$
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f_2	5	6	3	7	8	10	$(1\,2)$
f_3	6	5	3	7	8	10	$(2\,4)$
f_4	6	7	3	5	8	10	$(1\,2)$
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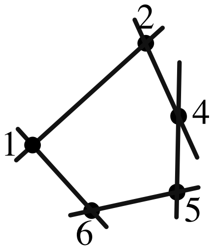

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	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
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f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
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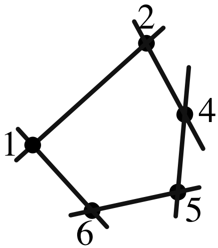

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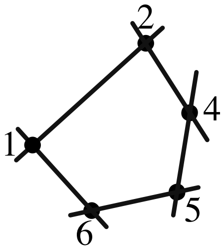

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f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
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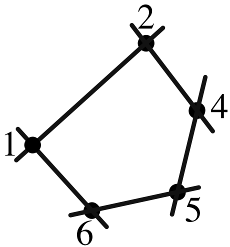

$$\begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ C \equiv & \begin{pmatrix} 1 & (\alpha_3+\alpha_5) & 0 & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & 0 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{pmatrix} \end{array}$$

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	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
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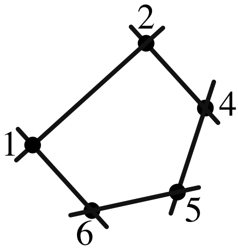

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‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
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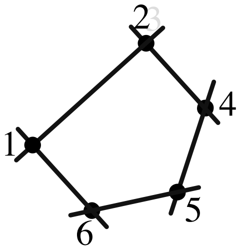

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$$(2\,3): \quad c_3 \mapsto c_3 + \alpha_2\, c_2$$

‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
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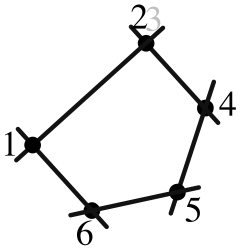

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f_4	6	7	3	5	8	10	$(1\,2)$
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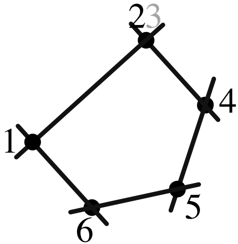

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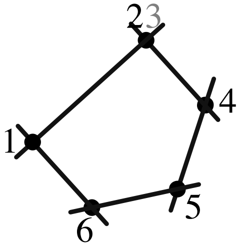

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‘Bridge’ Decomposition

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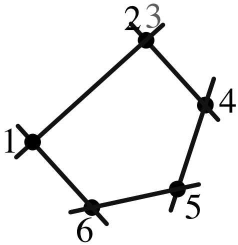

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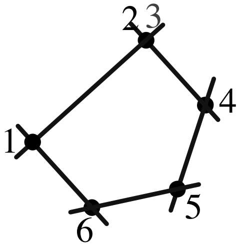

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'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
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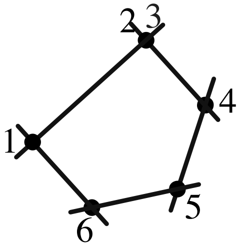

$$C \equiv \begin{array}{c} \\ \end{array}\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

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'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
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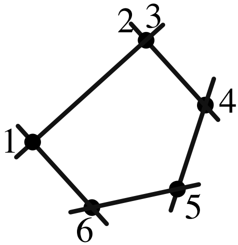
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	$(4\,6)$
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:



$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & (\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

$$(2\,3):\quad c_3 \mapsto c_3 + \alpha_2\, c_2$$

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
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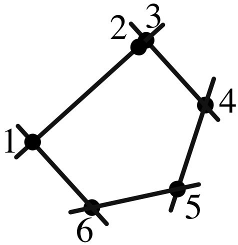

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

$$(1\,2): \quad c_2 \mapsto c_2 + \alpha_1\, c_1$$

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
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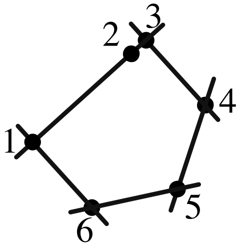

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‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
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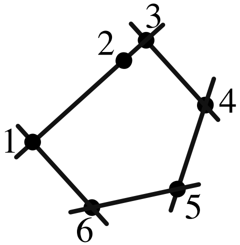

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'Bridge' Decomposition

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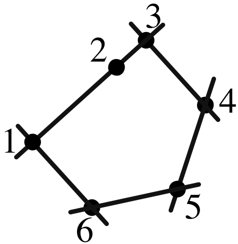

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f_0	{3	5	6	7	8	10}	$(1\,2)$
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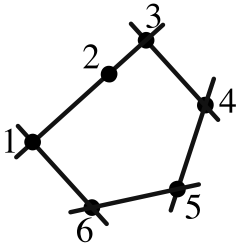

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

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‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
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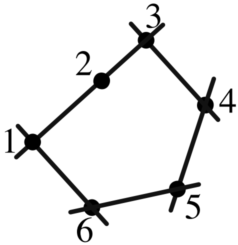

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'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
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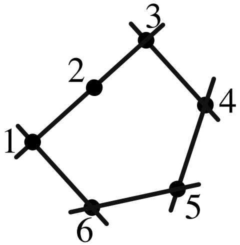

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'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	
							$(1\,2)$
f_1	{5	3	6	7	8	10}	
							$(2\,3)$
f_2	{5	6	3	7	8	10}	
							$(1\,2)$
f_3	{6	5	3	7	8	10}	
							$(2\,4)$
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							$(1\,2)$
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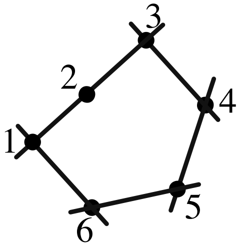

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$$(1\,2): \quad c_2 \mapsto c_2 + \alpha_1\, c_1$$

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
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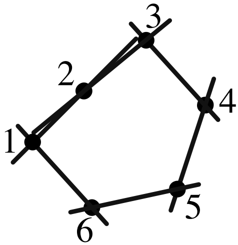

$$C \equiv \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

$$C \equiv \begin{pmatrix} 1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ \alpha_0\alpha_8 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{pmatrix}$$

$$(6\,1): \quad c_1 \mapsto c_1 + \alpha_0\, c_6$$

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
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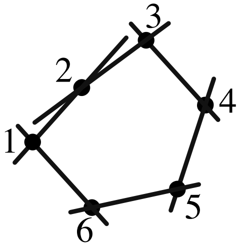

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‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
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f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
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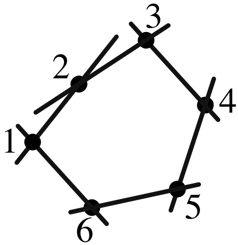

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'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	
							$(1\,2)$
f_1	{5	3	6	7	8	10}	
							$(2\,3)$
f_2	{5	6	3	7	8	10}	
							$(1\,2)$
f_3	{6	5	3	7	8	10}	
							$(2\,4)$
f_4	{6	7	3	5	8	10}	
							$(1\,2)$
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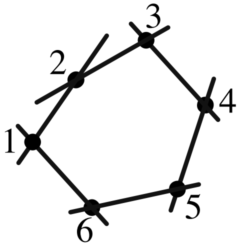

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'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
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f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
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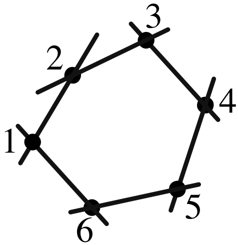

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$$C \equiv \left(\begin{array}{cccccc} 1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ \alpha_0\alpha_8 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)$$

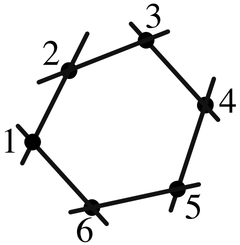
$$(6\,1): \quad c_1 \mapsto c_1 + \alpha_0\, c_6$$

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	$(4\,6)$
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\, b)$ such that $\sigma(a) < \sigma(b)$:



$$C \equiv \begin{array}{c} \\ \\ \end{array}\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

$$C \equiv \left(\begin{array}{cccccc} 1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ \alpha_0\alpha_8 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)$$

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'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	
							(1 2)
f_1	{5	3	6	7	8	10}	
							(2 3)
f_2	{5	6	3	7	8	10}	
							(1 2)
f_3	{6	5	3	7	8	10}	
							(2 4)
f_4	{6	7	3	5	8	10}	
							(1 2)
f_5	{7	6	3	5	8	10}	
							(4 5)
f_6	{7	6	3	8	5	10}	
							(2 4)
f_7	{7	8	3	6	5	10}	
							(4 6)
f_8	{7	8	3	10	5	6}	

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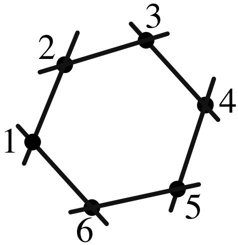

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ \alpha_0\alpha_8 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

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'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
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f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

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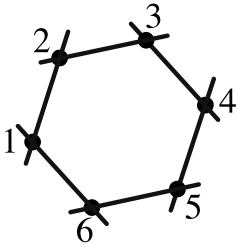

$$C \equiv \begin{array}{c} \\ \end{array}\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

$$C \equiv \left(\begin{array}{cccccc} 1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ \alpha_0\alpha_8 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)$$

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'Bridge' Decomposition

	1	2	3	4	5	6	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
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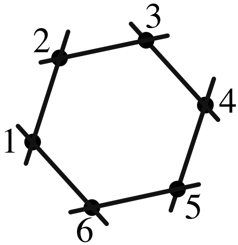

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'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
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f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
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f_8	{7	8	3	10	5	6}	

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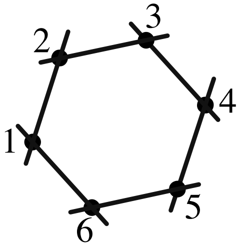

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ \alpha_0\alpha_8 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	
							$(1\,2)$
f_1	{5	3	6	7	8	10}	
							$(2\,3)$
f_2	{5	6	3	7	8	10}	
							$(1\,2)$
f_3	{6	5	3	7	8	10}	
							$(2\,4)$
f_4	{6	7	3	5	8	10}	
							$(1\,2)$
f_5	{7	6	3	5	8	10}	
							$(4\,5)$
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							$(2\,4)$
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							$(4\,6)$
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

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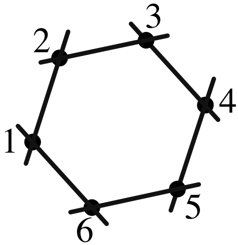


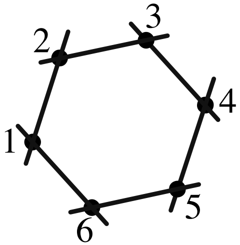
$$C \equiv \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

$$C \equiv \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ \alpha_0\alpha_8 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)$$

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	3	5	6	7	8	10	$(1\,2)$
f_1	5	3	6	7	8	10	$(2\,3)$
f_2	5	6	3	7	8	10	$(1\,2)$
f_3	6	5	3	7	8	10	$(2\,4)$
f_4	6	7	3	5	8	10	$(1\,2)$
f_5	7	6	3	5	8	10	$(4\,5)$
f_6	7	6	3	8	5	10	$(2\,4)$
f_7	7	8	3	6	5	10	$(4\,6)$
f_8	7	8	3	10	5	6	

Canonical Coordinates for Computing On-Shell Functions



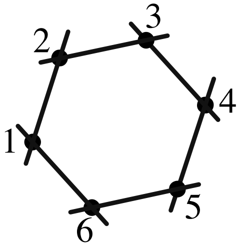
$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ \alpha_0\alpha_8 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

$$\mathcal{L}_{6,3} \equiv \frac{d\alpha_0}{\alpha_0} \cdots \frac{d\alpha_8}{\alpha_8}$$



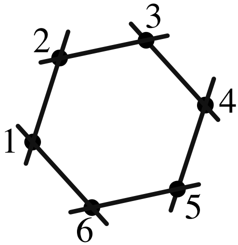
$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ \alpha_0\alpha_8 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

$$\mathcal{L}_{6,3} \equiv \frac{d\alpha_0}{\alpha_0} \cdots \frac{d\alpha_8}{\alpha_8}$$



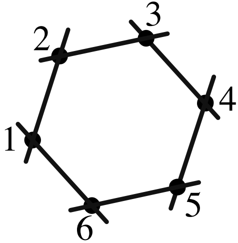
$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ \alpha_0\alpha_8 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	
							(1 2)
f_1	{5	3	6	7	8	10}	
							(2 3)
f_2	{5	6	3	7	8	10}	
							(1 2)
f_3	{6	5	3	7	8	10}	
							(2 4)
f_4	{6	7	3	5	8	10}	
							(1 2)
f_5	{7	6	3	5	8	10}	
							(4 5)
f_6	{7	6	3	8	5	10}	
							(2 4)
f_7	{7	8	3	6	5	10}	
							(4 6)
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

$$\mathcal{L}_{6,3} \equiv \frac{d\alpha_0}{\alpha_0} \cdots \frac{d\alpha_8}{\alpha_8} = \frac{d^{3\times 6}C}{\mathrm{vol}(GL(3))} \frac{1}{(123)(234)(345)(456)(561)(612)}$$



$$C \equiv \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \left(\begin{matrix} 1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ \alpha_0\alpha_8 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{matrix}\right) \end{array}$$

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
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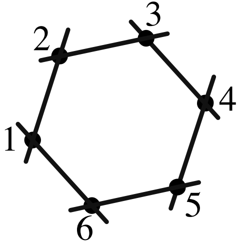
Canonical Coordinates for Computing On-Shell Functions

$$\mathcal{L}_{n,k} \equiv \frac{d\alpha_1}{\alpha_1} \cdots \frac{d\alpha_{k(n-k)}}{\alpha_{k(n-k)}} = \frac{d^{k\times n}C}{\mathrm{vol}(GL(k))} \frac{1}{(1\cdots k)\,(2\cdots k{+}1)\cdots(n\cdots k{-}1)}$$



$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ \alpha_0\alpha_8 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{pmatrix} \end{array}$$

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	
							(1 2)
f_1	{5	3	6	7	8	10}	
							(2 3)
f_2	{5	6	3	7	8	10}	
							(1 2)
f_3	{6	5	3	7	8	10}	
							(2 4)
f_4	{6	7	3	5	8	10}	
							(1 2)
f_5	{7	6	3	5	8	10}	
							(4 5)
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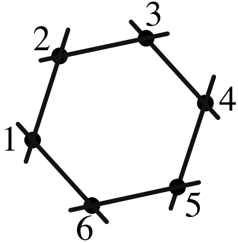
Canonical Coordinates for Computing On-Shell Functions

$$\mathcal{L}_{n,k} \equiv \frac{d\alpha_1}{\alpha_1} \cdots \frac{d\alpha_{k(n-k)}}{\alpha_{k(n-k)}} = \frac{d^{k\times n}C}{\mathrm{vol}(GL(k))} \frac{1}{(1\cdots k)\,(2\cdots k{+}1)\cdots(n\cdots k{-}1)}$$



$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ \alpha_0\alpha_8 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
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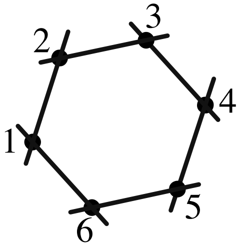
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

$$\mathcal{L}_{n,k} \equiv \frac{d\alpha_1}{\alpha_1} \cdots \frac{d\alpha_{k(n-k)}}{\alpha_{k(n-k)}} = \frac{d^{k\times n}C}{\mathrm{vol}(GL(k))} \frac{1}{(1\cdots k)\,(2\cdots k{+}1)\cdots(n\cdots k{-}1)}$$



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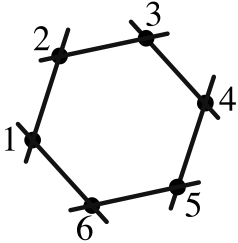
'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

$$\mathcal{L}_{n,k} \equiv \frac{d\alpha_1}{\alpha_1}\cdots\frac{d\alpha_{k(n-k)}}{\alpha_{k(n-k)}} = \frac{d^{k\times n}C}{\mathrm{vol}(GL(k))}\frac{1}{(1\cdots k)\,(2\cdots k{+}1)\cdots(n\cdots k{-}1)}$$



$$C \equiv \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

$$C \equiv \left(\begin{array}{cccccc} 1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ \alpha_0\alpha_8 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)$$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
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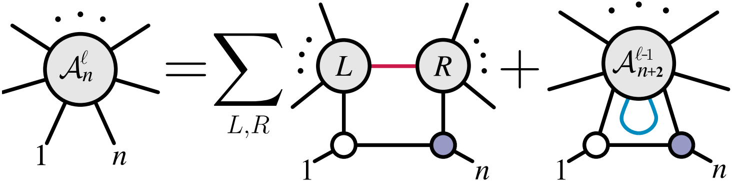

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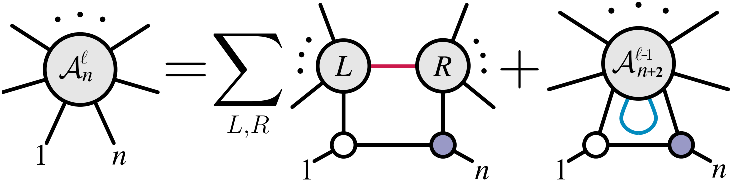

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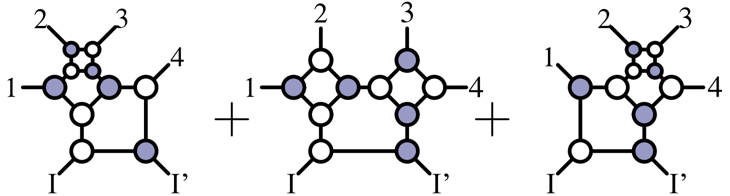

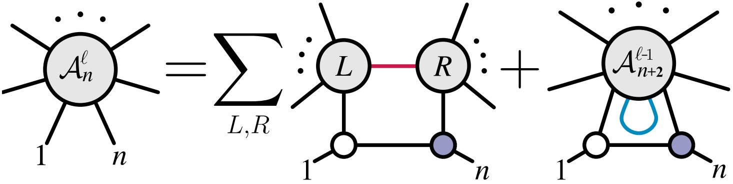

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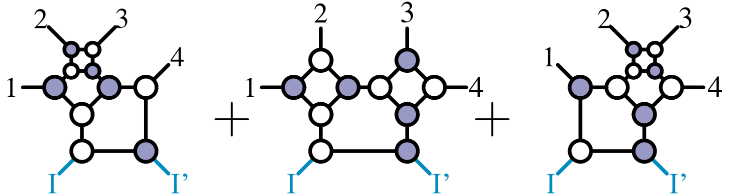

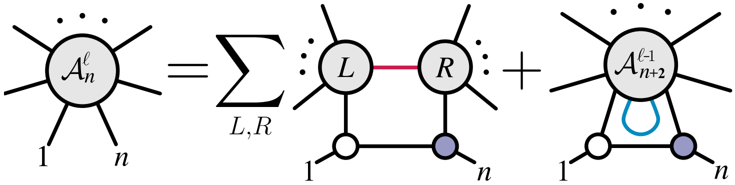

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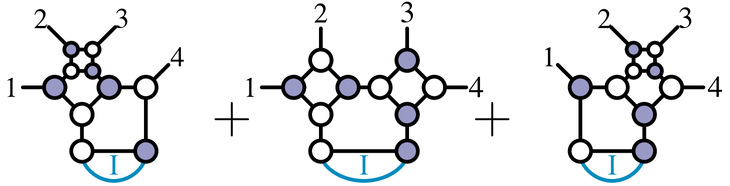

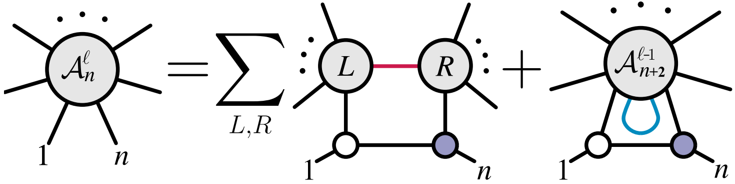

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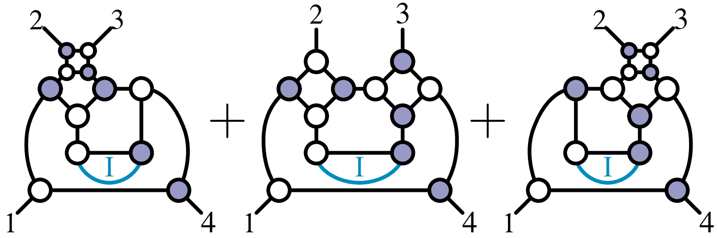

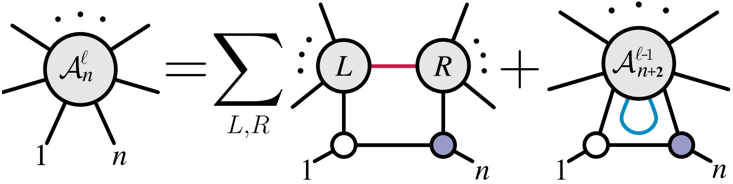

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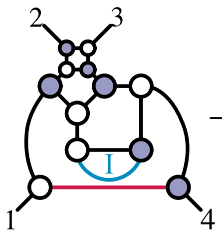

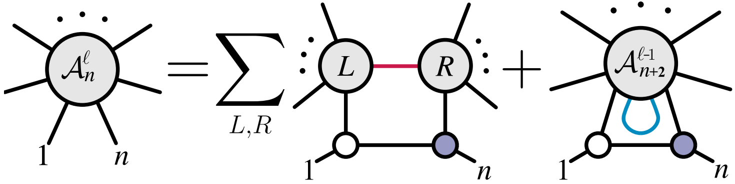

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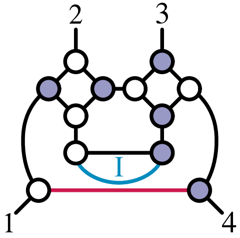

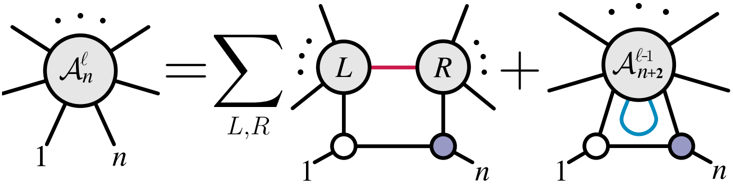

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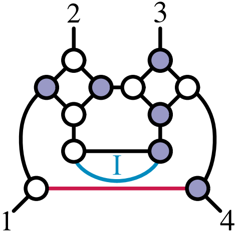

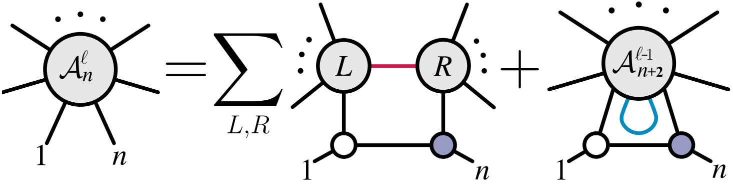

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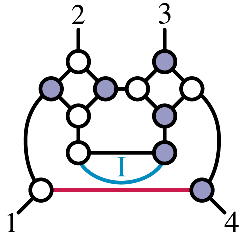
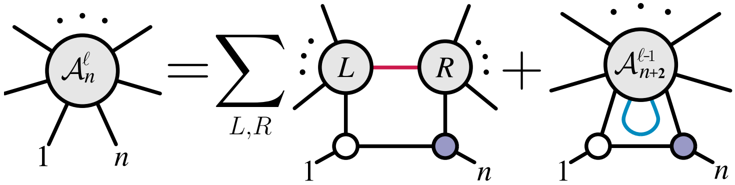


$$\int_{\ell\in\mathbb{R}^{3,1}} d^4\ell$$



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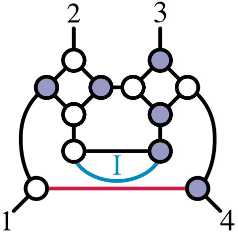

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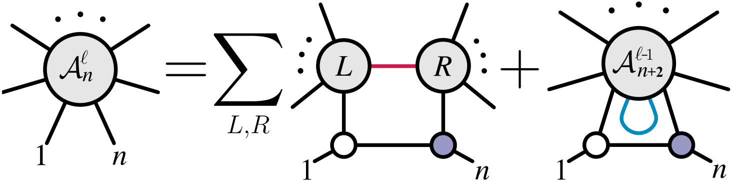

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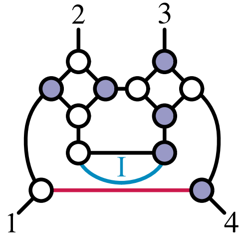

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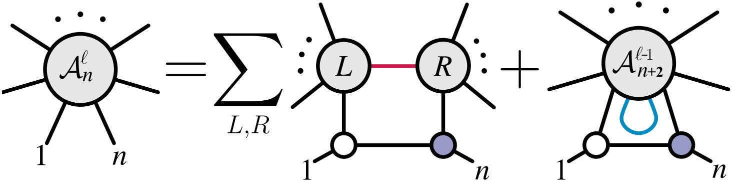

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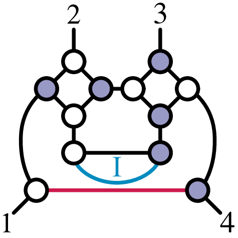

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$$\mathcal{A}_4^{(2),0}\times\int_{\ell\in\mathbb{R}^{3,1}} d\log\left(\frac{\ell^2}{(\ell-\ell^*)^2}\right) d\log\left(\frac{(\ell+p_1)^2}{(\ell-\ell^*)^2}\right) d\log\left(\frac{(\ell+p_1+p_2)^2}{(\ell-\ell^*)^2}\right) d\log\left(\frac{(\ell-p_4)^2}{(\ell-\ell^*)^2}\right)$$

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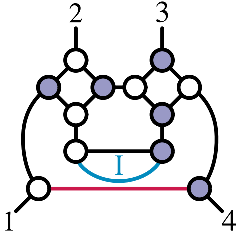

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$$=\mathcal{A}_4^{(2),0}\times\int\limits_{\ell\in\mathbb{R}^{3,1}} d^4\ell\,\frac{(p_1+p_2)^2(p_3+p_4)^2}{\ell^2(\ell+p_1)^2(\ell+p_1+p_2)^2(\ell-p_4)^2}$$